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A DECOMPOSITION THEOREM FOR PARTIALLY ORDERED SETS

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1. Introduction

Let P be a partially ordered set. Two elements a and b of P are *camparable* if either $a \geqq b$ or $b \geqq a$. Otherwise a and b are *non-comparable*. A subset S of P is *independent* if every two distinct elements of S are non-comparable. S is *dependent* if it contains two distinct elements which are comparable. A subset C of P is a *chain* if every two of its elements are comparable.

This paper will be devoted to the proof of the following theorem and some of its applications.

THEOREM 1.1. *Let every set of $k + 1$ elements of a partially ordered set P be dependent while at least one set of k elements is independent. Then P is a set sum of k disjoint chains.*[1]

It should be noted that the first part of the hypothesis of the theorem is also necessary. For if P is a set sum of k chains and S is any subset containing $k + 1$ elements, then at least one pair must belong to the same chain and hence be comparable.

Theorem 1.1 contains as a very special case the Radó-Hall theorem on representatives of sets (Hall [1]). Indeed, we shall derive from Theorem 1.1 a general theorem on representatives of subsets which contains the Kreweras (Kreweras [2]) generalization of the Radó-Hall theorem.

As a further application, Theorem 1.1 is used to prove the following imbedding theorem for distributive lattices.

THEOREM 1.2. *Let D be a finite distributive lattice. Let $k(a)$ be the number of distinct elements in D which cover a and let k be the largest of the numbers $k(a)$. Then D is a sublattice of a direct union of k chains and k is the smallest number for which such an imbedding holds.*

2. Proof of Theorem 1.1.

We shall prove the theorem first for the case where P is finite. The theorem in the general case will then follow by a transfinite argument. Hence let P be a finite partially ordered set and let k be the maximal number of independent elements. If $k = 1$, then every two elements of P are comparable and P is thus

[1] This theorem has a certain formal resemblance to a theorem of Menger on graphs (D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, (1936)). Menger's theorem, however, is concerned with the characterization of the maximal number of disjoint, *complete* chains. Another type of representation of partially ordered sets in terms of chains has been considered by Dushnik and Miller [3] (see also Komm [4]). It can be shown that if n is the maximal number of non-comparable elements, then the dimension of P in the sense of Dushnik and Miller is at most n. Except for this fact, there seems to be little connection between the two representations.

a chain. Hence the theorem is trivial in this case and we may make an argument by induction. Let us assume, then, that the theorem holds for all finite partially ordered sets for which the maximal number of independent elements is less than k. Now it will be sufficient to show that if $C_1, \cdots, C_k$ are k disjoint chains of P and if a is an element belonging to none of the C_i, then $C_1 + \cdots + C_k + a$ is a set sum of k disjoint chains. For beginning with a set $a_1, \cdots, a_k$ of independent elements (which exist by hypothesis) we may add one new element at a time and be sure that at each stage we have a set sum of k disjoint chains. Since P is finite, we finally have P itself represented as a set sum of k chains.

Let, then, $C_1, \cdots, C_k$ be k disjoint chains and let a be an element not belonging to $C_1 + \cdots + C_k$. Let U_i be the set of all elements of C_i which contain a, let L_i be the set of all elements of C_i which are contained in a, and let N_i be the set of all elements of C_i which are non-comparable with a. Finally let

$$U = U_1 + \cdots + U_k$$
$$L = L_1 + \cdots + L_k$$
$$N = N_1 + \cdots + N_k$$
$$C = C_1 + \cdots + C_k.$$

Clearly $U_i + N_i + L_i = C_i$ and $U + N + L = C$.

We show now that for some m the maximal number of independent elements in $N + U - U_m$ is less than k. For suppose that for each j there exists a set S_j consisting of k independent elements of $N + U - U_j$. Since there are k elements in S_j and they belong to $C = C_1 + \cdots + C_k$, there is exactly one element of S_j in each of the chains C_i. Since S_j contains no elements of U_j it follows that S_j contains exactly one element of N_j. Thus $S = S_1 + \cdots + S_k$ contains at least one element of N_i for each i. Now let s_i be the minimal element of S which belongs to C_i. s_i exists since the intersection of S and C_i is a finite chain which we have proved to be non-empty. Furthermore, $s_i \in N_i$ since there is at least one element of N_i which belongs to S and all of the elements of U_i properly contain all of the elements of N_i. Hence $s_1, \cdots, s_k \in N$. Now if $s_i \geqq s_j$ for $i \neq j$, let $s_j \in S_r$. Since S_r contains an element t_i belonging to C_i, we have from the definition of s_i that $t_i \geqq s_i \geqq s_j$ and $t_i \neq s_j$ since $t_i \in C_i$ and $s_j \in C_j$. But this contradicts our assumption that the elements of S_r are independent. Hence we must have $s_j \ngeqq s_j$ for $i \neq j$ and $s_1, \cdots, s_k$ form an independent set. But since s_i belongs to N, s_i is non-comparable with a and hence $a, s_1, \cdots, s_k$ is an independent set containing $k + 1$ elements. But this contradicts the hypothesis of the theorem and hence we conclude that for some m, the maximal number of independent elements in $N + U - U_m$ is less than k.

In an exactly dual manner it follows that for some l, the maximal number of independent elements in $N + L - L_l$ is less than k.

Now let T be an independent subset of $C - U_m - L_l$. If T contains an element x belonging to $U - U_m$ and an element y belonging to $L - L_l$, then $x \geqq a \geqq y$ contrary to the independence of T. Since

$$(N + U - U_m) + (N + L - L_l) = C - U_m - L_l$$

it follows that T is either a subset of $N + U - U_m$ or of $N + L - L_l$. Hence the number of elements in T is less than k and thus the maximal number of independent elements in $C - U_m - L_l$ is less than k. Since $U_m + L_l$ is a chain there is at least one independent set of $k - 1$ elements in $C - U_m - L_l$. Hence by the induction hypothesis $C - U_m - L_l = C_1' + \cdots + C_{k-1}'$ where $C_1', \cdots, C_{k-1}'$ are disjoint chains. Let C_k' be the chain $U_m + a + L_l$. Then

$$C + a = C_1' + \cdots + C_k'$$

and our assertion is proved.

We turn now to the proof of the general case. Again when $k = 1$ the theorem is trivial and we may proceed by induction. Hence let the theorem hold for all partially ordered sets having at most $k - 1$ independent elements and let P satisfy the hypotheses of the theorem. A subset C of P is said to be *strongly dependent* if for every finite subset S of P, there is a representation of S as a set sum of k disjoint chains such that all of the elements of C which belong to S are members of the same chain. Clearly any strongly dependent subset is a chain. Also from the theorem in the finite case it follows that a set consisting of a single element is always strongly dependent. Since strong dependence is a finiteness property it follows from the Maximal Principle that P contains a maximal strongly dependent subset C_1. Suppose that $P - C_1$ contains k independent elements $a_1, \cdots, a_k$. Then from the maximal property of C_1 we conclude that $C_1 + a_i$ is not strongly dependent for each i. Hence there exists a finite subset S_i such that in any representation as a set sum of k chains there are at least two chains which contain elements of $C_1 + a_i$. S_i must clearly contain a_i since C_1 is strongly dependent. Let $S = S_1 + \cdots + S_k$. By the strong dependence of C_1, $S = K_1 + \cdots + K_k$ where $K_1, \cdots, K_k$ are disjoint chains such that for some $n \leqq k$ we have $S \cdot C_1 \subseteqq K_n$. Since S contains $a_1, \cdots, a_k$ which are independent, for some $m \leqq k$ we have $a_m \in K_n$. Let K_i' be the chain $S_m \cdot K_i$. Then $S_m = K_1' + \cdots + K_k'$ and $S_m \cdot C_1 \subseteqq S_m \cdot S \cdot C_1 \subseteqq S_m \cdot K_n = K_n'$. But by definition $a_m \in S_m$ and $a_m \in K_n$. Hence $S_m \cdot (C_1 + a_m) \subseteqq K_n'$ which contradicts the definition of S_m. We conclude that $P - C_1$ contains at most $k - 1$ independent elements. But since C_1 is a chain and P contains a set of k independent elements, it follows that $P - C_1$ contains a set of $k - 1$ independent elements. Thus by the induction hypothesis we have $P - C_1 = C_2 + \cdots + C_k$. Hence

$$P = C_1 + \cdots + C_k$$

and the proof of the theorem is complete.

3. Application to representatives of sets.

G. Kreweras has proved the following extension of the Radó-Hall theorem on representatives of sets:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two partitions of a set into n parts and let h be the smallest number such that for any r, r parts of $\mathfrak{A}$ contain at most $r + h$ parts of $\mathfrak{B}$. Let k be the smallest number such that $n + k$ elements serve to represent both partitions. Then $h = k$.

To show the power of Theorem 1.1 we shall prove an even more general theorem in which the partition requirement is dropped. Now if $\mathfrak{A}$ is any finite collection of subsets of a set S we shall say that a set of n elements (repetitions being counted) represents $\mathfrak{A}$ if there exists a one-to-one correspondence of the sets of $\mathfrak{A}$ onto a subset of the n elements such that each set contains its corresponding element. For example, the set $\{1, 1, 1\}$ represents the three sets $\{1, 2\}$, $\{1, 3\}$, and $\{1, 4\}$. The theorem can then be stated as follows:

THEOREM 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finite collections of subsets of some set. Let $\mathfrak{A}$ and $\mathfrak{B}$ contain m and n sets respectively. Let h be the smallest number such that for every r, the union of any $r + h$ sets of $\mathfrak{A}$ intersects at least r sets of $\mathfrak{B}$. Let k be the smallest number such that $n + k$ elements serve to represent both collections $\mathfrak{A}$ and $\mathfrak{B}$. Then $h = k$.*

It can be easily verified that if $\mathfrak{A}$ and $\mathfrak{B}$ are partitions of a set, then h as defined in Theorem 2.1 is equivalent to the definition given in the theorem of Kreweras.

For the proof let $\mathfrak{A}$ consist of sets $A_1, \cdots, A_m$ and $\mathfrak{B}$ consist of sets $B_1, \cdots, B_n$. We make the sets $A_1, \cdots, A_m, B_1, \cdots, B_n$ into a partially ordered set P as follows:

$$A_i \geqq A_i \quad i = 1, \cdots, m$$

$$B_j \geqq B_j \quad j = 1, \cdots, n.$$

$$A_i \geqq B_j \text{ if and only if } A_i \text{ and } B_j \text{ intersect.}$$

It is obvious that P is a partially ordered set under this ordering. Now let w be the maximal number of independent elements of P. Since the union of any $r + h$ sets of $\mathfrak{A}$ intersects at least r sets of $\mathfrak{B}$, it follows that any independent subset of P can have at most $r + h + (n - r) = n + h$ elements. Hence $w \leqq n + h$. On the other hand for some r there are $r + h$ sets of $\mathfrak{A}$ whose union intersects precisely r sets of $\mathfrak{B}$. Hence these $r + h$ sets of $\mathfrak{A}$ and the remaining $n - r$ sets of $\mathfrak{B}$ form an independent subset of P containing $n + h$ elements. Thus $w = n + h$. By Theorem 1.1, P is the set sum of w chains $C_1, \cdots, C_w$. Now if a chain C_i contains two sets they have a non-null intersection by definition. Hence for each C_i there is an element a_i common to the sets of C_i. But since $A_1, \cdots, A_m$ are independent in P it follows that they belong to different chains and hence the w elements $a_1, \cdots, a_w$ represent $\mathfrak{A}$. Similarly, $a_1, \cdots, a_w$ represent $\mathfrak{B}$ and thus $n + k \leqq w$. But since P cannot be represented as a set sum of less than w chains, it follows that $n + k = w = n + h$. Hence $h = k$ and the theorem is proved.

4. Proof of Theorem 1.2.

Let us recall that an element q of a finite distributive lattice D is (*union*) *irreducible* if $q = x \cup y$ implies $q = x$ or $q = y$. It can be easily verified that if q is irreducible, then $q \leqq x \cup y$ implies $q \leqq x$ or $q \leqq y$. From the finiteness[2] of S it

[2] L is assumed to be finite for sake of simplicity. The theorem holds without this restriction. In the proof, "elements covered by a" must be replaced by "maximal ideals in a" and "irreducible elements" must be replaced by "prime ideals."

follows that every element of D can be expressed as a union of irreducible elements. From this fact we conclude that if $x > y$, there exists at least one irreducible q such that $x \geqq q$ and $y \ngeqq q$.

Now let P be the partially ordered set of union irreducible elements of D. Let a be such that $k = k(a)$. Then there are k elements $a_1, \cdots, a_k$ which cover a. Let q_i be an irreducible such that $a_i \geqq q_i$ and $a \ngeqq q_i$. Then if $q_i \geqq q_j$ where $i \neq j$ we have $a = a_i \cap a_j \geqq q_i \cap q_j \geqq q_j$ which contradicts $a \ngeqq q_j$. Hence $q_1, \cdots, q_k$ are an independent set of elements of P.

Next let $q_1', \cdots, q_l'$ be an arbitrary independent subset of P. Let $a' = q_1' \cup \cdots \cup q_l'$ and for each i let $p_i' = q_1' \cup \cdots \cup q_{i-1}' \cup q_{i+1}' \cup \cdots \cup q_l'$. Now if $p_i' = a'$ for some i, then

$$q_i' = q_i' \cap a' = q_i' \cap p_i'$$
$$= (q_i' \cap q_1') \cup \cdots \cup (q_i' \cap q_{i-1}') \cup (q_i' \cap q_{i+1}') \cup \cdots \cup \text{M}\ (q_i' \cap q_l')$$

and hence $q_i' = q_i' \cap q_j'$ for some $j \neq i$. But then $q_j' \geqq q_i'$ contrary to independence. Thus $a' > p_i'$ for each i and $p_i' \cup p_j' = a'$ for $i \neq j$. Let $a = p_1' \cap \cdots \cap p_l'$ and for each i let $p_i = p_1' \cap \cdots \cap p_{i-1}' \cap p_{i+1}' \cap \cdots \cap p_l'$. If $p_i = a$, then $p_i' = p_i' \cup a = p_i' \cup p_i = (p_i' \cup p_1') \cap \cdots \cap (p_i' \cup p_{i-1}') \cap (p_i' \cup p_{i+1}') \cap \cdots \cap (p_i' \cup p_l') = a'$ which contradicts $p_i' < a'$. Hence $p_i > a$ and $p_i \cap p_j = a$ for $i \neq j$. Let $p_i \geqq a_i$ where a_i covers a. Then $a \leqq a_i \cap a_j \leqq p_i \cap p_j = a$ for $i \neq j$ and hence $a_i \cap a_j = a, i \neq j$. Thus $a_1, \cdots, a_l$ are distinct elements of D covering a. It follows that $l \leqq k$ and hence k is the maximal number of independent elements of P.

Now by Theorem 1.1 P is the set sum of k disjoint chains $C_1, \cdots, C_k$. We adjoin the null element z of D to each of the chains C_i. Then for each $x \in D$, there is a unique maximal element x_i in C_i which is contained in x. Now suppose $x > x_1 \cup \cdots \cup x_k$ in D. Then there exists an irreducible q such that $x \geqq q$ and $x_1 \cup \cdots \cup x_k \ngeqq q$. But $q \in C_i$ for some i and hence $x_1 \cup \cdots \cup x_k \geqq x_i \geqq q$ contrary to the definition of q. Hence $x = x_1 \cup \cdots \cup x_k$. Consider the mapping of D into the direct union of $C_1, \cdots, C_k$ given by

$$x \rightarrow \{x_1, \cdots, x_k\}.$$

Now if $x_i = y_i$ for $i = 1, \cdots, k$, then $x = x_1 \cup \cdots \cup x_k = y_1 \cup \cdots \cup y_k = y$ and the mapping is thus one-to-one. Since $x \cup y \geqq x_i \cup y_i$ we have $(x \cup y)_i \geqq x_i \cup y_i$. But since $(x \cup y)_i$ is union irreducible we get $x \cup y \geqq (x \cup y)_i \rightarrow x \geqq (x \cup y)_i$ or $y \geqq (x \cup y)_i \rightarrow x_i \geqq (x \cup y)_i$ or $y_i \geqq (x \cup y)_i \rightarrow x_i \cup y_i \geqq (x \cup y)_i$. Thus $(x \cup y)_i = x_i \cup y_i$ and we have

$$x \cup y \rightarrow \{x_1 \cup y_1, \cdots, x_k \cup y_k\}.$$

Similarly $x \cap y \geqq x_i \cap y_i \rightarrow (x \cap y)_i \geqq x_i \cap y_i$. But $x \geqq x \cap y \rightarrow x_i \geqq (x \cap y)_i$ and $y \geqq x \cap y \rightarrow y_i \geqq (x \cap y)_i$. Hence $x_i \cap y_i \geqq (x \cap y)_i$. Thus $(x \cap y)_i = x_i \cap y_i$ and we have

$$x \cap y \rightarrow \{x_1 \cap y_1, \cdots, x_k \cap y_k\}.$$

This completes the proof that D is isomorphic to a sublattice of a direct union of k chains.

Now suppose that D is a sublattice of the direct union of l chains $C_1', \cdots, C_l'$ where $l < k$. Again let a be such that $k(a) = k$ and let $a_1, \cdots, a_k$ be the k distinct elements covering a. Define $a' = a_1 \cup \cdots \cup a_k$ and let $a_i' = a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_k$ for each i. Now $a_i' = q_1' \cup \cdots \cup q_l'$ where $q_i' \epsilon C_i'$. And if $q_i' = x' \cup y'$, then $q_i' = x_i' \cup y_i'$ where $x_i', y_i' \epsilon C_i'$. But then either $q_i' = x_i' \cup y_i' = x_i'$ or $q_i' = x_i' \cup y_i' = y_i'$ and hence either $q_i' = x'$ or $q_i' = y'$. Thus each q_i' is union irreducible. But $a_1 \cup \cdots \cup a_k = a' \geqq q_i'$ for $i = 1, \cdots, l$. Thus for each $i \leqq l$ there is a j such that $a_j \geqq q_i'$. Since $l < k$ there is some r such that $a_r' \geqq q_1' \cup \cdots \cup q_l' = a' \geqq a_r$. But then $a_r = a_r' \cap a_r = a$ which contradicts the fact that a_r covers a. Hence $l \geqq k$ and we conclude that k is the least number of chains whose direct union contains D as a sublattice. This completes the proof of Theorem 1.2.

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