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A DUAL OF DILWORTH'S DECOMPOSITION THEOREM

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Let P be a partially ordered set. A subset S of P will be called a **chain** if any two elements in S are comparable; it will be called an **antichain** if no two (distinct) elements in S are comparable. In particular, the empty set is both a chain and an antichain. A chain is said to be **maximal** if it is not a proper subset of any chain. An element x in S is said to be **maximal** if $y \leqq x$ for every element y in S which is comparable with x.

We owe to Dilworth [1] the following well-known and important decomposition theorem:

THEOREM 1. *Let* P *be a partially ordered set and m a natural number. If* P *possesses no antichain of cardinal* $m+1$, *then it can be expressed as the union of m chains.*

It may be of some interest to note that this statement remains valid if the roles of chains and antichains are interchanged. Thus we have the following result:

THEOREM 2. *Let* P *be a partially ordered set, and m a natural number. If* P *possesses no chain of cardinal* $m+1$, *then it can be expressed as the union of m antichains.*

Thus, in a formal sense, Theorem 2 may be regarded as a 'dual' of Theorem 1. However, as we shall see, the proof of the dual result is considerably easier than that of Dilworth's original theorem. In particular, to establish Theorem 1 we need first to deal with the case where P is finite (see Tverberg's elegant treatment in [5]) and then extend the conclusion to the general case, say by invoking Rado's selection principle (the details can be found, e.g., in [3]). By contrast, a single induction argument suffices to prove Theorem 2.

When $m=1$, the assertion holds trivially. Let $m \geqq 2$; assume that the assertion holds for $m-1$, and let P be a partially ordered set which has no chain of cardinal $m+1$. The antichain M consisting of all maximal elements in P is clearly non-empty since the maximal element of every maximal chain belongs to M. Further, no chain in P\M has cardinal m. For assume, on the contrary, that

$$x_1 < x_2 < \cdots < x_m, \quad x_k \in \mathrm{P} \backslash \mathrm{M} \qquad (1 \leqq k \leqq m).$$

Then, since this chain has cardinal m, it is maximal and so $x_m \in \mathrm{M}$, which contradicts the relation $x_m \in \mathrm{P}\backslash\mathrm{M}$. Since, then, no chain in P\M has cardinal m, it follows by the induction hypothesis that P\M can be expressed as the union of $m-1$ antichains. Hence P can be expressed as the union of m antichains.

We note an easy consequence of Theorem 2.

COROLLARY. *Let r, s be positive integers. Then a partially ordered set of $rs+1$ elements possesses a chain of cardinal $r+1$ or an antichain of cardinal $s+1$ or both.*

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If there is no chain of cardinal $r+1$, then the given set P can be expressed as the union of r antichains, which may be assumed to be pairwise disjoint, say $P=A_1\cup \cdots \cup A_r$. Hence, denoting by $|A|$ the cardinal of A, we have

$$rs+1 = |A_1| + \cdots + |A_r|.$$

Therefore

$$rs+1 \leqq r \max |A_i|$$

and so $s+1\leqq \max|A_i|$, as required. It should be noted that the corollary follows in just the same way from Theorem 1, and also that it is best possible in the sense that $rs+1$ cannot be replaced by rs.

In conclusion, we recall a result of Erdös and Szekeres [2] (see Seidenberg [4] for a very short proof) which is an easy consequence of the corollary: *Each sequence of* $rs+1$ *real terms possesses an increasing subsequence of* $r+1$ *terms or a decreasing subsequence of* $s+1$ *terms or both.* The deduction of this result from the corollary appears to be quite well known (or may be left as an exercise for the reader), and we omit the details.

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4. A. Seidenberg, A simple proof of a theorem of Erdös and Szekeres, J. London Math. Soc., 34 (1959) 352.
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TOPOLOGIES ON ORDERED SETS

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A recent problem in this Monthly [1] asks whether it is possible to topologize the integers in such a way that the connected sets are precisely the sets of consecutive integers. The object of this note is to point out that, for a suitable generalization of "sets of consecutive integers," there is a simple necessary and sufficient condition for any partially ordered set to have such a topology.

Let $\langle P; \leqq\rangle$ be a partially ordered set. For $a, b\in P$ we write aRb if and only if $a<b$ and $\{x\in P|a<x<b\}=\varnothing$, or $b<a$ and $\{x\in P|b<x<a\}=\varnothing$. We say that $\langle a_1, \cdots, a_n\rangle$ is an R-**chain of length** n connecting a_1 and a_n (n may be 1) if and only if a_iRa_{i+1} for $1\leqq i<n$; if $a_i\in A\subseteq P$ for each i, we say $\langle a_1, \cdots, a_n\rangle$ is an R-**chain** in A. Finally, we say that $A\subseteq P$ is a **set of consecutive elements** of P if and only if for all $a, b\in A$ there is an R-chain in A connecting a and b.

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