CHAPTER 7

Infinite Sets

In the previous chapter, we showed how to construct a bunch of things—integers, rationals, and reals—assuming some naïve set theory and the natural numbers. The question for this chapter is: Can we construct the set of natural numbers *itself* using set theory?

7.1 Hilbert's Hotel

The set of the natural numbers is obviously infinite. So, if we do not want to *help ourselves* to the natural numbers, our first step must be characterize an infinite set in terms that do not require mentioning the natural numbers themselves. Here is a nice approach, presented by Hilbert in a lecture from 1924. He asks us to imagine

> [...] a hotel with a finite number of rooms. All of these rooms should be occupied by exactly one guest. If the guests now swap their rooms somehow, [but] so that each room still contains no more than one person, then no rooms will become free, and the hotel-owner cannot in this way create a new place for a newly arriving guest [...¶...]
>
> Now we stipulate that the hotel shall have infinitely many numbered rooms 1, 2, 3, 4, 5, ..., each of which

> is occupied by exactly one guest. As soon as a new guest comes along, the owner only needs to move each of the old guests into the room associated with the number one higher, and room 1 will be free for the newly-arriving guest.
>
> $$\begin{tikzcd}[column sep=small, row sep=small]
1 \arrow[dr] & 2 \arrow[dr] & 3 \arrow[dr] & 4 \arrow[dr] & 5 \arrow[dr] & 6 \arrow[dr] & 7 \arrow[dr] & 8 \arrow[dr] & 9 \arrow[dr] & \cdots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots
\end{tikzcd}$$
>
> (published in Hilbert 2013, 730; our translation)

The crucial point is that Hilbert's Hotel has infinitely many rooms; and we can take his explanation to define what it means to say this. Indeed, this was Dedekind's approach (presented here, of course, with massive anachronism; Dedekind's definition is from 1888):

Definition 7.1. A set A is *Dedekind infinite* iff there is an injection from A to a proper subset of A. That is, there is some $o \in A$ and an injection $f : A \to A$ such that $o \notin \text{ran}(f)$.

7.2 Dedekind Algebras

We not only want natural numbers to be infinite; we want them to have certain (algebraic) properties: they need to behave well under addition, multiplication, and so forth.

Dedekind's idea was to take the idea of the *successor function* as basic, and then characterise the numbers as those with the following properties:

1. There is a number, 0, which is not the successor of any number
   i.e., $0 \notin \text{ran}(s)$
   i.e., $\forall x\, s(x) \neq 0$

2. Distinct numbers have distinct successors
   i.e., s is an injection
   i.e., $\forall x \forall y(s(x) = s(y) \to x = y)$

3. Every number is obtained from 0 by repeated applications of the successor function.

The first two conditions are easy to deal with using first-order logic (see above). But we cannot deal with (3) just using first-order logic. Dedekind's breakthrough was to reformulate condition (3), set-theoretically, as follows:

3′. The natural numbers are the smallest set that is *closed under the successor function*: that is, if we apply s to any member of the set, we obtain another member of the set.

But we shall need to spell this out slowly.

Definition 7.2. For any function f, the set X is f-*closed* iff $(\forall x \in X) f(x) \in X$. Now define, for any o:

$$\text{clo}_f(o) = \bigcap\{X : o \in X \text{ and } X \text{ is } f\text{-closed}\}$$

So $\text{clo}_f(o)$ is the intersection of all the f-closed sets with o as a member. Intuitively, then, $\text{clo}_f(o)$ is the *smallest* f-closed set with o as a member. This next result makes that intuitive thought precise;

Lemma 7.3. *For any function f and any $o \in A$:*

1. *$o \in \text{clo}_f(o)$; and*

2. *$\text{clo}_f(o)$ is f-closed; and*

3. *if X is f-closed and $o \in X$, then $\text{clo}_f(o) \subseteq X$*

Proof. Note that there is at least one f-closed set with o as a member, namely $\text{ran}(f) \cup \{o\}$. So $\text{clo}_f(o)$, the intersection of *all* such sets, exists. We must now check (1)–(3).

Concerning (1): $o \in \mathrm{clo}_f(o)$ as it is an intersection of sets which all have o as a member.

Concerning (2): suppose $x \in \mathrm{clo}_f(o)$. So if $o \in X$ and X is f-closed, then $x \in X$, and now $f(x) \in X$ as X is f-closed. So $f(x) \in \mathrm{clo}_f(o)$.

Concerning (3): quite generally, if $X \in C$ then $\bigcap C \subseteq X$. □

Using this, we can say:

Definition 7.4. A *Dedekind algebra* is a set A together with a function $f\colon A \to A$ and some $o \in A$ such that:

1. $o \notin \mathrm{ran}(f)$
2. f is an injection
3. $A = \mathrm{clo}_f(o)$

Since $A = \mathrm{clo}_f(o)$, our earlier result tells us that A is the smallest f-closed set with o as a member. Clearly a Dedekind algebra is Dedekind infinite; just look at clauses (1) and (2) of the definition. But the more exciting fact is that any Dedekind infinite set can be turned into a Dedekind algebra.

Theorem 7.5. *If there is a Dedekind infinite set, then there is a Dedekind algebra.*

Proof. Let D be Dedekind infinite. So there is an injection $g\colon D \to D$ and an element $o \in D \setminus \mathrm{ran}(g)$. Now let $A = \mathrm{clo}_g(o)$; by Lemma 7.3, A exists and $o \in A$. Let $f = g{\restriction}_A$. We will show that A, f, o comprise a Dedekind algebra.

Concerning (1): $o \notin \mathrm{ran}(g)$ and $\mathrm{ran}(f) \subseteq \mathrm{ran}(g)$ so $o \notin \mathrm{ran}(f)$.

Concerning (2): g is an injection on D; so $f \subseteq g$ must be an injection.

Concerning (3): by Lemma 7.3, A is g-closed; a fortiori, A is f-closed. So $\mathrm{clo}_f(o) \subseteq A$ by Lemma 7.3. Since also $\mathrm{clo}_f(o)$

is f-closed and $f = g\restriction_A$, it follows that $\text{clo}_f(o)$ is g-closed. So $A \subseteq \text{clo}_f(o)$ by Lemma 7.3. □

7.3 Dedekind Algebras and Arithmetical Induction

Crucially, now, a Dedekind algebra—indeed, *any* Dedekind algebra—will serve as a surrogate for the natural numbers. This is thanks to the following trivial consequence:

Theorem 7.6 (Arithmetical induction). *Let N, s, o comprise a Dedekind algebra. Then for any set X:*

$$\textit{if } o \in X \textit{ and } (\forall n \in N \cap X)s(n) \in X, \textit{ then } N \subseteq X.$$

Proof. By the definition of a Dedekind algebra, $N = \text{clo}_s(o)$. Now if both $o \in X$ and $(\forall n \in N)(n \in X \rightarrow s(n) \in X)$, then $N = \text{clo}_s(o) \subseteq X$. □

Since induction is characteristic of the natural numbers, the point is this. Given any Dedekind infinite set, we can form a Dedekind algebra, and use that algebra as our surrogate for the natural numbers.

Admittedly, Theorem 7.6 formulates induction in *set-theoretic* terms. But we can easily put the principle in terms which might be more familiar:

Corollary 7.7. *Let N, s, o comprise a Dedekind algebra. Then for any formula $\varphi(x)$, which may have parameters:*

$$\textit{if } \varphi(o) \textit{ and } (\forall n \in N)(\varphi(n) \rightarrow \varphi(s(n))), \textit{ then } (\forall n \in N)\varphi(n)$$

Proof. Let $X = \{n \in N : \varphi(n)\}$, and now use Theorem 7.6 □

In this result, we spoke of a formula "having parameters". What this means, roughly, is that for any objects $c_1, \ldots, c_k$, we can work with $\varphi(x, c_1, \ldots, c_k)$. More precisely, we can state the result without mentioning "parameters" as follows. For any formula $\varphi(x, v_1, \ldots, v_k)$, whose free variables are all displayed, we have:

$$\forall v_1 \ldots \forall v_k ((\varphi(o, v_1, \ldots, v_k) \land (\forall x \in N)(\varphi(x, v_1, \ldots, v_k) \to \varphi(s(x), v_1, \ldots, v_k))) \to (\forall x \in N)\varphi(x, v_1, \ldots, v_k))$$

Evidently, speaking of "having parameters" can make things much easier to read. (In part III, we will use this device rather frequently.)

Returning to Dedekind algebras: given any Dedekind algebra, we can also define the usual arithmetical functions of addition, multiplication and exponentiation. This is non-trivial, however, and it involves the technique of *recursive definition.* That is a technique which we shall introduce and justify much later, and in a much more general context. (Enthusiasts might want to revisit this after chapter 13, or perhaps read an alternative treatment, such as Potter 2004, pp. 95–8.) But, where N, s, o comprise a Dedekind algebra, we will ultimately be able to stipulate the following:

$$\begin{array}{ccc} a + o = a & a \times o = o & a^o = s(o) \\ a + s(b) = s(a + b) & a \times s(b) = (a \times b) + a & a^{s(b)} = a^b \times a \end{array}$$

and show that these behave as one would hope.

7.4 Dedekind's "Proof" of the Existence of an Infinite Set

In this chapter, we have offered a set-theoretic treatment of the natural numbers, in terms of Dedekind algebras. In section 6.5, we reflected on the philosophical significance of the arithmetisation of analysis (among other things). Now we should reflect on the significance of what we have achieved here.

Throughout chapter 6, we took the natural numbers as given, and used them to construct the integers, rationals, and reals, explicitly. In this chapter, we have not given an explicit construction of the natural numbers. We have just shown that, *given any Dedekind infinite set*, we can define a set which will behave just like we want $\mathbb{N}$ to behave.

Obviously, then, we cannot claim to have answered a metaphysical question, such as *which objects are the natural numbers*. But that's a good thing. After all, in section 6.5, we emphasized that we would be wrong to think of the definition of $\mathbb{R}$ as the set of Dedekind cuts as a *discovery*, rather than a convenient stipulation. The crucial observation is that the Dedekind cuts exemplify the key mathematical properties of the real numbers. So too here: the crucial observation is that *any* Dedekind algebra exemplifies the key mathematical properties of the natural numbers. (Indeed, Dedekind pushed this point home by proving that all Dedekind algebras are *isomorphic* (1888, Theorems 132–3). It is no surprise, then, that many contemporary "structuralists" cite Dedekind as a forerunner.)

Moreover, we have shown how to embed the theory of the natural numbers into a naïve simple set theory, which itself still remains rather informal, but which doesn't (apparently) assume the natural numbers as given. So, we may be on the way to realising Dedekind's own ambitious project, which he explained thus:

> In science nothing capable of proof ought to be believed without proof. Though this demand seems reasonable, I cannot regard it as having been met even in the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers. In speaking of arithmetic (algebra, analysis) as merely a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time—that I rather consider it an immediate product of the pure laws of thought. (Dedekind, 1888, preface)

Dedekind's bold idea is this. We have just shown how to build the natural numbers using (naïve) set theory alone. In chapter 6, we saw how to construct the reals given the natural numbers and some set theory. So, perhaps, "arithmetic (algebra, analysis)" turn out to be "merely a part of logic" (in Dedekind's extended sense of the word "logic").

That's the idea. But hold on for a moment. Our construction of a Dedekind algebra (our surrogate for the natural numbers) is conditional on the existence of a Dedekind infinite set. (Just look back to Theorem 7.5.) Unless the existence of a Dedekind infinite set can be established via "logic" or "the pure laws of thought", the project stalls.

So, *can* the existence of a Dedekind infinite set be established by "the pure laws of thought"? Here was Dedekind's effort:

> My own realm of thoughts, i.e., the totality S of all things which can be objects of my thought, is infinite. For if s signifies an element of S, then the thought s' that s can be an object of my thought, is itself an element of S. If we regard this as an image $\varphi(s)$ of the element s, then ... S is [Dedekind] infinite, which was to be proved. (Dedekind, 1888, §66)

This is quite an astonishing thing to find in the middle of a book which largely consists of highly rigorous mathematical proofs. Two remarks are worth making.

First: this "proof" scarcely has what we would now recognize as a "mathematical" character. It speaks of psychological objects (thoughts), and merely *possible* ones at that.

Second: at least as we have presented Dedekind algebras, this "proof" has a straightforward technical shortcoming. If Dedekind's argument is successful, it establishes only that there are infinitely many things (specifically, infinitely many thoughts). But Dedekind also needs to give us a reason to regard S as a single *set*, with infinitely many members, rather than thinking of S as *some things* (in the plural).

The fact that Dedekind did not see a gap here might suggest that his use of the word "totality" does not precisely track *our* use of the word "set".[1] But this would not be too surprising. The project we have pursued in the last two chapters—a "construction" of the naturals, and from them a "construction" of the integers, reals and rationals—has all been carried out naïvely. We have helped ourselves to this set, or that set, as and when we have needed them, without laying down many general principles concerning exactly which sets exist, and when. But we know that we need *some* general principles, for otherwise we will fall into Russell's Paradox.

The time has come for us to outgrow our naïvety.

7.5 Appendix: Proving Schröder-Bernstein

Before we depart from naïve set theory, we have one last naïve (but sophisticated!) proof to consider. This is a proof of Schröder-Bernstein (Theorem 5.17): if $A \preceq B$ and $B \preceq A$ then $A \approx B$; i.e., given injections $f\colon A \to B$ and $g\colon B \to A$ there is a bijection $h\colon A \to B$.

In this chapter, we followed Dedekind's notion of *closures*. In fact, Dedekind provided a lovely proof of Schröder-Bernstein using this notion, and we will present it here. The proof closely follows Potter (2004, pp. 157–8), if you want a slightly different but essentially similar treatment. A little googling will also convince you that this is a theorem—rather like the irrationality of $\sqrt{2}$—for which *many* interesting and different proofs exist.

Using similar notation as Definition 7.2, let

$$\mathrm{Clo}_f(B) = \bigcap\{X : B \subseteq X \text{ and } X \text{ is } f\text{-closed}\}$$

for each set B and function f. Defined thus, $\mathrm{Clo}_f(B)$ is the smallest f-closed set containing B, in that:

[1] Indeed, we have other reasons to think it did not; see Potter (2004, p. 23).

Lemma 7.8. *For any function f, and any B:*

1. $B \subseteq \mathrm{Clo}_f(B)$*; and*
2. $\mathrm{Clo}_f(B)$ *is f-closed; and*
3. *if X is f-closed and $B \subseteq X$, then* $\mathrm{Clo}_f(B) \subseteq X$.

Proof. Exactly as in Lemma 7.3. □

We need one last fact to get to Schröder-Bernstein:

Proposition 7.9. *If $A \subseteq B \subseteq C$ and $A \approx C$, then $A \approx B \approx C$.*

Proof. Given a bijection $f : C \to A$, let $F = \mathrm{Clo}_f(C \setminus B)$ and define a function g with domain C as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ x & \text{otherwise} \end{cases}$$

We'll show that g is a bijection from $C \to B$, from which it will follow that $g \circ f^{-1} : A \to B$ is a bijection, completing the proof.

First we claim that if $x \in F$ but $y \notin F$ then $g(x) \neq g(y)$. For reductio suppose otherwise, so that $y = g(y) = g(x) = f(x)$. Since $x \in F$ and F is f-closed by Lemma 7.8, we have $y = f(x) \in F$, a contradiction.

Now suppose $g(x) = g(y)$. So, by the above, $x \in F$ iff $y \in F$. If $x, y \in F$, then $f(x) = g(x) = g(y) = f(y)$ so that $x = y$ since f is a bijection. If $x, y \notin F$, then $x = g(x) = g(y) = y$. So g is an injection.

It remains to show that $\mathrm{ran}(g) = B$. So fix $x \in B \subseteq C$. If $x \notin F$, then $g(x) = x$. If $x \in F$, then $x = f(y)$ for some $y \in F$, since otherwise $F \setminus \{x\}$ would be f-closed and extend $C \setminus B$, which is impossible by Lemma 7.8; now $g(y) = f(y) = x$. □

Finally, here is the proof of the main result. Recall that given a function h and set D, we define $h[D] = \{h(x) : x \in D\}$.

Proof of Schröder-Bernstein. Let $f\colon A \to B$ and $g\colon B \to A$ be injections. Since $f[A] \subseteq B$ we have that $g[f[A]] \subseteq g[B] \subseteq A$. Also, $g \circ f\colon A \to g[f[A]]$ is an injection since both g and f are; and indeed $g \circ f$ is a bijection, just by the way we defined its codomain. So $g[f[A]] \approx A$, and hence by Proposition 7.9 there is a bijection $h\colon A \to g[B]$. Moreover, g^{-1} is a bijection $g[B] \to B$. So $g^{-1} \circ h\colon A \to B$ is a bijection. □