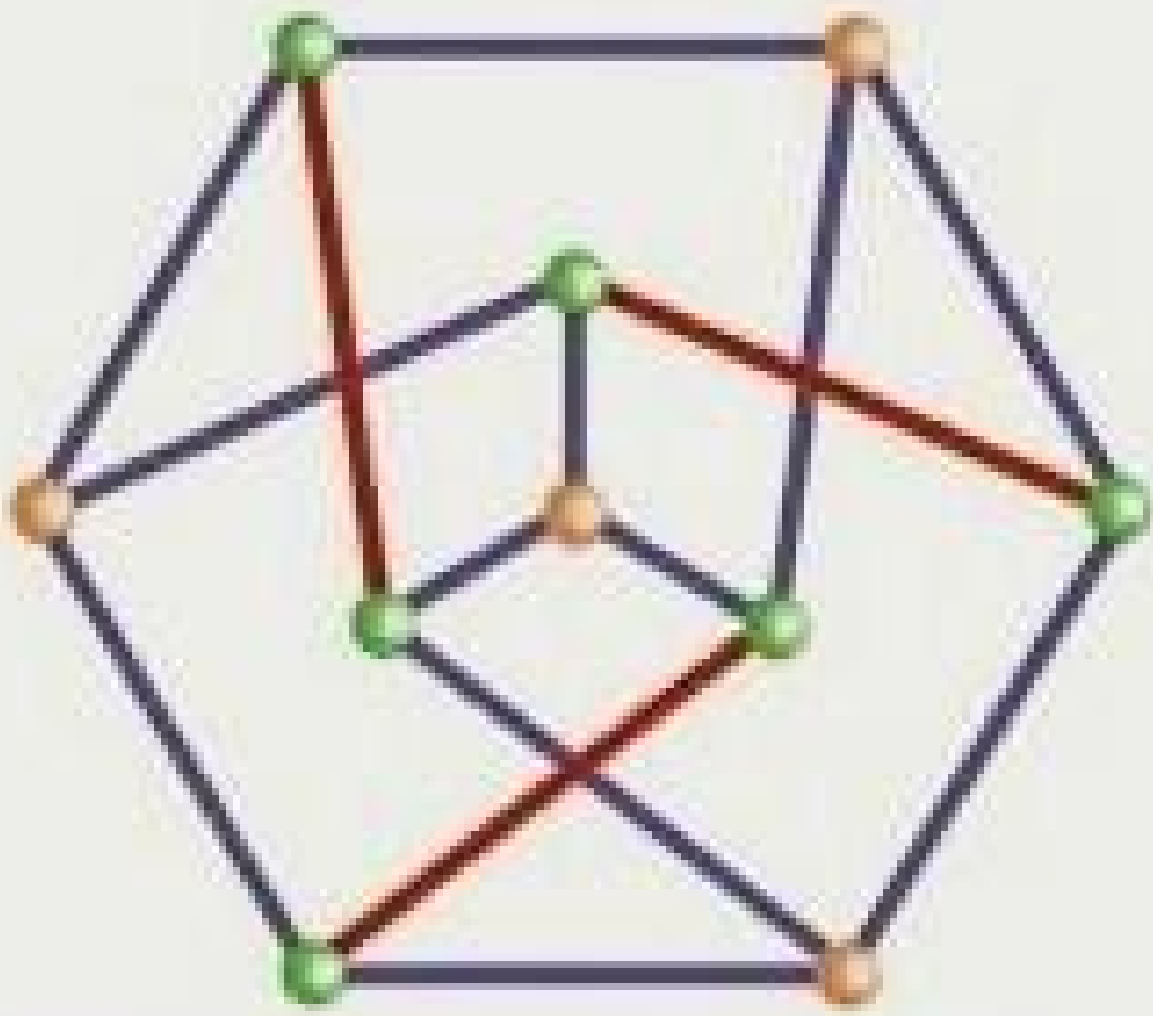
INTRODUCTION TO GRAPH THEORY

SECOND EDITION



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Introduction to Graph Theory

Second Edition

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ISBN 81-7808-830-4

First Indian Reprint, 2002

This edition is manufactured in India and is authorized for sale only in India, Bangladesh, Pakistan, Nepal, Sri Lanka and the Maldives.

Published by Pearson Education (Singapore) Pte. Ltd., Indian Branch, 482 F.I.E. Patparganj,
Delhi 110 092, India

Printed in India by Rashtriya Printers.

For my dear wife Ching

and for all lovers of graph theory

Contents

Glossary of Notation

Non-alphabetic notation

$\leftrightarrow$	adjacency relation
$\to$	successor relation (digraph)
$\cong$	isomorphism relation
$a \equiv b \bmod n$	congruence relation
$\Rightarrow$	implication
$\lfloor x \rfloor$	floor of number
$\lceil x \rceil$	ceiling of number
$[n]$	$\{1, \ldots, n\}$
$\|x\|$	absolute value of number
$\|S\|$	size of set
$\{x\colon P(x)\}$	set description
∞	infinity
$\varnothing$	empty set
$\cup$	union
$\cap$	intersection
$A \subseteq B$	subset
$G \subseteq H$	subgraph
$G[S]$	subgraph of G induced by S
$\overline{G}, \overline{X}$	complement of graph or set
G^*	(planar) dual
G^k	kth power of graph
S^k	set of k-tuples from S
$[S, \bar{S}]$	edge cut
$[S, T]$	source-sink cut
$G - v$	deletion of vertex
$G - e$	deletion of edge
$G \cdot e$	contraction of edge
$G + H$	disjoint union of graphs
$G \vee H$	join of graphs
$G \,\square\, H$	cartesian product of graphs
$G \triangle H$, $A \triangle B$	symmetric difference
$G \circ x$	vertex duplication
$G \circ h$	vertex multiplication
$A \times B$	cartesian product of sets
$A - B$	difference of sets
$\binom{n}{k}$	binomial coefficient
$\binom{n}{n_1 \cdots n_k}$	multinomial coefficient
$\mathbf{1}_n$	n-vector with all entries 1
$Y \| X$	conditional variable or event

Roman alphabet

$A(G)$	adjacency matrix
$\mathrm{Adj}\,A$	adjugate matrix
$B(G)$	bandwidth
$\mathbf{B}_M$	bases of matroid
$\mathbf{C}_M$	circuits of matroid
C_n	cycle with n vertices
C_n^d	power of a cycle
$c(G)$	number of components
$c(G)$	circumference
$C(G)$	(Hamiltonian) closure
$c(e)$	cost or capacity
$\mathrm{cap}(S, T)$	capacity of a cut
$d_1, \ldots, d_n$	degree sequence
$d(v), d_G(v)$	degree of vertex
$d^+(v), d^-(v)$	out-degree, in-degree
D	digraph
$D(G)$	distance sum
$d(u, v)$	distance from u to v
$\mathrm{diam}\, G$	diameter
$\det A$	determinant
$E(G)$	edge set
$E(X)$	expected value
$e(G)$	size (number of edges)
$f^+(v), f^+(S)$	total exiting flow
$f^-(v), f^-(S)$	total entering flow
f	function, flow
f	number of faces
G	graph (or digraph)
G^p	random graph in Model A
$H_{k,n}$	Harary graph
$\mathbf{I}_M$	independent sets of matroid
I	identity matrix
J	matrix of all 1's
K_n	complete graph
$K_{r,s}$	complete bipartite graph
$L(G)$	line graph
$l(e)$	lower bound on flow

continued on inside back cover

Preface

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics, and its results have applications in many areas of the computing, social, and natural sciences. The design of this book permits usage in a one-semester introduction at the undergraduate or beginning graduate level, or in a patient two-semester introduction. No previous knowledge of graph theory is assumed. Many algorithms and applications are included, but the focus is on understanding the structure of graphs and the techniques used to analyze problems in graph theory.

Many textbooks have been written about graph theory. Due to its emphasis on both proofs and applications, the initial model for this book was the elegant text by J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications* (Macmillan/North-Holland [1976]). Graph theory is still young, and no consensus has emerged on how the introductory material should be presented. Selection and order of topics, choice of proofs, objectives, and underlying themes are matters of lively debate. Revising this book dozens of times has taught me the difficulty of these decisions. This book is my contribution to the debate.

The Second Edition

The revision for the second edition emphasizes making the text easier for the students to learn from and easier for the instructor to teach from. There have not been great changes in the overall content of the book, but the presentation has been modified to make the material more accessible, especially in the early parts of the book. Some of the changes are discussed in more detail later in this preface; here I provide a brief summary.

- Optional material within non-optional sections is now designated by $(*)$; such material is not used later and can be skipped. Most of it is *intended* to be skipped in a one-semester course. When a subsection is marked "optional", the entire subsection is optional, and hence no individual items are starred.
- For less-experienced students, Appendix A has been added as a reference summary of helpful material on sets, logical statements, induction, counting arguments, binomial coefficients, relations, and the pigeonhole principle.

- Many proofs have been reworded in more patient language with additional details, and more examples have been added.
- More than 350 exercises have been added, mostly easier exercises in Chapters 1–7. There are now more than 1200 exercises.
- More than 100 illustrations have been added; there are now more than 400. In illustrations showing several types of edges, the switch to bold and solid edges instead of solid and dashed edges has increased clarity.
- Easier problems are now grouped at the beginning of each exercise section, usable as warm-ups. Statements of some exercises have been clarified.
- In addition to hints accompanying the exercise statements, there is now an appendix of supplemental hints.
- For easier access, terms being defined are in bold type, and the vast majority of them appear in Definition items.
- For easier access, the glossary of notation has been placed on the inside covers.
- Material involving Eulerian circuits, digraphs, and Turán's Theorem has been relocated to facilitate more efficient learning.
- Chapters 6 and 7 have been switched to introduce the idea of planarity earlier, and the section on complexity has become an appendix.
- The glossary has been improved to eliminate errors and to emphasize items more directly related to the text.

Features

Various features of this book facilitate students' efforts to understand the material. There is discussion of proof techniques, more than 1200 exercises of varying difficulty, more than 400 illustrations, and many examples. Proofs are presented in full in the text.

Many undergraduates begin a course in graph theory with little exposure to proof techniques. Appendix A provides background reading that will help them get started. Students who have difficulty understanding or writing proofs in the early material should be encouraged to read this appendix in conjunction with Chapter 1. Some discussion of proof techniques still appears in the early sections of the text (especially concerning induction), but an expanded treatment of the basic background (especially concerning sets, functions, relations, and elementary counting) is now in Appendix A.

Most of the exercises require proofs. Many undergraduates have had little practice at presenting explanations, and this hinders their appreciation of graph theory and other mathematics. The intellectual discipline of justifying an argument is valuable independently of mathematics; I hope that students will appreciate this. In writing solutions to exercises, students should be careful in their use of language ("say what you mean"), and they should be intellectually honest ("mean what you say").

Although many terms in graph theory suggest their definitions, the quantity of terminology remains an obstacle to fluency. Mathematicians like to gather definitions at the start, but most students succeed better if they use a concept before receiving the next. This, plus experience and requests from reviewers, has led me to postpone many definitions until they are needed. For example, the definition of cartesian product appears in Section 5.1 with coloring problems. Line graphs are defined in Section 4.2 with Menger's Theorem and in Section 7.1 with edge-coloring. The definitions of induced subgraph and join have now been postponed to Section 1.2 and Section 3.1, respectively.

I have changed the treatment of digraphs substantially by postponing their introduction to Section 1.4. Introducing digraphs at the same time as graphs tends to confuse or overwhelm students. Waiting to the end of Chapter 1 allows them to become comfortable with basic concepts in the context of a single model. The discussion of digraphs then reinforces some of those concepts while clarifying the distinctions. The two models are still discussed together in the material on connectivity.

This book contains more material than most introductory texts in graph theory. Collecting the advanced material as a final optional chapter of "additional topics" permits usage at different levels. The undergraduate introduction consists of the first seven chapters (omitting most optional material), leaving Chapter 8 as topical reading for interested students. A graduate course can treat most of Chapters 1 and 2 as recommended reading, moving rapidly to Chapter 3 in class and reaching some topics in Chapter 8. Chapter 8 can also be used as the basis for a second course in graph theory, along with material that was optional in earlier chapters.

Many results in graph theory have several proofs; illustrating this can increase students' flexibility in trying multiple approaches to a problem. I include some alternative proofs as remarks and others as exercises.

Many exercises have hints, some given with the exercise statement and others in Appendix C. Exercises marked "(−)" or "(+)" are easier or more difficult, respectively, than unmarked problems. Those marked "(+)" should *not* be assigned as homework in a typical undergraduate course. Exercises marked "(!)" are especially valuable, instructive, or entertaining. Those marked "(∗)" use material labeled optional in the text.

Each exercise section begins with a set of "(−)" exercises, ordered according to the material in the section and ending with a line of bullets. These exercises either check understanding of concepts or are immediate applications of results in the section. I recommend some of these to my class as "warmup" exercises to check their understanding before working the main homework problems, most of which are marked "(!)". Most problems marked "(−)" are good exam questions. When using other exercises on exams, it may be a good idea to provide hints from Appendix C.

Exercises that relate several concepts appear when the last is introduced. Many pointers to exercises appear in the text where relevant concepts are discussed. An exercise in the current section is cited by giving only its item number among the exercises of that section. Other cross-references are by Chapter.Section.Item.

Organization and Modifications

In the first edition, I sought a development that was intellectually coherent and displayed a gradual (not monotonic) increase in difficulty of proofs and in algorithmic complexity.

Carrying this further in the second edition, Eulerian circuits and Hamiltonian cycles are now even farther apart. The simple characterization of Eulerian circuits is now in Section 1.2 with material closely related to it. The remainder of the former Section 2.4 has been dispersed to relevant locations in other sections, with Fleury's Algorithm dropped.

Chapter 1 has been substantially rewritten. I continue to avoid the term "multigraph"; it causes more trouble than it resolves, because many students assume that a multigraph *must* have multiple edges. It is less distracting to append the word "simple" where needed and keep "graph" as the general object, with occasional statements that in particular contexts it makes sense to consider only simple graphs.

The treatment of definitions in Chapter 1 has been made more friendly and precise, particularly those involving paths, trails, and walks. The informal groupings of basic definitions in Section 1.1 have been replaced by Definition items that help students find definitions more easily.

In addition to the material on isomorphism, Section 1.1 now has a more precise treatment of the Petersen graph and an explicit introduction of the notions of decomposition and girth. This provides language that facilitates later discussion in various places, and it permits interesting explicit questions other than isomorphism.

Sections 1.2–1.4 have become more coherent. The treatment of Eulerian circuits motivates and completes Section 1.2. Some material has been removed from Section 1.3 to narrow its focus to degrees and counting, and this section has acquired the material on vertex degrees that had been in Section 1.4. Section 1.4 now provides the introduction to digraphs and can be treated lightly.

Trees and distance appear together in Chapter 2 due to the many relations between these topics. Many exercises combine these notions, and algorithms to compute distances produce or use trees.

Most graph theorists agree that the König-Egerváry Theorem deserves an independent proof without network flow. Also, students have trouble distinguishing "k-connected" from "connectivity k", which have the same relationship as "k-colorable" and "chromatic number k". I therefore treat matching first and later use matching to prove Menger's Theorem. Both matching and connectivity are used in the coloring material.

In response to requests from a number of users, I have added a short optional subsection on dominating sets at the end of Section 3.1. The material on weighted bipartite matching has been clarified by emphasis on vertex cover instead of augmenting path and by better use of examples.

Turán's Theorem uses only elementary ideas about vertex degrees and induction and hence appeared in Chapter 1 in the first edition. This caused some difficulties, because it was the most abstract item up to that point and students felt somewhat overwhelmed by it. Thus I have kept the simple triangle-free case (Mantel's Theorem) in Section 1.3 and have moved the full theorem to Section 5.2 under the viewpoint of extremal problems related to coloring.

The chapter on planarity now comes before that on "Edges and Cycles". When an instructor is short of time, planarity is more important to reach than the material on edge-coloring and Hamiltonian cycles. The questions involved in planarity appeal intuitively to students due to their visual aspects, and many students have encountered these questions before. Also, the ideas involved in discussing planar graphs seem more intellectually broadening in relation to the earlier material of the course than the ideas used to prove the basic results on edge-coloring and Hamiltonian cycles.

Finally, discussing planarity first makes the material of Chapter 7 more coherent. The new arrangement permits a more thorough discussion of the relationships among planarity, edge-coloring, and Hamiltonian cycles, leading naturally beyond the Four Color Theorem to the optional new material on nowhere-zero flows as a dual concept to coloring.

When students discover that the coloring and Hamiltonian cycle problems lack good algorithms, many become curious about NP-completeness. Appendix B satisfies this curiosity. Presentation of NP-completeness via formal languages can be technically abstract, so some students appreciate a presentation in the context of graph problems. NP-completeness proofs also illustrate the variety and usefulness of "graph transformation" arguments.

The text explores relationships among fundamental results. Petersen's Theorem on 2-factors (Chapter 3) uses Eulerian circuits and bipartite matching. The equivalence between Menger's Theorem and the Max Flow-Min Cut Theorem is explored more fully than in the first edition, and the "Baseball Elimination" application is now treated in more depth. The $k-1$-connectedness of k-color-critical graphs (Chapter 5) uses bipartite matching. Section 5.3 offers a brief introduction to perfect graphs, emphasizing chordal graphs. Additional features of this text in comparison to some others include the algorithmic proof of Vizing's Theorem and the proof of Kuratowski's Theorem by Thomassen's methods.

There are various other additions and improvements in the first seven chapters. There is now a brief discussion of Heawood's Formula and the Robertson–Seymour Theorem at the end of Chapter 6. In Section 7.1, a proof of Shannon's bound on the edge-chromatic number has been added. In Section 5.3, the characterization of chordal graphs is somewhat simpler than before by proving a stronger result about simplicial vertices. In Section 6.3, the proof of the reducibility of the Birkhoff diamond has been eliminated, but a brief discussion of discharging has been added. The material discussing issues in the proof of the theorem is optional, and the aim is to give the flavor of the approach without getting into detailed arguments. From this viewpoint the reducibility proof seemed out of focus.

Chapter 8 contains highlights of advanced material and is not intended for an undergraduate course. It assumes more sophistication than earlier chapters and is written more tersely. Its sections are independent; each selects appealing results from a large topic that merits a chapter of its own. Some of these sections become more difficult near the end; an instructor may prefer to sample early material in several sections rather than present one completely.

There may be occasional relationships between items in Chapter 8 and items marked optional in the first seven chapters, but generally cross-references indicate the connections. The material of Chapter 8 has not changed substantially since the first edition, although many corrections have been made and the presentation has been clarified in many places.

I will treat advanced graph theory more thoroughly in *The Art of Combinatorics*. Volume I is devoted to extremal graph theory and Volume II to structure of graphs. Volume III has chapters on matroids and integer programming (including network flow). Volume IV emphasizes methods in combinatorics and discusses various aspects of graphs, especially random graphs.

Design of Courses

I intend the 22 sections in Chapters 1–7 for a pace of slightly under two lectures per section when most optional material (starred items and optional subsections) is skipped. When I teach the course I spend eight lectures on Chapter 1, six lectures each on Chapters 4 and 5, and five lectures on each of Chapters 2, 3, 6, and 7. This presents the fundamental material in about 40 lectures. Some instructors may want to spend more time on Chapter 1 and omit more material from later chapters.

In chapters after the first, the most fundamental material is concentrated in the first section. Emphasizing these sections (while skipping the optional items) still illustrates the scope of graph theory in a slower-paced one-semester course. From the second sections of Chapters 2, 4, 5, 6, and 7, it would be beneficial to include Cayley's Formula, Menger's Theorem, Mycielski's construction, Kuratowski's Theorem, and Dirac's Theorem (spanning cycles), respectively.

Some optional material is particularly appealing to present in class. For example, I always present the optional subsections on Disjoint Spanning Trees (in Section 2.1) and Stable Matchings (in Section 3.2), and I usually present the optional subsection on f-factors (in Section 3.3). Subsections are marked optional when no later material in the first seven chapters requires them and they are not part of the central development of elementary graph theory, but these are nice applications that engage students' interest. In one sense, the "optional" marking indicates to students that the final examination is unlikely to have questions on these topics.

Graduate courses skimming the first two chapters might include from them such topics as graphic sequences, kernels of digraphs, Cayley's Formula, the Matrix Tree Theorem, and Kruskal's algorithm.

Courses that introduce graph theory in one term under the quarter system must aim for highlights; I suggest the following rough syllabus: 1.1: adjacency matrix, isomorphism, Petersen graph. 1.2: all. 1.3: degree-sum formula, large bipartite subgraphs. 1.4: up to strong components, plus tournaments. 2.1: up to centers of trees. 2.2: up to statement of Matrix Tree Theorem. 2.3: Kruskal's algorithm. 3.1: almost all. 3.2: none. 3.3: statement of Tutte's Theorem, proof of Petersen's results. 4.1: up to definition of blocks, omitting Harary graphs. 4.2: up to open ear decomposition, plus statement of Menger's Theorem(s). 4.3: duality between flows and cuts, statement of Max-flow = Min-cut. 5.1: up to Szekeres-Wilf theorem. 5.2: Mycielski's construction, possibly Turán's Theorem. 5.3: up to chromatic recurrence, plus perfection of chordal graphs. 6.1: non-planarity of K_5 and $K_{3,3}$, examples of dual graphs, Euler's formula with applications. 6.2: statement and examples of Kuratowski's Theorem and Tutte's Theorem. 6.3: 5-color Theorem, plus the idea of crossing number. 7.1: up to Vizing's Theorem. 7.2: up to Ore's condition, plus the Chvátal-Erdös condition. 7.3: Tait's Theorem, Grinberg's Theorem.

Further Pedagogical Aspects

In the revision I have emphasized some themes that arise naturally from the material; underscoring these in lecture helps provide continuity.

More emphasis has been given to the theme of TONCAS—"The obvious necessary condition is also sufficient." Explicit mention has been added that many of the fundamental results can be viewed in this way. This both provides a theme for the course and clarifies the distinction between the easy direction and the hard direction in an equivalence.

Another theme that underlies much of Chapters 3–5 and Section 7.1 is that of dual maximization and minimization problems. In a graph theory course one does not want to delve deeply into the nature of duality in linear optimization. It suffices to say that two optimization problems form a dual pair when every feasible solution to the maximization problem has value at most the value of every feasible solution to the minimization problem. When feasible solutions with the same value are given for the two problems, this duality implies that both solutions are optimal. A discussion of the linear programming context appears in Section 8.1.

Other themes can be identified among the proof techniques. One is the use of extremality to give short proofs and avoid the use of induction. Another is the paradigm for proving conditional statements by induction, as described explicitly in Remark 1.3.25.

The development leading to Kuratowski's Theorem is somewhat long. Nevertheless, it is preferable to present the proof in a single lecture. The preliminary lemmas reducing the problem to the 3-connected case can be treated lightly to save time. Note that the induction paradigm leads naturally to the two lemmas proved for the 3-connected case. Note also that the proof uses the notion of S-lobe defined in Section 5.2.

The first lecture in Chapter 6 should not belabor technical definitions of drawings and regions. These are better left as intuitive notions unless students ask for details; the precise statements appear in the text.

The motivating applications of digraphs in Section 1.4 have been marked optional because they are not needed in the rest of the text, but they help clarify that the appropriate model (graph or digraph) depends on the application.

Due to its reduced emphasis on numerical computation and increased emphasize on techniques of proof and clarity of explanations, graph theory is an excellent subject in which to encourage students to improve their habits of communication, both written and oral. In addition to assigning written homework that requires carefully presented arguments, I have found it productive to organize optional "collaborative study" sessions in which students can work together on problems while I circulate, listen, and answer questions. It should not be forgotten that one of the best ways to discover whether one understands a proof is to try to explain it to someone else. The students who participate find these sessions very beneficial.

Acknowledgments

This text benefited from classroom testing of gradually improving prepublication versions at many universities. Instructors who used the text on this experimental basis were, roughly in chronological order, Ed Scheinerman (Johns Hopkins), Kathryn Fraughnaugh (Colorado-Denver), Paul Weichsel / Paul Schupp / Xiaoyun Lu (Illinois), Dean Hoffman / Pete Johnson / Chris Rodger (Auburn), Dan Ullman (George Washington), Zevi Miller / Dan Pritikin (Miami-Ohio), David Matula (Southern Methodist), Pavol Hell (Simon Fraser), Grzegorz Kubicki (Louisville), Jeff Smith (Purdue), Ann Trenk (Wellesley), Ken Bogart (Dartmouth), Kirk Tolman (Brigham Young), Roger Eggleton (Illinois State), Herb Kasube (Bradley), and Jeff Dinitz (Vermont). Many of these (or their students) provided suggestions or complaints that led to improvements.

I thank George Lobell at Prentice Hall for his continuing commitment to this project and for finding diligent reviewers. Reviewers Paul Edelman, Renu Laskar, Gary MacGillivray, Joseph Neggers, Joseph Malkevitch, James Oxley, Sam Stueckle, and Barry Tesman made helpful comments. Reviewers for early versions of sections of Chapter 8 included Mike Albertson, Sanjoy Barvah, Dan Kleitman, James Oxley, Chris Rodger, and Alan Tucker. Reviewers for the second edition included Nate Dean, Dalibor Froncek, Renu Laskar, Michael Molloy, David Sumner, and Daniel Ullman.

Additional comments and corrections between the first and second editions came from many readers. These items range from typographical errors to additional exercises to simplifications in proofs. The accumulated effect of these contributions is substantial. Most of these suggestions don't lend themselves to attribution within the text, so I express my gratitude here for their willingness to contribute their observations, opinions, and expertise. These people include Troy Barcume, Stephan Brandt, Gerard Chang, Scott Clark, Dave Gunderson, Dean Hoffman, John D'Angelo, Charles Delzell, Thomas Emden-Weinert, Shimon Even, Fred Galvin, Alfio Giarlotta, Don Greenwell, Jing Huang, Garth

Isaak, Steve Kilner, Alexandr Kostochka, André Kündgen, Peter Kwok, Jean-Marc Lanlignel, Francois Margot, Alan Mehlenbacher, Joel Miller, Zevi Miller, Wendy Myrvold, Charles Parry, Robert Pratt, Dan Pritikin, Radhika Ramamurthi, Craig Rasmussen, Bruce Reznick, Jian Shen, Tom Shermer, Warren Shreve, Alexander Strehl, Tibor Szabó, Vitaly Voloshin, and C.Q. Zhang.

Several students found numerous typographical errors in the pre-publication version of the second edition (thereby earning extra credit!): Jaspreet Bagga, Brandon Bowersox, Mark Chabura, John Chuang, Greg Harfst, Shalene Melo, Charlie Pikscher, and Josh Reed.

The cover drawing for the first edition was produced by Ed Scheinerman using BRL-CAD, a product of the U.S. Army Ballistic Research Laboratory. For the second edition, the drawing was produced by Maria Muyot using CorelDraw.

Chris Hartman contributed vital assistance in preparing the bibliography for the first edition; additional references have now been added. Ted Harding helped resolve typesetting difficulties in other parts of the first edition.

Students who helped gather data for the index of the first edition included Maria Axenovich, Nicole Henley, André Kündgen, Peter Kwok, Kevin Leuthold, John Jozwiak, Radhika Ramamurthi, and Karl Schmidt. Raw data for the index of the second edition was gathered using scripts I wrote in `perl`; Maria Muyot and Radhika Ramamurthi assisted with processing of the index and the bibliography.

I prepared the second edition in TeX, the typesetting system for which the scientific world owes Donald E. Knuth eternal gratitude. The figures were generated using `gpic`, a product of the Free Software Foundation.

Feedback

I welcome corrections and suggestions, including comments on topics, attributions of results, updates, suggestions for exercises, typographical errors, omissions from the glossary or index, etc. Please send these to me at

west@math.uiuc.edu

In particular, I apologize in advance for missing references; please inform me of the proper citations! Also, no changes other than corrections of errors will be made between printings of this edition.

I maintain a web site containing a syllabus, errata, updates, and other *material. Please visit!*

http://www.math.uiuc.edu/~west/igt

I have corrected all typographical and mathematical errors known to me before the time of printing. Nevertheless, the robustness of the set of errors and the substantial rewriting and additions make me confident that some error remains. Please find it and tell me so I can correct it!

Douglas B. West
Urbana, Illinois

Chapter 1

Fundamental Concepts

1.1. What Is a Graph?

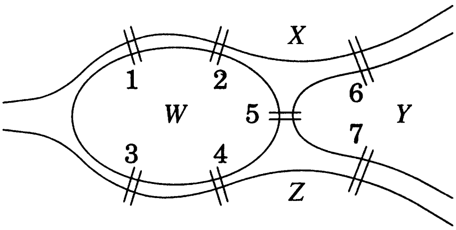
How can we lay cable at minimum cost to make every telephone reachable from every other? What is the fastest route from the national capital to each state capital? How can n jobs be filled by n people with maximum total utility? What is the maximum flow per unit time from source to sink in a network of pipes? How many layers does a computer chip need so that wires in the same layer don't cross? How can the season of a sports league be scheduled into the minimum number of weeks? In what order should a traveling salesman visit cities to minimize travel time? Can we color the regions of every map using four colors so that neighboring regions receive different colors?

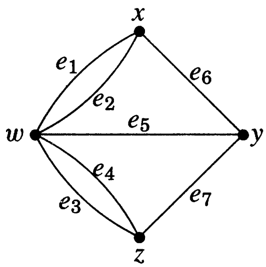
These and many other practical problems involve graph theory. In this book, we develop the theory of graphs and apply it to such problems. Our starting point assumes the mathematical background in Appendix A, where basic objects and language of mathematics are discussed.

THE DEFINITION

The problem that is often said to have been the birth of graph theory will suggest our basic definition of a graph.

1.1.1. Example. *The Königsberg Bridge Problem.* The city of Königsberg was located on the Pregel river in Prussia. The city occupied two islands plus areas on both banks. These regions were linked by seven bridges as shown on the left below. The citizens wondered whether they could leave home, cross every bridge exactly once, and return home. The problem reduces to traversing the figure on the right, with heavy dots representing land masses and curves representing bridges.





The model on the right makes it easy to argue that the desired traversal does not exist. Each time we enter and leave a land mass, we use two bridges ending at it. We can also pair the first bridge with the last bridge on the land mass where we begin and end. Thus existence of the desired traversal requires that each land mass be involved in an even number of bridges. This necessary condition did not hold in Königsberg. ■

The Königsberg Bridge Problem becomes more interesting when we show in Section 1.2 which configurations have traversals. Meanwhile, the problem suggests a general model for discussing such questions.

1.1.2. Definition. A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its **endpoints**.

We **draw** a graph on paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints.

1.1.3. Example. In the graph in Example 1.1.1, the vertex set is $\{x, y, z, w\}$, the edge set is $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, and the assignment of endpoints to edges can be read from the picture.

Note that edges e_1 and e_2 have the same endpoints, as do e_3 and e_4. Also, if we had a bridge over an inlet, then its ends would be in the same land mass, and we would draw it as a curve with both ends at the same point. We have appropriate terms for these types of edges in graphs. ■

1.1.4. Definition. A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints.

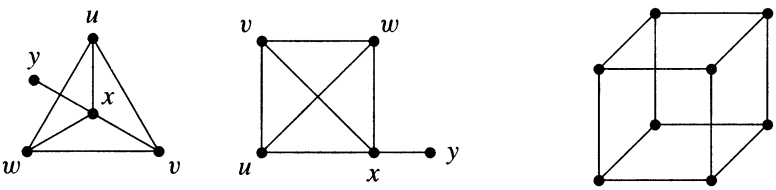
A **simple graph** is a graph having no loops or multiple edges. We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with endpoints u and v.

When u and v are the endpoints of an edge, they are **adjacent** and are **neighbors**. We write $u \leftrightarrow v$ for "u is adjacent to v".

In many important applications, loops and multiple edges do not arise, and we restrict our attention to simple graphs. In this case an edge is determined by its endpoints, so we can *name* the edge by its endpoints, as stated in Definition 1.1.4. Thus in a *simple* graph we view an edge as an unordered pair of vertices and can ignore the formality of the relation associating endpoints to edges. This book emphasizes simple graphs.

1.1.5. Example. On the left below are two drawings of a simple graph. The vertex set is $\{u, v, w, x, y\}$, and the edge set is $\{uv, uw, ux, vx, vw, xw, xy\}$.

The terms "vertex" and "edge" arise from solid geometry. A cube has vertices and edges, and these form the vertex set and edge set of a graph. It is drawn on the right below, omitting the names of vertices and edges. ∎



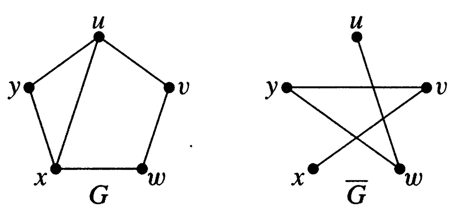
A graph is **finite** if its vertex set and edge set are finite. We adopt the convention that **every graph mentioned in this book is finite**, unless explicitly constructed otherwise.

1.1.6.* Remark. The **null graph** is the graph whose vertex set and edge set are empty. Extending general theorems to the null graph introduces unnecessary distractions, so we ignore it. All statements and exercises should be considered only for graphs with a nonempty set of vertices. ∎

GRAPHS AS MODELS

Graphs arise in many settings. The applications suggest useful concepts and terminology about the structure of graphs.

1.1.7. Example. *Acquaintance relations and subgraphs.* Does every set of six people contain three mutual acquaintances or three mutual strangers? Since "acquaintance" is symmetric, we model it using a simple graph with a vertex for each person and an edge for each acquainted pair. The "nonacquaintance" relation on the same set yields another graph with the "complementary" set of edges. We introduce terms for these concepts. ∎



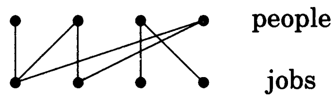
1.1.8. Definition. The **complement** $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A **clique** in a graph is a set of pairwise adjacent vertices. An **independent set** (or **stable set**) in a graph is a set of pairwise nonadjacent vertices.

In the graph G of Example 1.1.7, $\{u, x, y\}$ is a clique of size 3 and $\{u, w\}$ is an independent set of size 2, and these are the largest such sets. These values reverse in the complement $\overline{G}$, since cliques become independent sets (and vice versa) under complementation. The question in Example 1.1.7 asks whether it is true that every 6-vertex graph has a clique of size 3 or an independent set of size 3 (Exercise 29). Deleting edge ux from G yields a 5-vertex graph having no clique or independent set of size 3.

1.1.9. Example. *Job assignments and bipartite graphs.* We have m jobs and n people, but not all people are qualified for all jobs. Can we fill the jobs with qualified people? We model this using a simple graph H with vertices for the jobs and people; job j is adjacent to person p if p can do j.

Each job is to be filled by exactly one person, and each person can hold at most one of the jobs. Thus we seek m pairwise disjoint edges in H (viewing edges as pairs of vertices). Chapter 3 shows how to test for this; it can't be done in the graph below.

The use of graphs to model relations between two disjoint sets has many important applications. These are the graphs whose vertex sets can be partitioned into two independent sets; we need a name for them. ■

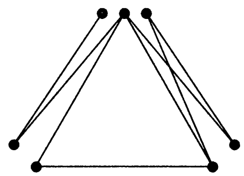


1.1.10. Definition. A graph G is **bipartite** if $V(G)$ is the union of two disjoint (possibly empty) independent sets called **partite sets** of G.

1.1.11. Example. *Scheduling and graph coloring.* Suppose we must schedule Senate committee meetings into designated weekly time periods. We cannot assign two committees to the same time if they have a common member. How many different time periods do we need?

We create a vertex for each committee, with two vertices adjacent when their committees have a common member. We must assign labels (time periods) to the vertices so the endpoints of each edge receive different labels. In the graph below, we can use one label for each of the three independent sets of vertices grouped closely together. The members of a clique must receive distinct labels, so in this example the minimum number of time periods is three.

Since we are only interested in partitioning the vertices, and the labels have no numerical value, it is convenient to call them **colors**. ■



1.1.12. Definition. The **chromatic number** of a graph G, written $\chi(G)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. A graph G is **k-partite** if $V(G)$ can be expressed as the union of k (possibly empty) independent sets.

This generalizes the idea of bipartite graphs, which are 2-partite. Vertices given the same color must form an independent set, so $\chi(G)$ is the minimum number of independent sets needed to partition $V(G)$. A graph is k-partite if and only if its chromatic number is at most k. We use the term "partite set" when referring to a set in a partition into independent sets.

We study chromatic number and graph colorings in Chapter 5. The most (in)famous problem in graph theory involves coloring of "maps".

1.1.13. Example. *Maps and coloring.* Roughly speaking, a **map** is a partition of the plane into connected regions. Can we color the regions of every map using at most four colors so that neighboring regions have different colors?

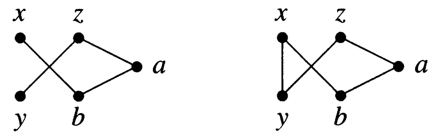
To relate map coloring to graph coloring, we introduce a vertex for each region and an edge for regions sharing a boundary. The map question asks whether the resulting graph must have chromatic number at most 4. The graph can be drawn in the plane without crossing edges; such graphs are **planar**. The graph before Definition 1.1.12 is planar; that drawing has a crossing, but another drawing has no crossings. We study planar graphs in Chapter 6. ■

1.1.14. Example. *Routes in road networks.* We can model a road network using a graph with edges corresponding to road segments between intersections. We can assign edge weights to measure distance or travel time. In this context edges do represent physical links. How can we find the shortest route from x to y? We show how to compute this in Chapter 2.

If the vertices of the graph represent our house and other places to visit, then we may want to follow a route that visits every vertex exactly once, so as to visit everyone without overstaying our welcome. We consider the existence of such a route in Chapter 7.

We need terms to describe these two types of routes in graphs. ■

1.1.15. Definition. A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.



Above we show a path and a cycle, as demonstrated by listing the vertices in the order x, b, a, z, y. Dropping one edge from a cycle produces a path. In studying the routes in road networks, we think of paths and cycles *contained* in the graph. Also, we hope that every vertex in the network can be reached from every other. The next definition makes these concepts precise.

1.1.16. Definition. A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that "G **contains** H".

A graph G is **connected** if each pair of vertices in G belongs to a path; otherwise, G is **disconnected**.

The graph before Definition 1.1.12 has three subgraphs that are cycles. It is a connected graph, but the graph in Example 1.1.9 is not.

MATRICES AND ISOMORPHISM

How do we specify a graph? We can list the vertices and edges (with endpoints), but there are other useful representations. Saying that a graph is **loopless** means that multiple edges are allowed but loops are not.

1.1.17. Definition. Let G be a loopless graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G) = \{e_1, \dots, e_m\}$. The **adjacency matrix** of G, written $A(G)$, is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$. The **incidence matrix** $M(G)$ is the n-by-m matrix in which entry $m_{i,j}$ is 1 if v_i is an endpoint of e_j and otherwise is 0.

If vertex v is an endpoint of edge e, then v and e are **incident**. The **degree** of vertex v (in a loopless graph) is the number of incident edges.

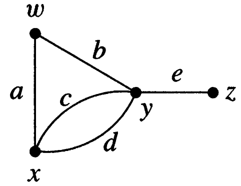
The appropriate way to define adjacency matrix, incidence matrix, or vertex degrees for graphs with loops depends on the application; Sections 1.2 and 1.3 discuss this.

1.1.18. Remark. An adjacency matrix is determined by a vertex ordering. Every adjacency matrix is **symmetric** ($a_{i,j} = a_{j,i}$ for all i, j). An adjacency matrix of a simple graph G has entries 0 or 1, with 0s on the diagonal. The degree of v is the sum of the entries in the row for v in either $A(G)$ or $M(G)$. ■

1.1.19. Example. For the loopless graph G below, we exhibit the adjacency matrix and incidence matrix that result from the vertex ordering w, x, y, z and the edge ordering a, b, c, d, e. The degree of y is 4, by viewing the graph or by summing the row for y in either matrix. ■

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$A(G)$



G

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$M(G)$

Presenting an adjacency matrix for a graph implicitly names the vertices by the order of the rows; the ith vertex corresponds to the ith row and column. Storing a graph in a computer requires naming the vertices.

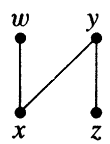
Nevertheless, we want to study properties (like connectedness) that do not depend on these names. Intuitively, the structural properties of G and H will be the same if we can rename the vertices of G using the vertices in H so that G will actually become H. We make the definition precise for simple graphs. The renaming is a function from $V(G)$ to $V(H)$ that assigns each element of $V(H)$ to one element of $V(G)$, thus pairing them up. Such a function is a *one-to-one correspondence* or **bijection** (see Appendix A). Saying that the renaming turns G into H is saying that the vertex bijection preserves the adjacency relation.

1.1.20. Definition. An **isomorphism** from a simple graph G to a simple graph H is a bijection $f\colon V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is **isomorphic to** H", written $G \cong H$, if there is an isomorphism from G to H.

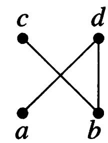
1.1.21. Example. The graphs G and H drawn below are 4-vertex paths. Define the function $f\colon V(G) \to V(H)$ by $f(w) = a$, $f(x) = d$, $f(y) = b$, $f(z) = c$. To show that f is an isomorphism, we check that f preserves edges and non-edges. Note that rewriting $A(G)$ by placing the rows in the order w, y, z, x and the columns also in that order yields $A(H)$, as illustrated below; this verifies that f is an isomorphism.

Another isomorphism maps w, x, y, z to c, b, d, a, respectively. ■

G



H



$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ w \\ y \\ z \\ x \end{array} \begin{array}{c} \begin{array}{cccc} w & y & z & x \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

1.1.22. Remark. *Finding isomorphisms.* As suggested in Example 1.1.21, presenting the adjacency matrices with vertices ordered so that the matrices are identical is one way to prove that two graphs are isomorphic. Applying a permutation σ to both the rows and the columns of $A(G)$ has the effect of reordering the vertices of G. If the new matrix equals $A(H)$, then the permutation yields an isomorphism. One can also verify preservation of the adjacency relation without writing out the matrices.

In order for an explicit vertex bijection to be an isomorphism from G to H, the image in H of a vertex v in G must behave in H as v does in G. For example, they must have the same degree. ■

1.1.23.* Remark. *Isomorphism for non-simple graphs.* The definition of isomorphism extends to graphs with loops and multiple edges, but the precise statement needs the language of Definition 1.1.2.

> An **isomorphism** from G to H is a bijection f that maps $V(G)$ to $V(H)$ and $E(G)$ to $E(H)$ such each edge of G with endpoints u and v is mapped to an edge with endpoints $f(u)$ and $f(v)$.

This technicality will not concern us, because we will study isomorphism only in the context of simple graphs. ■

Since H is isomorphic to G whenever G is isomorphic to H, we often say "G and H are isomorphic" (meaning to each other). The adjective "isomorphic" applies only to pairs of graphs; "G is isomorphic" by itself has no meaning (we respond, "isomorphic to what?"). Similarly, we may say that a set of graphs is "pairwise isomorphic" (taken two at a time), but it doesn't make sense to say "this set of graphs is isomorphic".

A **relation** on a set S is a collection of ordered pairs from S. An **equivalence relation** is a relation that is reflexive, symmetric, and transitive (see Appendix A). For example, the adjacency relation on the set of vertices of a graph is symmetric, but it is not reflexive and rarely is transitive. On the other hand, the **isomorphism relation**, consisting of the set of ordered pairs (G, H) such that G is isomorphic to H, does have all three properties.

1.1.24. Proposition. The isomorphism relation is an equivalence relation on the set of (simple) graphs.

Proof: *Reflexive property.* The identity permutation on $V(G)$ is an isomorphism from G to itself. Thus $G \cong G$.

Symmetric property. If $f\colon V(G) \to V(H)$ is an isomorphism from G to H, then f^{-1} is an isomorphism from H to G, because the statement "$uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$" yields "$xy \in E(H)$ if and only if $f^{-1}(x)f^{-1}(y) \in E(H)$". Thus $G \cong H$ implies $H \cong G$.

Transitive property. Suppose that $f\colon V(F) \to V(G)$ and $g\colon V(G) \to V(H)$ are isomorphisms. We are given "$uv \in E(F)$ if and only if $f(u)f(v) \in E(G)$" and "$xy \in E(G)$ if and only if $g(x)g(y) \in E(H)$". Since f is an isomorphism, for every $xy \in E(G)$ we can find $uv \in E(F)$ such that $f(u) = x$ and $f(v) = y$. This yields "$uv \in E(F)$ if and only if $g(f(u))g(f(v)) \in E(H)$". Thus the composition $g \circ f$ is an isomorphism from F to H. We have proved that $F \cong G$ and $G \cong H$ together imply $F \cong H$. ■

An equivalence relation partitions a set into **equivalence classes**; two elements satisfy the relation if and only if they lie in the same class.

1.1.25. Definition. An **isomorphism class** of graphs is an equivalence class of graphs under the isomorphism relation.

Paths with n vertices are pairwise isomorphic; the set of all n-vertex paths forms an isomorphism class.

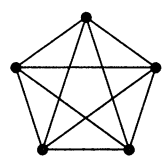
1.1.26. Remark. *"Unlabeled" graphs and isomorphism classes.* When discussing a graph G, we have a fixed vertex set, but our structural comments apply also to every graph isomorphic to G. Our conclusions are independent of the names (labels) of the vertices. Thus, we use the informal expression "unlabeled graph" to mean an isomorphism class of graphs.

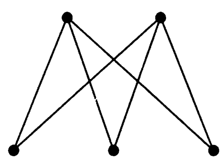
When we draw a graph, its vertices are named by their physical locations, even if we give them no other names. Hence a drawing of a graph is a member of its isomorphism class, and we just call it a graph. When we redraw a graph to display some structural aspect, we have chosen a more convenient member of the isomorphism class, still discussing the same "unlabeled graph". ■

When discussing structure of graphs, it is convenient to have names and notation for important isomorphism classes. We want the flexibility to refer to the isomorphism class or to any representative of it.

1.1.27. Definition. The (unlabeled) path and cycle with n vertices are denoted P_n and C_n, respectively; an n-**cycle** is a cycle with n vertices. A **complete graph** is a simple graph whose vertices are pairwise adjacent; the (unlabeled) complete graph with n vertices is denoted K_n. A **complete bipartite graph** or **biclique** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes r and s, the (unlabeled) biclique is denoted $K_{r,s}$.

K_5





$K_{2,3}$

1.1.28.* Remark. We have defined a *complete graph* as a graph whose vertices are pairwise adjacent, while a *clique* is a set of pairwise adjacent vertices in a graph. Many authors use the terms interchangeably, but the distinction allows us to discuss cliques in the same language as independent sets.

In the bipartite setting, we simply use "biclique" to abbreviate "complete bipartite graph". The alternative name "biclique" is a reminder that a complete bipartite graph is generally *not* a complete graph (Exercise 1). ■

1.1.29. Remark. *A graph by any other name . . .* When we name a graph without naming its vertices, we often mean its isomorphism class. Technically, "H is a subgraph of G" means that some subgraph of G is isomorphic to H (we say "G contains a **copy** of H"). Thus C_3 is a subgraph of K_5 (every complete graph with 5 vertices has 10 subgraphs isomorphic to C_3) but not of $K_{2,3}$.

Similarly, asking whether G "is" C_n means asking whether G is isomorphic to a cycle with n vertices. ■

The structural properties of a graph are determined by its adjacency relation and hence are preserved by isomorphism. We can prove that G and H are *not* isomorphic by finding some structural property in which they differ. If they have different number of edges, or different subgraphs, or different complements, etc., then they cannot be isomorphic.

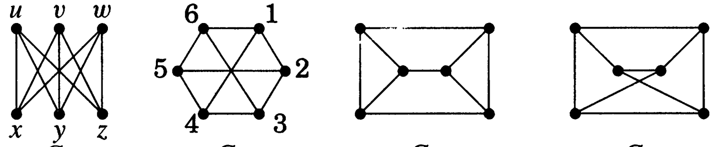
On the other hand, checking that a few structural properties are the same does not imply that $G \cong H$. To prove that $G \cong H$, we must present a bijection $f\colon V(G) \to V(H)$ that preserves the adjacency relation.

1.1.30. Example. *Isomorphic or not?* Each graph below has six vertices and nine edges and is connected, but these graphs are not pairwise isomorphic.

To prove that $G_1 \cong G_2$, we give names to the vertices, specify a bijection, and check that it preserves the adjacency relation. As labeled below, the bijection that sends u, v, w, x, y, z to $1, 3, 5, 2, 4, 6$, respectively, is an isomorphism from G_1 to G_2. The map sending u, v, w, x, y, z to $6, 4, 2, 1, 3, 5$, respectively, is another isomorphism.

Both G_1 and G_2 are bipartite; they are drawings of $K_{3,3}$ (as is G_4). The graph G_3 contains K_3, so its vertices cannot be partitioned into two independent sets. Thus G_3 is not isomorphic to the others.

Sometimes we can test isomorphism quickly using the complements. Simple graphs G and H are isomorphic if and only if their complements are isomorphic (Exercise 4). Here $\overline{G_1}, \overline{G_2}, \overline{G_4}$ all consist of two disjoint 3-cycles and are not connected, but $\overline{G_3}$ is a 6-cycle and is connected. ■

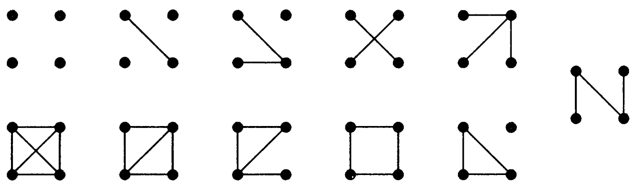


G_1 G_2 G_3 G_4

1.1.31. Example. *The number of n-vertex graphs.* When choosing two vertices from a set of size n, we can pick one and then the other but don't care about the order, so the number of ways is $n(n-1)/2$. (The notation for the number of ways to choose k elements from n elements is $\binom{n}{k}$, read "n choose k". These numbers are called **binomial coefficients**; see Appendix A for further background.)

In a simple graph with a vertex set X of size n, each vertex pair may form an edge or may not. Making the choice for each pair specifies the graph, so the number of simple graphs with vertex set X is $2^{\binom{n}{2}}$.

For example, there are 64 simple graphs on a fixed set of four vertices. These graphs form only 11 isomorphism classes. The classes appear below in complementary pairs; only P_4 is isomorphic to its complement. Isomorphism classes have different sizes, so we cannot count the isomorphism classes of n-vertex simple graphs by dividing $2^{\binom{n}{2}}$ by the size of a class. ■



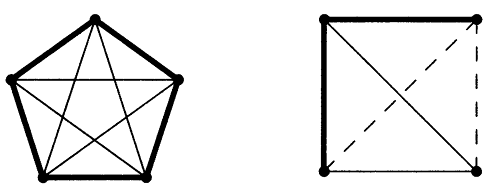
DECOMPOSITION AND SPECIAL GRAPHS

The example $P_4 \cong \overline{P}_4$ suggests a family of graph problems.

1.1.32. Definition. A graph is **self-complementary** if it is isomorphic to its complement. A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

An n-vertex graph H is self-complementary if and only if K_n has a decomposition consisting of two copies of H.

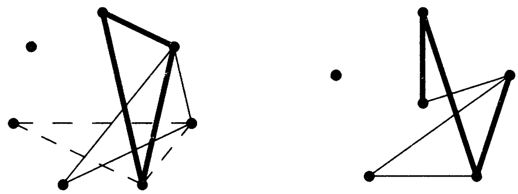
1.1.33. Example. We can decompose K_5 into two 5-cycles, and thus the 5-cycle is self-complementary. Any n-vertex graph and its complement decompose K_n. Also $K_{1,n-1}$ and K_{n-1} decompose K_n, even though one of these subgraphs omits a vertex. On the right below we show a decomposition of K_4 using three copies of P_3. Exercises 31–39 consider graph decompositions. ■



1.1.34.* Example. The question of which complete graphs decompose into copies of K_3 is a fundamental question in the theory of combinatorial designs.

On the left below we suggest such a decomposition for K_7. Rotating the triangle through seven positions uses each edge exactly once.

On the right we suggest a decomposition of K_6 into copies of P_4. Placing one vertex in the center groups the edges into three types: the outer 5-cycle, the inner (crossing) 5-cycle on those vertices, and the edges involving the central vertex. Each 4-vertex path in the decomposition uses one edge of each type; we rotate the picture to get the next path. ■

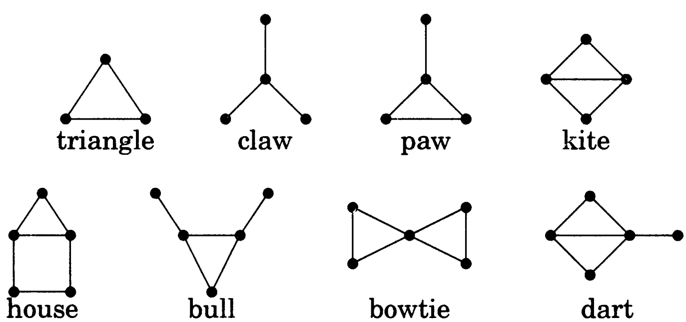


We referred to a copy of K_3 as a *triangle*. Short names for graphs that arise frequently in structural discussions can be convenient.

1.1.35. Example. *The Graph Menagerie.* A catchy "name" for a graph usually comes from some drawing of the graph. We also use such a name for all other drawings, and hence it is best viewed as a name for the isomorphism class. Below we give names to several graphs with at most five vertices.

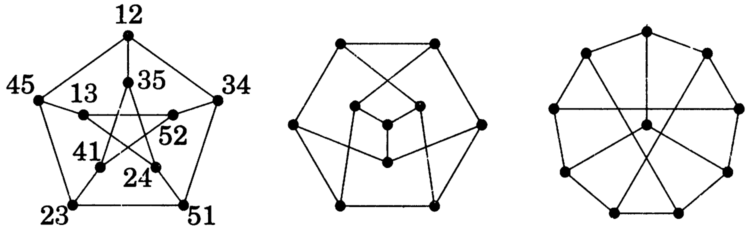
Among these the most important are the **triangle** (K_3) and the **claw** ($K_{1,3}$). We also sometimes discuss the **paw** ($K_{1,3} + e$)and the **kite** ($K_4 - e$); the others arise less frequently.

The complements of the graphs in the first row are disconnected. The complement of the house is P_5, and the bull is self-complementary. Exercise 39 asks which of these graphs can be used to decompose K_6. ■



In order to decompose H into copies of G, the number of edges of G must divide the number of edges of H. This is not sufficient, since K_5 does not decompose into two copies of the kite.

1.1.36. Definition. The **Petersen graph** is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.



We have drawn the Petersen graph in three ways above. It is a useful example so often that an entire book was devoted to it (Holton–Sheehan [1993]). Its properties follow from the statement of its adjacency relation that we have used as the definition.

1.1.37. Example. *Structure of the Petersen graph.* Using $[5] = \{1, 2, 3, 4, 5\}$ as our 5-element set, we write the pair $\{a, b\}$ as ab or ba. Since 12 and 34 are disjoint, they are adjacent vertices when we form the graph, but 12 and 23 are not. For each 2-set ab, there are three ways to pick a 2-set from the remaining three elements of [5], so every vertex has degree 3.

The Petersen graph consists of two disjoint 5-cycles plus edges that pair up vertices on the two 5-cycles. The disjointness definition tells us that 12, 34, 51, 23, 45 in order are the vertices of a 5-cycle, and similarly this holds for the remaining vertices 13, 52, 41, 35, 24. Also 13 is adjacent to 45, and 52 is adjacent to 34, and so on, as shown on the left above.

We use this name even when we do not specify the vertex labeling; in essence, we use "Petersen graph" to name an isomorphism class. To show that the graphs above are pairwise isomorphic, it suffices to name the vertices of each using the 2-element subsets of [5] so that in each case the adjacency relation is disjointness (Exercise 24). ■

1.1.38. Proposition. If two vertices are nonadjacent in the Petersen graph, then they have exactly one common neighbor.

Proof: Nonadjacent vertices are 2-sets sharing one element; their union S has size 3. A vertex adjacent to both is a 2-set disjoint from both. Since the 2-sets are chosen from $\{1, 2, 3, 4, 5\}$, there is exactly one 2-set disjoint from S. ■

1.1.39. Definition. The **girth** of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.

1.1.40. Corollary. The Petersen graph has girth 5.

Proof: The graph is simple, so it has no 1-cycle or 2-cycle. A 3-cycle would require three pairwise-disjoint 2-sets, which can't occur among 5 elements.

A 4-cycle in the absence of 3-cycles would require nonadjacent vertices with two common neighbors, which Proposition 1.1.38 forbids. Finally, the vertices 12, 34, 51, 23, 45 yield a 5-cycle, so the girth is 5. ■

The Petersen graph is highly symmetric. Every permutation of $\{1, 2, 3, 4, 5\}$ generates a permutation of the 2-subsets that preserves the disjointness relation. Thus there are at least $5! = 120$ isomorphisms from the Petersen graph to itself. Exercise 43 confirms that there are no others.

1.1.41.* Definition. An **automorphism** of G is an isomorphism from G to G. A graph G is **vertex-transitive** if for every pair $u, v \in V(G)$ there is an automorphism that maps u to v.

The automorphisms of G are the permutations of $V(G)$ that can be applied to both the rows and the columns of $A(G)$ without changing $A(G)$.

1.1.42.* Example. *Automorphisms.* Let G be the path with vertex set $\{1, 2, 3, 4\}$ and edge set $\{12, 23, 34\}$. This graph has two automorphisms: the identity permutation and the permutation that switches 1 with 4 and switches 2 with 3. Interchanging vertices 1 and 2 is not an automorphism of G, although G is isomorphic to the graph with vertex set $\{1, 2, 3, 4\}$ and edge set $\{21, 13, 34\}$.

In $K_{r,s}$, permuting the vertices of one partite set does not change the adjacency matrix; this leads to $r!s!$ automorphisms. When $r = s$, we can also interchange the partite sets; $K_{t,t}$ has $2(t!)^2$ automorphisms.

The biclique $K_{r,s}$ is vertex-transitive if and only if $r = s$. If $n > 2$, then P_n is not vertex-transitive, but every cycle is vertex-transitive. The Petersen graph is vertex-transitive. ■

We can prove a statement for every vertex in a vertex-transitive graph by proving it for one vertex. Vertex-transitivity guarantees that the graph "looks the same" from each vertex.

EXERCISES

Solutions to problems generally require clear explanations written in sentences. The designations on problems have the following meanings:

"(−)" = easier or shorter than most,

"(+)" = harder or longer than most,

"(!)" = particularly useful or instructive,

"(∗)" = involves concepts marked optional in the text.

The exercise sections begin with easier problems to check understanding, ending with a line of dots. The remaining problems roughly follow the order of material in the text.

1.1.1. (−) Determine which complete bipartite graphs are complete graphs.

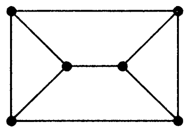
1.1.2. (−) Write down all possible adjacency matrices and incidence matrices for a 3-vertex path. Also write down an adjacency matrix for a path with six vertices and for a cycle with six vertices.

1.1.3. (−) Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$.

1.1.4. (−) From the definition of isomorphism, prove that $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

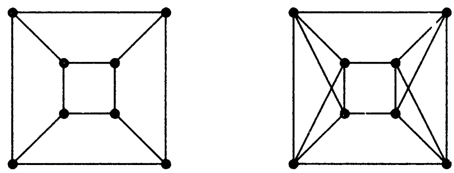
1.1.5. (−) Prove or disprove: If every vertex of a simple graph G has degree 2, then G is a cycle.

1.1.6. (−) Determine whether the graph below decomposes into copies of P_4.



1.1.7. (−) Prove that a graph with more than six vertices of odd degree cannot be decomposed into three paths.

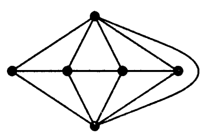
1.1.8. (−) Prove that the 8-vertex graph on the left below decomposes into copies of $K_{1,3}$ and also into copies of P_4.



1.1.9. (−) Prove that the graph on the right above is isomorphic to the complement of the graph on the left.

1.1.10. (−) Prove or disprove: The complement of a simple disconnected graph must be connected.

1.1.11. Determine the maximum size of a clique and the maximum size of an independent set in the graph below.



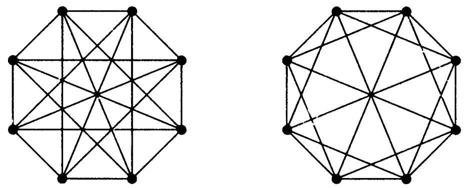
1.1.12. Determine whether the Petersen graph is bipartite, and find the size of its largest independent set.

1.1.13. Let G be the graph whose vertex set is the set of k-tuples with coordinates in $\{0, 1\}$, with x adjacent to y when x and y differ in exactly one position. Determine whether G is bipartite.

1.1.14. (!) Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be partitioned into 1-by-2 and 2-by-1 rectangles. Using the same argument, make a general statement about all bipartite graphs.

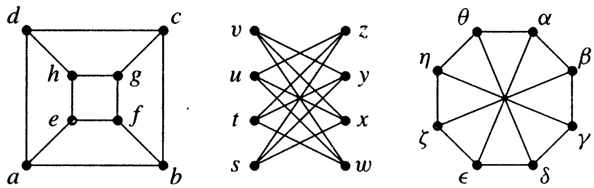
1.1.15. Consider the following four families of graphs: $A = \{\text{paths}\}$, $B = \{\text{cycles}\}$, $C = \{\text{complete graphs}\}$, $D = \{\text{bipartite graphs}\}$. For each pair of these families, determine all isomorphism classes of graphs that belong to both families.

1.1.16. Determine whether the graphs below are isomorphic.

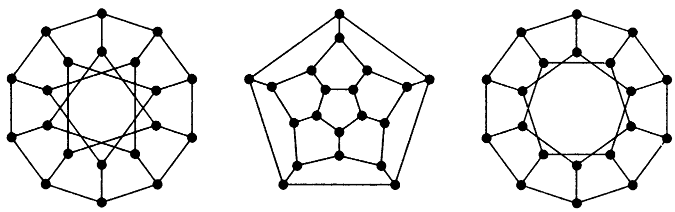


1.1.17. Determine the number of isomorphism classes of simple 7-vertex graphs in which every vertex has degree 4.

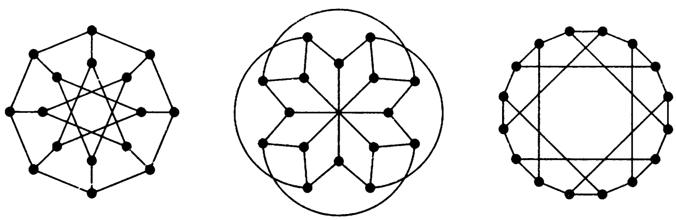
1.1.18. Determine which pairs of graphs below are isomorphic.



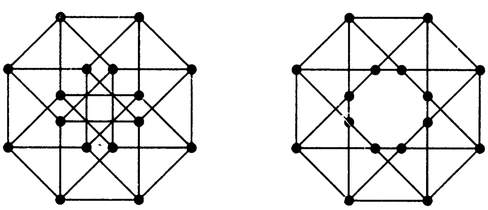
1.1.19. Determine which pairs of graphs below are isomorphic.



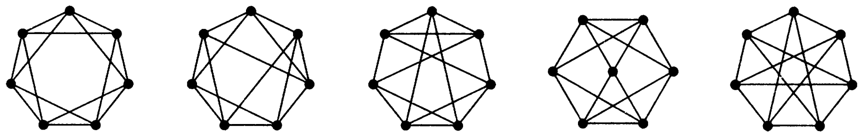
1.1.20. Determine which pairs of graphs below are isomorphic.



1.1.21. Determine whether the graphs below are bipartite and whether they are isomorphic. (The graph on the left appears on the cover of Wilson–Watkins [1990].)



1.1.22. (!) Determine which pairs of graphs below are isomorphic, presenting the proof by testing the smallest possible number of pairs.

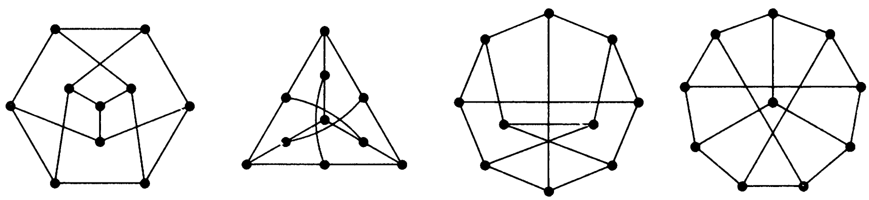


1.1.23. In each class below, determine the smallest n such that there exist nonisomorphic n-vertex graphs having the same list of vertex degrees.

(a) all graphs, (b) loopless graphs, (c) simple graphs.

(Hint: Since each class contains the next, the answers form a nondecreasing triple. For part (c), use the list of isomorphism classes in Example 1.1.31.)

1.1.24. (!) Prove that the graphs below are all drawings of the Petersen graph (Definition 1.1.36). (Hint: Use the disjointness definition of adjacency.)



1.1.25. (!) Prove that the Petersen graph has no cycle of length 7.

1.1.26. (!) Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least $2k$ vertices. Determine all such graphs with exactly $2k$ vertices.

1.1.27. (!) Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with exactly $k^2 + 1$ vertices.

1.1.28. (+) *The Odd Graph O_k.* The vertices of the graph O_k are the k-element subsets of $\{1, 2, \ldots, 2k + 1\}$. Two vertices are adjacent if and only if they are disjoint sets. Thus O_2 is the Petersen graph. Prove that the girth of O_k is 6 if $k \geq 3$.

1.1.29. Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers.

1.1.30. Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry in A^2 and in MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

1.1.31. (!) Prove that a self-complementary graph with n vertices exists if and only if n or $n - 1$ is divisible by 4. (Hint: When n is divisible by 4, generalize the structure of P_4 by splitting the vertices into four groups. For $n \equiv 1$ mod 4, add one vertex to the graph constructed for $n - 1$.)

1.1.32. Determine which bicliques decompose into two isomorphic subgraphs.

1.1.33. For $n = 5$, $n = 7$, and $n = 9$, decompose K_n into copies of C_n.

1.1.34. (!) Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of P_4.

1.1.35. (!) Prove that K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n+1$ is not divisible by 3. (Hint: For the case where n is divisible by 3, split the vertices into three sets of equal size.)

1.1.36. Prove that if K_n decomposes into triangles, then $n-1$ or $n-3$ is divisible by 6.

1.1.37. Let G be a graph in which every vertex has degree 3. Prove that G has no decomposition into paths that each have at least 5 vertices.

1.1.38. (!) Let G be a simple graph in which every vertex has degree 3. Prove that G decomposes into claws if and only if G is bipartite.

1.1.39. (+) Determine which of the graphs in Example 1.1.35 can be used to form a decomposition of K_6 into pairwise-isomorphic subgraphs. (Hint: Each graph that is not excluded by some divisibility condition works.)

1.1.40. (∗) Count the automorphisms of P_n, C_n, and K_n.

1.1.41. (∗) Construct a simple graph with six vertices that has only one automorphism. Construct a simple graph that has exactly three automorphisms. (Hint: Think of a rotating triangle with appendages to prevent flips.)

1.1.42. (∗) Verify that the set of automorphisms of G has the following properties:

a) The composition of two automorphisms is an automorphism.
b) The identity permutation is an automorphism.
c) The inverse of an automorphism is also an automorphism.
d) Composition of automorphisms satisfies the associative property.

(Comment: Thus the set of automorphisms satisfies the defining properties for a *group*.)

1.1.43. (∗) *Automorphisms of the Petersen graph.* Consider the Petersen graph as defined by disjointness of 2-sets in $\{1,2,3,4,5\}$. Prove that every automorphism maps the 5-cycle with vertices $12, 34, 51, 23, 45$ to a 5-cycle with vertices ab, cd, ea, bc, de determined by a permutation of $\{1,2,3,4,5\}$ taking elements 1,2,3,4,5 to a, b, c, d, e, respectively. (Comment: This implies that there are only 120 automorphisms.)

1.1.44. (∗) The Petersen graph has even more symmetry than vertex-transitivity. Let $P = (u_0, u_1, u_2, u_3)$ and $Q = (v_0, v_1, v_2, v_3)$ be paths with three edges in the Petersen graph. Prove that there is exactly one automorphism of the Petersen graph that maps u_i into v_i for $i = 0, 1, 2, 3$. (Hint: Use the disjointness description.)

1.1.45. (∗) Construct a graph with 12 vertices in which every vertex has degree 3 and the only automorphism is the identity.

1.1.46. (∗) *Edge-transitivity.* A graph G is **edge-transitive** if for all $e, f \in E(G)$ there is an automorphism of G that maps the endpoints of e to the endpoints of f (in either order). Prove that the graphs of Exercise 1.1.21 are vertex-transitive and edge-transitive. (Comment: Complete graphs, bicliques, and the Petersen graph are edge-transitive.)

1.1.47. (∗) *Edge-transitive versus vertex-transitive.*

a) Let G be obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2. Prove that G is edge-transitive but not vertex-transitive.

b) Suppose that G is edge-transitive but not vertex-transitive and has no vertices of degree 0. Prove that G is bipartite.

c) Prove that the graph in Exercise 1.1.6 is vertex-transitive but not edge-transitive.

1.2. Paths, Cycles, and Trails

In this section we return to the Königsberg Bridge Problem, determining when it is possible to traverse all the edges of a graph. We also we develop useful properties of connection, paths, and cycles.

Before embarking on this, we review an important technique of proof. Many statements in graph theory can be proved using the principle of induction. Readers unfamiliar with induction should read the material on this proof technique in Appendix A. Here we describe the form of induction that we will use most frequently, in order to familiarize the reader with a template for proof.

1.2.1. Theorem. (Strong Principle of Induction). Let $P(n)$ be a statement with an integer parameter n. If the following two conditions hold, then $P(n)$ is true for each positive integer n.
1) $P(1)$ is true.
2) For all $n > 1$, "$P(k)$ is true for $1 \le k < n$" implies "$P(n)$ is true".

Proof: We ASSUME the **Well Ordering Property** for the positive integers: every nonempty set of positive integers has a least element. Given this, suppose that $P(n)$ fails for some n. By the Well Ordering Property, there is a least n such that $P(n)$ fails. Statement (1) ensures that this value cannot be 1. Statement (2) ensures that this value cannot be greater than 1. The contradiction implies that $P(n)$ holds for every positive integer n. ■

In order to apply induction, we verify (1) and (2) for our sequence of statements. Verifying (1) is the **basis step** of the proof; verifying (2) is the **induction step**. The statement "$P(k)$ is true for all $k < n$" is the **induction hypothesis**, because it is the hypothesis of the implication proved in the induction step. The variable that indexes the sequence of statements is the **induction parameter**.

The induction parameter may be any integer function of the instances of our problem, such as the number of vertices or edges in a graph. We say that we are using "induction on n" when the induction parameter is n.

There are many ways to phrase inductive proofs. We can start at 0 to prove a statement for nonnegative integers. When our proof of $P(n)$ in the induction step makes use only of $P(n-1)$ from the induction hypothesis, the technique is called "ordinary" induction; making use of all previous statements is "strong" induction. We seldom distinguish between strong induction and ordinary induction; they are equivalent (see Appendix A).

Most students first learn ordinary induction in the following phrasing: 1) verify that $P(n)$ is true when $n = 1$, and 2) prove that if $P(n)$ is true when n is k, then $P(n)$ is also true when n is $k+1$. Proving $P(k+1)$ from $P(k)$ for $k \ge 1$ is equivalent to proving $P(n)$ from $P(n-1)$ for $n > 1$.

When we focus on proving the statement for the parameter value n in the induction step, we need not decide at the outset whether we are using strong induction or ordinary induction. The language is also simpler, since we avoid introducing a new name for the parameter. In Section 1.3 we will explain why this phrasing is also less prone to error.

CONNECTION IN GRAPHS

As defined in Definition 1.1.15, paths and cycles are graphs; a path *in* a graph G is a subgraph of G that is a path (similarly for cycles). We introduce further definitions to model other movements in graphs. A tourist wandering in a city (or a Königsberg pedestrian) may want to allow vertex repetitions but avoid edge repetitions. A mail carrier delivers mail to houses on both sides of the street and hence traverses each edge twice.

1.2.2. Definition. A **walk** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i. A **trail** is a walk with no repeated edge. A u, v-**walk** or u, v-**trail** has first vertex u and last vertex v; these are its **endpoints**. A u, v-**path** is a path whose vertices of degree 1 (its **endpoints**) are u and v; the others are **internal vertices**.

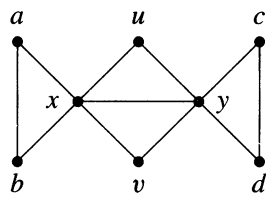
The **length** of a walk, trail, path, or cycle is its number of edges. A walk or trail is **closed** if its endpoints are the same.

1.2.3. Example. In the Königsberg graph (Example 1.1.1), the list $x, e_2, w, e_5, y, e_6, x, e_1, w, e_2, x$ is a closed walk of length 5; it repeats edge e_2 and hence is not a trail. Deleting the last edge and vertex yields a trail of length 4; it repeats vertices but not edges. The subgraph consisting of edges e_1, e_5, e_6 and vertices x, w, y is a cycle of length 3; deleting one of its edges yields a path. Two edges with the same endpoints (such as e_1 and e_2) form a cycle of length 2. A loop is a cycle of length 1. ■

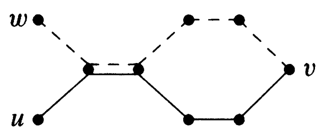
The reason for listing the edges in a walk is to distinguish among multiple edges when a graph is not simple. In a simple graph, a walk (or trail) is completely specified by its ordered list of vertices. We usually name a path, cycle, trail, or walk in a simple graph by listing only its vertices in order, even though it consists of both vertices and edges. When discussing a cycle, we can start at any vertex and do not repeat the first vertex at the end. We can use parentheses to clarify that this is a cycle and not a path.

1.2.4. Example. We illustrate the simplified notation in a simple graph. In the graph below, $a, x, a, x, u, y, c, d, y, v, x, b, a$ specifies a closed walk of length 12. Omitting the first two steps yields a closed trail.

The graph has five cycles: (a, b, x), (c, y, d), (u, x, y), (x, y, v), (u, x, v, y). The u, v-trail u, y, c, d, y, x, v contains the edges of the u, v-path u, y, x, v, but not of the u, v-path u, y, v. ■



Suppose we follow a path from u to v in a graph and then follow a path from v to w. The result need not be a u, w-path, because the u, v-path and v, w-path may have a common internal vertex. Nevertheless, the list of vertices and edges that we visit does form a u, w-walk. In the illustration below, the u, w-walk contains a u, w-path. Saying that a walk W **contains** a path P means that the vertices and edges of P occur as a sublist of the vertices and edges of W, in order but not necessarily consecutive.

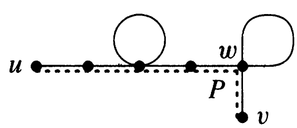


1.2.5. Lemma. Every u, v-walk contains a u, v-path.

Proof: We prove the statement by induction on the length l of a u, v-walk W.

Basis step: $l = 0$. Having no edge, W consists of a single vertex ($u = v$). This vertex is a u, v-path of length 0.

Induction step: $l \geq 1$. We suppose that the claim holds for walks of length less than l. If W has no repeated vertex, then its vertices and edges form a u, v-path. If W has a repeated vertex w, then deleting the edges and vertices between appearances of w (leaving one copy of w) yields a shorter u, v-walk W' contained in W. By the induction hypothesis, W' contains a u, v-path P, and this path P is contained in W. ■



Exercise 13b develops a shorter proof. We apply the lemma to properties of connection.

1.2.6. Definition. A graph G is **connected** if it has a u, v-path whenever $u, v \in V(G)$ (otherwise, G is **disconnected**). If G has a u, v-path, then u is **connected to** v in G. The **connection relation** on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.

"Connected" is an adjective we apply only to graphs and to pairs of vertices (we never say "v is disconnected" when v is a vertex). The phrase "u is connected to v" is convenient when writing proofs, but in adopting it we must clarify the distinction between connection and adjacency:

G has a u, v-path	$uv \in E(G)$
u and v are connected	u and v are adjacent
u is connected to v	u is joined to v
	u is adjacent to v

1.2.7. Remark. By Lemma 1.2.5, we can prove that a graph is connected by showing that from each vertex there is a walk to one particular vertex.

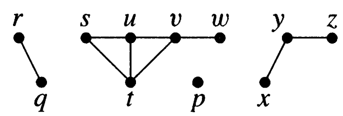
By Lemma 1.2.5, the connection relation is transitive: if G has a u, v-path and a v, w-path, then G has a u, w-path. It is also reflexive (paths of length 0) and symmetric (paths are reversible), so it is an equivalence relation. ■

Our next definition leads us to the equivalence classes of the connection relation. A *maximal* connected subgraph of G is a subgraph that is connected and is not contained in any other connected subgraph of G.

1.2.8. Definition. The **components** of a graph G are its maximal connected subgraphs. A component (or graph) is **trivial** if it has no edges; otherwise it is **nontrivial**. An **isolated vertex** is a vertex of degree 0.

The equivalence classes of the connection relation on $V(G)$ are the vertex sets of the components of G. An isolated vertex forms a trivial component, consisting of one vertex and no edge.

1.2.9. Example. The graph below has four components, one being an isolated vertex. The vertex sets of the components are $\{p\}$, $\{q, r\}$, $\{s, t, u, v, w\}$, and $\{x, y, z\}$; these are the equivalence classes of the connection relation. ■



1.2.10. Remark. Components are pairwise disjoint; no two share a vertex. Adding an edge with endpoints in distinct components combines them into one component. Thus adding an edge decreases the number of components by 0 or 1, and deleting an edge increases the number of components by 0 or 1. ■

1.2.11. Proposition. Every graph with n vertices and k edges has at least $n - k$ components.

Proof: An n-vertex graph with no edges has n components. By Remark 1.2.10, each edge added reduces this by at most 1, so when k edges have been added the number of components is still at least $n - k$. ■

Deleting a vertex or an edge can increase the number of components. Although deleting an edge can only increase the number of components by 1, deleting a vertex can increase it by many (consider the biclique $K_{1,m}$). When we obtain a subgraph by deleting a vertex, it must be a graph, so deleting the vertex also deletes all edges incident to it.

1.2.12. Definition. A **cut-edge** or **cut-vertex** of a graph is an edge or vertex whose deletion increases the number of components. We write $G - e$ or $G - M$ for the subgraph of G obtained by deleting an edge e or set of edges M. We write $G - v$ or $G - S$ for the subgraph obtained by deleting a vertex v or set of vertices S. An **induced subgraph** is a subgraph obtained by deleting a set of vertices. We write $G[T]$ for $G - \overline{T}$, where $\overline{T} = V(G) - T$; this is the subgraph of G **induced by** T.

When $T \subseteq V(G)$, the induced subgraph $G[T]$ consists of T and all edges whose endpoints are contained in T. The full graph is itself an induced subgraph, as are individual vertices. A set S of vertices is an independent set if and only if the subgraph induced by it has no edges.

1.2.13. Example. The graph of Example 1.2.9 has cut-vertices v and y. Its cut-edges are qr, vw, xy, and yz. (When we delete an edge, its endpoints remain.)

This graph has C_4 and P_5 as subgraphs but *not* as induced subgraphs. The subgraph induced by $\{s, t, u, v\}$ is a kite; the 4-vertex paths on these vertices are not induced subgraphs. The graph P_4 does occur as an induced subgraph; it is the subgraph induced by $\{s, t, v, w\}$ (also by $\{s, u, v, w\}$). ■

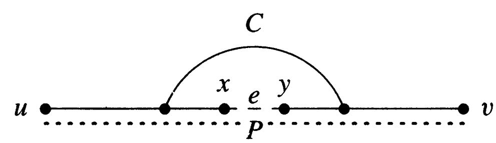
Next we characterize cut-edges in terms of cycles.

1.2.14. Theorem. An edge is a cut-edge if and only if it belongs to no cycle.

Proof: Let e be an edge in a graph G (with endpoints x, y), and let H be the component containing e. Since deletion of e affects no other component, it suffices to prove that $H - e$ is connected if and only if e belongs to a cycle.

First suppose that $H - e$ is connected. This implies that $H - e$ contains an x, y-path, and this path completes a cycle with e.

Now suppose that e lies in a cycle C. Choose $u, v \in V(H)$. Since H is connected, H has a u, v-path P. If P does not contain e, then P exists in $H - e$. If P contains e, suppose by symmetry that x is between u and y on P. Since $H - e$ contains a u, x-path along P, an x, y-path along C, and a y, v-path along P, the transitivity of the connection relation implies that $H - e$ has a u, v-path. We did this for all $u, v \in V(H)$, so $H - e$ is connected. ■



BIPARTITE GRAPHS

Our next goal is to characterize bipartite graphs using cycles. Characterizations are equivalence statements, like Theorem 1.2.14. When two conditions are equivalent, checking one also yields the other for free.

Characterizing a class **G** by a condition P means proving the equivalence "$G \in \mathbf{G}$ if and only if G satisfies P". In other words, P is both a **necessary** and a **sufficient** condition for membership in **G**.

Necessity	Sufficiency
$G \in \mathbf{G}$ *only if* G satisfies P	$G \in \mathbf{G}$ *if* G satisfies P
$G \in \mathbf{G} \Rightarrow G$ satisfies P	G satisfies P $\Rightarrow G \in \mathbf{G}$

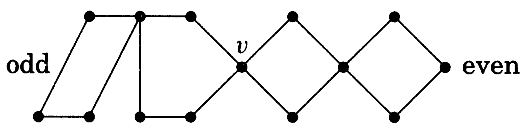
Recall that a loop is a cycle of length 1; also two distinct edges with the same endpoints form a cycle of length 2. A walk is **odd** or **even** as its length is odd or even. As in Lemma 1.2.5, a closed walk **contains** a cycle C if the vertices and edges of C occur as a sublist of W, in cyclic order but not necessarily consecutive. We can think of a closed walk or a cycle as starting at any vertex; the next lemma requires this viewpoint.

1.2.15. Lemma. Every closed odd walk contains an odd cycle.

Proof: We use induction on the length l of a closed odd walk W.

Basis step: $l = 1$. A closed walk of length 1 traverses a cycle of length 1.

Induction step: $l > 1$. Assume the claim for closed odd walks shorter than W. If W has no repeated vertex (other than first = last), then W itself forms a cycle of odd length. If vertex v is repeated in W, then we view W as starting at v and break W into two v, v-walks. Since W has odd length, one of these is odd and the other is even. The odd one is shorter than W. By the induction hypothesis, it contains an odd cycle, and this cycle appears in order in W. ■



1.2.16. Remark. A closed even walk need not contain a cycle; it may simply repeat. Nevertheless, if an edge e appears *exactly once* in a closed walk W, then W does contain a cycle through e. Let x, y be the endpoints of e. Deleting e from W leaves an x, y-walk that avoids e. By Lemma 1.2.5, this walk contains an x, y-path, and this path completes a cycle with e. (See Exercises 15–16.) ■

Lemma 1.2.15 will help us characterize bipartite graphs.

1.2.17. Definition. A **bipartition** of G is a specification of two disjoint independent sets in G whose union is $V(G)$. The statement "Let G be a bipartite graph with bipartition X, Y" specifies one such partition. An X, Y**-bigraph** is a bipartite graph with bipartition X, Y.

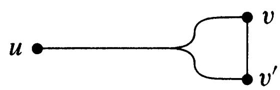
The sets of a bipartition are partite sets (Definition 1.1.10). A disconnected bipartite graph has more than one bipartition. A connected bipartite graph has only one bipartition, except for interchanging the two sets (Exercise 7).

1.2.18. Theorem. (König [1936]) A graph is bipartite if and only if it has no odd cycle.

Proof: *Necessity.* Let G be a bipartite graph. Every walk alternates between the two sets of a bipartition, so every return to the original partite set happens after an even number of steps. Hence G has no odd cycle.

Sufficiency. Let G be a graph with no odd cycle. We prove that G is bipartite by constructing a bipartition of each nontrivial component. Let u be a vertex in a nontrivial component H. For each $v \in V(H)$, let $f(v)$ be the minimum length of a u,v-path. Since H is connected, $f(v)$ is defined for each $v \in V(H)$.

Let $X = \{v \in V(H)\colon f(v) \text{ is even}\}$ and $Y = \{v \in V(H)\colon f(v) \text{ is odd}\}$. An edge v, v' within X or Y would create a closed odd walk using a shortest u,v-path, the edge vv', and the reverse of a shortest u,v'-path. By Lemma 1.2.15, such a walk must contain an odd cycle, which contradicts our hypothesis. Hence X and Y are independent sets. Also $X \cup Y = V(H)$, so H is an X,Y-bigraph. ■

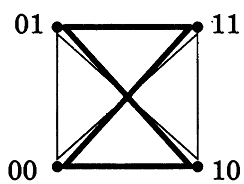


1.2.19. Remark. *Testing whether a graph is bipartite.* Theorem 1.2.18 implies that whenever a graph G is not bipartite, we can prove this statement by presenting an odd cycle in G. This is much easier than examining all possible bipartitions to prove that none work. When we want to prove that G *is* bipartite, we define a bipartition and prove that the two sets are independent; this is easier than examining all cycles. ■

We consider one application.

1.2.20. Definition. The **union** of graphs $G_1, \dots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.

1.2.21. Example. Below we show K_4 as the union of two 4-cycles. When a graph G is expressed as the union of two or more subgraphs, an edge of G can belong to many of them. This distinguishes union from decomposition, where each edge belongs to only one subgraph in the list. ■



1.2.22. Example. Consider an air traffic system with k airlines. Suppose that

1) direct service between two cities means round-trip direct service, and
2) each pair of cities has direct service from at least one airline.

Suppose also that no airline can schedule a cycle through an odd number of cities. In terms of k, what is the maximum number of cities in the system?

By Theorem 1.2.18, we seek the largest n such that K_n can be expressed as the union of k bipartite graphs, one for each airline. The answer is 2^k. ■

1.2.23. Theorem. The complete graph K_n can be expressed as the union of k bipartite graphs if and only if $n \le 2^k$.

Proof: We use induction on k. Basis step: $k = 1$. Since K_3 has an odd cycle and K_2 does not, K_n is itself a bipartite graph if and only if $n \le 2$.

Induction step: $k > 1$. We prove each implication using the induction hypothesis. Suppose first that $K_n = G_1 \cup \cdots \cup G_k$, where each G_i is bipartite. We partition the vertex set into two sets X, Y such that G_k has no edge within X or within Y. The union of the other $k-1$ bipartite subgraphs must cover the complete subgraphs induced by X and by Y. Applying the induction hypothesis to each yields $|X| \le 2^{k-1}$ and $|Y| \le 2^{k-1}$, so $n \le 2^{k-1} + 2^{k-1} = 2^k$.

Conversely, suppose that $n \le 2^k$. We partition the vertex set into subsets X, Y, each of size at most 2^{k-1}. By the induction hypothesis, we can cover the complete subgraph induced by either subset with $k-1$ bipartite subgraphs. The union of the ith such subgraph on X with the ith such subgraph on Y is a bipartite graph. Hence we obtain $k-1$ bipartite graphs whose union consists of the complete subgraphs induced by X and Y. The remaining edges are those of the biclique with bipartition X, Y. Letting this be the kth bipartite subgraph completes the construction. ■

This theorem can also be proved without induction by encoding the vertices as binary k-tuples (Exercise 31).

EULERIAN CIRCUITS

We return to our analysis of the Königsberg Bridge Problem. What the people of Königsberg wanted was a closed trail containing all the edges in a graph. As we have observed, a necessary condition for existence of such a trail is that all vertex degrees be even. Also it is necessary that all edges belong to the same component of the graph.

The Swiss mathematician Leonhard Euler (pronounced "oiler") stated [1736] that these conditions are also sufficient. In honor of his contribution, we associate his name with such graphs. Euler's paper appeared in 1741 but gave no proof that the obvious necessary conditions are sufficient. Hierholzer [1873] gave the first complete published proof. The graph we drew in Example 1.1.1 to model the city did not appear in print until 1894 (see Wilson [1986] for a discussion of the historical record).

1.2.24. Definition. A graph is **Eulerian** if it has a closed trail containing all edges. We call a closed trail a **circuit** when we do not specify the first vertex but keep the list in cyclic order. An **Eulerian circuit** or **Eulerian trail** in a graph is a circuit or trail containing all the edges.

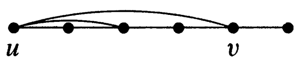
An **even graph** is a graph with vertex degrees all even. A vertex is **odd** [**even**] when its degree is odd [even].

Our discussion of Eulerian circuits applies also to graphs with loops; we extend the notion of vertex degree to graphs with loops by letting each loop contribute 2 to the degree of its vertex. This does not change the parity of the degree, and the presence of a loop does not affect whether a graph has an Eulerian circuit unless it is a loop in a component with one vertex.

Our proof of the characterization of Eulerian graphs uses a lemma. A **maximal path** in a graph G is a path P in G that is not contained in a longer path. When a graph is finite, no path can extend forever, so maximal (non-extendible) paths exist.

1.2.25. Lemma. If every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof: Let P be a maximal path in G, and let u be an endpoint of P. Since P cannot be extended, every neighbor of u must already be a vertex of P. Since u has degree at least 2, it has a neighbor v in $V(P)$ via an edge not in P. The edge uv completes a cycle with the portion of P from v to u. ■



Note the importance of finiteness. If $V(G) = \mathbb{Z}$ and $E(G) = \{ij\colon |i-j| = 1\}$, then every vertex of G has degree 2, but G has no cycle (and no non-extendible path). We avoid such examples by assuming that all graphs in this book are finite, with rare explicit exceptions.

1.2.26. Theorem. A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.

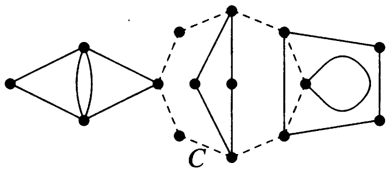
Proof: *Necessity.* Suppose that G has an Eulerian circuit C. Each passage of C through a vertex uses two incident edges, and the first edge is paired with the last at the first vertex. Hence every vertex has even degree. Also, two edges can be in the same trail only when they lie in the same component, so there is at most one nontrivial component.

Sufficiency. Assuming that the condition holds, we obtain an Eulerian circuit using induction on the number of edges, m.

Basis step: $m = 0$. A closed trail consisting of one vertex suffices.

Induction step: $m > 0$. With even degrees, each vertex in the nontrivial component of G has degree at least 2. By Lemma 1.2.25, the nontrivial component has a cycle C. Let G' be the graph obtained from G by deleting $E(C)$.

Since C has 0 or 2 edges at each vertex, each component of G' is also an even graph. Since each component also is connected and has fewer than m edges, we can apply the induction hypothesis to conclude that each component of G' has an Eulerian circuit. To combine these into an Eulerian circuit of G, we traverse C, but when a component of G' is entered for the first time we detour along an Eulerian circuit of that component. This circuit ends at the vertex where we began the detour. When we complete the traversal of C, we have completed an Eulerian circuit of G. ■



Perhaps as important as the characterization of Eulerian graphs is what the method of proof says about even graphs.

1.2.27. Proposition. Every even graph decomposes into cycles.

Proof: In the proof of Theorem 1.2.26, we noted that every even nontrivial graph has a cycle, and that the deletion of a cycle leaves an even graph. Thus this proposition follows by induction on the number of edges. ■

In the characterization of Eulerian circuits, the necessity of the condition is easy to see. This also holds for the characterization of bipartite graphs by absence of odd cycles and for many other characterizations. Nash-Williams and others popularized a mnemonic for such theorems: **TONCAS**, meaning "The Obvious Necessary Conditions are Also Sufficient".

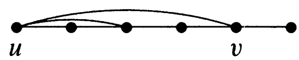
The proof of Lemma 1.2.25 is an example of an important technique of proof in graph theory that we call **extremality**. When considering structures of a given type, choosing an example that is extreme in some sense may yield useful additional information. For example, since a maximal path P cannot be extended, we obtain the extra information that every neighbor of an endpoint of P belongs to $V(P)$.

In a sense, making an extremal choice goes directly to the important case. In Lemma 1.2.25, we could start with any path. If it is extendible, then we extend it. If not, then something important happens. We illustrate the technique with several examples, and Exercises 37–42 also use extremality. We begin by strengthening Lemma 1.2.25 for simple graphs.

1.2.28. Proposition. If G is a simple graph in which every vertex has degree at least k, then G contains a path of length at least k. If $k \geq 2$, then G also contains a cycle of length at least $k + 1$.

Proof: Let u be an endpoint of a maximal path P in G. Since P does not extend, every neighbor of u is in $V(P)$. Since u has at least k neighbors and G is simple,

P therefore has at least k vertices other than u and has length at least k. If $k \geq 2$, then the edge from u to its farthest neighbor v along P completes a sufficiently long cycle with the portion of P from v to u. ■



1.2.29. Proposition. Every graph with a nonloop edge has at least two vertices that are not cut-vertices.

Proof: If u is an endpoint of a maximal path P in G, then the neighbors of u lie on P. Since $P - u$ is connected in $G - u$, the neighbors of u belong to a single component of $G - u$, and u is not a cut-vertex. ■

1.2.30. Remark. Note the difference between "maximal" and "maximum". As adjectives, **maximum** means "maximum-sized", and **maximal** means "no larger one contains this one". Every maximum path is a maximal path, but maximal paths need not have maximum length. Similarly, the biclique $K_{r,s}$ has two maximal independent sets, but when $r \neq s$ it has only one maximum independent set. When describing numbers rather than containment, the meanings are the same; maximum vertex degree = maximal vertex degree.

Besides maximal or maximum paths or independent sets, other extremal aspects include vertices of minimum or maximum degree, the first vertex where two paths diverge, maximal connected subgraphs (components), etc. In a connected graph G with disjoint sets $S, T \subset V(G)$, we can obtain a path from S to T having only its endpoints in $S \cup T$ by choosing a shortest path from S to T; Exercise 40 applies this. Exercise 37 uses extremality for a short proof of the transitivity of the connection relation. ■

Many proofs using induction can be phrased using extremality, and many proofs using extremality can be done by induction. To underscore the interplay, we reprove the characterization of Eulerian graphs using extremality directly.

1.2.31. Lemma. In an even graph, every maximal trail is closed.

Proof: Let T be a maximal trail in an even graph. Every passage of T through a vertex v uses two edges at v, none repeated. Thus when arriving at a vertex v other than its initial vertex, T has used an odd number of edges incident to v. Since v has even degree, there remains an edge on which T can continue.

Hence T can only end at its initial vertex. In a finite graph, T must indeed end. We conclude that a maximal trail must be closed. ■

1.2.32. Theorem 1.2.26—Second Proof. We prove TONCAS. In a graph G satisfying the conditions, let T be a trail of maximum length; T must also be a maximal trail. By Lemma 1.2.31, T is closed.

Suppose that T omits some edge e of G. Since G has only one nontrivial component, G has a shortest path from e to the vertex set of T. Hence some edge e' not in T is incident to some vertex v of T.

Since T is closed, there is a trail T' that starts and ends at v and uses the same edges as T. We now extend T' along e' to obtain a longer trail than T. This contradicts the choice of T, and hence T traverses all edges of G. ■

This proof and the resulting construction procedure (Exercise 12) are similar to those of Hierholzer [1873]. Exercise 35 develops another proof.

Later chapters contain several applications of the statement that every connected even graph has an Eulerian circuit. Here we give a simple one. When drawing a figure G on paper, how many times must we stop and move the pen? We are not allowed to repeat segments of the drawing, so each visit to the paper contributes a trail. Thus we seek a decomposition of G into the minimum number of trails. We may reduce the problem to connected graphs, since the number of trails needed to draw G is the sum of the number needed to draw each component.

For example, the graph G below has four odd vertices and decomposes into two trails. Adding the dashed edges on the right makes it Eulerian.

1.2.33. Theorem. For a connected nontrivial graph with exactly $2k$ odd vertices, the minimum number of trails that decompose it is $\max\{k, 1\}$.

Proof: A trail contributes even degree to every vertex, except that a non-closed trail contributes odd degree to its endpoints. Therefore, a partition of the edges into trails must have some non-closed trail ending at each odd vertex. Since each trail has only two ends, we must use at least k trails to satisfy $2k$ odd vertices. We also need at least one trail since G has an edge, and Theorem 1.2.26 implies that one trail suffices when $k = 0$.

It remains to prove that k trails suffice when $k > 0$. Given such a graph G, we pair up the odd vertices in G (in any way) and form G' by adding for each pair an edge joining its two vertices, as illustrated above. The resulting graph G' is connected and even, so by Theorem 1.2.26 it has an Eulerian circuit C. As we traverse C in G', we start a new trail in G each time we traverse an edge of $G' - E(G)$. This yields k trails decomposing G. ■

We prove theorems in general contexts to avoid work. The proof of Theorem 1.2.33 illustrates this; by transforming G into a graph where Theorem 1.2.26 applies, we avoid repeating the basic argument of Theorem 1.2.26. Exercise 33 requests a proof of Theorem 1.2.33 directly by induction.

Note that Theorem 1.2.33 considers only graphs having an even number of vertices of odd degree. Our first result in the next section explains why.

EXERCISES

Most problems in this book require proofs. Words like "construct", "show", "obtain", "determine", etc., explicitly state that proof is required. Disproof by providing a counterexample requires confirming that it is a counterexample.

1.2.1. (−) Determine whether the statements below are true or false.
a) Every disconnected graph has an isolated vertex.
b) A graph is connected if and only if some vertex is connected to all other vertices.
c) The edge set of every closed trail can be partitioned into edge sets of cycles.
d) If a maximal trail in a graph is not closed, then its endpoints have odd degree.

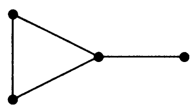
1.2.2. (−) Determine whether K_4 contains the following (give an example or a proof of non-existence).
a) A walk that is not a trail.
b) A trail that is not closed and is not a path.
c) A closed trail that is not a cycle.

1.2.3. (−) Let G be the graph with vertex set $\{1, \ldots, 15\}$ in which i and j are adjacent if and only if their greatest common factor exceeds 1. Count the components of G and determine the maximum length of a path in G.

1.2.4. (−) Let G be a graph. For $v \in V(G)$ and $e \in E(G)$, describe the adjacency and incidence matrices of $G - v$ and $G - e$ in terms of the corresponding matrices for G.

1.2.5. (−) Let v be a vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Conclude that no graph has a cut-vertex of degree 1.

1.2.6. (−) In the graph below (the paw), find all the maximal paths, maximal cliques, and maximal independent sets. Also find all the maximum paths, maximum cliques, and maximum independent sets.



1.2.7. (−) Prove that a bipartite graph has a unique bipartition (except for interchanging the two partite sets) if and only if it is connected.

1.2.8. (−) Determine the values of m and n such that $K_{m,n}$ is Eulerian.

1.2.9. (−) What is the minimum number of trails needed to decompose the Petersen graph? Is there a decomposition into this many trails using only paths?

1.2.10. (−) Prove or disprove:
a) Every Eulerian bipartite graph has an even number of edges.
b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

1.2.11. (−) Prove or disprove: If G is an Eulerian graph with edges e, f that share a vertex, then G has an Eulerian circuit in which e, f appear consecutively.

1.2.12. (−) Convert the proof at 1.2.32 to an procedure for finding an Eulerian circuit in a connected even graph.

• • • • •

1.2.13. *Alternative proofs that every u, v-walk contains a u, v-path (Lemma 1.2.5).*

a) (ordinary induction) Given that every walk of length $l - 1$ contains a path from its first vertex to its last, prove that every walk of length l also satisfies this.

b) (extremality) Given a u, v-walk W, consider a shortest u, v-walk contained in W.

1.2.14. Prove or disprove the following statements about simple graphs. (Comment: "Distinct" does not mean "disjoint".)

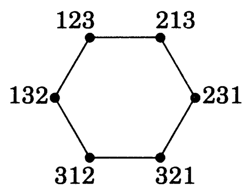
a) The union of the edge sets of distinct u, v-walks must contain a cycle.

b) The union of the edge sets of distinct u, v-paths must contain a cycle.

1.2.15. (!) Let W be a closed walk of length at least 1 that does not contain a cycle. Prove that some edge of W repeats immediately (once in each direction).

1.2.16. Let e be an edge appearing an odd number of times in a closed walk W. Prove that W contains the edges of a cycle through e.

1.2.17. (!) Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$, with two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ adjacent if they differ by interchanging a pair of adjacent entries (G_3 shown below). Prove that G_n is connected.



1.2.18. (!) Let G be the graph whose vertex set is the set of k-tuples with elements in $\{0, 1\}$, with x adjacent to y if x and y differ in exactly two positions. Determine the number of components of G.

1.2.19. Let r and s be natural numbers. Let G be the simple graph with vertex set $v_0, \ldots, v_{n-1}$ such that $v_i \leftrightarrow v_j$ if and only if $|j - i| \in \{r, s\}$. Prove that S has exactly k components, where k is the greatest common divisor of $\{n, r, s\}$.

1.2.20. (!) Let v be a cut-vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

1.2.21. Let G be a self-complementary graph. Prove that G has a cut-vertex if and only if G has a vertex of degree 1. (Akiyama–Harary [1981])

1.2.22. Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

1.2.23. For each statement below, determine whether it is true for every connected simple graph G that is not a complete graph.

a) Every vertex of G belongs to an induced subgraph isomorphic to P_3.

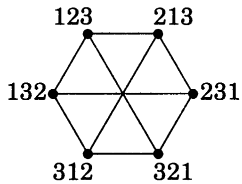
b) Every edge of G belongs to an induced subgraph isomorphic to P_3.

1.2.24. Let G be a simple graph having no isolated vertex and no induced subgraph with exactly two edges. Prove that G is a complete graph.

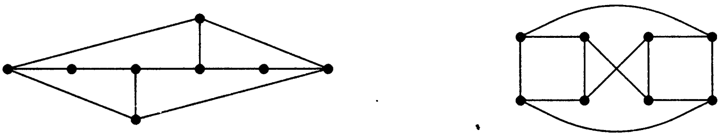
1.2.25. (!) Use ordinary induction on the number of edges to prove that absence of odd cycles is a sufficient condition for a graph to be bipartite.

1.2.26. (!) Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.

1.2.27. Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$, with two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ adjacent if they differ by switching two entries. Prove that G_n is bipartite (G_3 shown below). (Hint: For each permutation a, count the pairs i, j such that $i < j$ and $a_i > a_j$; these are called **inversions**.)



1.2.28. (!) In each graph below, find a bipartite subgraph with the maximum number of edges. Prove that this is the maximum, and determine whether this is the only bipartite subgraph with this many edges.



1.2.29. (!) Let G be a connected simple graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique (complete bipartite graph).

1.2.30. Let G be a simple graph with vertices $v_1, \ldots, v_n$. Let A^k denote the kth power of the adjacency matrix of G under matrix multiplication. Prove that entry i, j of A^k is the number of v_i, v_j-walks of length k in G. Prove that G is bipartite if and only if, for the odd integer r nearest to n, the diagonal entries of A^r are all 0. (Reminder: A walk is an **ordered** list of vertices and edges.)

1.2.31. (!) *Non-inductive proof of Theorem 1.2.23* (see Example 1.2.21).

a) Given $n \le 2^k$, encode the vertices of K_n as distinct binary k-tuples. Use this to construct k bipartite graphs whose union is K_n.

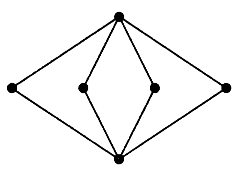
b) Given that K_n is a union of bipartite graphs $G_1, \ldots, G_k$, encode the vertices of K_n as distinct binary k-tuples. Use this to prove that $n \le 2^k$.

1.2.32. The statement below is false. Add a hypothesis to correct it, and prove the corrected statement.

"Every maximal trail in an even graph is an Eulerian circuit."

1.2.33. Use ordinary induction on k or on the number of edges (one by one) to prove that a connected graph with $2k$ odd vertices decomposes into k trails if $k > 0$. Does this remain true without the connectedness hypothesis?

1.2.34. Two Eulerian circuits are *equivalent* if they have the same unordered pairs of consecutive edges, viewed cyclically (the starting point and direction are unimportant). A cycle, for example, has only one equivalence class of Eulerian circuits. How many equivalence classes of Eulerian circuits are there in the graph drawn below?



1.2.35. *Tucker's Algorithm.* Let G be a connected even graph. At each vertex, partition the incident edges into pairs (each edge appears in a pair for each of its endpoints). Starting along a given edge e, form a trail by leaving each vertex along the edge paired with the edge just used to enter it, ending with the edge paired with e. This decomposes G into closed trails. As long as there is more than one trail in the decomposition, find two trails with a common vertex and combine them into a longer trail by changing the pairing at a common vertex. Prove that this procedure works and produces an Eulerian circuit as its final trail. (Tucker [1976])

1.2.36. (+) *Alternative characterization of Eulerian graphs.*

a) Prove that if G is Eulerian and $G' = G - uv$, then G' has an odd number of u, v-trails that visit v only at the end. Prove also that the number of the trails in this list that are not paths is even. (Toida [1973])

b) Let v be a vertex of odd degree in a graph. For each edge e incident to v, let $c(e)$ be the number of cycles containing e. Use $\sum_e c(e)$ to prove that $c(e)$ is even for some e incident to v. (McKee [1984])

c) Use part (a) and part (b) to conclude that a nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles.

1.2.37. (!) Use extremality to prove that the connection relation is transitive. (Hint: Given a u, v-path P and a v, w-path Q, consider the first vertex of P in Q.)

1.2.38. (!) Prove that every n-vertex graph with at least n edges contains a cycle.

1.2.39. Suppose that every vertex of a loopless graph G has degree at least 3. Prove that G has a cycle of even length. (Hint: Consider a maximal path.) (P. Kwok)

1.2.40. (!) Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

1.2.41. Let G be a connected graph with at least three vertices. Prove that G has two vertices x, y such that 1) $G - \{x, y\}$ is connected and 2) x, y are adjacent or have a common neighbor. (Hint: Consider a longest path.) (Chung [1978a])

1.2.42. Let G be a connected simple graph that does not have P_4 or C_4 as an induced subgraph. Prove that G has a vertex adjacent to all other vertices. (Hint: Consider a vertex of maximum degree.) (Wolk [1965])

1.2.43. (+) Use induction on k to prove that every connected simple graph with an even number of edges decomposes into paths of length 2. Does the conclusion remain true if the hypothesis of connectedness is omitted?

1.3. Vertex Degrees and Counting

The degrees of the vertices are fundamental parameters of a graph. We repeat the definition in order to introduce important notation.

1.3.1. Definition. The **degree** of vertex v in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v, except that each loop at v counts twice. The maximum degree is $\Delta(G)$, the minimum degree is $\delta(G)$, and G is **regular** if $\Delta(G) = \delta(G)$. It is k-**regular** if the common degree is k. The **neighborhood** of v, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v.

1.3.2. Definition. The **order** of a graph G, written $n(G)$, is the number of vertices in G. An **n-vertex graph** is a graph of order n. The **size** of a graph G, written $e(G)$, is the number of edges in G. For $n \in \mathbb{N}$, the notation $[n]$ indicates the set $\{1, \ldots, n\}$.

Since our graphs are finite, $n(G)$ and $e(G)$ are well-defined nonnegative integers. We also often use "e" by itself to denote an edge. When e denotes a particular edge, it is not followed by the name of a graph in parentheses, so the context indicates the usage. We have used "n-cycle" to denote a cycle with n vertices; this is consistent with "n-vertex graph".

COUNTING AND BIJECTIONS

We begin with counting problems about subgraphs in a graph. The first such problem is to count the edges; we do this using the vertex degrees. The resulting formula is an essential tool of graph theory, sometimes called the "First Theorem of Graph Theory" or the "Handshaking Lemma".

1.3.3. Proposition. (Degree-Sum Formula) If G is a graph, then

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Proof: Summing the degrees counts each edge twice, since each edge has two ends and contributes to the degree at each endpoint. ■

The proof holds even when G has loops, since a loop contributes 2 to the degree of its endpoint. For a loopless graph, the two sides of the formula count the set of pairs (v, e) such that v is an endpoint of e, grouped by vertices or grouped by edges. "Counting two ways" is an elegant technique for proving integer identities (see Exercise 31 and Appendix A).

The degree-sum formula has several immediate corollaries. Corollary 1.3.5 applies in Exercises 9–13 and in many arguments of later chapters.

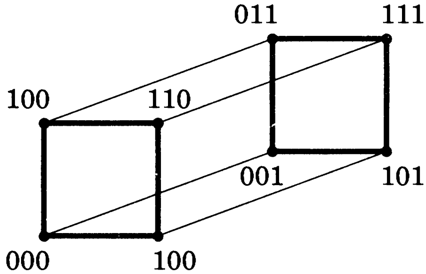
1.3.4. Corollary. In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence $\delta(G) \le \frac{2e(G)}{n(G)} \le \Delta(G)$. ■

1.3.5. Corollary. Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree. ■

1.3.6. Corollary. A k-regular graph with n vertices has $nk/2$ edges. ■

We next introduce an important family of graphs.

1.3.7. Definition. The **k-dimensional cube** or **hypercube** Q_k is the simple graph whose vertices are the k-tuples with entries in $\{0, 1\}$ and whose edges are the pairs of k-tuples that differ in exactly one position. A j-**dimensional subcube** of Q_k is a subgraph of Q_k isomorphic to Q_j.



Above we show Q_3. The hypercube is a natural computer architecture. Processors can communicate directly if they correspond to adjacent vertices in Q_k. The k-tuples that name the vertices serve as addresses for the processors.

1.3.8. Example. *Structure of hypercubes.* The *parity* of a vertex in Q_k is the parity of the number of 1s in its name, even or odd. Each edge of Q_k has an even vertex and an odd vertex as endpoints. Hence the even vertices form an independent set, as do the odd vertices, and Q_k is bipartite.

Each position in the k-tuples can be specified in two ways, so $n(Q_k) = 2^k$. A neighbor of a vertex is obtained by changing one of the k positions in its name to the other value. Thus Q_k is k-regular. By Corollary 1.3.6, $e(Q_k) = k2^{k-1}$.

The bold edges above show two subgraphs of Q_3 isomorphic to Q_2, formed by keeping the last coordinate fixed at 0 or at 1. We can form a j-dimensional subcube by keeping any $k - j$ coordinates fixed and letting the values in the remaining j coordinates range over all 2^j possible j-tuples. The subgraph induced by such a set of vertices is isomorphic to Q_j. Since there are $\binom{k}{j}$ ways to pick j coordinates to vary and 2^{k-j} ways to specify the values in the fixed coordinates, this specifies $\binom{k}{j}2^{k-j}$ such subcubes. In fact, there are no other j-dimensional subcubes (Exercise 29).

The copies of Q_1 are simply the edges in Q_k. Our formula reduces to $k2^{k-1}$ when $j = 1$, so we have found another counting argument to compute $e(Q_k)$.

When $j = k - 1$, our discussion suggests a recursive description of Q_k. Append 0 to the vertex names in a copy of Q_{k-1}; append 1 in another copy. Add edges joining vertices from the two copies whose first $k - 1$ coordinates are equal. The result is Q_k. The basis of the construction is the 1-vertex graph Q_0. This description leads to inductive proofs for many properties of hypercubes, including $e(Q_k) = k2^{k-1}$ (Exercise 23). ■

A hypercube is a regular bipartite graph. A simple counting argument proves a fundamental observation about such graphs.

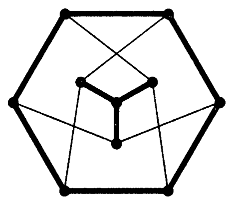
1.3.9. Proposition. If $k > 0$, then a k-regular bipartite graph has the same number of vertices in each partite set.

Proof: Let G be an X,Y-bigraph. Counting the edges according to their endpoints in X yields $e(G) = k\,|X|$. Counting them by their endpoints in Y yields $e(G) = k\,|Y|$. Thus $k\,|X| = k\,|Y|$, which yields $|X| = |Y|$ when $k > 0$. ■

Another technique for counting a set is to establish a bijection from it to a set of known size. Our next example uses this approach. Other examples of combinatorial arguments for counting problems appear in Appendix A. Exercises 18–35 involve counting.

1.3.10. Example. *The Petersen graph has ten 6-cycles.* Let G be the Petersen graph. Being 3-regular, G has ten claws (copies of $K_{1,3}$). We establish a one-to-one correspondence between the 6-cycles and the claws.

Since G has girth 5, every 6-cycle F is an induced subgraph. Each vertex of F has one neighbor outside F. Since nonadjacent vertices have exactly one common neighbor (Proposition 1.1.38), opposite vertices on F have a common neighbor outside F. Since G is 3-regular, the resulting three vertices outside F are distinct. Thus deleting $V(F)$ leaves a subgraph with three vertices of degree 1 and one vertex of degree 3; it is a claw.



We show that each claw H in G arises exactly once in this way. Let S be the set of vertices with degree 1 in H; S is an independent set. The central vertex of H is already a common neighbor, so the six other edges from S reach distinct vertices. Thus $G - V(H)$ is 2-regular. Since G has girth 5, $G - V(H)$ must be a 6-cycle. This 6-cycle yields H when its vertices are deleted. ■

We present one more counting argument related to a long-standing conjecture. Subgraphs obtained by deleting a single vertex are called **vertex-deleted subgraphs**. These subgraphs need not all be distinct; for example, the n vertex-deleted subgraphs of C_n are all isomorphic to P_{n-1}.

1.3.11.* Proposition. For a simple graph G with vertices $v_1, \ldots, v_n$ and $n \ge 3$,

$$e(G) = \frac{\Sigma e(G - v_i)}{n-2} \qquad \text{and} \qquad d_G(v_j) = \frac{\Sigma e(G - v_i)}{n-2} - e(G - v_j).$$

Proof: An edge e of G appears in $G - v_i$ if and only if v_i is not an endpoint of e. Thus $\sum(G - v_i)$ counts each edge exactly $n - 2$ times.

Once we know $e(G)$, the degree of v_j can be computed as the number of edges lost when deleting v_j to form $G - v_j$. ■

Typically, we are given the vertex-deleted subgraphs as unlabeled graphs; we know only the list of isomorphism classes, not which vertex of $G - v_i$ corresponds to which vertex in G. This can make it very difficult to tell what G is. For example, K_2 and its complement have the same list of vertex-deleted subgraphs. For larger graphs we have the **Reconstruction Conjecture**, formulated in 1942 by Kelly and Ulam.

1.3.12.* Conjecture. (Reconstruction Conjecture) If G is a simple graph with at least three vertices, then G is uniquely determined by the list of (isomorphism classes of) its vertex-deleted subgraphs. ∎

The list of vertex-deleted subgraphs of G has $n(G)$ items. Proposition 1.3.11 shows that $e(G)$ and the list of vertex degrees can be reconstructed. The latter implies that regular graphs can be reconstructed (Exercise 37). We can also determine whether G is connected (Exercise 38); using this, disconnected graphs can be reconstructed (Exercise 39). Other sufficient conditions for reconstructibility are known, but the general conjecture remains open.

EXTREMAL PROBLEMS

An **extremal problem** asks for the maximum or minimum value of a function over a class of objects. For example, the maximum number of edges in a simple graph with n vertices is $\binom{n}{2}$.

1.3.13. Proposition. The minimum number of edges in a connected graph with n vertices is $n - 1$.

Proof: By Proposition 1.2.11, every graph with n vertices and k edges has at least $n - k$ components. Hence every n-vertex graph with fewer than $n - 1$ edges has at least two components and is disconnected. The contrapositive of this is that every connected n-vertex graph has at least $n - 1$ edges. This lower bound is achieved by the path P_n. ∎

1.3.14. Remark. Proving that β is the minimum of $f(G)$ for graphs in a class **G** requires showing two things:

1) $f(G) \geq \beta$ for all $G \in \mathbf{G}$.

2) $f(G) = \beta$ for some $G \in \mathbf{G}$.

The proof of the bound must apply to every $G \in \mathbf{G}$. For equality it suffices to obtain an example in **G** with the desired value of f.

Changing "$\geq$" to "$\leq$" yields the criteria for a maximum. ∎

Next we solve a maximization problem that is not initially phrased as such.

1.3.15. Proposition. If G is a simple n-vertex graph with $\delta(G) \geq (n - 1)/2$, then G is connected.

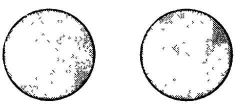
Proof: Choose $u, v \in V(G)$. It suffices to show that u, v have a common neighbor if they are not adjacent. Since G is simple, we have $|N(u)| \geq \delta(G) \geq (n-1)/2$, and similarly for v. When $u \not\leftrightarrow v$, we have $|N(u) \cup N(v)| \leq n-2$, since u and v are not in the union. Using Remark A.13 of Appendix A, we thus compute

$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq \tfrac{n-1}{2} + \tfrac{n-1}{2} - (n-2) = 1. \blacksquare$$

We say that a result is **best possible** or **sharp** when there is some aspect of it that cannot be strengthened without the statement becoming false. As shown by the next example, this holds for Proposition 1.3.15; when $\delta(G)$ is smaller than $(n(G)-1)/2$, we cannot still conclude that G must be connected.

1.3.16. Example. Let G be the n-vertex graph with components isomorphic to $K_{\lfloor n/2 \rfloor}$ and $K_{\lceil n/2 \rceil}$, where the **floor** $\lfloor x \rfloor$ of x is the largest integer at most x and the **ceiling** $\lceil x \rceil$ of x is the smallest integer at least x. Since $\delta(G) = \lfloor n/2 \rfloor - 1$ and G is disconnected, the inequality in Proposition 1.3.15 is sharp.

We use the floor and ceiling functions here in order to describe a single family of graphs providing an example for each n. ■

$K_{\lfloor n/2 \rfloor}$  $K_{\lceil n/2 \rceil}$

By providing a family of examples to show that the bound is best possible, we have solved an extremal problem. Together, Proposition 1.3.15 and Example 1.3.16 prove "The minimum value of $\delta(G)$ that forces an n-vertex simple graph G to be connected is $\lfloor n/2 \rfloor$," or "The maximum value of $\delta(G)$ among disconnected n-vertex simple graphs is $\lfloor n/2 \rfloor - 1$."

We introduce compact notation to describe the graph of Example 1.3.16.

1.3.17. Definition. The graph obtained by taking the union of graphs G and H with disjoint vertex sets is the **disjoint union** or **sum**, written $G + H$. In general, mG is the graph consisting of m pairwise disjoint copies of G.

1.3.18. Example. If G and H are connected, then $G+H$ has components G and H, so the graph in Example 1.3.16 is $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$. This notation is convenient when we have not named the vertices. Note that $K_m + K_n = \overline{K}_{m,n}$.

The graph mK_2 consists of m pairwise disjoint edges. ■

In graph theory, we use "extremal problem" for finding an optimum over a class of graphs. When seeking extremes in a single graph, such as the maximum size of an independent set, or maximum size of a bipartite subgraph, we have a different problem for each graph. To distinguish these from the earlier type of problem, we call them **optimization problems**.

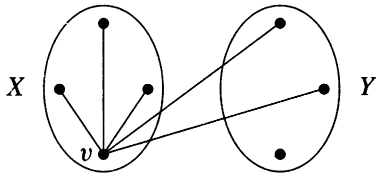
Since an optimization problem has an instance for each graph, we usually can't list all solutions. We may seek a solution procedure or bounds on the answer in terms of other aspects of the input graph. In this light we consider the problem of finding a large bipartite subgraph. It allows us to introduce the technique of constructive or "algorithmic" proof. (An **algorithm** is a procedure for performing some task.)

One way to prove that something exists is to build it. Such proofs can be viewed as algorithms. To complete an algorithmic proof, we must prove that the algorithm terminates and yields the desired result. This may involve induction,contradiction, finiteness, etc. We prove that every graph has a large bipartitesubgraph by providing an algorithm to find one. Exercises 45–49 arerelated to finding large bipartite subgraphs.

1.3.19. Theorem. Every loopless graph G has a bipartite subgraph with at least $e(G)/2$ edges.

Proof: We start with any partition of $V(G)$ into two sets X, Y. Using the edges having one endpoint in each set yields a bipartite subgraph H with bipartition X, Y. If H contains fewer than half the edges of G incident to a vertex v, then v has more edges to vertices in its own class than in the other class, as illustrated below. Moving v to the other class gains more edges of G than it loses.

We move one vertex in this way as long as the current bipartite subgraph captures less than half of the edges at some vertex. Each such switch increases the size of the subgraph, so the process must terminate. When it terminates, we have $d_H(v) \geq d_G(v)/2$ for every $v \in V(G)$. Summing this and applying the degree-sum formula yields $e(H) \geq e(G)/2$. ∎

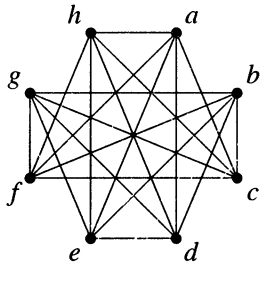


Algorithmic proofs often correspond to proofs by induction or extremality. Such proofs are shorter and may be easier to find, so we may seek such a proof and later convert it to an algorithm. For example, here is the proof of Theorem 1.3.19 in the language of extremality and contradiction; in effect, the extremal choice of H goes directly to the end of the algorithm:

> Let H be the bipartite subgraph of G that has the most edges. If $d_H(v) \geq d_G(v)/2$ for all $v \in V(G)$, then the degree-sum formula yields $e(H) \geq e(G)/2$. Otherwise, $d_H(v) < d_G(v)/2$ for some $v \in V(G)$, and then switching v in the bipartition contradicts the choice of H.

1.3.20. Example. *Local maximum.* The algorithm in Theorem 1.3.19 need not produce a bipartite subgraph with the most edges, merely one with at least half the edges. The graph below is 5-regular with 8 vertices and hence has 20 edges. The bipartition $X = \{a, b, c, d\}$ and $Y = \{e, f, g, h\}$ yields a 3-regular bipartite subgraph with 12 edges. The algorithm terminates here; switching one vertex would pick up two edges but lose three.

Nevertheless, the bipartition $X = \{a, b, g, h\}$ and $Y = \{c, d, e, f\}$ yields a 4-regular bipartite subgraph with 16 edges. An algorithm seeking the maximum by local changes may get stuck in a local maximum. ■



1.3.21. Remark. In a graph G, the (global) maximum number of edges in a bipartite subgraph is $e(G)$ minus the minimum number of edges needed to obtain at least one edge from every odd cycle. ■

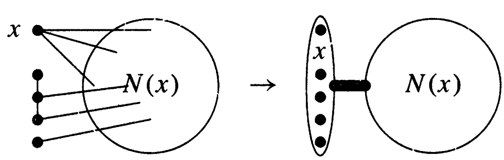
Our next extremal problem doesn't start with bipartite graphs, but it winds up there. In politics and warfare, seldom do two enemies have a common enemy; usually two of the three combine against the third. Given n factions, how many pairs of enemies can there be if no two enemies have a common enemy?

In the language of graphs, we are asking for the maximum number of edges in a simple n-vertex graph with no triangle. Bipartite graphs have no triangles, but also many non-bipartite graphs (such as the Petersen graph) have no triangles. Using extremality (by choosing a vertex of maximum degree), we will prove that the maximum is indeed achieved by a complete bipartite graph.

1.3.22. Definition. A graph G is H**-free** if G has no induced subgraph isomorphic to H.

1.3.23. Theorem. (Mantel [1907]) The maximum number of edges in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$.

Proof: Let G be an n-vertex triangle-free simple graph. Let x be a vertex of maximum degree, with $k = d(x)$. Since G has no triangles, there are no edges among neighbors of x. Hence summing the degrees of x and its nonneighbors counts at least one endpoint of every edge: $\sum_{v \notin N(x)} d(v) \geq e(G)$. We sum over $n - k$ vertices, each having degree at most k, so $e(G) \leq (n - k)k$.



Since $(n - k)k$ counts the edges in $K_{n-k,k}$, we have now proved that $e(G)$ is bounded by the size of some biclique with n vertices. Moving a vertex of $K_{n,k,k}$

from the set of size k to the set of size $n-k$ gains $k-1$ edges and loses $n-k$ edges. The net gain is $2k-1-n$, which is positive for $2k > n+1$ and negative for $2k < n+1$. Thus $e(K_{n-k,k})$ is maximized when k is $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$. The product is then $n^2/4$ for even n and $(n^2-1)/4$ for odd n. Thus $e(G) \le \lfloor n^2/4 \rfloor$.

To prove that the bound is best possible, we exhibit a triangle-free graph with $\lfloor n^2/4 \rfloor$ edges: $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. ■

Although $(n-k)k$ can be maximized over k using calculus, the discrete approach is preferable in some ways. It directly restricts k to be an integer and generalizes easily to more variables. The switching idea used is that of Theorem 1.3.19; here we have used it to find the largest bipartite subgraph of K_n. In Theorem 5.2.9 we generalize Theorem 1.3.23 to K_{r+1}-free graphs.

Mantel's result leads us to another reason for phrasing inductive proofs in the format that we have used. The reason is safety.

1.3.24. Example. *A failed proof.* Let us try to prove Theorem 1.3.23 by induction on n. Basis step: $n \le 2$. Here the complete graph K_n has the most edges and has no triangles.

Induction step: $n > 2$. We try "Suppose that the claim is true when $n = k$, so $K_{\lfloor k/2 \rfloor, \lceil k/2 \rceil}$ is the largest triangle-free graph with k vertices. We add a new vertex x to form a triangle-free graph with $k+1$ vertices. Making x adjacent to vertices from both partite sets would create a triangle. Hence we add the most edges by making x adjacent to all the vertices in the larger partite set of $K_{\lfloor k/2 \rfloor, \lceil k/2 \rceil}$. Doing so creates $K_{\lfloor (k+1)/2 \rfloor, \lceil (k+1)/2 \rceil}$. This completes the proof."

This argument is wrong, because we did not consider all triangle-free graphs with $k+1$ vertices. We considered only those containing the extremal k-vertex graph as an induced subgraph. This graph does appear in the extremal graph with $k+1$ vertices, but we cannot use that fact before proving it. It remains possible that the largest example with $k+1$ vertices arises by adding a new vertex of high degree to a non-maximal example with k vertices.

Exercise 51 develops a correct proof by induction on n. ■

The error in Example 1.3.24 was that our induction step did not consider all instances of the statement for the new larger value of the parameter. We call this error the **induction trap**. If the induction step grows an instance with the new value of the parameter from a smaller instance, then we must prove that all instances with the new value have been considered.

When there is only one instance for each value of the induction parameter (as in summation formulas), this does not cause trouble. With more than one instance, it is safer and simpler to start with an arbitrary instance for the larger parameter value. This explicitly considers each instance G for the larger value, so we don't need to prove that we have generated them all.

However, when we obtain from G a smaller instance, we must confirm that the induction hypothesis applies to it. For example, in the inductive proof of the characterization of Eulerian circuits (Theorem 1.2.26), we must apply the induction hypothesis to each component of the graph obtained by deleting the edges of a cycle, not to the entire graph at once.

1.3.25. Remark. *A template for induction.* Often the statement we want to prove by induction on n is an implication: $A(n) \Rightarrow B(n)$. We must prove that every instance G satisfying $A(n)$ also satisfies $B(n)$. Our induction step follows a typical format. From G we obtain some (smaller) G'. If we show that G' satisfies $A(n-1)$ (for ordinary induction), then the induction hypothesis implies that G' satisfies $B(n-1)$. Now we use the information that G' satisfies $B(n-1)$ to prove that G satisfies $B(n)$.

$$\begin{array}{ccc} G \text{ satisfies } A(n) & & G \text{ satisfies } B(n) \\ \Downarrow & & \Uparrow \\ G' \text{ satisfies } A(n-1) & \Rightarrow & G' \text{ satisfies } B(n-1) \end{array}$$

Here the central implication is the statement of the induction hypothesis, and the others are the work we must do. Our induction proofs have followed this format. ■

1.3.26.* Example. *The induction trap.* The induction trap can lead to a *false* conclusion. Let us try to prove by induction on the number of vertices that every 3-regular connected simple graph has no cut-edge.

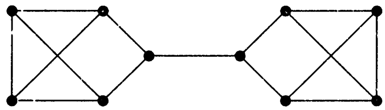
By the degree-sum formula, every regular graph with odd degree has even order, so we consider graphs with $2m$ vertices. The smallest 3-regular simple graph, K_4, is connected and has no cut-edge; this proves the basis step with $m = 2$. Now consider the induction step.

Given a simple 3-regular graph G with $2k$ vertices, we can obtain a simple 3-regular graph G' with $2(k+1)$ vertices (the next larger possible order) by "expansion": take two edges of G, replace them by paths of length 2 through new vertices, and add an edge joining the two new vertices. As illustrated below, $K_{3,3}$ arises from K_4 by one expansion on two disjoint edges.

If G is connected, then the expanded graph G' is also connected: a path between old vertices that traversed a replaced edge has merely lengthened, and a path to a new vertex in G' is obtained from a path in G to a neighbor.

If G has no cut-edge, then every edge lies on a cycle (Theorem 1.2.14). These cycles remain in G' (those using replaced edges become longer). The edge joining the two new vertices in G' also lies on a cycle using a path in G between the edges that were replaced. Theorem 1.2.14 now implies that G' has no cut-edge.

We have proved that if G is connected and has no cut-edge, then the same holds for G'. We might think we have proved by induction on m that every 3-regular simple connected graph with $2m$ vertices has no cut-edge, but the graph below is a counterexample. The proof fails because we cannot build every 3-regular simple connected graph from K_4 by expansions. We cannot even obtain all those without cut-edges, as shown in Exercise 66. ■



Appendix A presents another example of the induction trap.

GRAPHIC SEQUENCES

Next we consider all the vertex degrees together.

1.3.27. Definition. The **degree sequence** of a graph is the list of vertex degrees, *usually* written in nonincreasing order, as $d_1 \geq \cdots \geq d_n$.

Every graph has a degree sequence, but which sequences occur? That is, given nonnegative integers $d_1, \ldots, d_n$, is there a graph with these as the vertex degrees? The degree-sum formula implies that $\sum d_i$ must be even. When we allow loops and multiple edges, TONCAS.

1.3.28. Proposition. The nonnegative integers $d_1, \ldots, d_n$ are the vertex degrees of some graph if and only if $\sum d_i$ is even.

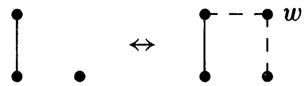
Proof: *Necessity.* When some graph G has these numbers as its vertex degrees, the degree-sum formula implies that $\sum d_i = 2e(G)$, which is even.

Sufficiency. Suppose that $\sum d_i$ is even. We construct a graph with vertex set $v_1, \ldots, v_n$ and $d(v_i) = d_i$ for all i. Since $\sum d_i$ is even, the number of odd values is even. First form an arbitrary pairing of the vertices in $\{v_i\colon d_i \text{ is odd}\}$. For each resulting pair, form an edge having these two vertices as its endpoints. The remaining degree needed at each vertex is even and nonnegative; satisfy this for each i by placing $\lfloor d_i/2 \rfloor$ loops at v_i. ■

This proof is constructive; we could also use induction (Exercise 56). The construction is easy with loops available. Without them, (2, 0, 0) is not realizable and the condition is not sufficient. Exercise 63 characterizes the degree sequences of loopless graphs. We next characterize degree sequences of simple graphs by a recursive condition that readily yields an algorithm. Many other characterizations are known; Sierksma–Hoogeveen [1991] lists seven.

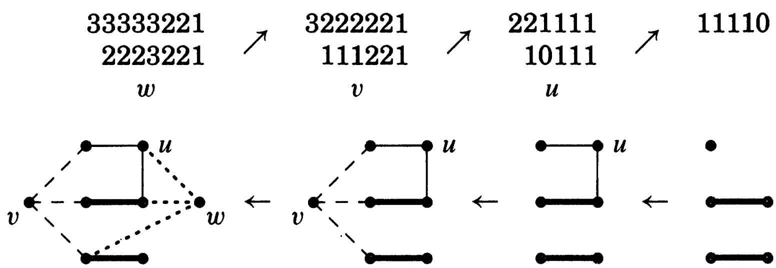
1.3.29. Definition. A **graphic sequence** is a list of nonnegative numbers that is the degree sequence of some simple graph. A simple graph with degree sequence d "realizes" d.

1.3.30. Example. *A recursive condition.* The lists $2, 2, 1, 1$ and $1, 0, 1$ are graphic. The graph $K_2 + K_1$ realizes $1, 0, 1$. Adding a new vertex adjacent to vertices of degrees 1 and 0 yields a graph with degree sequence $2, 2, 1, 1$, as shown below. Conversely, if a graph realizing $2, 2, 1, 1$ has a vertex w with neighbors of degrees 2 and 1, then deleting w yields a graph with degrees $1, 0, 1$.



Similarly, to test 33333221, we seek a realization with a vertex w of degree 3 having three neighbors of degree 3. This exists if and only if 2223221 is graphic. We reorder this and test 3222221. We continue deleting and reordering until we can tell whether the remaining list is realizable. If it is, then we insert vertices with the desired neighbors to work back to a realization of the original list. The realization is not unique.

The next theorem implies that this recursive test works. ■



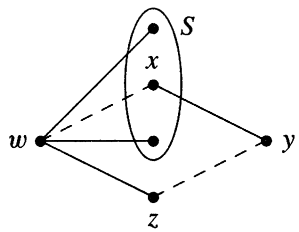
1.3.31. Theorem. (Havel [1955], Hakimi [1962]) For $n > 1$, an integer list d of size n is graphic if and only if d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1-element graphic sequence is $d_1 = 0$.

Proof: For $n = 1$, the statement is trivial. For $n > 1$, we first prove that the condition is sufficient. Given d with $d_1 \geq \cdots \geq d_n$ and a simple graph G' with degree sequence d', we add a new vertex adjacent to vertices in G' with degrees $d_2 - 1, \ldots, d_{\Delta+1} - 1$. These d_i are the Δ largest elements of d after (one copy of) Δ itself, but $d_2 - 1, \ldots, d_{\Delta+1} - 1$ need not be the Δ largest numbers in d'.

To prove necessity, we begin with a simple graph G realizing d and produce a simple graph G' realizing d'. Let w be a vertex of degree Δ in G. Let S be a set of Δ vertices in G having the "desired degrees" $d_2, \ldots, d_{\Delta+1}$. If $N(w) = S$, then we delete w to obtain G'.

Otherwise, some vertex of S is missing from $N(w)$. In this case, we modify G to increase $|N(w) \cap S|$ without changing any vertex degree. Since $|N(w) \cap S|$ can increase at most Δ times, repeating this converts G into another graph G^* that realizes d and has S as the neighborhood of w. From G^* we then delete w to obtain the desired graph G' realizing d'.

To find the modification when $N(w) \neq S$, we choose $x \in S$ and $z \notin S$ so that $w \leftrightarrow z$ and $w \not\leftrightarrow x$. We want to add wx and delete wz, but we must preserve vertex degrees. Since $d(x) \geq d(z)$ and already w is a neighbor of z but not x, there must be a vertex y adjacent to x but not to z. Now we delete $\{wz, xy\}$ and add $\{wx, yz\}$ to increase $|N(w) \cap S|$. ■

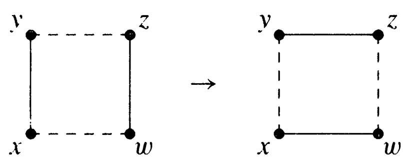


Theorem 1.3.31 tests a list of n numbers by testing a list of $n-1$ numbers; it yields a recursive algorithm to test whether d is graphic. The necessary condition "$\sum d_i$ even" holds implicitly: $\sum d_i' = (\sum d_i) - 2\Delta$ implies that $\sum d_i'$ and $\sum d_i$ have the same parity.

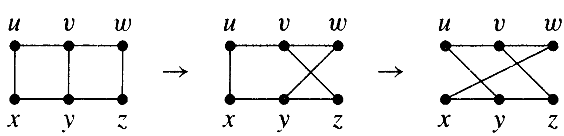
An algorithmic proof using "local change" pushes an object toward a desired condition. This can be phrased as proof by induction, where the induction parameter is the "distance" from the desired condition. In the proof of Theorem 1.3.31, this distance is the number of vertices in S that are missing from $N(w)$.

We used edge switches to transform an arbitrary graph with degree sequence d into a graph satisfying the desired condition. Next we will show that every simple graph with degree sequence d can be transformed by such switches into every other.

1.3.32. Definition. A **2-switch** is the replacement of a pair of edges xy and zw in a simple graph by the edges yz and wx, given that yz and wx did not appear in the graph originally.



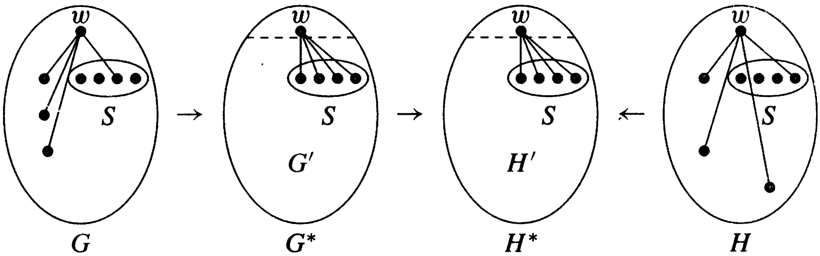
The dashed lines above indicate nonadjacent pairs. If $y \leftrightarrow z$ or $w \leftrightarrow x$, then the 2-switch cannot be performed, because the resulting graph would not be simple. A 2-switch preserves all vertex degrees. If some 2-switch turns H into H^*, then a 2-switch on the same four vertices turns H^* into H. Below we illustrate two successive 2-switches.



1.3.33.* Theorem. (Berge [1973, p153–154]) If G and H are two simple graphs with vertex set V, then $d_G(v) = d_H(v)$ for every $v \in V$ if and only if there is a sequence of 2-switches that transforms G into H.

Proof: Every 2-switch preserves vertex degrees, so the condition is sufficient. Conversely, when $d_G(v) = d_H(v)$ for all $v \in V$, we obtain an appropriate sequence of 2-switches by induction on the number of vertices, n. If $n \leq 3$, then for each $d_1, \ldots, d_n$ there is at most one simple graph with $d(v_i) = d_i$. Hence we can use $n = 3$ as the basis step.

Consider $n \geq 4$, and let w be a vertex of maximum degree, Δ. Let $S = \{v_1, \ldots, v_\Delta\}$ be a fixed set of vertices with the Δ highest degrees other than w. As in the proof of Theorem 1.3.31, some sequence of 2-switches transforms G to a graph G^* such that $N_{G^*}(w) = S$, and some such sequence transforms H to a graph H^* such that $N_{H^*}(w) = S$.



Since $N_{G^*}(w) = N_{H^*}(w)$, deleting w leaves simple graphs $G' = G^* - w$ and $H' = H^* - w$ with $d_{G'}(v) = d_{H'}(v)$ for every vertex v. By the induction hypothesis, some sequence of 2-switches transforms G' to H'. Since these do not involve w, and w has the same neighbors in G^* and H^*, applying this sequence transforms G^* to H^*. Hence we can transform G to H by transforming G to G^*, then G^* to H^*, then (in reverse order) the transformation of H to H^*. ■

We could also phrase this using induction on the number of edges appearing in exactly one of G and H, which is 0 if and only if they are already the same. In this approach, it suffices to find a 2-switch in G that makes it closer to H or a 2-switch in H that makes it closer to G.

EXERCISES

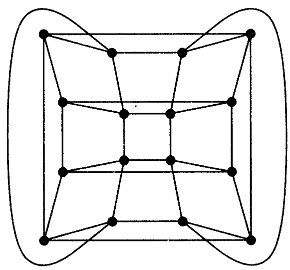
A statement with a parameter must be proved for all values of the parameter; it cannot be proved by giving examples. Counting a set includes providing proof.

1.3.1. (−) Prove or disprove: If u and v are the only vertices of odd degree in a graph G, then G contains a u, v-path.

1.3.2. (−) In a class with nine students, each student sends valentine cards to three others. Determine whether it is possible that each student receives cards from the same three students to whom he or she sent cards.

1.3.3. (−) Let u and v be adjacent vertices in a simple graph G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

1.3.4. (−) Prove that the graph below is isomorphic to Q_4.



1.3.5. (−) Count the copies of P_3 and C_4 in Q_k.

1.3.6. (−) Given graphs G and H, determine the number of components and maximum degree of $G + H$ in terms of the those parameters for G and H.

1.3.7. (−) Determine the maximum number of edges in a bipartite subgraph of P_n, of C_n, and of K_n.

1.3.8. (−) Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.

a) (5,5,4,3,2,2,2,1), c) (5,5,5,3,2,2,1,1),
b) (5,5,4,4,2,2,1,1), d) (5,5,5,4,2,1,1,1).

• • • • •

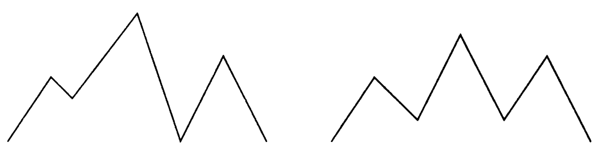
1.3.9. In a league with two divisions of 13 teams each, determine whether it is possible to schedule a season with each team playing nine games against teams within its division and four games against teams in the other division.

1.3.10. Let l, m, n be nonnegative integers with $l + m = n$. Find necessary and sufficient conditions on l, m, n such that there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree.

1.3.11. Let W be a closed walk in a graph G. Let H be the subgraph of G consisting of edges used an odd number of times in W. Prove that $d_H(v)$ is even for every $v \in V(G)$.

1.3.12. (!) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $2k + 1$-regular simple graph having a cut-edge.

1.3.13. (+) A **mountain range** is a polygonal curve from $(a, 0)$ to $(b, 0)$ in the upper half-plane. Hikers A and B begin at $(a, 0)$ and $(b, 0)$, respectively. Prove that A and B can meet by traveling on the mountain range in such a way that at all times their heights above the horizontal axis are the same. (Hint: Define a graph to model the movements, and use Corollary 1.3.5.) (Communicated by D.G. Hoffman.)



1.3.14. Prove that every simple graph with at least two vertices has two vertices of equal degree. Is the conclusion true for loopless graphs?

1.3.15. For each $k \geq 3$, determine the smallest n such that
a) there is a simple k-regular graph with n vertices.
b) there exist nonisomorphic simple k-regular graphs with n vertices.

1.3.16. (+) For $k \geq 2$ and $g \geq 2$, prove that there exists an k-regular graph with girth g. (Hint: To construct such a graph inductively, make use of an $k-1$-regular graph H with girth g and a graph with girth $\lceil g/2 \rceil$ that is $n(H)$-regular. Comment: Such a graph with minimum order is a (k, g)-**cage**.) (Erdős–Sachs [1963])

1.3.17. (!) Let G be a graph with at least two vertices. Prove or disprove:
a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.
b) Deleting a vertex of degree $\delta(G)$ cannot reduce the average degree.

1.3.18. (!) For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.

1.3.19. Let G be a claw-free graph. Prove that if $\Delta(G) \geq 5$, then G has a 4-cycle. For all $n \in \mathbb{N}$, construct a 4-regular claw-free graph of order at least n that has no 4-cycle.

1.3.20. (!) Count the cycles of length n in K_n and the cycles of length $2n$ in $K_{n,n}$.

1.3.21. Count the 6-cycles in $K_{m,n}$.

1.3.22. (!) Let G be a nonbipartite graph with n vertices and minimum degree k. Let l be the minimum length of an odd cycle in G.

a) Let C be a cycle of length l in G. Prove that every vertex not in $V(C)$ has at most two neighbors in $V(C)$.

b) By counting the edges joining $V(C)$ and $G - V(C)$ in two ways, prove that $n \geq kl/2$ (and thus $l \leq 2n/k$). (Campbell–Staton [1991])

c) When k is even, prove that the inequality of part (b) is best possible. (Hint: form a graph having $k/2$ pairwise disjoint l-cycles.)

1.3.23. Use the recursive description of Q_k (Example 1.3.8) to prove that $e(Q_k) = k2^{k-1}$.

1.3.24. Prove that $K_{2,3}$ is not contained in any hypercube Q_k.

1.3.25. (!) Prove that every cycle of length $2r$ in a hypercube is contained in a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r?

1.3.26. (!) Count the 6-cycles in Q_3. Prove that every 6-cycle in Q_k lies in exactly one 3-dimensional subcube. Use this to count the 6-cycles in Q_k for $k \geq 3$.

1.3.27. Given $k \in \mathbb{N}$, let G be the subgraph of Q_{2k+1} induced by the vertices in which the number of ones and zeros differs by 1. Prove that G is regular, and compute $n(G)$, $e(G)$, and the girth of G.

1.3.28. Let V be the set of binary k-tuples. Define a simple graph Q'_k with vertex set V by putting $u \leftrightarrow v$ if and only if u and v *agree* in exactly one coordinate. Prove that Q'_k is isomorphic to the hypercube Q_k if and only if k is even. (D.G. Hoffman)

1.3.29. (∗+) *Automorphisms of the k-dimensional cube Q_k.*

a) Prove that every copy of Q_j in Q_k is a subgraph induced by a set of 2^j vertices having specified values on a fixed set of $k-j$ coordinates. (Hint: Prove that a copy of Q_j must have two vertices differing in j coordinates.)

b) Use part (a) to count the automorphisms of Q_k.

1.3.30. Prove that every edge in the Petersen graph belongs to exactly four 5-cycles, and use this to show that the Petersen graph has exactly twelve 5-cycles. (Hint: For the first part, extend the edge to a copy of P_4 and apply Proposition 1.1.38.)

1.3.31. (!) Use complete graphs and counting arguments (not algebra!) to prove that

a) $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$ for $0 \le k \le n$.

b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \le \binom{n}{2}$.

1.3.32. (!) Prove that the number of simple even graphs with vertex set $[n]$ is $2^{\binom{n-1}{2}}$. (Hint: Establish a bijection to the set of all simple graphs with vertex set $[n-1]$.)

1.3.33. (+) Let G be a triangle-free simple n-vertex graph such that every pair of nonadjacent vertices has exactly two common neighbors.

a) Prove that $n(G) = 1 + \binom{d(x)}{2}$, where $x \in V(G)$. Conclude that G is regular.

b) When $k = 5$, prove that deleting any one vertex and its neighbors from G leaves the Petersen graph. (Comment: When $k = 5$, the graph G is in fact the graph obtained from Q_4 by adding edges joining complementary vertices.)

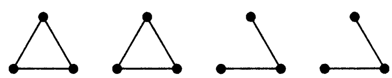
1.3.34. (+) Let G be a kite-free simple n-vertex graph such that every pair of nonadjacent vertices has exactly two common neighbors. Prove that G is regular. (Galvin)

1.3.35. (+) Let n and k be integers such that $1 < k < n-1$. Let G be a simple n-vertex graph such that every k-vertex induced subgraph of G has m edges.

a) Let G' be an induced subgraph of G with l vertices, where $l > k$. Prove that $e(G') = m\binom{l}{k}/\binom{l-2}{k-2}$.

b) Use part (a) to prove that $G \in \{K_n, \overline{K}_n\}$. (Hint: Use part (a) to prove that the number of edges with endpoints u, v is independent of the choice of u and v.)

1.3.36. Let G be a 4-vertex graph whose list of subgraphs obtained by deleting one vertex appears below. Determine G.



1.3.37. Let H be a graph formed by deleting a vertex from a loopless regular graph G with $n(G) \ge 3$. Describe (and justify) a method for obtaining G from H.

1.3.38. Let G be a graph with at least 3 vertices. Prove that G is connected if and only if at least two of the subgraphs obtained by deleting one vertex of G are connected. (Hint: Use Proposition 1.2.29.)

1.3.39. (∗+) Prove that every disconnected graph G with at least three vertices is reconstructible. (Hint: Having used Exercise 1.3.38 to determine that G is disconnected, use $G_1, \dots, G_n$ to find a component M of G that occurs the most times among the components with the maximum number of vertices, use Proposition 1.2.29 to choose v so that $L = M - v$ is connected, and reconstruct G by finding some $G - v_i$ in which a copy of M became a copy of L.)

1.3.40. (!) Let G be an n-vertex simple graph, where $n \ge 2$. Determine the maximum possible number of edges in G under each of the following conditions.

a) G has an independent set of size a.

b) G has exactly k components.

c) G is disconnected.

1.3.41. (!) Prove or disprove: If G is an n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

1.3.42. Let S be a set of vertices in a k-regular graph G such that no two vertices in S are adjacent or have a common neighbor. Use the pigeonhole principle to prove that $|S| \le \lfloor n(G)/(k+1) \rfloor$. Show that the bound is best possible for the cube Q_3. (Comment: The bound is not best possible for Q_4.)

1.3.43. (+) Let G be a simple graph with no isolated vertices, and let $a = 2e(G)/n(G)$ be the average degree in G. Let $t(v)$ denote the average of the degrees of the neighbors of v. Prove that $t(v) \ge a$ for some $v \in V(G)$. Construct an infinite family of connected graphs such that $t(v) > a$ for every vertex v. (Hint: For the first part, compute the average of $t(v)$, using that $x/y + y/x \ge 2$ when $x, y > 0$.) (Ajtai–Komlós–Szemerédi [1980])

1.3.44. (!) Let G be a loopless graph with average vertex degree $a = 2e(G)/n(G)$.

a) Prove that $G - x$ has average degree at least a if and only if $d(x) \le a/2$.

b) Use part (a) to give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.

c) Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible. (Hint: Use $K_{1,n-1}$.)

1.3.45. Determine the maximum number of edges in a bipartite subgraph of the Petersen graph.

1.3.46. Prove or disprove: Whenever the algorithm of Theorem 1.3.19 is applied to a bipartite graph, it finds the bipartite subgraph with the most edges (the full graph).

1.3.47. Use induction on $n(G)$ to prove that every nontrivial loopless graph G has a bipartite subgraph H such that H has *more* than $e(G)/2$ edges.

1.3.48. Construct graphs $G_1, G_2, \ldots$, with G_n having $2n$ vertices, such that $\lim_{n\to\infty} f_n = 1/2$, where f_n is the fraction of $E(G_n)$ belonging to the largest bipartite subgraph of G_n.

1.3.49. For each $k \in \mathbb{N}$ and each loopless graph G, prove that G has a k-partite subgraph H (Definition 1.1.12) such that $e(H) \ge (1 - 1/k)e(G)$.

1.3.50. (+) For $n \ge 3$, determine the minimum number of edges in a connected n-vertex graph in which every edge belongs to a triangle. (Erdős [1988])

1.3.51. (+) Let G be a simple n-vertex graph, where $n > 3$.

a) Use Proposition 1.3.11 to prove that if G has more than $n^2/4$ edges, then G has a vertex whose deletion leaves a graph with more than $(n-1)^2/4$ edges. (Hint: In every graph, the number of edges is an integer.)

b) Use part (a) to prove by induction that G contains a triangle if $e(G) > n^2/4$.

1.3.52. Prove that every n-vertex triangle-free simple graph with the maximum number of edges is isomorphic to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. (Hint: Strengthen the proof of Theorem 1.3.23.)

1.3.53. (!) Each game of *bridge* involves two teams of two partners each. Consider a club in which four players cannot play a game if two of them have previously been partners that night. Suppose that 15 members arrive, but one decides to study graph theory. The other 14 people play until each has been a partner with four others. Next they succeed in playing six more games (12 partnerships), but after that they cannot find four players containing no pair of previous partners. Prove that if they can convince the graph theorist to play, then at least one more game can be played. (Adapted from Bondy–Murty [1976, p111]).

1.3.54. (+) Let G be a simple graph with n vertices. Let $t(G)$ be the total number of triangles in G and $\overline{G}$ together.

a) Prove that $t(G) = \binom{n}{3} - (n-2)e(G) + \sum_{v \in V(G)} \binom{d(v)}{2}$ triangles. (Hint: Consider the contribution made to each side by each triple of vertices.

b) Prove that $t(G) \ge n(n-1)(n-5)/24$. (Hint: Use a lower bound on $\sum_{v \in V(G)} \binom{d(v)}{2}$ in terms of average degree.)

c) When $n-1$ is divisible by 4, construct a graph achieving equality in part (b). (Goodman [1959])

1.3.55. (+) *Maximum size with no induced P_4.*

a) Let G be the complement of a disconnected simple graph. Prove that $e(G) \le \Delta(G)^2$, with equality only for $K_{\Delta(G),\Delta(G)}$.

b) Let G be a simple connected P_4-free graph with maximum degree k. Prove that $e(G) \le k^2$. (Seinsche [1974], Chung–West [1993])

1.3.56. Use induction (on n or on $\sum d_i$) to prove that if $d_1, \dots, d_n$ are nonnegative integers and $\sum d_i$ is even, then there is an n-vertex graph with vertex degrees $d_1, \dots, d_n$. (Comment: This requests an alternative proof of Proposition 1.3.28.)

1.3.57. (!) Let n be a positive integer. Let d be a list of n nonnegative integers with even sum whose largest entry is less than n and differs from the smallest entry by at most 1. Prove that d is graphic. (Hint: Use the Havel–Hakimi Theorem. Example: 443333 is such a list, as is 33333322.)

1.3.58. *Generalization of Havel–Hakimi Theorem.* Given a nonincreasing list d of nonnegative integers, let d' be obtained by deleting d_k and subtracting 1 from the k largest elements remaining in the list. Prove that d is graphic if and only if d' is graphic. (Hint: Mimic the proof of Theorem 1.3.31.) (Wang–Kleitman [1973])

1.3.59. Define $d = (d_1, \dots, d_{2k})$ by $d_{2i} = d_{2i-1} = i$ for $1 \le i \le k$. Prove that d is graphic. (Hint: Do not use the Havel–Hakimi Theorem.)

1.3.60. (+) Let d be a list of integers consisting of k copies of a and $n-k$ copies of b, with $a \ge b \ge 0$. Determine necessary and sufficient conditions for d to be graphic.

1.3.61. (!) Suppose that $G \cong \overline{G}$ and that $n(G) \equiv 1 \bmod 4$. Prove that G has at least one vertex of degree $(n(G)-1)/2$.

1.3.62. Suppose that n is congruent to 0 or 1 modulo 4. Construct an n-vertex simple graph G with $\frac{1}{2}\binom{n}{2}$ edges such that $\Delta(G) - \delta(G) \le 1$.

1.3.63. (!) Let $d_1, \dots, d_n$ be integers such that $d_1 \ge \cdots \ge d_n \ge 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \dots, d_n$ if and only if $\sum d_i$ is even and $d_1 \le d_2 + \cdots + d_n$. (Hakimi [1962])

1.3.64. (!) Let $d_1 \le \cdots \le d_n$ be the vertex degrees of a simple graph G. Prove that G is connected if $d_j \ge j$ when $j \le n-1-d_n$. (Hint: Consider a component that omits some vertex of maximum degree.)

1.3.65. (+) Let $a_1 < \cdots < a_k$ be distinct positive integers. Prove that there is a simple graph with $a_k + 1$ vertices whose set of distinct vertex degrees is $a_1, \dots, a_k$. (Hint: Use induction on k to construct such a graph.) (Kapoor–Polimeni–Wall [1977])

1.3.66. (∗) *Expansion of 3-regular graphs* (see Example 1.3.26). For $n = 4k$, where $k \ge 2$, construct a connected 3-regular simple graph with n vertices that has no cut-edge but cannot be obtained from a smaller 3-regular simple graph by expansion. (Hint:

The desired graph must have no edge to which the inverse "erasure" operation can be applied to obtain a smaller simple graph.)

1.3.67. (∗) *Construction of 3-regular simple graphs*

a) Prove that a 2-switch can be performed by performing a sequence of expansions and erasures; these operations are defined in Example 1.3.26. (Caution: Erasure is not allowed when it would produce multiple edges.)

b) Use part (a) to prove that every 3-regular simple graph can be obtained from K_4 by a sequence of expansions and erasures. (Batagelj [1984])

1.3.68. (∗) Let G and H be two simple bipartite graphs, each with bipartition X, Y. Prove that $d_G(v) = d_H(v)$ for all $v \in X \cup Y$ if and only if there is a sequence of 2-switches that transforms G into H without ever changing the bipartition (each 2-switch replaces two edges joining X and Y by two other edges joining X and Y).

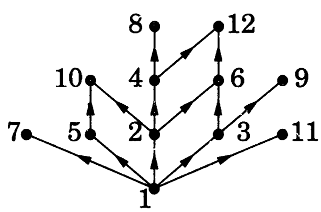
1.4. Directed Graphs

We have used graphs to model symmetric relations. Relation need not be symmetric; in general, a relation on S can be any set of ordered pairs in $S \times S$ (see Appendix A). For such relations, we need a more general model.

DEFINITIONS AND EXAMPLES

Seeking a graphical representation of the information in a general relation on S leads us to a model of directed graphs.

1.4.1. Example. For natural numbers x, y, we say that x is a "maximal divisor" of y if y/x is a prime number. For $S \subseteq \mathbb{N}$, the set $R = \{(x, y) \in S^2 : x \text{ is a maximal divisor of } y\}$ is a relation on S. To represent it graphically, we name a point in the plane for each element of S and draw an arrow from x to y whenever $(x, y) \in R$. Below we show the result when $S = [12]$. ■



1.4.2. Definition. A **directed graph** or **digraph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the **tail** of the edge, and the second is the **head**; together, they are the **endpoints**. We say that an edge is an edge **from** its tail **to** its head.

The terms "head" and "tail" come from the arrows used to draw digraphs. As with graphs, we assign each vertex a point in the plane and each edge a curve joining its endpoints. When drawing a digraph, we give the curve a direction from the tail to the head.

When a digraph models a relation, each ordered pair is the (head, tail) pair for at most one edge. In this setting as with simple graphs, we ignore the technicality of a function assigning endpoints to edges and simply treat an edge as an ordered pair of vertices.

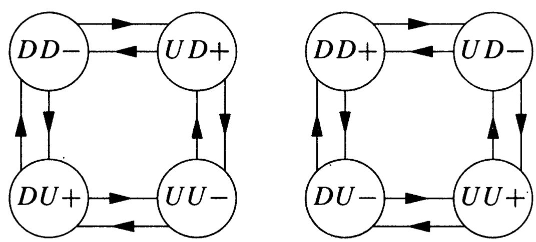
1.4.3. Definition. In a digraph, a **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same ordered pair of endpoints. A digraph is **simple** if each ordered pair is the head and tail of at most one edge; one loop may be present at each vertex.

In a simple digraph, we write uv for an edge with tail u and head v. If there is an edge from u to v, then v is a **successor** of u, and u is a **predecessor** of v. We write $u \to v$ for "there is an edge from u to v".

1.4.4. Application. A **finite state machine** (also called **finite automaton** or **discrete system**) has a number of possible "states". Such a system can be modeled using a digraph in which vertices represent the states and edges represent the possible transitions between states.

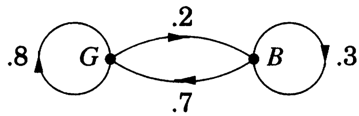
Transitions inherently move in one direction, so digraphs provide the appropriate model. Labels on the edges can be used to record the events that cause the transitions. When an event causes the system to remain in the same state, we have a loop. When two types of events can cause a particular transition, we might use multiple edges.

Consider a light controlled by two switches, often called a "three-way switch". The first switch can be up or down, the second switch can be up or down, and the light can be on (+) or off (−). Thus there are eight states. Transitions between states result by flipping switches. In the drawing below, the horizontal edges represent transitions caused by flipping the first switch, and the vertical edges represent transitions caused by flipping the second switch. (Drawing vertices large enough to put labels inside is not uncommon when discussing finite state machines, but we will stick with filled dots.) ■



1.4.5.* Application. Edge labels can be used to record transition probabilities when a system operates randomly. The probabilities on the edges leaving a vertex sum to 1, and the system is called a **Markov chain**. Methods of linear algebra can be used to compute the long-term fraction of time spent in each state.

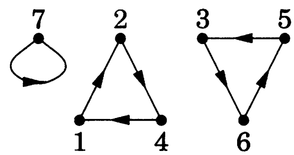
For example, suppose that weather has two states: good and bad. Air masses move slowly enough that tomorrow's weather tends to be like today's. In most places, storm systems don't linger long, so we might have transition probabilities as shown below. If we record states hourly instead of daily, then the probably of remaining in the same state is much higher. ■



1.4.6. Definition. A digraph is a **path** if it is a simple digraph whose vertices can be linearly ordered so that there is an edge with tail u and head v if and only if v immediately follows u in the vertex ordering. A **cycle** is defined similarly using an ordering of the vertices on a circle.

1.4.7. Example. *Functional digraphs.* We can study a function $f\colon A \to A$ using digraphs. The **functional digraph** of f is the simple digraph with vertex set A and edge set $\{(x, f(x)\colon x \in A\}$. For each x, the single edge with tail x points to the image of x under f.

Following a path in a functional digraph corresponds to iterating the function. In a permutation, each element is the image of exactly one element, so the functional digraph has one head and one tail at each vertex. Hence the functional digraph of a permutation consists of disjoint cycles. Below we show the functional digraph for a permutation of [7]. ■

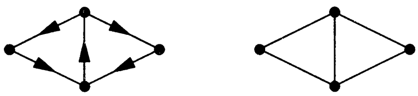


1.4.8.* Remark. We often use the same names for corresponding concepts in the graph and digraph models. Many authors replace "vertex" and "edge" with "node" and "arc" to discuss digraphs, but this obscures the analogies. Some results have the same statements and proofs; it would be wasteful to repeat them just to change terminology (especially in Chapter 4).

Also, a graph G can be modeled using a digraph D in which each edge $uv \in E(G)$ is replaced with $uv, vu \in E(D)$. In this way, results about digraphs can be applied to graphs. Since the notion of "edge" in digraphs extends the notion of "edge" in graphs, using the same name makes sense.

Some authors write "directed path" and "directed cycle" for our concepts of path and cycle in digraphs, but the distinction is unnecessary; for the "weak" version that does not follow the arrows, we can speak of a path or cycle in the graph obtained by ignoring the directions, which we define next. ■

1.4.9. Definition. The **underlying graph** of a digraph D is the graph G obtained by treating the edges of D as unordered pairs; the vertex set and edge set remain the same, and the endpoints of an edge are the same in G as in D, but in G they become an unordered pair.



D $\qquad$ G

Most ideas and methods of graph theory arise in the study of ordinary graphs. Digraphs can be a useful additional tool, especially in applications, as we have tried to suggest. We hope that describing the analogies and contrasts between graphs and digraphs will help clarify the concepts.

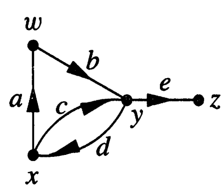
When comparing a digraph with a graph, we usually use G for the graph and D for the digraph. When discussing a single digraph, we often use G.

1.4.10. Definition. The definitions of **subgraph, isomorphism, decomposition**, and **union** are the same for graphs and digraphs. In the **adjacency matrix** $A(G)$ of a digraph G, the entry in position i, j is the number of edges from v_i to v_j. In the **incidence matrix** $M(G)$ of a loopless digraph G, we set $m_{i,j} = +1$ if v_i is the tail of e_j and $m_{i,j} = -1$ if v_i is the head of e_j.

1.4.11. Example. The underlying graph of the digraph below is the graph of Example 1.1.19; note the similarities and differences in their matrices. ■

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$A(G)$



G

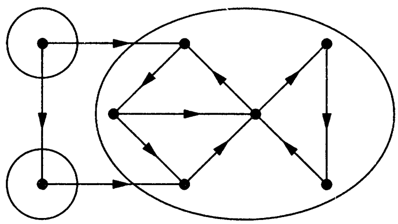
$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & +1 & -1 & 0 \\ 0 & -1 & -1 & +1 & +1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$M(G)$

To define connected digraphs, two options come to mind. We could require only that the underlying graph be connected. However, this does not capture the most useful sense of connection for digraphs.

1.4.12. Definition. A digraph is **weakly connected** if its underlying graph is connected. A digraph is **strongly connected** or **strong** if for each *ordered pair* u, v of vertices, there is a path from u to v. The **strong components** of a digraph are its maximal strong subgraphs.

1.4.13. Example. The 2-vertex digraph consisting only of the edge xy has an x, y-path but no y, x-path and is not strongly connected. As a digraph, an n-vertex path has n strong components, but a cycle has only one. In the digraph below, the three circled subdigraphs are the strong components. Properties of strong components are discussed in Exercises 10–13. ■



1.4.14.* Application. *Games.* Many games with two players can be described as finite state machines. The vertex set is the set of possible states of the game. There is an edge from state x to state y if some move can be made (by the player whose turn it is to play) to reach state y from state x.

Let W be the set of vertices for winning positions; a player who brings the game to such a state wins. No edges leave W. A player who brings the game to a state with an edge to W loses, since the other player then reaches W. One way to analyze the game is to seek a set S of pairwise nonadjacent vertices containing W such that every vertex outside S has an edge to a vertex in S. A player who can bring the game to a position in S wins, but one who must move from a position in S loses.

For example, consider a game with two piles of pennies. At his or her turn, each player can remove any portion of a single pile. The player who removes the last coin wins. The possible game positions are the nonnegative integer pairs (r, s). The definition of the game specifies $(0, 0)$ as the only winning position. However, the set S of desirable positions is $\{(r, r)\colon r \ge 0\}$. Since only one coordinate can decrease on a move, there is no edge within S. For each vertex $(r, s) \notin S$, a player can remove $|r - s|$ from the larger pile to reach S.

The general game of Nim starts with an arbitrary number of piles with arbitrary sizes, but otherwise the rules of the game are the same as this. Exercise 18 guarantees that Nim always has a winning strategy set S, since the digraph for this game has no cycles. If the initial position is in S, then the second player wins (assuming optimal play). Otherwise, the first player wins. ■

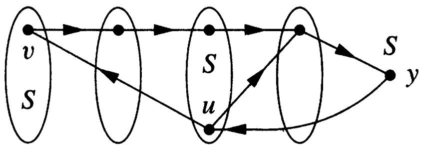
1.4.15.* Definition. A **kernel** in the digraph D is a set $S \subseteq V(D)$ such that S induces no edges and every vertex outside S has a successor in S.

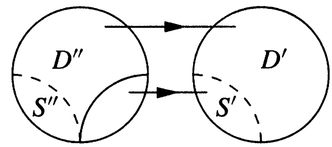
A digraph that is an odd cycle has no kernel (Exercise 17), but forbidding odd cycles as subdigraphs always yields a kernel. In proving this, all uses of paths, cycles, and walks are in the directed sense. We need several statements about movement in digraphs that hold by the same proofs as in graphs. For example, every u, v-walk in a digraph contains a u, v-path (Exercise 3), and every closed odd walk in a digraph contains an odd cycle (Exercise 4). The concept of **distance** from x to y will be explored more fully in Section 2.1; it is the least length of an x, y-path.

1.4.16.* Theorem. (Richardson [1953]) Every digraph having no odd cycle has a kernel.

Proof: Let D be such a digraph. We first consider the case that D is strongly connected; see the figure on the left below. Given an arbitrary vertex $y \in V(D)$, let S be the set of vertices with even distance to y. Every vertex with odd distance to y has a successor in S, as desired.

If the vertices of S are not pairwise nonadjacent, then there is an edge uv with $u, v \in S$. By the definition of S, there is a u, y-path P of even length and a v, y-path P' of even length. Adding uv at the start of P' yields a u, y-walk W of odd length. Because D is strong, D has a y, u-path Q. Combining Q with one of P or W yields a closed odd walk in D. This is impossible, since a closed odd walk contains an odd cycle. Thus S is a kernel in D.





For the general case, we use induction on $n(D)$.

Basis step: $n(D) = 1$. The only example is a single vertex with no loop. This vertex is a kernel by itself.

Induction step: $n(D) > 1$. Since we have already proved the claim for strong digraphs, we may assume that D is not strong. For some strong component D' of D, there is no edge from a vertex of D' to a vertex not in D' (Exercise 11). We have shown that D' has a kernel; let S' be a kernel of D'.

Let D'' be the subdigraph obtained from D by deleting D' and all the predecessors of S'. By the induction hypothesis, D'' has a kernel; let S'' be a kernel of D''. We claim that $S' \cup S''$ is a kernel of D. Since D'' has no predecessor of S', there is no edge within $S' \cup S''$. Every vertex in $D'' - S''$ has a successor in S'', and all other vertices not in $S' \cup S''$ have a successor in S'. ■

VERTEX DEGREES

In a digraph, we use the same notation for number of vertices and number of edges as in graphs. The notation for vertex degrees incorporates the distinction between heads and tails of edges.

1.4.17. Definition. Let v be a vertex in a digraph. The **outdegree** $d^+(v)$ is the number of edges with tail v. The **indegree** $d^-(v)$ is the number of edges with head v. The **out-neighborhood** or **successor set** $N^+(v)$ is $\{x \in V(G): v \to x\}$. The **in-neighborhood** or **predecessor set** $N^-(v)$ is $\{x \in V(G): x \to v\}$. The minimum and maximum indegree are $\delta^-(G)$ and $\Delta^-(G)$; for outdegree we use $\delta^+(G)$ and $\Delta^+(G)$.

The digraph analogue of the degree-sum formula for graphs is easy.

1.4.18. Proposition. In a digraph G, $\sum_{v \in V(G)} d^+(v) = e(G) = \sum_{v \in V(G)} d^-(v)$.

Proof: Every edge has exactly one tail and exactly one head. ■

The digraph analogue of degree sequence is the list of "degree pairs" $(d^+(v_i), d^-(v_i))$. When is a list of pairs realizable as the degree pairs of a digraph? As with graphs, this is easy when we allow multiple edges.

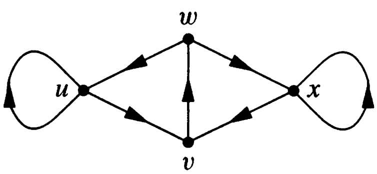
1.4.19.* Proposition. A list of pairs of nonnegative integers is realizable as the degree pairs of a digraph if and only if the sum of the first coordinates equals the sum of the second coordinates.

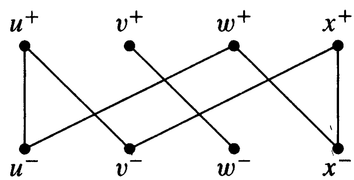
Proof: The condition is necessary because every edge has one tail and one head, contributing once to each sum.

For sufficiency, consider the pairs $\{(d_i^+, d_i^-): 1 \le i \le n\}$ and vertices $v_1, \ldots, v_n$. Let $m = \sum d_i^+ = \sum d_j^-$. Consider m dots. Give the dots positive labels, with d_i^+ of them having label i. Also give the dots negative labels, with d_j^- of them having label $-j$. For each dot with labels i and $-j$, place an edge from v_i to v_j. This creates a digraph with $d^+(v_i) = d_i^+$ and $d^-(v_i) = d_i^-$. ■

The analogous question for simple digraphs is harder. The question can be rephrased in terms of bipartite graphs via a transformation that is useful in many problems about digraphs.

1.4.20.* Definition. The **split** of a digraph D is a bipartite graph G whose partite sets V^+, V^- are copies of $V(D)$. For each vertex $x \in V(D)$, there is one vertex $x^+ \in V^+$ and one vertex $x^- \in V^-$. For each edge from u to v in D, there is an edge with endpoints u^+, v^- in G.





1.4.21.* Remark. The degrees of the vertices in the split of D are the indegrees and outdegrees of the vertices in D.

Furthermore, an X, Y-bigraph G with $|X| = |Y| = n$ can be transformed into an n-vertex digraph D by putting an edge $v_i v_j$ in D for each edge $x_i y_j$ in G; now G is the split of D. (This is one reason to allow loops in simple digraphs.)

Thus there is a simple digraph with degree pairs $\{(d_i^+, d_i^-): 1 \le i \le n\}$ if and only if there is a simple bipartite graph G in which the vertex degrees are $d_1^+, \ldots, d_n^+$ in one partite set and $d_1^-, \ldots, d_n^-$ in the other partite set. Exercise 32 obtains a recursive test for existence of such a bipartite graph. The statement and proof are like that of the Havel–Hakimi Theorem, so we leave further discussion to the exercise. ■

EULERIAN DIGRAPHS

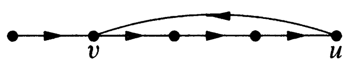
The definitions of **trail**, **walk**, **circuit**, and the **connection relation** are the same in graphs and digraphs when we list edges as ordered pairs of vertices. In a digraph, the successive edges must "follow the arrows". In a walk $v_0, e_1, \ldots, e_k, v_k$, the edge e_i has tail v_{i-1} and head v_i.

1.4.22. Definition. An **Eulerian trail** in a digraph (or graph) is a trail containing all edges. An **Eulerian circuit** is a closed trail containing all edges. A digraph is **Eulerian** if it has an Eulerian circuit.

The characterization of Eulerian digraphs is analogous to the characterization of Eulerian graphs. The proof is essentially the same as for graphs, so we leave it to the exercises.

1.4.23. Lemma. If G is a digraph with $\delta^+(G) \geq 1$, then G contains a cycle. The same conclusion holds when $\delta^-(G) \geq 1$.

Proof: Let P be a maximal path in G, and let u be the last vertex of P. Since P cannot be extended, every successor of u must already be a vertex of P. Since $\delta^+(G) \geq 1$, u has a successor v on P. The edge uv completes a cycle with the portion of P from v to u. ■



1.4.24. Theorem. A digraph is Eulerian if and only if $d^+(v) = d^-(v)$ for each vertex v and the underlying graph has at most one nontrivial component.

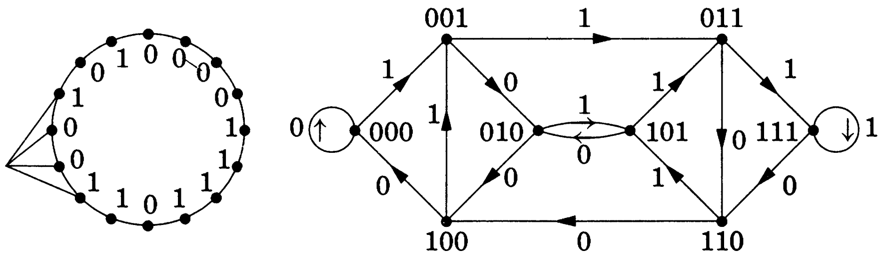
Proof: See Exercise 19 or Exercise 20. ■

Every Eulerian digraph with no isolated vertices is strongly connected, although the characterization states that being weakly connected is sufficient.

1.4.25. Application. *de Bruijn cycles.* There are 2^n binary strings of length n. Is there a cyclic arrangement of 2^n binary digits such that the 2^n strings of n consecutive digits are all distinct? For $n = 4$, (0000111101100101) works.

We can use such an arrangement to keep track of the position of a rotating drum (Good [1946]). Our drum has 2^n rotational positions. A band around the circumference is split into 2^n portions that can be coded 0 or 1. Sensors read n consecutive portions. If the coding has the property specified above, then the position of the drum is determined by the string read by the sensors.

To obtain such a circular arrangement, define a digraph D_n whose vertices are the binary $(n-1)$-tuples. Put an edge from a to b if the last $n-2$ entries of a agree with the first $n-2$ entries of b. Label the edge with the last entry of b. Below we show D_4. We next prove that D_n is Eulerian and show how an Eulerian circuit yields the desired circular arrangement. ■



1.4.26. Theorem. The digraph D_n of Application 1.4.25 is Eulerian, and the edge labels on the edges in any Eulerian circuit of D_n form a cyclic arrangement in which the 2^n consecutive segments of length n are distinct.

Proof: We show first that D_n is Eulerian. Every vertex has outdegree 2, because we can append a 0 or a 1 to its name to obtain the name of a successor vertex. Similarly, every vertex has indegree 2, because the same argument applies when moving in reverse and putting a 0 or a 1 on the front of the name. Also, D_n is strongly connected, because we can reach the vertex $b = (b_1, \ldots, b_{n-1})$ from any vertex by successively following the edges labeled $b_1, \ldots, b_{n-1}$. Thus D_n satisfies the hypotheses of Theorem 1.4.24 and is Eulerian.

Let C be an Eulerian circuit of D_n. Arrival at vertex $a = (a_1, \ldots, a_{n-1})$ must be along an edge with label a_{n-1}, because the label on an edge entering a vertex agrees with the last entry of the name of the vertex. Since we delete the front and shift the rest to obtain the rest of the name at the head, the successive earlier labels (looking backward) must have been $a_{n-2}, \ldots, a_1$ in order. If C next uses an edge with label a_n, then the list consisting of the n most recent edge labels at that time is $a_1, \ldots, a_n$.

Since the 2^{n-1} vertex labels are distinct, and the two edges leaving each vertex have distinct labels, and we traverse each edge from each vertex exactly once along C, we have shown that the 2^n strings of length n in the circular arrangement given by the edge labels along C are distinct. ■

The digraph D_n is the **de Bruijn graph** of order n on an alphabet of size 2. It is useful for other purposes, because it has many vertices and few edges (only twice the number of vertices) and yet we can reach each vertex from any other by a short path. We can reach any desired vertex in $n - 1$ steps by introducing the bits in its name in order from the current vertex.

ORIENTATIONS AND TOURNAMENTS

There are n^2 ordered pairs of elements that can be formed from a vertex set of size n. A simple digraph allows loops but uses each ordered pair at most once as an edge. Thus there are n^2 ordered pairs that may or may not be present as edges, and there are 2^{n^2} simple digraphs with vertex set $v_1, \ldots, v_n$.

Sometimes we want to forbid loops.

1.4.27. Definition. An **orientation** of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$. An **oriented graph** is an orientation of a simple graph. A **tournament** is an orientation of a complete graph.

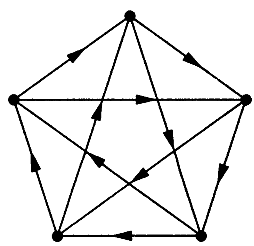
An oriented graph is the same thing as a loopless simple digraph. When the edges of a graph represent comparisons to be performed among items corresponding to the vertices, we can record the results by putting $x \to y$ when x does better than y in the comparison. The outcome is an orientation of G.

The number of oriented graphs with vertices $v_1, \ldots, v_n$ is $3^{\binom{n}{2}}$; the number of tournaments is $2^{\binom{n}{2}}$.

1.4.28. Example. Orientations of complete graphs model "round-robin tournaments". Consider an n-team league where each team plays every other exactly once. For each pair u, v, we include the edge uv if u wins or vu if v wins. At the end of the season we have an orientation of K_n. The "score" of a team is its outdegree, which equals its number of wins.

We therefore call the outdegree sequence of a tournament its **score sequence**. The outdegrees determine the indegrees, since $d^+(v) + d^-(v) = n - 1$ for every vertex v. It is easier to characterize the score sequences of tournaments than the degree sequences of simple graphs (Exercise 35). ■

A tournament may have more than one vertex with maximum outdegree, so there may be no clear "winner"—in the example below, every vertex has outdegree 2 and indegree 2. Choosing a champion when several teams have the maximum number of wins can be difficult. Although there need not be a clear winner, we show next that there must always be a team x such that, for every other team z, either x beats z or x beats some team that beats z.

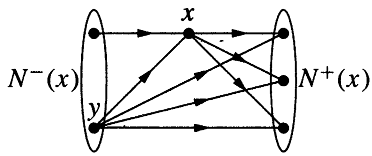


1.4.29. Definition. In a digraph, a **king** is a vertex from which every vertex is reachable by a path of length at most 2.

1.4.30. Proposition. (Landau [1953]) Every tournament has a king.

Proof: Let x be a vertex in a tournament T. If x is not a king, then some vertex y is not reachable from x by a path of length at most 2. Hence no successor of x is a predecessor of y. Since T is an orientation of a clique, every successor of x must therefore be a successor of y. Also $y \to x$. Hence $d^+(y) > d^+(x)$.

If y is not a king, then we repeat the argument to find z with yet larger outdegree. Since T is finite, we cannot forever obtain vertices of successively higher outdegree. The procedure must terminate, and it can terminate only when we have found a king. ∎



In the language of extremality, we have proved that every vertex of maximum outdegree in a tournament is a king. Exercises 36–38 ask further questions about kings (see also Maurer [1980]). Exercise 39 generalizes the result to arbitrary digraphs.

EXERCISES

1.4.1. (−) Describe a relation in the real world whose digraph has no cycles. Describe another that has cycles but is not symmetric.

1.4.2. (−) In the lightswitch system of Application 1.4.4, suppose the first switch becomes disconnected from the wiring. Draw the digraph that models the resulting system.

1.4.3. (−) Prove that every u,v-walk in a digraph contains a u,v-path.

1.4.4. (−) Prove that every closed walk of odd length in a digraph contains the edges of an odd cycle. (Hint: Follow Lemma 1.2.15.)

1.4.5. (−) Let G be a digraph in which indegree equals outdegree at each vertex. Prove that G decomposes into cycles.

1.4.6. (−) Draw the de Bruijn graphs D_2 and D_4.

1.4.7. (−) Prove or disprove: If D is an orientation of a simple graph with 10 vertices, then the vertices of D cannot have distinct outdegrees.

1.4.8. (−) Prove that there is an n-vertex tournament with indegree equal to outdegree at every vertex if and only if n is odd.

• • • • •

1.4.9. For each $n \geq 1$, prove or disprove: Every simple digraph with n vertices has two vertices with the same outdegree or two vertices with the same indegree.

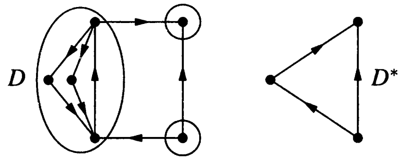
1.4.10. (!) Prove that a digraph is strongly connected if and only if for each partition of the vertex set into nonempty sets S and T, there is an edge from S to T.

1.4.11. (!) Prove that in every digraph, some strong component has no entering edges, and some strong component has no exiting edges.

1.4.12. Prove that in a digraph the connection relation is an equivalence relation, and its equivalence classes are the vertex sets of the strong components.

1.4.13. a) Prove that the strong components of a digraph are pairwise disjoint.

b) Let $D_1, \ldots, D_k$ be the strong components of a digraph D. Let D^* be the loopless digraph with vertices $v_1, \ldots, v_k$ such that $v_i \to v_j$ if and only if $i \neq j$ and D has an edge from D_i to D_j. Prove that D^* has no cycle.



1.4.14. (!) Let G be an n-vertex digraph with no cycles. Prove that the vertices of G can be ordered as $v_1, \ldots, v_n$ so that if $v_i v_j \in E(G)$, then $i < j$.

1.4.15. Let G be the simple digraph with vertex set $\{(i, j) \in \mathbb{Z}^2\colon 0 \le i \le m \text{ and } 0 \le n\}$ and an edge from (i, j) to (i', j') if and only if (i', j') is obtained from (i, j) by adding 1 to one coordinate. Prove that the number of paths from $(0, 0)$ to (m, n) in G is $\binom{m+n}{n}$.

1.4.16. (+) *Fermat's Little Theorem.* Let $\mathbb{Z}_n$ denote the set of congruence classes of integers modulo n (see Appendix A). Let a be a natural number having no common prime factors with n; multiplication by a defines a permutation of $\mathbb{Z}_n$. Let l be the least natural number such that $a^l \equiv a \bmod n$.

a) Let G be the functional digraph with vertex set $\mathbb{Z}_n$ for the permutation defined by multiplication by a. Prove that all cycles in G (except the loop on n) have length $l - 1$.

b) Conclude from part (a) that $a^{n-1} \equiv 1 \bmod n$.

1.4.17. (∗) Prove that a (directed) odd cycle is a digraph with no kernel. Construct a digraph that has an odd cycle as an induced subgraph but does have a kernel.

1.4.18. (∗) Prove that a digraph having no cycle has a unique kernel.

1.4.19. Use Lemma 1.4.23 and induction on the number of edges to prove the characterization of Eulerian digraphs (Theorem 1.4.24). (Hint: Follow Theorem 1.2.26.)

1.4.20. Prove the characterization of Eulerian digraphs (Theorem 1.4.24) using the notion of maximal trails. (Hint: Follow 1.2.32, the second proof of Theorem 1.2.26.)

1.4.21. Theorem 1.4.24 establishes necessary and sufficient conditions for a digraph to have an Eulerian circuit. Determine (with proof), the necessary and sufficient conditions for a digraph to have an Eulerian trail (Definition 1.4.22). (Good [1946])

1.4.22. Let D be a digraph with $d^-(v) = d^+(v)$ for every vertex v, except that $d^+(x) - d^-(x) = k = d^-(y) - d^+(y)$. Use the characterization of Eulerian digraphs to prove that D contains k pairwise edge-disjoint x, y-paths.

1.4.23. Prove that every graph G has an orientation D that is "balanced" at each vertex, meaning that $\left|d_D^+(v) - d_D^-(v)\right| \le 1$ for every $v \in V(G)$.

1.4.24. Prove or disprove: Every graph G has an orientation such that for every $S \subseteq V(G)$, the number of edges entering S and leaving S differ by at most 1.

1.4.25. (!) *Orientations and P_3-decomposition.*

a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1. (Rotman [1991])

b) Use part (a) to conclude that a simple connected graph with an even number of edges can be decomposed into paths with two edges.

1.4.26. Arrange seven 0's and seven 1's cyclically so that the 14 strings of four consecutive bits are all the 4-digit binary strings other than 0101 and 1010.

1.4.27. *DeBruijn sequence for any alphabet and length.* Let A be an alphabet of size k. Prove that there exists a cyclic arrangement of k^l characters chosen from A such that the k^l strings of length l in the sequence are all distinct. (Good [1946], Rees [1946])

1.4.28. Let S be an alphabet of size m. Explain how to produce a cyclic arrangement of $m^4 - m$ letters from S such that all four-letter strings of consecutive letters are different and contain at least two distinct letters.

1.4.29. (!) Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle. (Hint: Consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle $(v_1, \ldots, v_k)$ of G.)

1.4.30. (+) Given a strong digraph D, let $f(D)$ be the length of the shortest closed walk visiting every vertex. Prove that the maximum value of $f(D)$ over all strong digraphs with n vertices is $\lfloor (n+1)^2/4 \rfloor$ if $n \geq 2$. (Cull [1980])

1.4.31. Determine the minimum n such that there is a pair of nonisomorphic n-vertex tournaments with the same list of outdegrees.

1.4.32. Let $p = p_1, \ldots, p_m$ and $q = q_1, \ldots, q_n$ be lists of nonnegative integers. The pair (p, q) is **bigraphic** if there is a simple bipartite graph in which $p_1, \ldots, p_m$ are the degrees for one partite set and $q_1, \ldots, q_n$ are the degrees for the other. When p has positive sum, prove that (p, q) is bigraphic if and only if (p', q') is bigraphic, where (p', q') is obtained from (p, q) by deleting the largest element Δ from p and subtracting 1 from each of the Δ largest elements of q. (Hint: Follow the method of Theorem 1.3.31.)

1.4.33. (∗) Let A and B be two m by n matrices with entries in $\{0, 1\}$. An *exchange* operation substitutes a submatrix of the form $\binom{01}{10}$ for a submatrix of the form $\binom{10}{01}$ or vice versa. Prove that if A and B have the same list of row sums and have the same list of column sums, then A can be transformed into B by a sequence of exchange operations. Interpret this conclusion in the context of bipartite graphs. (Ryser [1957])

1.4.34. (!) Let G and H be two tournaments on a vertex set V. Prove that $d_G^+(v) = d_H^+(v)$ for all $v \in V$ if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3. (Hint: Consider a vertex of maximum outdegree in the subgraph of G consisting of edges oriented oppositely in H.) (Ryser [1964])

1.4.35. (+) Let $p_1, \ldots, p_n$ be nonnegative integers with $p_1 \leq \cdots \leq p_n$. Let $p'_k = \sum_{i=1}^k p_i$. Prove that there exists a tournament with outdegrees $p_1, \ldots, p_n$ if and only if $p'_k \geq \binom{k}{2}$ for $1 \leq k < n$ and $p'_n = \binom{n}{2}$. (Hint: Use induction on $\sum_{k=1}^n [p'_k - \binom{k}{2}]$.) (Landau [1953])

1.4.36. By Proposition 1.4.30, every tournament has a king. Let T be a tournament having no vertex with indegree 0.

a) Prove that if x is a king in T, then T has another king in $N^-(x)$.

b) Use part (a) to prove that T has at least three kings.

c) For each $n \geq 3$, construct a tournament T with $\delta^-(T) > 0$ and only 3 kings.

(Comment: There exists an n-vertex tournament having exactly k kings whenever $n \geq k \geq 1$ except when $k = 2$ and when $n = k = 4$.) (Maurer [1980])

1.4.37. Consider the following algorithm whose input is a tournament T.

1) Select a vertex x in T.

2) If x has indegree 0, call x a king of T and stop.

3) Otherwise, delete $\{x\} \cup N^+(x)$ from T to form T'.
4) Run the algorithm on T'; call the output a king in T and stop.
Prove that this algorithm terminates and produces a king in T

1.4.38. (+) For $n \in \mathbb{N}$, prove that there is an n-vertex tournament in which every vertex is a king if and only if $n \notin \{2, 4\}$.

1.4.39. (+) Prove that every loopless digraph D has a set S of pairwise nonadjacent vertices such that every vertex outside S is reached from S by a path of length at most 2. (Hint: Use strong induction on $n(D)$. Comment: This generalizes Proposition 1.4.30.) (Chvátal–Lovász [1974])

1.4.40. A directed graph is **unipathic** if for every pair of vertices x, y there is at most one (directed) x, y-path. Let T_n be the tournament on n vertices with the edge between v_i and v_j directed toward the vertex with larger index. What is the maximum number of edges in a unipathic subgraph of T_n? How many unipathic subgraphs are there with the maximum number of edges? (Hint: Show that the underlying graph has no triangles.) (Maurer–Rabinovitch–Trotter [1980])

1.4.41. Let G be a tournament. Let L_0 be a listing of $V(G)$ in some order. If y immediately follows x in L_0 but $y \to x$ in G, then yx is a **reverse edge**. We can interchange x and y in the order when yx is a reverse edge (this may increase the number of reverse edges). Suppose that a sequence $L_0, L_1, \cdots$ is produced by successively switching one reverse edge in the current order. Prove that this always leads to a list with no reverse edges. Determine the maximum number of steps to termination. (Comment: In the special case where the vertices are numbers and each edge points to the higher number of the pair, the result says that successively switching adjacent numbers that are out of order always eventually sorts the list.) (Locke [1995])

1.4.42. (!) Given an ordering $\sigma = v_1, \ldots, v_n$ of the vertices of a tournament, let $f(\sigma)$ be the sum of the lengths of the feedback edges, meaning the sum of $j - i$ over edges $v_j v_i$ such that $j > i$. Prove that every ordering minimizing $f(\sigma)$ places the vertices in nonincreasing order of outdegree. (Hint: Determine how $f(\sigma)$ changes when consecutive elements of σ are exchanged.) (Kano–Sakamoto [1983], Isaak–Tesman [1991])

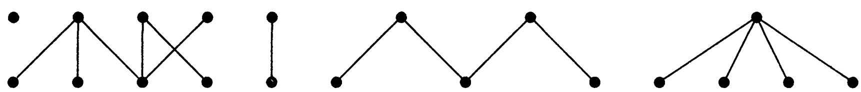
Chapter 2

Trees and Distance

2.1. Basic Properties

The word "tree" suggests branching out from a root and never completing a cycle. Trees as graphs have many applications, especially in data storage, searching, and communication.

2.1.1. Definition. A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph. A **leaf** (or **pendant vertex**) is a vertex of degree 1. A **spanning subgraph** of G is a subgraph with vertex set $V(G)$. A **spanning tree** is a spanning subgraph that is a tree.



2.1.2. Example. A tree is a connected forest, and every component of a forest is a tree. A graph with no cycles has no odd cycles; hence trees and forests are bipartite.

Paths are trees. A tree is a path if and only if its maximum degree is 2. A **star** is a tree consisting of one vertex adjacent to all the others. The n-vertex star is the biclique $K_{1,n-1}$.

A graph that is a tree has exactly one spanning tree; the full graph itself. A spanning subgraph of G need not be connected, and a connected subgraph of G need not be a spanning subgraph. For example:

If $n(G) > 1$, then the empty subgraph with vertex set $V(G)$ and edge set $\emptyset$ is spanning but not connected.

If $n(G) > 2$, then a subgraph consisting of one edge and its endpoints is connected but not spanning. ∎

PROPERTIES OF TREES

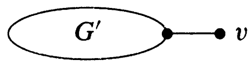
Trees have many equivalent characterizations, any of which could be taken as the definition. Such characterizations are useful because we need only verify that a graph satisfies any one of them to prove that it is a tree, after which we can use all the other properties.

We first prove that deleting a leaf from a tree yields a smaller tree.

2.1.3. Lemma. Every tree with at least two vertices has at least two leaves. Deleting a leaf from an n-vertex tree produces a tree with $n-1$ vertices.

Proof: A connected graph with at least two vertices has an edge. In an acyclic graph, an endpoint of a maximal nontrivial path has no neighbor other than its neighbor on the path. Hence the endpoints of a such a path are leaves.

Let v be a leaf of a tree G, and let $G' = G - v$. A vertex of degree 1 belongs to no path connecting two other vertices. Therefore, for $u, w \in V(G')$, every u, w-path in G is also in G'. Hence G' is connected. Since deleting a vertex cannot create a cycle, G' also is acyclic. Thus G' is a tree with $n-1$ vertices. ■



Lemma 2.1.3 implies that every tree with more than one vertex arises from a smaller tree by adding a vertex of degree 1 (all our graphs are finite). This rescues some proofs from the induction trap: growing an $n+1$-vertex tree from an arbitrary n-vertex tree by adding a new neighbor at an arbitrary old vertex generates all trees with $n+1$ vertices. The word "arbitrary" means that the discussion considers all ways of making the choice.

Our proof of equivalence of characterizations of trees uses induction, prior results, a counting argument, extremality, and contradiction.

2.1.4. Theorem. For an n-vertex graph G (with $n \geq 1$), the following are equivalent (and characterize the trees with n vertices).

A) G is connected and has no cycles.
B) G is connected and has $n-1$ edges.
C) G has $n-1$ edges and no cycles.
D) For $u, v \in V(G)$, G has exactly one u, v-path.

Proof: We first demonstrate the equivalence of A, B, and C by proving that any two of {connected, acyclic, $n-1$ edges} together imply the third.

A $\Rightarrow$ {B, C}. We use induction on n. For $n = 1$, an acyclic 1-vertex graph has no edge. For $n > 1$, we suppose that the implication holds for graphs with fewer than n vertices. Given an acyclic connected graph G, Lemma 2.1.3 provides a leaf v and states that $G' = G - v$ also is acyclic and connected (see figure above). Applying the induction hypothesis to G' yields $e(G') = n-2$. Since only one edge is incident to v, we have $e(G) = n-1$.

B $\Rightarrow$ {A, C}. Delete edges from cycles of G one by one until the resulting graph G' is acyclic. Since no edge of a cycle is a cut-edge (Theorem 1.2.14), G' is connected. Now the preceding paragraph implies that $e(G') = n - 1$. Since we are given $e(G) = n - 1$, no edges were deleted. Thus $G' = G$, and G is acyclic.

C $\Rightarrow$ {A, B}. Let $G_1, \ldots, G_k$ be the components of G. Since every vertex appears in one component, $\sum_i n(G_i) = n$. Since G has no cycles, each component satisfies property A. Thus $e(G_i) = n(G_i) - 1$. Summing over i yields $e(G) = \sum_i [n(G_i) - 1] = n - k$. We are given $e(G) = n - 1$, so $k = 1$, and G is connected.

A $\Rightarrow$ D. Since G is connected, each pair of vertices is connected by a path. If some pair is connected by more than one, we choose a shortest (total length) pair P, Q of distinct paths with the same endpoints. By this extremal choice, no internal vertex of P or Q can belong to the other path (see figure below). This implies that $P \cup Q$ is a cycle, which contradicts the hypothesis A.

D $\Rightarrow$ A. If there is a u, v-path for every $u, v \in V(G)$, then G is connected. If G has a cycle C, then G has two u, v-paths for $u, v \in V(C)$; hence G is acyclic (this also forbids loops). ■

2.1.5. Corollary. a) Every edge of a tree is a cut-edge.

b) Adding one edge to a tree forms exactly one cycle.

c) Every connected graph contains a spanning tree.

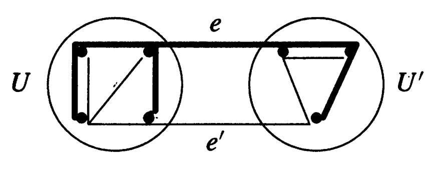
Proof: (a) A tree has no cycles, so Theorem 1.2.14 implies that every edge is a cut-edge. (b) A tree has a unique path linking each pair of vertices (Theorem 2.1.4D), so joining two vertices by an edge creates exactly one cycle. (c) As in the proof of B$\Rightarrow$A,C in Theorem 2.1.4, iteratively deleting edges from cycles in a connected graph yields a connected acyclic subgraph. ■

We apply Corollary 2.1.5 to prove two results about pairs of spanning trees. We use subtraction and addition to indicate deletion and inclusion of edges.

2.1.6. Proposition. If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.

Proof: By Corollary 2.1.5a, every edge of T is a cut-edge of T. Let U and U' be the two components of $T - e$. Since T' is connected, T' has an edge e' with endpoints in U and U'. Now $T - e + e'$ is connected, has $n(G) - 1$ edges, and is a spanning tree of G.

(In the figure below, T is bold, T' is solid, and they share two edges.) ■



2.1.7. Proposition. If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree of G.

Proof: By Corollary 2.1.5b, The graph $T' + e$ contains a unique cycle C. Since T is acyclic, there is an edge $e' \in E(C) - E(T)$. Deleting e' breaks the only cycle in $T' + e$. Now $T' + e - e'$ is connected and acyclic and is a spanning tree of G.

(In the figure above, adding e to T creates a cycle C of length five; all four edges of $C - e$ belong to $E(T) - E(T')$ and can serve as e'.) ■

The edge e' can be chosen to satisfy the conclusions of Propositions 2.1.6–2.1.7 simultaneously, as illustrated in the figure between them (Exercise 37).

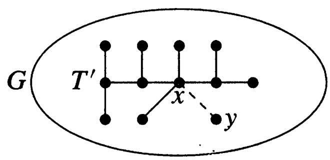
The next result illustrates proof by induction using deletion of a leaf.

2.1.8. Proposition. If T is a tree with k edges and G is a simple graph with $\delta(G) \geq k$, then T is a subgraph of G.

Proof: We use induction on k. Basis step: $k = 0$. Every simple graph contains K_1, which is the only tree with no edges.

Induction step: $k > 0$. We assume that the claim holds for trees with fewer than k edges. Since $k > 0$, Lemma 2.1.3 allows us to choose a leaf v in T; let u be its neighbor. Consider the smaller tree $T' = T - v$. By the induction hypothesis, G contains T' as a subgraph, since $\delta(G) \geq k > k - 1$.

Let x be the vertex in this copy of T' that corresponds to u (see illustration). Because T' has only $k - 1$ vertices other than u and $d_G(x) \geq k$, x has a neighbor y in G that is not in this copy of T'. Adding the edge xy expands this copy of T' into a copy of T in G, with y playing the role of v. ■



The inequality of Proposition 2.1.8 is sharp; the graph K_k has minimum degree $k - 1$, but it contains no tree with k edges. The proposition implies that every n-vertex simple graph G with more than $n(k - 1)$ edges has T as a subgraph (Exercise 34). Erdős and Sós conjectured the stronger statement that $e(G) > n(k - 1)/2$ forces T as a subgraph (Erdős [1964]). This has been proved for graphs without 4-cycles (Saclé–Woźniak [1997]). Ajtai, Komlós, and Szemerédi proved an asymptotic version, as reported in Soffer [2000].

DISTANCE IN TREES AND GRAPHS

When using graphs to model communication networks, we want vertices to be close together to avoid communication delays. We measure distance using lengths of paths.

2.1.9. Definition. If G has a u, v-path, then the **distance** from u to v, written $d_G(u, v)$ or simply $d(u, v)$, is the least length of a u, v-path. If G has no such path, then $d(u, v) = \infty$. The **diameter** (diam G) is $\max_{u,v \in V(G)} d(u, v)$.

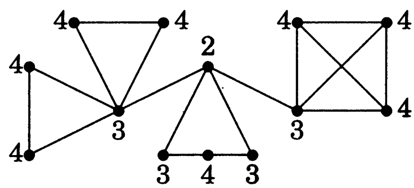
The **eccentricity** of a vertex u, written $\epsilon(u)$, is $\max_{v \in V(G)} d(u, v)$. The **radius** of a graph G, written rad G, is $\min_{u \in V(G)} \epsilon(u)$.

The diameter equals the maximum of the vertex eccentricities. In a disconnected graph, the diameter and radius (and every eccentricity) are infinite, because distance between vertices in different components is infinite. We use the word "diameter" due to its use in geometry, where it is the greatest distance between two elements of a set.

2.1.10. Example. The Petersen graph has diameter 2, since nonadjacent vertices have a common neighbor. The hypercube Q_k has diameter k, since it takes k steps to change all k coordinates. The cycle C_n has diameter $\lfloor n/2 \rfloor$. In each of these, every vertex has the same eccentricity, and diam G = rad G.

For $n \geq 3$, the n-vertex tree of least diameter is the star, with diameter 2 and radius 1. The one of largest diameter is the path, with diameter $n - 1$ and radius $\lceil (n-1)/2 \rceil$. Every path in a tree is the shortest (the only!) path between its endpoints, so the diameter of a tree is the length of its longest path.

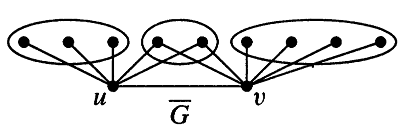
In the graph below, each vertex is labeled with its eccentricity. The radius is 2, the diameter is 4, and the length of the longest path is 7. ■



To have large diameter, many edges must be missing. Thus we expect the complement of a graph with large diameter to have small diameter. We use the simple observation that a graph has diameter at most 2 if and only if nonadjacent vertices always have common neighbors (see also Exercise 15).

2.1.11. Theorem. If G is a simple graph, then diam $G \geq 3 \Rightarrow$ diam $\overline{G} \leq 3$.

Proof: When diam $G > 2$, there exist nonadjacent vertices $u, v \in V(G)$ with no common neighbor. Hence every $x \in V(G) - \{u, v\}$ has at least one of $\{u, v\}$ as a nonneighbor. This makes x adjacent in $\overline{G}$ to at least one of $\{u, v\}$ in $\overline{G}$. Since also $uv \in E(\overline{G})$, for every pair x, y there is an x, y-path of length at most 3 in $\overline{G}$ through $\{u, v\}$. Hence diam $\overline{G} \leq 3$. ■



2.1.12. Definition. The **center** of a graph G is the subgraph induced by the vertices of minimum eccentricity.

The center of a graph is the full graph if and only if the radius and diameter are equal. We next describe the centers of trees. In the induction step, we delete *all* leaves instead of just one.

2.1.13. Theorem. (Jordan [1869]) The center of a tree is a vertex or an edge.

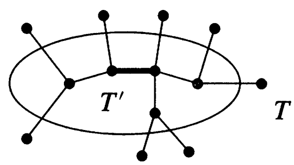
Proof: We use induction on the number of vertices in a tree T.

Basis step: $n(T) \leq 2$. With at most two vertices, the center is the entire tree.

Induction step: $n(T) > 2$. Form T' by deleting every leaf of T. By Lemma 2.1.3, T' is a tree. Since the internal vertices on paths between leaves of T remain, T' has at least one vertex.

Every vertex at maximum distance in T from a vertex $u \in V(T)$ is a leaf (otherwise, the path reaching it from u can be extended farther). Since all the leaves have been removed and no path between two other vertices uses a leaf, $\epsilon_{T'}(u) = \epsilon_T(u) - 1$ for every $u \in V(T')$. Also, the eccentricity of a leaf in T is greater than the eccentricity of its neighbor in T. Hence the vertices minimizing $\epsilon_T(u)$ are the same as the vertices minimizing $\epsilon_{T'}(u)$.

We have shown that T and T' have the same center. By the induction hypothesis, the center of T' is a vertex or an edge. ■



In a communication network, large diameter may be acceptable if most pairs can communicate via short paths. This leads us to study the average distance instead of the maximum. Since the average is the sum divided by $\binom{n}{2}$ (the number of vertex pairs), it is equivalent to study $D(G) = \sum_{u,v \in V(G)} d_G(u, v)$.

The sum $D(G)$ has been called the **Wiener index** of G (also written $W(G)$). Wiener used it to study the boiling point of paraffin. Molecules can be modeled by graphs with vertices for atoms and edges for atomic bonds. Many chemical properties of molecules are related to the Wiener index of the corresponding graphs. We study the extreme values of $D(G)$.

2.1.14. Theorem. Among trees with n vertices, the Wiener index $D(T) = \sum_{u,v} d(u, v)$ is minimized by stars and maximized by paths, both uniquely.

Proof: Since a tree has $n - 1$ edges, it has $n - 1$ pairs of vertices at distance 1, and all other pairs have distance at least 2. The star achieves this and hence minimizes $D(T)$. To show that no other tree achieves this, consider a leaf x in T, and let v be its neighbor. If all other vertices have distance 2 from x, then they must be neighbors of v, and T is the star. The value is $D(K_{1,n-1}) = (n - 1) + 2\binom{n-1}{2} = (n - 1)^2$.

For the maximization, consider first $D(P_n)$. This equals the sum of the distances from an endpoint u to the other vertices, plus $D(P_{n-1})$. We have $\sum_{v\in V(P_n)} d(u,v) = \sum_{i=0}^{n-1} i = \binom{n}{2}$. Thus $D(P_n) = D(P_{n-1}) + \binom{n}{2}$. With Pascal's Formula $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ (see Appendix A), induction yields $D(P_n) = \binom{n+1}{3}$.

We prove by induction on n that among n-vertex tree, P_n is the only tree that maximizes $D(T)$.

Basis step: $n = 1$. The only tree with one vertex is P_1.

Induction step: $n > 1$. Let u be a leaf of an n-vertex tree T. Now $D(T) = D(T-u) + \sum_{v\in V(T)} d(u,v)$. By the induction hypothesis, $D(T-u) \le D(P_{n-1})$, with equality if and only if $T-u$ is a path. Thus it suffices to show that $\sum_{v\in V(T)} d(u,v)$ is maximized only when T is a path and u is an endpoint of T.

Consider the list of distances from u. In P_n, this list is $1, 2, \ldots, n-1$, all distinct. A shortest path from u to a vertex farthest from u contains vertices at all distances from u, so in any tree the set of distances from u to other vertices has no gaps. Thus any repetition makes $\sum_{v\in V(T)} d(u,v)$ smaller than when u is a leaf of a path. When T is not a path, such a repetition occurs. ∎

Over all connected n-vertex graphs, $D(G)$ is minimized by K_n. The maximization problem reduces to what we have already done with trees.

2.1.15. Lemma. If H is a subgraph of G, then $d_G(u,v) \le d_H(u,v)$.

Proof: Every u,v-path in H appears also in G, so the shortest u,v-path in G is no longer than the shortest u,v-path in H. ∎

2.1.16. Corollary. If G is a connected n-vertex graph, then $D(G) \le D(P_n)$.

Proof: Let T be a spanning tree of G. By Lemma 2.1.15, $D(G) \le D(T)$. By Theorem 2.1.14, $D(T) \le D(P_n)$. ∎

DISJOINT SPANNING TREES (optional)

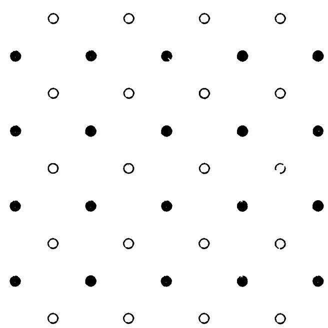
We have seen that every connected graph has a spanning tree. Edge-disjoint spanning trees provide alternate routes when an edge in the primary tree fails. Tutte [1961a] and Nash-Williams [1961] independently characterized graphs having k pairwise edge-disjoint spanning trees (see Exercise 67).

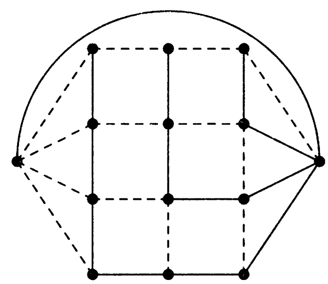
We describe one application of edge-disjoint spanning trees. David Gale devised a game marketed under the name "Bridg-it" (copyright 1960 by Hassenfeld Bros., Inc.—"Hasbro Toys"). Each of two players owns a rectangular grid of posts. The players move alternately, at each move joining two of their own posts by a unit-length bridge. The figure on the left below illustrates the board; Player 1's posts are solid, and Player 2's are hollow. The object of Player 1 is to construct a path of bridges from the left column to the right column; Player 2 wants a path of bridges from the top row to the bottom row.

Bridges cannot cross. Therefore, every bridge that is played eliminates a potential move for the other player. Since every path from left to right crosses every path from top to bottom, the players cannot both win. Note also that the layout of the board is symmetric in the two players.

We argue that Player 2 cannot have a winning strategy. Suppose otherwise. Because the board is symmetric, Player 1 can start with any move and then followthe strategy of Player 2, making an arbitrary move if the strategy of Player 2 ever calls for a bridge that has already been played. Before Player 2 can win, Player 1 wins by using the same strategy.

If the game is played until no further moves are possible, then some player must have won (Exercise 70). Since Player 2 has no winning strategy, this implies that Player 1 has a winning strategy. Here we give an explicit strategythat Player 1 can use to win. (The argument holds more generally in the contextof "matroids"—see Theorem 8.2.46.)





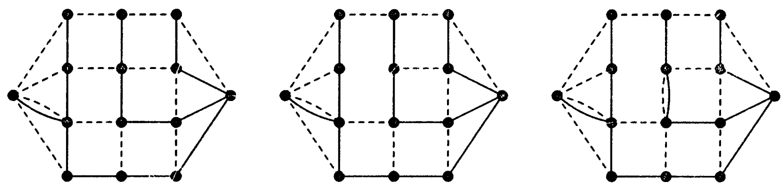
2.1.17. Theorem. Player 1 has a winning strategy in Bridg-it.

Proof: We form a graph of the potential connections for Player 1. Posts on the same end are equivalent, so we collect the (solid) posts from the end columns as single vertices. We add an auxiliary edge between the ends. The figure on the right above illustrates that this graph is the union of two edge-disjoint spanning trees; we omit a technical description of the two trees.

Together, the two trees contain edge-disjoint paths between the goal vertices. Since the auxiliary edge doesn't really exist, we pretend Player 2 moved first and took that edge. A move by Player 2 cuts one edge e in the graph and makes it no longer available. This cuts one of the trees into two components. By Proposition 2.1.6, some edge e' from the other tree reconnects it.

Player 1 chooses such an edge e'. This makes e' uncuttable, in effect putting e' in both spanning trees. After deleting e and making e' a double edge with one copy in each tree, our graph still consists of two edge-disjoint spanning trees. Since Player 2 cannot cut a double edge, Player 2 cannot cut both trees. Thus Player 1 can always defend. The figure below illustrates the strategy.

The process stops when Player 1 has won or when no single edges remain to be cut. In the latter case the remaining edges are double edges and form a spanning tree of bridges built by Player 1. Thus in either case Player 1 has constructed a path connecting the special vertices. ∎



Player 1 reconnects Player 2 cuts Player 1 reconnects

EXERCISES

2.1.1. (−) For each k, list the isomorphism classes of trees with maximum degree k and at most six vertices. Do the same for diameter k. (Explain why there are no others.)

2.1.2. (−) Let G be a graph.

a) Prove that G is a tree if and only if G is connected and every edge is a cut-edge.

b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

2.1.3. (−) Prove that a graph is a tree if and only if it is loopless and has exactly one spanning tree.

2.1.4. (−) Prove or disprove: Every graph with fewer edges than vertices has a component that is a tree.

2.1.5. (−) Let G be a graph. Prove that a maximal acyclic subgraph of G consists of a spanning tree from each component of G.

2.1.6. (−) Let T be a tree with average degree a. In terms of a, determine $n(T)$.

2.1.7. (−) Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles.

2.1.8. (−) Prove that each property below characterizes forests.

a) Every induced subgraph has a vertex of degree at most 1.

b) Every connected subgraph is an induced subgraph.

c) The number of components is the number of vertices minus the number of edges.

2.1.9. (−) For $2 \le k \le n-1$, prove that the n-vertex graph formed by adding one vertex adjacent to every vertex of P_{n-1} has a spanning tree with diameter k.

2.1.10. (−) Let u and v be vertices in a connected n-vertex simple graph. Prove that if $d(u, v) > 2$, then $d(u) + d(v) \le n + 1 - d(u, v)$. Construct examples to show that this can fail whenever $n \ge 3$ and $d(u, v) \le 2$.

2.1.11. (−) Let x and y be adjacent vertices in a graph G. For all $z \in V(G)$, prove that $|d_G(x, z) - d_G(y, z)| \le 1$.

2.1.12. (−) Compute the diameter and radius of the biclique $K_{m,n}$.

2.1.13. (−) Prove that every graph with diameter d has an independent set with at least $\lceil (1 + d)/2 \rceil$ vertices.

2.1.14. (−) Suppose that the processors in a computer are named by binary k-tuples, and pairs can communicate directly if and only if their names are adjacent in the k-dimensional cube Q_k. A processor with name u wants to send a message to the processor with name v. How can it find the first step on a shortest path to v?

2.1.15. (−) Let G be a simple graph with diameter at least 4. Prove that $\overline{G}$ has diameter at most 2. (Hint: Use Theorem 2.1.11.)

2.1.16. (−) Given a simple graph G, define G' to be the simple graph on the same vertex set such that $xy \in E(G')$ if and only if x and y are adjacent in G or have a common neighbor in G. Prove that diam $(G') = \lceil \text{diam}\,(G)/2 \rceil$.

• • • • •

2.1.17. (!) Prove C $\Rightarrow$ {A, B} in Theorem 2.1.4 by adding edges to connect components.

2.1.18. (!) Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1. Show that this is best possible by constructing an n-vertex tree with exactly Δ leaves, for each choice of n, Δ with $n > \Delta \geq 2$.

2.1.19. Prove or disprove: If n_i denotes the number of vertices of degree i in a tree T, then $\sum i n_i$ depends only on the number of vertices in T.

2.1.20. A *saturated hydrocarbon* is a molecule formed from k carbon atoms and l hydrogen atoms by adding bonds between atoms such that each carbon atom is in four bonds, each hydrogen atom is in one bond, and no sequence of bonds forms a cycle of atoms. Prove that $l = 2k + 2$. (Bondy–Murty [1976, p27])

2.1.21. Let G be an n-vertex simple graph having a decomposition into k spanning trees. Suppose also that $\Delta(G) = \delta(G) + 1$. For $2k \geq n$, show that this is impossible. For $2k < n$, determine the degree sequence of G in terms of n and k.

2.1.22. Let T be an n-vertex tree having one vertex of each degree i with $2 \leq i \leq k$; the remaining $n - k + 1$ vertices are leaves. Determine n in terms of k.

2.1.23. Let T be a tree in which every vertex has degree 1 or degree k. Determine the possible values of $n(T)$.

2.1.24. Prove that every nontrivial tree has at least two maximal independent sets, with equality only for stars. (Note: maximal $\neq$ maximum.)

2.1.25. Prove that among trees with n vertices, the star has the most independent sets.

2.1.26. (!) For $n \geq 3$, let G be an n-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine $e(G)$, and use this to determine G itself.

2.1.27. (!) Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

2.1.28. Let $d_1 \geq \cdots \geq d_n$ be nonnegative integers. Prove that there exists a connected graph (loops and multiple edges allowed) with degree sequence $d_1, \ldots, d_n$ if and only if $\sum d_i$ is even, $d_n \geq 1$, and $\sum d_i \geq 2n - 2$. (Hint: Consider a realization with the fewest components.) Is the statement true for simple graphs?

2.1.29. (!) Every tree is bipartite. Prove that every tree has a leaf in its larger partite set (in both if they have equal size).

2.1.30. Let T be a tree in which all vertices adjacent to leaves have degree at least 3. Prove that T has some pair of leaves with a common neighbor.

2.1.31. Prove that a simple connected graph having exactly two vertices that are not cut-vertices is a path.

2.1.32. Prove that an edge e of a connected graph G is a cut-edge if and only if e belongs to every spanning tree. Prove that e is a loop if and only if e belongs to no spanning tree.

2.1.33. (!) Let G be a connected n-vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

2.1.34. (!) Let T be a tree with k edges, and let G be a simple n-vertex graph with more than $n(k-1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n > k$.

2.1.35. (!) Let T be a tree. Prove that the vertices of T all have odd degree if and only if for all $e \in E(T)$, both components of $T - e$ have odd order.

2.1.36. (!) Let T be a tree of even order. Prove that T has exactly one spanning subgraph in which every vertex has odd degree.

2.1.37. (!) Let T, T' be two spanning trees of a connected graph G. For $e \in E(T) - E(T')$, prove that there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of G.

2.1.38. Let T, T' be two trees on the same vertex set such that $d_T(v) = d'_T(v)$ for each vertex v. Prove that T' can be obtained from T' using 2-switches (Definition 1.3.32) so that every graph along the way is also a tree. (Kelmans [1998])

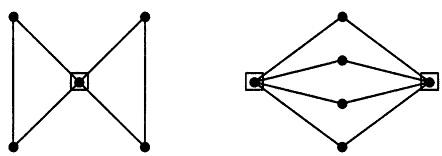
2.1.39. (!) Let G be a tree with $2k$ vertices of odd degree. Prove that G decomposes into k paths. (Hint: Prove the stronger result that the claim holds for all forests.)

2.1.40. (!) Let G be a tree with k leaves. Prove that G is the union of paths $P_1, \ldots, P_{\lceil k/2 \rceil}$ such that $P_i \cap P_j \neq \emptyset$ for all $i \neq j$. (Ando–Kaneko–Gervacio [1996])

2.1.41. For $n \geq 4$, let G be a simple n-vertex graph with $e(G) \geq 2n-3$. Prove that G has two cycles of equal length. (Chen–Jacobson–Lehel–Shreve [1999] strengthens this.)

2.1.42. Let G be a connected Eulerian graph with at least three vertices. A vertex v in G is *extendible* if every trail beginning at v can be extended to form an Eulerian circuit of G. For example, in the graphs below only the marked vertices are extendible. Prove the following statements about G (adapted from Chartrand–Lesniak [1986, p61]).

a) A vertex $v \in V(G)$ is extendible if and only if $G - v$ is a forest. (Ore [1951])
b) If v is extendible, then $d(v) = \Delta(G)$. (Bäbler [1953])
c) All vertices of G are extendible if and only if G is a cycle.
d) If G is not a cycle, then G has at most two extendible vertices.

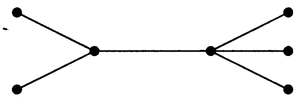


2.1.43. Let u be a vertex in a connected graph G. Prove that it is possible to select shortest paths from u to all other vertices of G so that the union of the paths is a tree.

2.1.44. (!) Prove or disprove: If a simple graph with diameter 2 has a cut-vertex, then its complement has an isolated vertex.

2.1.45. Let G be a graph having spanning trees with diameter 2 and diameter l. For $2 < k < l$, prove that G also has a spanning tree with diameter k. (Galvin)

2.1.46. (!) Prove that the trees with diameter 3 are the **double-stars** (two central vertices plus leaves). Count the isomorphism classes of double-stars with n vertices.



2.1.47. (!) *Diameter and radius.*

a) Prove that the distance function $d(u, v)$ on pairs of vertices of a graph satisfies the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$.

b) Use part (a) to prove that diam $G \leq 2$rad G for every graph G.

c) For all positive integers r and d that satisfy $r \leq d \leq 2r$, construct a simple graph with radius r and diameter d. (Hint: Build a suitable graph with one cycle.)

2.1.48. (!) For $n \geq 4$, prove that the minimum number of edges in an n-vertex graph with diameter 2 and maximum degree $n - 2$ is $2n - 4$.

2.1.49. Let G be a simple graph. Prove that rad $G \geq 3 \Rightarrow$ rad $\overline{G} \leq 2$.

2.1.50. *Radius and eccentricity.*

a) Prove that the eccentricities of adjacent vertices differ by at most 1.

b) In terms of the radius r, determine the maximum possible distance from a vertex of eccentricity $r + 1$ to the center of G. (Hint: Use a graph with one cycle.)

2.1.51. Let x and y be distinct neighbors of a vertex v in a graph G.

a) Prove that if G is a tree, then $2\epsilon(v) \leq \epsilon(x) + \epsilon(y)$.

b) Determine the smallest graph where this inequality can fail.

2.1.52. Let x be a vertex in a graph G, and suppose that $\epsilon(x) >$ rad G.

a) Prove that if G is a tree, then x has a neighbor with eccentricity $k - 1$.

b) Show that part (a) does not hold for all graphs by constructing, for each even r that is at least 4, a graph with radius r in which x has eccentricity $r + 2$ and has no neighbor with eccentricity $r + 1$. (Hint: Use a graph with one cycle.)

2.1.53. Prove that the center of a graph can be disconnected and can have components arbitrarily far apart by constructing a graph where the center consists of two vertices and the distance between these two vertices is k.

2.1.54. *Centers of trees.* Let T be a tree.

a) Give a noninductive proof that the center of T is a vertex or an edge.

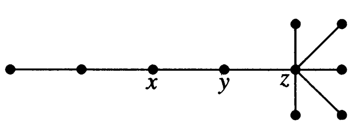
b) Prove that the center of T is one vertex if and only if diam $T = 2$rad T.

c) Use part (a) to prove that if $n(T)$ is odd, then every automorphism of T maps some vertex to itself.

2.1.55. Given $x \in V(G)$, let $s(x) = \sum_{v \in V(G)} d(x, v)$. The **barycenter** of G is the subgraph induced by the set of vertices minimizing $s(x)$ (the set is also called the **median**).

a) Prove that the barycenter of a tree is a single vertex or an edge. (Hint: Study $s(u) - s(v)$ when u and v are adjacent.) (Jordan [1869])

b) Determine the maximum distance between the center and the barycenter in a tree of diameter d. (Example: in the tree below, the center is the edge xy, the barycenter contains only z, and the distance between them is 1.)



2.1.56. Let T be a tree. Prove that T has a vertex v such that for all $e \in E(T)$, the component of $T - e$ containing v has at least $\lceil n(T)/2 \rceil$ vertices. Prove that either v is unique or there are just two adjacent such vertices.

2.1.57. Let $n_1, \ldots, n_k$ be positive integers with sum $n - 1$.

a) By counting edges in complete graphs, prove that $\sum_{i=1}^{k} \binom{n_i}{2} \le \binom{n-1}{2}$.

b) Use part (a) to prove that $\sum_{v \in V(T)} d(u, v) \le \binom{n}{2}$ when u is a vertex of a tree T. (Hint: Use strong induction on the number of vertices.)

2.1.58. (+) Let S and T be trees with leaves $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$, respectively. Suppose that $d_S(x_i, x_j) = d_T(y_i, y_j)$ for each pair i, j. Prove that S and T are isomorphic. (Smolenskii [1962])

2.1.59. (!) Let G be a tree with n vertices, k leaves, and maximum degree k.

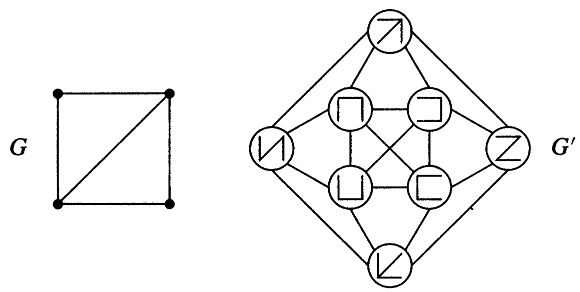
a) Prove that G is the union of k paths with a common endpoint.

b) Determine the maximum and minimum possible values of diam G.

2.1.60. Let G be a graph with diameter d and maximum degree k. Prove that $n(G) \le 1 + [(k-1)^d - 1]k/(k-2)$. (Comment: Equality holds for the Petersen graph.)

2.1.61. (+) Let G be a graph with smallest order among k-regular graphs with girth at least g (Exercise 1.3.16 establishes the existence of such graphs). Prove that G has diameter at most g. (Hint: If $d_G(x, y) > g$, modify G to obtain a smaller k-regular graph with girth at least g.) (Erdős–Sachs [1963])

2.1.62. (!) Let G be a connected graph with n vertices. Define a new graph G' having one vertex for each spanning tree of G, with vertices adjacent in G' if and only if the corresponding trees have exactly $n(G) - 2$ common edges. Prove that G' is connected. Determine the diameter of G'. An example appears below.



2.1.63. (!) Prove that every n-vertex graph with $n + 1$ edges has girth at most $\lfloor (2n+2)/3 \rfloor$. For each n, construct an example achieving this bound.

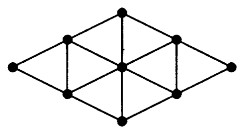
2.1.64. (!) Let G be a connected graph that is not a tree. Prove that some cycle in G has length at most $2(\text{diam } G) + 1$. For each $k \in \mathbb{N}$, show that this is best possible by exhibiting a graph with diameter k and girth $2k + 1$.

2.1.65. (+) Let G be a connected n-vertex graph with minimum degree k, where $k \ge 2$ and $n - 2 \ge 2(k+1)$. Prove that diam $G \le 3(n-2)/(k+1) - 1$. Whenever $k \ge 2$ and $(n-2)/(k+1)$ is an integer greater than 1, construct a graph where the bound holds with equality. (Moon [1965b])

2.1.66. Let $F_1, \ldots, F_m$ be forests whose union is G. Prove that $m \ge \max_{H \subseteq G} \left\lceil \frac{e(H)}{n(H)-1} \right\rceil$. (Comment: Nash-Williams [1964] and Edmonds [1965b] proved that this bound is always achieveable—Corollary 8.2.57).

2.1.67. Prove that the following is a necessary condition for the existence of k pairwise edge-disjoint spanning trees in G: for every partition of the vertices of G into r sets, there are at least $k(r-1)$ edges of G whose endpoints are in different sets of the partition. (Comment: Corollary 8.2.59 shows that this condition is also sufficient - Tutte [1961a], Nash-Williams [1961], Edmonds [1965c].)

2.1.68. Can the graph below be decomposed into edge-disjoint spanning trees? Into isomorphic edge-disjoint spanning trees?



2.1.69. (∗) Consider the graph before Theorem 2.1.17 with 12 vertical edges and 16 edges that are horizontal or slanted. Let $g_{i,j}$ be the ith edge from the top in the jth column of vertical edges. Let $h_{i,j}$ be the jth edge from the left in the ith row of horizontal/diagonal edges. Suppose that Player 1 follows the strategy of Theorem 2.1.17 and first takes $h_{1,1}$. Player 2 deletes $g_{2,2}$, and Player 1 takes $h_{2,3}$. Next Player 2 deletes $v_{3,2}$, and Player 1 takes $h_{4,2}$. Draw the two spanning trees at this point. Given that Player 2 next deletes $g_{2,1}$, list all moves available to Player 1 within the strategy. (Pritikin)

2.1.70. (∗) Prove that Bridg-it cannot end in a tie no matter how the moves are made. That is, prove that when no further moves can be made, one of the players must have built a path connecting his/her goals.

2.1.71. (∗) The players change the rules of Bridg-it so that a player with path between friendly ends is the *loser*. It is forbidden to stall by building a bridge joining end posts or joining posts already connected by a path. Show that Player 2 has a strategy that forces Player 1 to lose. (Hint: Use Proposition 2.1.7 instead of Proposition 2.1.6.) (Pritikin)

2.1.72. (+) Prove that if $G_1, \dots, G_k$ are pairwise-intersecting subtrees of a tree G, then G has a vertex that belongs to all of $G_1, \dots, G_k$. (Hint: Use induction on k. Comment: This result is the **Helly property** for trees.)

2.1.73. (+) Prove that a simple graph G is a forest if and only if for every pairwise intersecting family of paths in G, the paths have a common vertex. (Hint: For sufficiency, use induction on the size of the family of paths.)

2.1.74. Let G be a simple n-vertex graph having $n-2$ edges. Prove that G has an isolated vertex or has two components that are nontrivial trees. Use this to prove inductively that G is a subgraph of $\overline{G}$. (Comment: The claim is not true for all graphs with $n-1$ edges.) (Burns–Schuster [1977])

2.1.75. (+) Prove that every n-vertex tree other than $K_{1,n-1}$ is contained in its complement. (Hint: Use induction on n to prove a stronger statement: if T is an n-vertex tree other than a star, then K_n contains two edge-disjoint copies of T in which the two copies of each non-leaf vertex of T appear at distinct vertices.) (Burns–Schuster [1978])

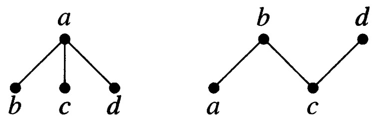
2.1.76. (+) Let S be an n-element set, and let $\{A_1, \dots, A_n\}$ be n distinct subsets of S. Prove that S has an element x such that the sets $A_1 \cup \{x\}, \dots, A_n \cup \{x\}$ are distinct. (Hint: Define a graph with vertices $a_1, \dots, a_n$ such that $a_i \leftrightarrow a_j$ if and only if one of $\{A_i, A_j\}$ is obtained from the other by adding a single element y. Use y as a label on the edge. Prove that there is a forest consisting of one edge with each label used. Use this to obtain the desired x.) (Bondy [1972a])

2.2. Spanning Trees and Enumeration

There are $2^{\binom{n}{2}}$ simple graphs with vertex set $[n] = \{1, \ldots, n\}$, since each pair may or may not form an edge. How many of these are trees? In this section, we solve this counting problem, count spanning trees in arbitrary graphs, and discuss several applications.

ENUMERATION OF TREES

With one or two vertices, only one tree can be formed. With three vertices there is still only one isomorphism class, but the adjacency matrix is determined by which vertex is the center. Thus there are three trees with vertex set [3]. With vertex set [4], there are four stars and 12 paths, yielding 16 trees. With vertex set [5], a careful study yields 125 trees.

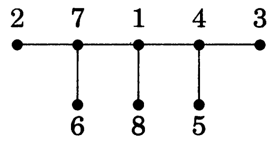


Now we may see a pattern. With vertex set $[n]$, there are n^{n-2} trees; this is **Cayley's Formula**. Prüfer, Kirchhoff, Pólya, Renyi, and others found proofs. J.W. Moon [1970] wrote a book about enumerating classes of trees. We present a bijective proof, establishing a one-to-one correspondence between the set of trees with vertex set $[n]$ and a set of known size.

Given a set S of n numbers, there are exactly n^{n-2} ways to form a list of length $n-2$ with entries in S. The set of lists is denoted S^{n-2} (see Appendix A). We use S^{n-2} to encode the trees with vertex set S. The list that results from a tree is its **Prüfer code**.

2.2.1. Algorithm. (Prüfer code) Production of $f(T) = (a_1, \ldots, a_{n-2})$.
Input: A tree T with vertex set $S \subseteq \mathbb{N}$.
Iteration: At the ith step, delete the least remaining leaf, and let a_i be the *neighbor* of this leaf. ■

2.2.2. Example. After $n-2$ iterations, only one of the original $n-1$ edges remains, and we have produced a list $f(T)$ of length $n-2$ with entries in S. In the tree below, the least leaf is 2; we delete it and record 7. After deleting 3 and 5 and recording 4 each time, the least leaf in the remaining 5-vertex tree is 4. The full code is (744171), and the vertices remaining at the end are 1 and 8. After the first step, the remainder of the Prüfer code is the Prüfer code of the subtree T' with vertex set $[8] - \{2\}$.



If we know the vertex set S, then we can retrieve the tree from the code a. The idea is to retrieve all the edges. We start with the set S of isolated vertices. At each step we create one edge and mark one vertex. When we are ready to consider a_i, there remain $n - i + 1$ unmarked vertices and $n - i - 1$ entries of a (including a_i). Thus at least two of the unmarked vertices do not appear among the remaining entries of a. Let x be the least of these, add xa_i to the list of edges, and mark x. After repeating this $n - 2$ times, two unmarked vertices remain; we join them to form the final edge.

In the example above, the least element of S not in the code is 2, so the first edge added joins 2 and 7, and we mark 2. Now the least unmarked element absent from the rest is 3, and we join it to 4, which is a_2. As we continue, we reconstruct edges in the order they were deleted to obtain a from T.

Throughout the process, each component of the graph we have grown has one unmarked vertex. This is true initially, and thus adding an edge with two unmarked endpoints combines two components. After marking one vertex of the new edge, again each component has one unmarked vertex. After $n - 2$ steps, we have two unmarked vertices and therefore two components. Adding the last edge yields a connected graph. We have built a connected graph with n vertices and $n - 1$ edges. By Theorem 2.1.4B, it is a tree, but we have not yet proved that its Prüfer code is a. ■

2.2.3. Theorem. (Cayley's Formula [1889]). For a set $S \subseteq \mathbb{N}$ of size n, there are n^{n-2} trees with vertex set S.

Proof: (Prüfer [1918]). This holds for $n = 1$, so we assume $n \geq 2$. We prove that Algorithm 2.2.1 defines a bijection f from the set of trees with vertex set S to the set S^{n-2} of lists of length $n - 2$ from S. We must show for each $a = (a_1, \ldots, a_{n-2}) \in S^{n-2}$ that exactly one tree T with vertex set S satisfies $f(T) = a$. We prove this by induction on n.

Basis step: $n = 2$. There is tree with two vertices. The Prüfer code is a list of length 0, and it is the only such list.

Induction step: $n > 2$. Computing $f(T)$ reduces each vertex to degree 1 and then possibly deletes it. Thus every nonleaf vertex in T appears in $f(T)$. No leaf appears, because recording a leaf as a neighbor of a leaf would require reducing the tree to one vertex. Hence the leaves of T are the elements of S not in $f(T)$. If $f(T) = a$, then the first leaf deleted is the least element of S not in a (call it x), and the neighbor of x is a_1.

We are given $a \in S^{n-2}$ and seek all solutions to $f(T) = a$. We have shown that every such tree has x as its least leaf and has the edge xa_1. Deleting x leaves a tree with vertex set $S' = S - \{x\}$. Its Prüfer code is $a' = (a_2, \ldots, a_{n-2})$, an $n - 3$-tuple formed from S'.

By the induction hypothesis, there exists exactly one tree T' having vertex set S' and Prüfer code a'. Since every tree with Prüfer code a is formed by adding the edge xa_1 to such a tree, there is at most one solution to $f(T) = a$. Furthermore, adding xa_1 to T' does create a tree with vertex set S and Prüfer code a, so there is at least one solution. ■

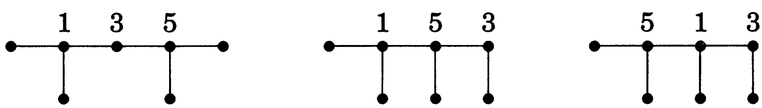
Cayley approached the problem algebraically and counted the trees by their vertex degrees. Prüfer's bijection also provides this information.

2.2.4. Corollary. Given positive integers $d_1, \ldots, d_n$ summing to $2n - 2$, there are exactly $\frac{(n-2)!}{\prod(d_i-1)!}$ trees with vertex set $[n]$ such that vertex i has degree d_i, for each i.

Proof: While constructing the Prüfer code of a tree T, we record x each time we delete a neighbor of x, until we delete x itself or leave x among the last two vertices. Thus each vertex x appears $d_T(x) - 1$ times in the Prüfer code.

Therefore, we count trees with these vertex degrees by counting lists of length $n - 2$ that for each i have $d_i - 1$ copies of i. If we assign subscripts to the copies of each i to distinguish them, then we are permuting $n - 2$ distinct objects and there are $(n - 2)!$ lists. Since the copies of i are not distinguishable, we have counted each desired arrangement $\prod(d_i - 1)!$ times, once for each way to order the subscripts on each type of label. (Appendix A discusses further aspects of this counting problem.) ■

2.2.5. Example. *Trees with fixed degrees.* Consider trees with vertices $\{1, 2, 3, 4, 5, 6, 7\}$ that have degrees $(3, 1, 2, 1, 3, 1, 1)$, respectively. We compute $\frac{(n-2)!}{\prod(d_i-1)!} = 30$; the trees are suggested below. Only the vertices $\{1, 3, 5\}$ are non-leaves. Deleting the leaves yields a subtree on $\{1, 3, 5\}$. There are three such subtrees, determined by which of the three is in the middle.



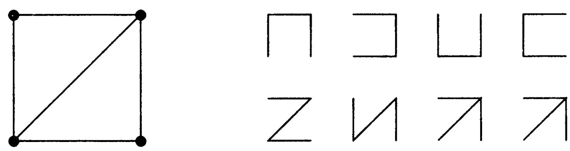
To complete each tree, we add the appropriate number of leaf neighbors for each non-leaf to give it the desired degree. There are six ways to complete the first tree (pick from the remaining four vertices the two adjacent to vertex 1) and twelve ways to complete each of the others (pick the neighbor of vertex 3 from the remaining four, and then pick the neighbor of the central vertex from the remaining three). ■

SPANNING TREES IN GRAPHS

We can interpret Cayley's Formula in another way. Since the complete graph with vertex set $[n]$ has all edges that can be used in forming trees with vertex set $[n]$, the number of trees with a specified vertex set of size n equals the number of spanning trees in a complete graph on n vertices.

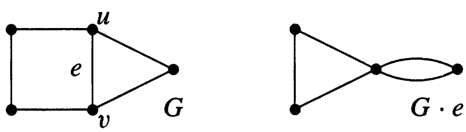
We now consider the more general problem of computing the number of spanning trees in any graph G. In general, G will not have as much symmetry as a complete graph, so it is unreasonable to expect as simple a formula as for K_n, but we can hope for an algorithm that provides a simple way to compute the answer when given a graph G.

2.2.6. Example. Below is the kite. To count the spanning trees, observe that four are paths around the outside cycle in the drawing. The remaining spanning trees use the diagonal edge. Since we must include an edge to each vertex of degree 2, we obtain four more spanning trees. The total is eight. ■



In Example 2.2.6, we counted separately the trees that did or did not contain the diagonal edge. This suggests a recursive procedure to count spanning trees. It is clear that the spanning trees of G not containing e are simply the spanning trees of $G - e$, but how do we count the trees that contain e? The answer uses an elementary operation on graphs.

2.2.7. Definition. In a graph G, **contraction** of edge e with endpoints u, v is the replacement of u and v with a single vertex whose incident edges are the edges other than e that were incident to u or v. The resulting graph $G \cdot e$ has one less edge than G.



In a drawing of G, contraction of e shrinks the edge to a single point. Contracting an edge can produce multiple edges or loops. To count spanning trees correctly, we must keep multiple edges (see Example 2.2.9). In other applications of contraction, the multiple edges may be irrelevant.

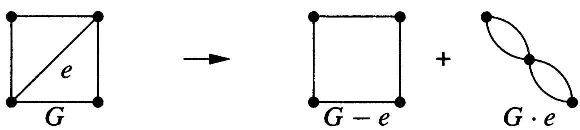
The recurrence applies for all graphs.

2.2.8. Proposition. Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not a loop, then $\tau(G) = \tau(G - e) + \tau(G \cdot e)$.

Proof: The spanning trees of G that omit e are precisely the spanning trees of $G - e$. To show that G has $\tau(G \cdot e)$ spanning trees containing e, we show that contraction of e defines a bijection from the set of spanning trees of G containing e to the set of spanning trees of $G \cdot e$.

When we contract e in a spanning tree that contains e, we obtain a spanning tree of $G \cdot e$, because the resulting subgraph of $G \cdot e$ is spanning and connected and has the right number of edges. The other edges maintain their identity under contraction, so no two trees are mapped to the same spanning tree of $G \cdot e$ by this operation. Also, each spanning tree of $G \cdot e$ arises in this way, since expanding the new vertex back into e yields a spanning tree of G. Since each spanning tree of $G \cdot e$ arises exactly once, the function is a bijection. ■

2.2.9. Example. *A step in the recurrence.* The graphs on the right each have four spanning trees, so Proposition 2.2.8 implies that the kite has eight spanning trees. Without the multiple edges, the computation would fail. ■



We can save some computation time by recognizing special graphs G where we know $\tau(G)$, such as the graph on the right above.

2.2.10. Remark. If G is a connected loopless graph with no cycle of length at least 3, then $\tau(G)$ is the product of the edge multiplicities. A disconnected graph has no spanning trees. ■

We cannot apply the recurrence of Proposition 2.2.8 when e is a loop. For example, a graph consisting of one vertex and one loop has one spanning tree, but deleting and contracting the loop would count it twice. Since loops do not affect the number of spanning trees, we can delete loops as they arise.

Counting trees recursively requires initial conditions for graphs in which all edges are loops. Such a graph has one spanning tree if it has only one vertex, and it has no spanning trees if it has more than one vertex. If a computer completes the computation by deleting or contracting every edge in a loopless graph G, then it may compute as many as $2^{e(G)}$ terms. Even with savings from Remark 2.2.10, the amount of computation grows exponentially with the size of the graph; this is impractical.

Another technique leads to a much faster computation. The Matrix Tree Theorem, implicit in the work of Kirchhoff [1847], computes $\tau(G)$ using a determinant. This is much faster, because determinants of n-by-n matrices can be computed using fewer than n^3 operations. Also, Cayley's Formula follows from the Matrix Tree Theorem with $G = K_n$ (Exercise 17), but it does not follow easily from Proposition 2.2.8.

Before stating the theorem, we illustrate the computation it specifies.

2.2.11. Example. *A Matrix Tree computation.* Theorem 2.2.12 instructs us to form a matrix by putting the vertex degrees on the diagonal and subtracting the adjacency matrix. We then delete a row and a column and take the determinant. When G is the kite of Example 2.2.9, the vertex degrees are 3, 3, 2, 2. We form the matrix on the left below and take the determinant of the matrix in the middle. The result is the number of spanning trees! ■

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \rightarrow 8$$

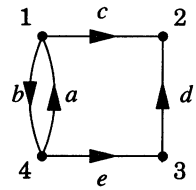
Loops don't affect spanning trees, so we delete them before the computation. The proof of the theorem uses properties of determinants.

2.2.12. Theorem. (Matrix Tree Theorem) Given a loopless graph G with vertex set $v_1, \dots, v_n$, let $a_{i,j}$ be the number of edges with endpoints v_i and v_j. Let Q be the matrix in which entry (i, j) is $-a_{i,j}$ when $i \neq j$ and is $d(v_i)$ when $i = j$. If Q^* is a matrix obtained by deleting row s and column t of Q, then $\tau(G) = (-1)^{s+t} \det Q^*$.

Proof*: We prove this only when $s = t$; the general statement follows from a result in linear algebra (when the columns of a matrix sum to the vector 0, the cofactors are constant in each row—Exercise 8.6.18).

Step 1. If D is an orientation of G, and M is the incidence matrix of D, then $Q = MM^T$. With edges $e_1, \dots, e_m$, the entries of M are $m_{i,j} = 1$ when v_i is the tail of e_j, $m_{i,j} = -1$ when v_i is the head of e_j, and $m_{i,j} = 0$ otherwise. Entry i, j in MM^T is the dot product of rows i and j of M. When $i \neq j$, the product counts -1 for every edge of G joining the two vertices; when $i = j$, it counts 1 for every incident edge and yields the degree.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$



$$Q = \begin{pmatrix} 3 & -1 & 0 & -2 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -2 & 0 & -1 & 3 \end{pmatrix}$$

Step 2. If B is an $(n-1)$-by-$(n-1)$ submatrix of M, then $\det B = \pm 1$ *if the corresponding $n-1$ edges form a spanning tree of G, and otherwise* $\det B = 0$. In the first case, we use induction on n to prove that $\det B = \pm 1$. For $n = 1$, by convention a 0×0 matrix has determinant 1. For $n > 1$, let T be the spanning tree whose edges are the columns of B. Since T has at least two leaves and only one row is deleted, B has a row corresponding to a leaf x of T. This row has only one nonzero entry in B. When computing the determinant by expanding along this row, the only submatrix B' with nonzero weight in the expansion corresponds to the spanning subtree of $G - x$ obtained by deleting x and its incident edge from T. Since B' is an $(n-2)$-by-$(n-2)$ submatrix of the incidence matrix for an orientation of $G - x$, the induction hypothesis yields $\det B' = \pm 1$. Since the nonzero entry in row x is ± 1, we obtain the same result for B.

If the $n-1$ edges corresponding to columns of B do not form a spanning tree, then by Theorem 2.1.4C they contain a cycle C. We form a linear combination of the columns with coefficient 0 if the corresponding edge is not in C, $+1$ if it is followed forward by C, and -1 if it is followed backward by C. The result is total weight 0 at each vertex, so the columns are linearly dependent, which yields $\det B = 0$.

Step 3. Computation of $\det Q^*$. Let M^* be the result of deleting row t of M, so $Q^* = M^*(M^*)^T$. If $m < n-1$, then the determinant is 0 and there are no spanning subtrees, so we assume that $m \geq n-1$. The Binet–Cauchy Formula (Exercise 8.6.19) computes the determinant of a product of non-square matrices using the determinants of square submatrices of the factors. When $m \geq p$, A is p-by-m, and B is m-by-p, it states that $\det AB = \sum_S \det A_S \det B_S$, where the summ runs over all p-sets S in $[m]$, A_S is the submatrix of A consisting of the columns indexed by S, and B_S is the submatrix of B consisting of the rows indexed by S. When we apply the formula to $Q^* = M^*(M^*)^T$, the submatrix A_S is an $(n-1)$-by-$(n-1)$ submatrix of M as discussed in Step 2, and $B_S = A_S^T$. Hence the summation counts $1 = (\pm 1)^2$ for each set of $n-1$ edges corresponding to a spanning tree and 0 for all other sets of $n-1$ edges. ■

DECOMPOSITION AND GRACEFUL LABELINGS

We consider another problem about graph decomposition (Definition 1.1.32). We can always decompose G into single edges; can we decompose G into copies of a larger tree T? This requires that $e(T)$ divides $e(G)$ and $\Delta(G) \geq \Delta(T)$; is that sufficient? Even when G is $e(T)$-regular, this may fail (Exercise 20); for example, the Petersen graph does not decompose into claws.

Häggkvist conjectured that if G is a $2m$-regular graph and T is a tree with m edges, then $E(G)$ decomposes into $n(G)$ copies of T. Even the "simplest" case when G is a clique is still unsettled and notorious.

2.2.13. Conjecture. (Ringel [1964]) If T is a fixed tree with m edges, then K_{2m+1} decomposes into $2m+1$ copies of T. ■

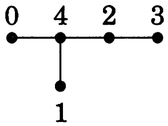
Attempts to prove Ringel's conjecture have focused on the stronger **Graceful Tree Conjecture**. This implies Ringel's conjecture and a similar statement about decomposing complete graphs of even order (Exercise 23).

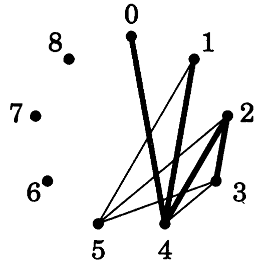
2.2.14. Definition. A **graceful labeling** of a graph G with m edges is a function $f\colon V(G) \to \{0, \ldots, m\}$ such that distinct vertices receive distinct numbers and $\{|f(u) - f(v)| : uv \in E(G)\} = \{1, \ldots, m\}$. A graph is **graceful** if it has a graceful labeling.

2.2.15. Conjecture. (Graceful Tree Conjecture—Kotzig, Ringel [1964]) Every tree has a graceful labeling. ■

2.2.16. Theorem. (Rosa [1967]) If a tree T with m edges has a graceful labeling, then K_{2m+1} has a decomposition into $2m + 1$ copies of T.

Proof: View the vertices of K_{2m+1} as the congruence classes modulo $2m + 1$, arranged circularly. The *difference* between two congruence classes is 1 if they are consecutive, 2 if one class is between them, and so on up to difference m. We group the edges of K_{2m+1} by the difference between the endpoints. For $1 \le j \le m$, there are $2m + 1$ edges with difference j.



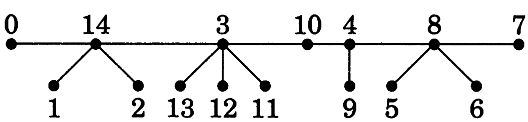


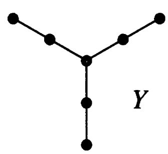
From a graceful labeling of T, we define copies of T in K_{2m+1}; the copies are $T_0, \dots, T_{2m}$. The vertices of T_k are $k, \dots, k+m$ (mod $2m + 1$), with $k+i$ adjacent to $k + j$ if and only if i is adjacent to j in the graceful labeling of T. The copy T_0 looks just like the graceful labeling and has one edge with each difference. Moving to the next copy shifts each edge to another having the same difference by adding one to the name of each endpoint. Each difference class of edges has one edge in each T_k, and thus $T_0, \dots, T_{2m}$ decompose K_{2m+1}. ■

Graceful labelings are known to exist for some types of trees and for some other families of graphs (see Gallian [1998]). It is easy to find graceful labelings for stars and paths. We next define a family of trees that generalizes both by permitting the addition of edges incident to a path.

2.2.17. Definition. A **caterpillar** is a tree in which a single path (the **spine**) is incident to (or contains) every edge.

2.2.18. Example. The vertices not on the spine of a caterpillar (the "feet") are leaves. Below we show a graceful labeling of a caterpillar; in fact, every caterpillar is graceful (Exercise 31). The tree Y below is not a caterpillar. ■





Y

2.2.19. Theorem. A tree is a caterpillar if and only if it does not contain the tree Y above.

Proof: Let G' denote the tree obtained from a tree G by deleting each leaf of G. Since all vertices that survive in G' are non-leaves in G, G' has a vertex of degree at least 3 if and only if Y appears in G. Hence G has no copy of Y if and only if $\Delta(G') \le 2$. This is equivalent to G' being a path, which is equivalent to G being a caterpillar. ■

BRANCHINGS AND EULERIAN DIGRAPHS (optional)

Tutte extended the Matrix Tree Theorem to digraphs. His theorem reduces to the Matrix Tree Theorem when the digraph is symmetric (a digraph is *symmetric* if its adjacency matrix is symmetric, and then it models a graph). There is a surprising connection between this theorem and Eulerian circuits.

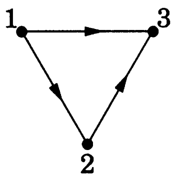
2.2.20. Definition. A **branching** or **out-tree** is an orientation of a tree having a root of indegree 0 and all other vertices of indegree 1. An **in-tree** is an out-tree with edges reversed.

A branching with root v is a union of paths from v (Exercise 33). Each vertex is reached by exactly one path. The analogous result holds for in-trees; an in-tree is a union of paths to the root, one from each vertex.

We state without proof Tutte's theorem to count branchings.

2.2.21. Theorem. (Directed Matrix Tree Theorem—Tutte [1948]) Given a loopless digraph G, let $Q^- = D^- - A'$ and $Q^+ = D^+ - A'$, where D^- and D^+ are the diagonal matrices of indegrees and outdegrees in G, and the i,jth-entry of A' is the number of edges from v_j to v_i. The number of spanning out-trees (in-trees) of G rooted at v_i is the value of each cofactor in the ith row of Q^- (ith column of Q^+). ■

$$Q^- = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$



$$Q^+ = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

2.2.22. Example. The digraph above has two spanning out-trees rooted at 1 and two spanning in-trees rooted at 3. Every cofactor in the first row of Q^- is 2, and every cofactor in the third column of Q^+ is 2. ■

Isolated vertices don't affect Eulerian circuits. After discarding these, a digraph is Eulerian if and only if indegree equals outdegree at every vertex and the underlying graph is connected (Theorem 1.4.24). Such a digraph also is strongly connected, which allows us to find a spanning in-tree. We will describe Eulerian circuits in terms of a spanning in-tree.

2.2.23. Lemma. In a strong digraph, every vertex is the root of an out-tree (and an in-tree).

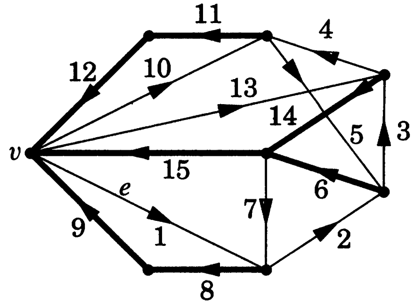
Proof: Consider a vertex v. We iteratively add edges to grow a branching from v. Let S_i be the set of vertices reached when i edges have been added; initialize $S_0 = \{v\}$. Because the digraph is strong, there is an edge leaving S_i (Exercise 1.4.10). We add one such edge to the branching and add its head to S_i to obtain S_{i+1}. This repeats until we have reached all vertices.

To obtain an in-tree of paths to v, reverse all edges and apply the same procedure; the reverse of a strong digraph is also strong. ■

The lemma constructively produces a *search tree* of paths from a root. The next section discusses search trees in more generality.

2.2.24. Algorithm. (Eulerian circuit in directed graph)
Input: An Eulerian digraph G without isolated vertices and a spanning in-tree T consisting of paths to a vertex v.
Step 1: For each $u \in V(G)$, specify an ordering of the edges that leave u, such that for $u \neq v$ the edge leaving u in T comes last.
Step 2: Beginning at v, construct an Eulerian circuit by always exiting the current vertex u along the next unused edge in the ordering specified at u. ■

2.2.25. Example. In the digraph below, the bold edges form an in-tree T of paths to v. The edges labeled in order starting with 1 form an Eulerian circuit. It leaves a vertex along an edge of T only where there is no alternative. If the ordering at v places 1 before 10 before 13, then the algorithm traverses the edges in the order indicated. ■



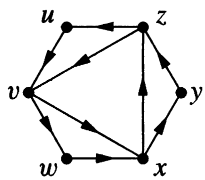
2.2.26. Theorem. Algorithm 2.2.24 always produces an Eulerian circuit.

Proof: Using Lemma 2.2.23, we construct an in-tree T to a vertex v. We then apply Algorithm 2.2.24 to construct a trail. It suffices to show that the trail can end only at v and does so only after traversing all edges.

When we enter a vertex $u \neq v$, the edge leaving u in T has not yet been used, since $d^+(u) = d^-(u)$. Thus whenever we enter u there is still a way out. Therefore the trail can only end at v.

We end when we cannot continue; we are at v and have used all exiting edges. Since $d^-(v) = d^+(v)$, we must also have used all edges entering v. Since we cannot use an edge of T until it is the only remaining edge leaving its tail, we cannot use all edges entering v until we have finished all the other vertices, since T contains a path from each vertex to v. ■

2.2.27. Example. In the digraph below, every in-tree to v contains all of uv, yz, wx, exactly one of $\{zu, zv\}$, and exactly one of $\{xy, xz\}$. There are four in-trees to v. For each in-tree, we consider $\prod(d_i - 1)! = (0!)^3(1!)^3 = 1$ orderings of the edges leaving the vertices. Hence we can obtain one Eulerian circuit from each in-tree, starting along the edge $e = vw$ from v. The four in-trees and the corresponding circuits appear below. ■



In–tree has	Circuit
zu & xy	$(v, w, x, z, v, x, y, z, u)$
zu & xz	$(v, w, x, y, z, v, x, z, u)$
zv & xy	$(v, w, x, z, u, v, x, y, z)$
zv & xz	$(v, w, x, y, z, u, v, x, z)$

Two Eulerian circuits are the same if the successive pairs of edges are the same. From each in-tree to v, Algorithm 2.2.24 generates $\prod_{u \in V(G)}(d^+(u) - 1)!$ different Eulerian circuits. The last out-edge is fixed by the tree for vertices other than v, and since we consider only the cyclic order of the edges we may also choose a particular edge e to start the ordering of edges leaving v. Any change in the exit orderings at vertices specifies at some point different choices for the next edge, so the circuits are distinct. Similarly, circuits obtained from distinct in-trees are distinct. Hence we have generated $c \prod_{u \in V(G)}(d^+(u) - 1)!$ distinct Eulerian circuits, where c is the number of in-trees to v.

In fact, these are all the Eulerian circuits. This yields a combinatorial proof that the number of in-trees to each vertex of an Eulerian digraph is the same. The graph obtained by reversing all the edges has the same number of Eulerian circuits, so the number of out-trees from any vertex also has this value, c. Theorem 2.2.21 provides a computation of c.

2.2.28. Theorem. (van Aardenne-Ehrenfest and de Bruijn [1951]). In an Eulerian digraph with $d_i = d^+(v_i) = d^-(v_i)$ the number of Eulerian circuits is $c \prod_i (d_i - 1)!$, where c counts the in-trees to or out-trees from any vertex.

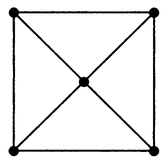
Proof: We have argued that Algorithm 2.2.24 generates this many distinct Eulerian circuits using in-trees to vertex v (starting from v alond e). We need only show that this produces all Eulerian circuits.

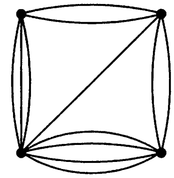
To find the tree and ordering that generates an Eulerian circuit C, follow C from e, and record the order of the edges leaving each vertex. Let T be the subdigraph consisting of the last edge on C leaving each vertex other than v. Since the last edge leaving a vertex occurs in C after all edges entering it, each edge in T extends to a path in T that reaches v. With $n - 1$ edges, T thus forms an in-tree to v. Furthermore, C is the circuit obtained by Algorithm 2.2.24 from T and the orderings of exiting edges that we recorded. ■

EXERCISES

2.2.1. (−) Determine which trees have Prüfer codes that (a) contain only one value, (b) contain exactly two values, or (c) have distinct values in all positions.

2.2.2. (−) Count the spanning trees in the graph on the left below. (Proposition 2.2.8 provides a systematic approach, and then Remark 2.2.10 and Example 2.2.6 can be used to shorten the computation.)



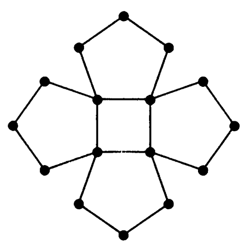


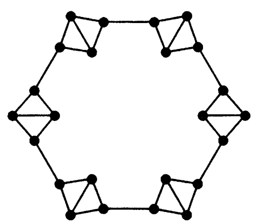
2.2.3. (−) Let G be the graph on the right above. Use the Matrix Tree Theorem to find a matrix whose determinant is $\tau(G)$. Compute $\tau(G)$.

2.2.4. (−) Let G be a simple graph with m edges. Prove that if G has a graceful labeling, then K_{2m+1} decomposes into copies of G. (Hint: Follow the proof of Theorem 2.2.16.)

• • • • •

2.2.5. The graph on the left below was the logo of the 9th Quadrennial International Conference in Graph Theory, held in Kalamazoo in 2000. Count its spanning trees.





2.2.6. (!) Let G be the 3-regular graph with $4m$ vertices formed from m pairwise disjoint kites by adding m edges to link them in a ring, as shown on the right above for $m = 6$. Prove that $\tau(G) = 2m8^m$

2.2.7. (!) Use Cayley's Formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.

2.2.8. Count the following sets of trees with vertex set $[n]$, giving two proofs for each: one using the Prüfer correspondence and one by direct counting arguments.

a) trees that have 2 leaves.

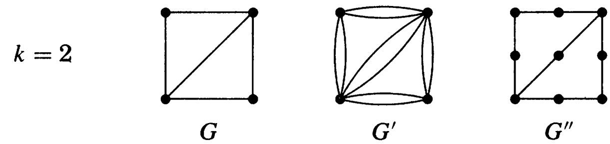
b) trees that have $n-2$ leaves.

2.2.9. Let $S(m, r)$ denote the number of partitions of an m-element set into r nonempty subsets. In terms of these numbers, count the trees with vertex set $\{v_1, \dots, v_n\}$ that have exactly k leaves. (Rényi [1959])

2.2.10. Compute $\tau(K_{2,m})$. Also compute the number of isomorphism classes of spanning trees of $K_{2,m}$.

2.2.11. (+) Compute $\tau(K_{3,m})$.

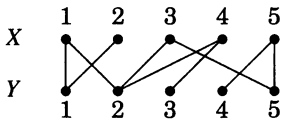
2.2.12. From a graph G we define two new graphs. Let G' be the graph obtained by replacing each edge of G with k copies of that edge. Let G'' be the graph obtained by replacing each edge $uv \in E(G)$ with a u, v-path of length k through $k-1$ new vertices. Determine $\tau(G')$ and $\tau(G'')$ in terms of $\tau(G)$ and k.



2.2.13. Consider $K_{n,n}$ with bipartition X, Y, where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. For each spanning tree T, we form a list $f(T)$ of ordered pairs (written vertically). Having generated part of the list, let u be the least-indexed leaf in X in the remaining subtree, and similarly let v be the least-indexed leaf in Y. Append the pair $\binom{a}{b}$ to the list, where a is the index of the neighbor of u and b is the index of the neighbor of v. Delete $\{u, v\}$. Iterate until $n-1$ pairs have been generated to form $f(T)$ (one edge remains). Part (a) shows that f is well-defined.

a) Prove that every spanning tree of $K_{n,n}$ has a leaf in each partite set.

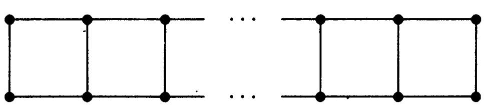
b) Prove that f is a bijection from the set of spanning trees of $K_{n,n}$ to $([n] \times [n])^{n-1}$ Thus $K_{n,n}$ has n^{2n-2} spanning trees. (Rényi [1966], Kelmans [1992], Pritikin [1995])



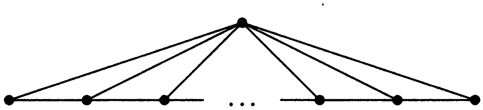
$\rightarrow \quad \binom{1}{4}, \binom{2}{1}, \binom{2}{5}, \binom{5}{3}$

2.2.14. (+) Let $f(r, s)$ be the number of trees with vertex $[n]$ that have partite sets of sizes r and s (with $r + s = n$). Prove that $f(r, s) = \binom{r+s}{s} s^{r-1} r^{s-1}$ if $r \neq s$. What is the formula when $r = s$? (Hint: First show that the Prüfer sequence for such a tree will have $r-1$ of its terms from the partite set of size s and $s-1$ of its terms from the partite set of size r.) (Scoins [1962], Glicksman [1963])

2.2.15. Let G_n be the graph with $2n$ vertices and $3n-2$ edges pictured below, for $n \geq 1$. Prove for $n > 2$ that $\tau(G_n) = 4\tau(G_{n-1}) - \tau(G_{n-2})$. (Kelmans [1967a])



2.2.16. For $n \geq 1$, let a_n be the number of spanning trees in the graph formed from P_n by adding one vertex adjacent to all of $V(P_n)$. For example, $a_1 = 1$, $a_2 = 3$, and $a_3 = 8$. Prove for $n > 1$ that $a_n = a_{n-1} + 1 + \sum_{i=1}^{n-1} a_i$. Use this to prove for $n > 2$ that $a_n = 3a_{n-1} - a_{n-2}$. (Comment: It is also possible to argue directly that $a_n = 3a_{n-1} - a_{n-2}$.)



2.2.17. Use the Matrix Tree Theorem to prove Cayley's Formula.

2.2.18. Use the Matrix Tree Theorem to compute $\tau(K_{r,s})$. (Lovász [1979, p223]—see Kelmans [1965] for a generalization)

2.2.19. (+) Prove combinatorially that the number t_n of trees with vertex set $[n]$ satisfies the recurrence $t_n = \sum_{k=1}^{n-1} k\binom{n-2}{k-1} t_k t_{n-k}$. (Comment: Since $t_n = n^{n-2}$, this proves the identity $n^{n-2} = \sum_{k=1}^{n-1} \binom{n-2}{k-1} k^{k-1}(n-k)^{n-k-2}$.) (Dziobek [1917]; see Lovász [1979, p219])

2.2.20. (!) Prove that a d-regular simple graph G has a decomposition into copies of $K_{1,d}$ if and only if it is bipartite.

2.2.21. (+) Prove that $K_{2m-1,2m}$ decomposes into m spanning paths.

2.2.22. Let G be an n-vertex simple graph that decomposes into k spanning trees. Given also that $\Delta(G) = \delta(G) + 1$, determine the degree sequence of G in terms of n and k.

2.2.23. (!) Prove that if the Graceful Tree Conjecture is true and T is a tree with m edges, then K_{2m} decomposes into $2m-1$ copies of T. (Hint: Apply the cyclically invariant decomposition of K_{2m-1} for trees with $m-1$ edges from the proof of Theorem 2.2.16.)

2.2.24. Of the n^{n-2} trees with vertex set $\{0, \ldots, n-1\}$, how many are gracefully labeled by their vertex names?

2.2.25. (!) Prove that if a graph G is graceful and Eulerian, then $e(G)$ is congruent to 0 or 3 mod 4. (Hint: Sum the absolute edge differences (mod 2) in two different ways.)

2.2.26. (+) Prove that C_n is graceful if and only if 4 divides n or $n+1$. (Frucht [1979])

2.2.27. (+) Let G be the graph consisting of k 4-cycles with one common vertex. Prove that G is graceful. (Hint: Put 0 at the vertex of degree $2k$.)

2.2.28. Let $d_1, \ldots, d_n$ be positive integers. Prove directly that there exists a caterpillar with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

2.2.29. Prove that every tree can be turned into a caterpillar with the same degree sequence using 2-switches (Definition 1.3.32) such that each intermediate graph is a tree.

2.2.30. A bipartite graph is *drawn on a channel* if the vertices of one partite set are placed on one line in the plane (in some order) and the vertices of the other partite set are placed on a line parallel to it and the edges are drawn as straight-line segments between them. Prove that a connected graph G can be drawn on a channel without edge crossings if and only if G is a caterpillar.

2.2.31. (!) An *up/down labeling* is a graceful labeling for which there exists a *critical value* α such that every edge joins vertices with labels above and below α. Prove that every caterpillar has an up/down labeling. Prove that the 7-vertex tree that is not a caterpillar has no up/down-labeling.

2.2.32. (+) Prove that the number of isomorphism classes of n-vertex caterpillars is $2^{n-4} + 2^{\lfloor n/2 \rfloor - 2}$ if $n \geq 3$. (Harary–Schwenk [1973], Kimble–Schwenk [1981])

2.2.33. (!) Let T be an orientation of a tree such that the heads of the edges are all distinct; the one vertex that is not a head is the *root*. Prove that T is a union of paths from the root. Prove that for each vertex of T, exactly one path reaches it from the root.

2.2.34. (∗) Use Theorem 2.2.26 to prove that the algorithm below generates a binary deBruijn cycle of length 2^n (the cycle in Application 1.4.25 arises in this way).

Start with n 0's. Subsequently, append a 1 if doing so does not repeat a previous string of length n, otherwise append a 0.

2.2.35. (∗) *Tarry's Algorithm* (as presented by D.G. Hoffman). Consider a castle with finitely many rooms and corridors. Each corridor has two ends; each end has a door into a room. Each room has door(s), each of which leads to a corridor. Each room can be reached from any other by traversing corridors and rooms. Initially, no doors have marks. A robot started in some room will explore the castle using the following rules.

1) After entering a corridor, traverse it and enter the room at the other end.
2) Upon entering a room with all doors unmarked, mark I on the door of entry.
3) In a room with an unmarked door, mark O on such a door and use it.
4) In a room with all doors marked, exit via a door not marked O if one exists.
5) In a room with all doors marked O, stop.

Prove that the robot traverses each corridor exactly twice, once in each direction, and then stops. (Hint: Prove that this holds for the corridors at every reached vertex, and prove that every vertex is reached. Comment: All decisions are completely local; the robot sees nothing other than the current room or corridor. Tarry's Algorithm [1895] and others are described by König [1936, p35–56] and by Fleischner [1983, 1991].)

2.3. Optimization and Trees

"The best spanning tree" may have various meanings. A **weighted graph** is a graph with numerical labels on the edges. When building links to connect locations, the costs of potential links yield a weighted graph. The minimum cost of connecting the system is the minimum total weight of its spanning trees.

Alternatively, the weights may represent distances. In these case we define the length of a path to be the sum of its edge weights. We may seek a spanning tree with small distances. When discussing weighted graphs, **we consider only nonnegative edge weights.**

We also study a problem about finding good trees to encode messages.

MINIMUM SPANNING TREE

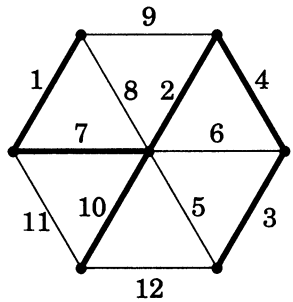
In a connected weighted graph of possible communication links, all spanning trees have $n-1$ edges; we seek one that minimizes or maximizes the sum of the edge weights. For these problems, the most naive heuristic quickly produces an optimal solution.

2.3.1. Algorithm. (Kruskal's Algorithm - for minimum spanning trees.)
Input: A weighted connected graph.
Idea: Maintain an acyclic spanning subgraph H, enlarging it by edges with low weight to form a spanning tree. Consider edges in nondecreasing order of weight, breaking ties arbitrarily.
Initialization: Set $E(H)=\emptyset$.
Iteration: If the next cheapest edge joins two components of H, then include it; otherwise, discard it. Terminate when H is connected. ∎

Theorem 2.3.3 verifies that Kruskal's Algorithm produces an optimal tree. Unsophisticated locally optimal heuristics are called **greedy algorithms**. They usually don't guarantee optimal solutions, but this one does.

In a computer, the weights appear in a matrix, with huge weight on "unavailable" edges. Edges of equal weight may be examined in any order; the resulting trees have the same cost. Kruskal's Algorithm begins with a forest of n isolated vertices. Each selected edge combines two components.

2.3.2. Example. Choices in Kruskal's Algorithm depend only on the order of the weights, not on their values. In the graph below we have used positive integers as weights to emphasize the order of examination of edges. The four cheapest edges are selected, but then we cannot take the edges of weight 5 or 6. We can take the edge of weight 7, but then not those of weight 8 or 9. ∎



2.3.3. Theorem. (Kruskal [1956]). In a connected weighted graph G, Kruskal's Algorithm constructs a minimum-weight spanning tree.

Proof: We show first that the algorithm produces a tree. It never chooses an edge that completes a cycle. If the final graph has more than one component, then we considered no edge joining two of them, because such an edge would be accepted. Since G is connected, some such edge exists and we considered it. Thus the final graph is connected and acyclic, which makes it a tree.

Let T be the resulting tree, and let T^* be a spanning tree of minimum weight. It $T = T^*$, we are done. If $T \neq T^*$, let e be the first edge chosen for T that is not in T^*. Adding e to T^* creates one cycle C. Since T has no cycle, C has an edge $e' \notin E(T)$. Consider the spanning tree $T^* + e - e'$.

Since T^* contains e' and all the edges of T chosen before e, both e' and e are available when the algorithm chooses e, and hence $w(e) \leq w(e')$. Thus $T^* + e - e'$ is a spanning tree with weight at most T^* that agrees with T for a longer initial list of edges than T^* does.

Repeating this argument eventually yields a minimum-weight spanning tree that agrees completely with T. Phrased extremally, we have proved that the minimum spanning tree agreeing with T the longest is T itself. ∎

2.3.4.* Remark. To implement Kruskal's Algorithm, we first sort the m edge weights. We then maintain for each vertex the label of the component containing it, accepting the next cheapest edge if its endpoints have different labels. We merge the two components by changing the label of each vertex in the smaller component to the label of the larger. Since the size of the component at least doubles when a label changes, each label changes at most $\lg n$ times, and the total number of changes is at most $n \lg n$ (we use lg for the base 2 logarithm).

With this labeling method, the running time for large graphs depends on the time to sort m numbers. With this cost included, other algorithms may be faster than Kruskal's Algorithm. In *Prim's Algorithm* (Exercise 10, due also to Jarník), a spanning tree is grown from a single vertex by iteratively adding the cheapest edge that incorporates a new vertex. Prim's and Kruskal's Algorithms have similar running times when edges are pre-sorted by weight.

Both Borůvka [1926] and Jarník [1930] posed and solved the minimum spanning tree problem. Borůvka's algorithm picks the next edge by considering the cheapest edge leaving each component of the current forest. Modern improvements use clever data structures to merge components quickly. Fast versions appear in Tarjan [1984] for when the edges are pre-sorted and in Gabow–Galil–Spencer–Tarjan [1986] for when they are not. Thorough discussion and further references appear in Ahuja–Magnanti–Orlin [1993, Chapter 13]. More recent developments appear in Karger–Klein–Tarjan [1995]. ■

SHORTEST PATHS

How can we find the shortest route from one location to another? How can we find the shortest routes from our home to every place in town? This requires finding shortest paths from one vertex to all other vertices in a weighted graph. Together, these paths form a spanning tree.

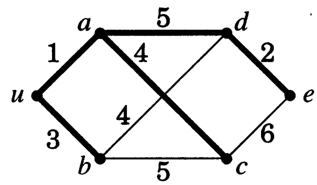
Dijkstra's Algorithm (Dijkstra [1959] and Whiting–Hillier [1960]) solves this problem quickly, using the observation that the u, v-portion of a shortest u, z-path must be a shortest u, v-path. It finds optimal routes from u to other vertices z in increasing order of $d(u, z)$. The **distance** $d(u, z)$ in a weighted graph is the minimum sum of the weights on the edges in a u, z-path (we consider only nonnegative weights).

2.3.5. Algorithm. (Dijkstra's Algorithm—distances from one vertex.)
Input: A graph (or digraph) with nonnegative edge weights and a starting vertex u. The weight of edge xy is $w(xy)$; let $w(xy) = \infty$ if xy is not an edge.
Idea: Maintain the set S of vertices to which a shortest path from u is known, enlarging S to include all vertices. To do this, maintain a tentative distance $t(z)$ from u to each $z \notin S$, being the length of the shortest u, z-path yet found.
Initialization: Set $S = \{u\}$; $t(u) = 0$; $t(z) = w(uz)$ for $z \neq u$.
Iteration: Select a vertex v outside S such that $t(v) = \min_{z \notin S} t(z)$. Add v to S. Explore edges from v to update tentative distances: for each edge vz with $z \notin S$, update $t(z)$ to $\min\{t(z), t(v) + w(vz)\}$.

The iteration continues until $S = V(G)$ or until $t(z) = \infty$ for every $z \notin S$. At the end, set $d(u, v) = t(v)$ for all v. ■

2.3.6. Example. In the weighted graph below, shortest paths from u are found to the other vertices in the order a, b, c, d, e, with distances 1,3,5,6,8, respectively. To reconstruct the paths, we only need the edge on which each shortest path arrives at its destination, because the earlier portion of a shortest u, z-path that reaches z on the edge vz is a shortest u, v-path.

The algorithm can maintain this information by recording the identity of the "selected vertex" whenever the tentative distance to z is updated. When z is selected, the vertex that was recorded when $t(z)$ was last updated is the predecessor of z on the u, z-path of length $d(u, z)$. In this example, the final edges on the paths to a, b, c, d, e generated by the algorithm are ua, ub, ac, ad, de, respectively, and these are the edges of the spanning tree generated from u. ■



With the phrasing given in Algorithm 2.3.5, Dijkstra's Algorithm works also for digraphs, generating an out-tree rooted at u if every vertex is reachable from u. The proof works for graphs and for digraphs. The technique of proving a stronger statement in order to make an inductive proof work is called "loading the induction hypothesis".

2.3.7. Theorem. Given a (di)graph G and a vertex $u \in V(G)$, Dijkstra's Algorithm computes $d(u, z)$ for every $z \in V(G)$.

Proof: We prove the stronger statement that at each iteration,

1) for $z \in S$, $t(z) = d(u, z)$, and

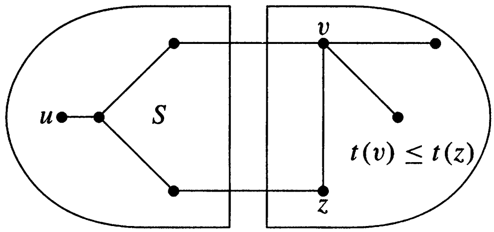
2) for $z \notin S$, $t(z)$ is the least length of a u, z-path reaching z directly from S.

We use induction on $k = |S|$. Basis step: $k = 1$. From the initialization, $S = \{u\}$, $d(u, u) = t(u) = 0$, and the least length of a u, z-path reaching z directly from S is $t(z) = w(u, z)$, which is infinite when uz is not an edge.

Induction step: Suppose that when $|S| = k$, (1) and (2) are true. Let v be a vertex among $z \notin S$ such that $t(z)$ is smallest. The algorithm now chooses v; let $S' = S \cup \{v\}$. We first argue that $d(u, v) = t(v)$. A shortest u, v-path must exit S before reaching v. The induction hypothesis states that the length of the shortest path going directly to v from S is $t(v)$. The induction hypothesis and choice of v also guarantee that a path visiting any vertex outside S and later reaching v has length at least $t(v)$. Hence $d(u, v) = t(v)$, and (1) holds for S'.

To prove (2) for S', let z be a vertex outside S other than v. By the hypothesis, the shortest u, z-path reaching z directly from S has length $t(z)$ (∞ if there is no such path). When we add v to S, we must also consider paths reaching z from v. Since we have now computed $d(u, v) = t(v)$, the shortest such path has length $t(v) + w(vz)$, and we compare this with the previous value of $t(z)$ to find the shortest path reaching z directly from S'.

We have verified that (1) and (2) hold for the new set S' of size $k + 1$; this completes the induction step. ■



The algorithm maintains the condition that $d(u, x) \leq t(z)$ for all $x \in S$ and $z \notin S$; hence it selects vertices in nondecreasing order of distance from u. It computes $d(u, v) = \infty$ when v is unreachable from u. The special case for unweighted graphs is **Breadth-First Search** from u. Here both the algorithm and the proof (Exercise 17) have simpler descriptions.

2.3.8. Algorithm. (Breadth-First Search—BFS)
Input: An unweighted graph (or digraph) and a start vertex u.
Idea: Maintain a set R of vertices that have been reached but not searched and a set S of vertices that have been searched. The set R is maintained as a First-In First-Out list (queue), so the first vertices found are the first vertices explored.
Initialization: $R = \{u\}$, $S = \emptyset$, $d(u, u) = 0$.
Iteration: As long as $R \neq \emptyset$, we search from the first vertex v of R. The neighbors of v not in $S \cup R$ are added to the back of R and assigned distance $d(u, v) + 1$, and then v is removed from the front of R and placed in S. ■

The largest distance from a vertex u to another vertex is the eccentricity $\epsilon(u)$. Hence we can compute the diameter of a graph by running Breadth-First Search from each vertex.

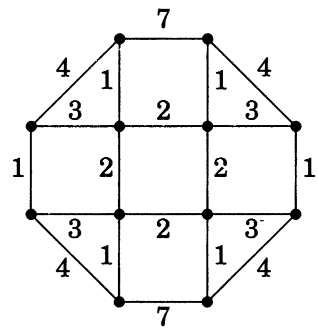
Like Dijkstra's Algorithm, BFS from u yields a tree T in which for each vertex v, the u, v-path is a shortest u, v-path. Thus the graph has no additional edges joining vertices of a u, v-path in T.

Dijkstra's Algorithm figures prominently in the solution of another well-known optimization problem.

2.3.9. Application. A mail carrier must traverse all edges in a road network, starting and ending at the Post Office. The edges have nonnegative weights representing distance or time. We seek a closed walk of minimum total length that uses all the edges. This is the **Chinese Postman Problem**, named in honor of the Chinese mathematician Guan Meigu [1962], who proposed it.

If every vertex is even, then the graph is Eulerian and the answer is the sum of the edge weights. Otherwise, we must repeat edges. Every traversal is an Eulerian circuit of a graph obtained by duplicating edges. Finding the shortest traversal is equivalent to finding the minimum total weight of edges whose duplication will make all vertex degrees even. We say "duplication" because we need not use an edge more than twice. If we use an edge three or more times in making all vertices even, then deleting two of those copies will leave all vertices even. There may be many ways to choose the duplicated edges. ∎

2.3.10. Example. In the example below, the eight outer vertices have odd degree. If we match them around the outside to make the degrees even, the extra cost is $4+4+4+4=16$ or $1+7+7+1=16$. We can do better by using all the vertical edges, which total only 10. ∎



Adding an edge from an odd vertex to an even vertex makes the even vertex odd. We must continue adding edges until we complete a trail to an odd vertex. The duplicated edges must consist of a collection of trails that pair the odd vertices. We may restrict our attention to paths pairing up the odd vertices (Exercise 24), but the paths may need to intersect.

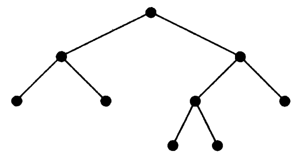
Edmonds and Johnson [1973] described a way to solve the Chinese Postman Problem. If there are only two odd vertices, then we can use Dijkstra's Algorithm to find the shortest path between them and solve the problem. If there are $2k$ odd vertices, then we still can use Dijkstra's Algorithm to find the shortest paths connecting each pair of odd vertices; these are candidates to use in the solution. We use these lengths as weights on the edges of K_{2k}, and then our problem is to find the minimum total weight of k edges that pair up these $2k$ vertices. This is a weighted version of the maximum matching problem discussed in Section 3.3. An exposition appears in Gibbons [1985, p163–165].

TREES IN COMPUTER SCIENCE (optional)

Most applications of trees in computer science use rooted trees.

2.3.11. Definition. A **rooted tree** is a tree with one vertex r chosen as **root**. For each vertex v, let $P(v)$ be the unique v,r-path. The **parent** of v is its neighbor on $P(v)$; its **children** are its other neighbors. Its **ancestors** are the vertices of $P(v)-v$. Its **descendants** are the vertices u such that $P(u)$

contains v. The **leaves** are the vertices with no children. A **rooted plane tree** or **planted tree** is a rooted tree with a left-to-right ordering specified for the children of each vertex.



After a BFS from u, we view the resulting tree T as rooted at u.

2.3.12. Definition. A **binary tree** is a rooted plane tree where each vertex has at most two children, and each child of a vertex is designated as its **left child** or **right child**. The subtrees rooted at the children of the root are the **left subtree** and the **right subtree** of the tree. A **k-ary tree** allows each vertex up to k children.

In many applications of binary trees, all non-leaves have exactly two children (Exercise 26). Binary trees permit storage of data for quick access. We store each item at a leaf and access it by following the path from the root. We encode the path by recording 0 when we move to a left child and 1 when we move to a right child. The search time is the length of this code word for the leaf. Given access probabilities among n items, we want to place them at the leaves of a rooted binary tree to minimize the expected search time.

Similarly, given large computer files and limited storage, we want to encode characters as binary lists to minimize total length. Dividing the frequencies by the total length of the file yields probabilities. This encoding problem then reduces to the problem above.

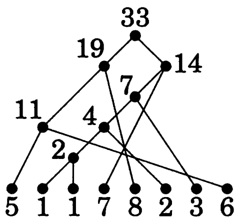
The length of code words may vary; we need a way to recognize the end of the current word. If no code word is an initial portion of another, then the current word ends as soon as the bits since the end of the previous word form a code word. Under this **prefix-free** condition, the binary code words correspond to the leaves of a binary tree using the left/right encoding described above. The expected length of a message is $\sum p_i l_i$, where the ith item has probability p_i and its code has length l_i. Constructing the optimal code is surprisingly easy.

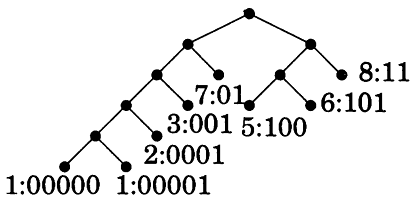
2.3.13. Algorithm. (Huffman's Algorithm [1952]—prefix-free coding)
Input: Weights (frequencies or probabilities) $p_1, \ldots, p_n$.
Output: Prefix-free code (equivalently, a binary tree).
Idea: Infrequent items should have longer codes; put infrequent items deeper by combining them into parent nodes.
Initial case: When $n = 2$, the optimal length is one, with 0 and 1 being the codes assigned to the two items (the tree has a root and two leaves; $n = 1$ can also be used as the initial case).

Recursion: When $n > 2$, replace the two least likely items p, p' with a single item q of weight $p + p'$. Treat the smaller set as a problem with $n - 1$ items. After solving it, give children with weights p, p' to the resulting leaf with weight q. Equivalently, replace the code computed for the combined item with its extensions by 1 and 0, assigned to the items that were replaced. ■

2.3.14. Example. *Huffman coding.* Consider eight items with frequencies $5, 1, 1, 7, 8, 2, 3, 6$. Algorithm 2.3.13 combines items according to the tree on the left below, working from the bottom up. First the two items of weight 1 combine to form one of weight 2. Now this and the original item of weight 2 are the least likely and combine to form an item of weight 4. The 3 and 4 now combine, after which the least likely elements are the original items of weights 5 and 6. The remaining combinations in order are $5 + 6 = 11$, $7 + 7 = 14$, $8 + 11 = 19$, and $14 + 19 = 33$.

From the drawing of this tree on the right, we obtain code words. In their original order, the items have code words 100, 00000, 00001, 01, 11, 0001, 001, and 101. The expected length is $\sum p_i l_i = 90/33$. This is less than 3, which would be the expected length of a code using the eight words of length 3. ■





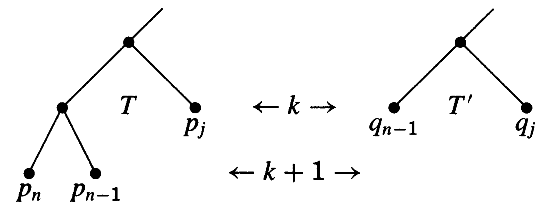
2.3.15. Theorem. Given a probability distribution $\{p_i\}$ on n items, Huffman's Algorithm produces the prefix-free code with minimum expected length.

Proof: We use induction on n. Basis step: $n = 2$. We must send a bit to send a message, and the algorithm encodes each item as a single bit, so the optimum is expected length 1.

Induction step: $n > 2$. Suppose that the algorithm computes the optimal code when given a distribution for $n - 1$ items. Every code assigns items to the leaves of a binary tree. Given a fixed tree with n leaves, we minimize the expected length by greedily assigning the messages with probabilities $p_1 \geq \cdots \geq p_n$ to leaves in increasing order of depth. Thus every optimal code has least likely messages assigned to leaves of greatest depth. Since every leaf at maximum depth has another leaf as its sibling and permuting the items at a given depth does not change the expected length, we may assume that two least likely messages appear as siblings at greatest depth.

Let T be an optimal tree for $p_1, \ldots, p_n$, with the least likely items p_n and p_{n-1} located as sibling leaves at greatest depth. Let T' be the tree obtained from T by deleting these leaves, and let $q_1, \ldots, q_{n-1}$ be the probability distribution obtained by replacing $\{p_{n-1}, p_n\}$ by $q_{n-1} = p_{n-1} + p_n$. The tree T' yields a code for $\{q_i\}$. The expected length for T is the expected length for T' plus q_{n-1}, since if k is the depth of the leaf assigned q_{n-1}, we lose kq_{n-1} and gain $(k+1)(p_{n-1}+p_n)$ in moving from T' to T.

This holds for each choice of T', so it is best to use the tree T' that is optimal for $\{q_i\}$. By the induction hypothesis, the optimal choice for T' is obtained by applying Huffman's Algorithm to $\{q_i\}$. Since the replacement of $\{p_{n-1}, p_n\}$ by q_{n-1} is the first step of Huffman's Algorithm for $\{p_i\}$, we conclude that Huffman's Algorithm generates the optimal tree T for $\{p_i\}$. ∎



Huffman's Algorithm computes an optimal prefix-free code, and its expected length is close to the optimum over all types of binary codes. Shannon [1948] proved that for every code with binary digits, the expected length is at least the **entropy** of the discrete probability distribution $\{p_i\}$, defined to be $-\sum p_i \lg p_i$ (Exercise 31). When each p_i is a power of $1/2$, the Huffman code meets this bound exactly (Exercise 30).

EXERCISES

2.3.1. $(-)$ Assign integer weights to the edges of K_n. Prove that the total weight on every cycle is even if and only if the total weight on every triangle is even.

2.3.2. $(-)$ Prove or disprove: If T is a minimum-weight spanning tree of a weighted graph G, then the u,v-path in T is a minimum-weight u,v-path in G.

2.3.3. $(-)$ There are five cities in a network. The cost of building a road directly between i and j is the entry $a_{i,j}$ in the matrix below. An infinite entry indicates that there is a mountain in the way and the road cannot be built. Determine the least cost of making all the cities reachable from each other.

$$\begin{pmatrix} 0 & 3 & 5 & 11 & 9 \\ 3 & 0 & 3 & 9 & 8 \\ 5 & 3 & 0 & \infty & 10 \\ 11 & 9 & \infty & 0 & 7 \\ 9 & 8 & 10 & 7 & 0 \end{pmatrix}$$

2.3.4. $(-)$ In the graph below, assign weights $(1,1,2,2,3,3,4,4)$ to the edges in two ways: one way so that the minimum-weight spanning tree is unique, and another way so that the minimum-weight spanning tree is not unique.

2.3.5. $(-)$ There are five cities in a network. The travel time for traveling directly from i to j is the entry $a_{i,j}$ in the matrix below. The matrix is not symmetric (use directed graphs), and $a_{i,j} = \infty$ indicates that there is no direct route. Determine the least travel time and quickest route from i to j for each pair i, j.

$$\begin{pmatrix} 0 & 10 & 20 & \infty & 17 \\ 7 & 0 & 5 & 22 & 33 \\ 14 & 13 & 0 & 15 & 27 \\ 30 & \infty & 17 & 0 & 10 \\ \infty & 15 & 12 & 8 & 0 \end{pmatrix}$$

• • • • •

2.3.6. (!) Assign integer weights to the edges of K_n. Prove that on every cycle the total weight is even if and only if the subgraph consisting of the edges with odd weight is a spanning complete bipartite subgraph. (Hint: Show that every component of the subgraph consisting of the edges with even weight is a complete graph.)

2.3.7. Let G be a weighted connected graph with distinct edge weights. Without using Kruskal's Algorithm, prove that G has only one minimum-weight spanning tree. (Hint: Use Exercise 2.1.34.)

2.3.8. Let G be a weighted connected graph. Prove that no matter how ties are broken in choosing the next edge for Kruskal's Algorithm, the list of weights of a minimum spanning tree (in nondecreasing order) is unique.

2.3.9. Let F be a spanning forest of a connected weighted graph G. Among all edges of G having endpoints in different components of F, let e be one of minimum weight. Prove that among all the spanning trees of G that contain F, there is one of minimum weight that contains e. Use this to give another proof that Kruskal's Algorithm works.

2.3.10. (!) **Prim's Algorithm** grows a spanning tree from a given vertex of a connected weighted graph G, iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all the vertices of G have been reached. (Ties are broken arbitrarily.) Prove that Prim's Algorithm produces a minimum-weight spanning tree of G. (Jarník [1930], Prim [1957], Dijkstra [1959], independently).

2.3.11. For a spanning tree T in a weighted graph, let $m(T)$ denote the maximum among the weights of the edges in T. Let x denote the minimum of $m(T)$ over all spanning trees of a weighted graph G. Prove that if T is a spanning tree in G with minimum total weight, then $m(T) = x$ (in other words, T also minimizes the maximum weight). Construct an example to show that the converse is false. (Comment: A tree that minimizes the maximum weight is called a **bottleneck** or **minimax** spanning tree.)

2.3.12. In a weighted complete graph, iteratively select the edge of least weight such that the edges selected so far form a disjoint union of paths. After $n-1$ steps, the result is a spanning path. Prove that this algorithm always gives a minimum-weight spanning path, or give an infinite family of counterexamples where it fails.

2.3.13. (!) Let T be a minimum-weight spanning tree in G, and let T' be another spanning tree in G. Prove that T' can be transformed into T by a list of steps that exchange one edge of T' for one edge of T, such that the edge set is always a spanning tree and the total weight never increases.

2.3.14. (!) Let C be a cycle in a connected weighted graph. Let e be an edge of maximum weight on C. Prove that there is a minimum spanning tree not containing e. Use this to prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces a minimum-weight spanning tree.

2.3.15. Let T be a minimum-weight spanning tree in a weighted connected graph G. Prove that T omits some heaviest edge from every cycle in G.

2.3.16. Four people must cross a canyon at night on a fragile bridge. At most two people can be on the bridge at once. Crossing requires carrying a flashlight, and there is only one flashlight (which can cross only by being carried). Alone, the four people cross in 10, 5, 2, 1 minutes, respectively. When two cross together, they move at the speed of the slower person. In 18 minutes, a flash flood coming down the canyon will wash away the bridge. Can the four people get across in time? Prove your answer without using graph theory and describe how the answer can be found using graph theory.

2.3.17. Given a starting vertex u in an unweighted graph or digraph G, prove directly (without Dijkstra's Algorithm) that Algorithm 2.3.8 computes $d(u, z)$ for all $z \in V(G)$.

2.3.18. Explain how to use Breadth-First Search to compute the girth of a graph.

2.3.19. (+) Prove that the following algorithm correctly finds the diameter of a tree. First, run BFS from an arbitrary vertex w to find a vertex u at maximum distance from w. Next, run BFS from u to reach a vertex v at maximum distance from u. Report diam $T = d(u, v)$. (Cormen–Leiserson–Rivest [1990, p476])

2.3.20. *Minimum diameter spanning tree.* An MDST is a spanning tree where the maximum length of a path is as small as possible. Intuition suggests that running Dijkstra's Algorithm from a vertex of minimum eccentricity (a center) will produce an MDST, but this may fail.

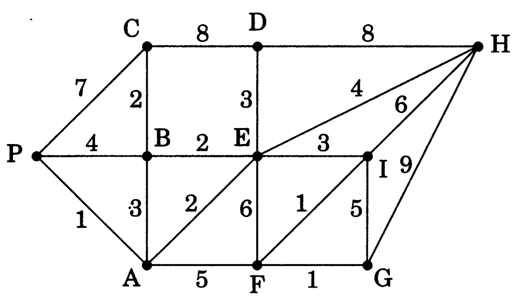
a) Construct a 5-vertex example of an unweighted graph (edge weights all equal 1) such that Dijkstra's Algorithm can be run from some vertex of minimum eccentricity and produce a spanning tree that does not have minimum diameter.

b) Construct a 4-vertex example of a weighted graph such that Dijkstra's algorithm cannot produce an MDST when run from any vertex.

2.3.21. Develop a fast algorithm to test whether a graph is bipartite. The graph is given by its adjacency matrix or by lists of vertices and their neighbors. The algorithm should not need to consider an edge more than twice.

2.3.22. (−) Solve the Chinese Postman Problem in the k-dimensional cube Q_k under the condition that every edge has weight 1.

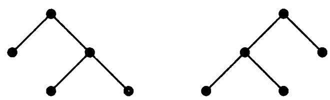
2.3.23. Every morning the Lazy Postman takes the bus to the Post Office. From there, he chooses a route to reach home as quickly as possible (**NOT** ending at the Post Office). Below is a map of the streets along which he must deliver mail, giving the number of minutes required to walk each block whether delivering or not. P denotes the post office and H denotes home. What must the edges traveled more than once satisfy? How many times will each edge be traversed in the optimal route?



2.3.24. $(-)$ Explain why the optimal trails pairing up odd vertices in an optimal solution to the Chinese Postman Problem may be assumed to be paths. Construct a weighted graph with four odd vertices where the optimal solution to the Chinese Postman Problem requires duplicating the edges on two paths that have a common vertex.

2.3.25. Let G be a rooted tree where every vertex has 0 or k children. Given k, for what values of $n(G)$ is this possible?

2.3.26. Find a recurrence relation to count the binary trees with $n+1$ leaves (here each non-leaf vertex has exactly two children, and the left-to-right order of children matters). When $n = 2$, the possibilities are the two trees below.



2.3.27. Find a recurrence relation for the number of rooted plane trees with n vertices. (As in a rooted binary tree, the subtrees obtained by deleting the root of a rooted plane tree are distinguished by their order from left to right.)

2.3.28. $(-)$ Compute a code with minimum expected length for a set of ten messages whose relative frequencies are $1, 2, 3, 4, 5, 5, 6, 7, 8, 9$. What is the expected length of a message in this optimal code?

2.3.29. $(-)$ The game of *Scrabble* has 100 tiles as listed below. This does not agree with English; "S" is less frequent here, for example, to improve the game. Pretend that these are the relative frequencies in English, and compute a prefix-free code of minimum expected length for transmitting messages. Give the answer by listing the relative frequency for each length of code word. Compute the expected length of the code (per text character). (Comment: ASCII coding uses five bits per letter; this code will beat that. Of course, ASCII suffers the handicap of including codes for punctuation.)

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*	*Ø*
9	2	2	4	12	2	3	2	9	1	1	4	2	6	8	2	1	6	4	6	4	2	2	1	2	1	2

2.3.30. Consider n messages occurring with probabilities $p_1, \ldots, p_n$, such that each p_i is a power of $1/2$ (each $p_i \ge 0$ and $\sum p_i = 1$).

a) Prove that the two least likely messages have equal probability.

b) Prove that the expected message length of the Huffman code for this distribution is $-\sum p_i \lg p_i$.

2.3.31. $(+)$ Suppose that n messages occur with probabilities $p_1, \ldots, p_n$ and that the words are assigned distinct binary code words. Prove that for every code, the expected length of a code word with respect to this distribution is at least $-\sum p_i \lg p_i$. (Hint: Use induction on n.) (Shannon [1948])

Chapter 3

Matchings and Factors

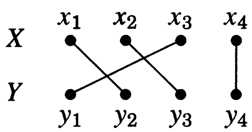
3.1. Matchings and Covers

Within a set of people, some pairs are compatible as roommates; under what conditions can we pair them all up? Many applications of graphs involve such pairings. In Example 1.1.9 we considered the problem of filling jobs with qualified applicants. Bipartite graphs have a natural vertex partition into two sets, and we want to know whether the two sets can be paired using edges. In the roommate question, the graph need not be bipartite.

3.1.1. Definition. A **matching** in a graph G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are **saturated** by M; the others are **unsaturated** (we say *M-saturated* and *M-unsaturated*). A **perfect matching** in a graph is a matching that saturates every vertex.

3.1.2. Example. *Perfect matchings in $K_{n,n}$.* Consider $K_{n,n}$ with partite sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. A perfect matching defines a bijection from X to Y. Successively finding mates for $x_1, x_2, \dots$ yields $n!$ perfect matchings.

Each matching is represented by a permutation of $[n]$, mapping i to j when x_i is matched to y_j. We can express the matchings as matrices. With X and Y indexing the rows and columns, we let position i, j be 1 for each edge $x_i y_j$ in a matching M to obtain the corresponding matrix. There is one 1 in each row and each column. ■

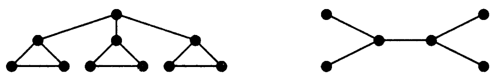


$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.1.3. Example. *Perfect matchings in complete graphs.* Since it has odd order, K_{2n+1} has no perfect matching. The number f_n of perfect matchings in K_{2n} is the number of ways to pair up $2n$ distinct people. There are $2n - 1$ choices for the partner of v_{2n}, and for each such choice there are f_{n-1} ways to complete the matching. Hence $f_n = (2n-1)f_{n-1}$ for $n \geq 1$. With $f_0 = 1$, it follows by induction that $f_n = (2n-1) \cdot (2n-3) \cdots (1)$.

There is also a counting argument for f_n. From an ordering of $2n$ people, we form a matching by pairing the first two, the next two, and so on. Each ordering thus yields one matching. Each matching is generated by $2^n n!$ orderings, since change the order of the pairs or the order within a pair does not change the resulting matching. Thus there are $f_n = (2n)!/(2^n n!)$ perfect matchings. ■

The usual drawing of the Petersen graph shows a perfect matching and two 5-cycles; counting the perfect matchings takes some effort (Exercise 14). The inductive construction of the hypercube Q_k readily yields many perfect matchings (Exercise 16), but counting them exactly is difficult. The graphs below have even order but no perfect matchings.



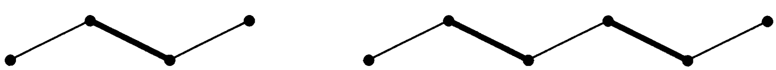
MAXIMUM MATCHINGS

A matching is a set of edges, so its **size** is the number of edges. We can seek a large matching by iteratively selecting edges whose endpoints are not used by the edges already selected, until no more are available. This yields a maximal matching but maybe not a maximum matching.

3.1.4. Definition. A **maximal matching** in a graph is a matching that cannot be enlarged by adding an edge. A **maximum matching** is a matching of maximum size among all matchings in the graph.

A matching M is maximal if every edge not in M is incident to an edge already in M. Every maximum matching is a maximal matching, but the converse need not hold.

3.1.5. Example. *Maximal $\neq$ maximum.* The smallest graph having a maximal matching that is not a maximum matching is P_4. If we take the middle edge, then we can add no other, but the two end edges form a larger matching. Below we show this phenomenon in P_4 and in P_6. ■



In Example 3.1.5, replacing the bold edges by the solid edges yields a larger matching. This gives us a way to look for larger matchings.

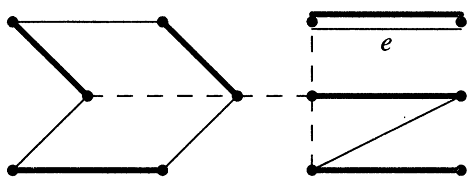
3.1.6. Definition. Given a matching M, an M-**alternating path** is a path that alternates between edges in M and edges not in M. An M-alternating path whose enpoints are unsaturated by M is an M-**augmenting path**.

Given an M-augmenting path P, we can replace the edges of M in P with the other edges of P to obtain a new matching M' with one more edge. Thus when M is a maximum matching, there is no M-augmenting path.

In fact, we prove next that maximum matchings are characterized by the absence of augmenting paths. We prove this by considering two matchings and examining the set of edges belonging to exactly one of them. We define this operation for any two graphs with the same vertex set. (The operation is defined in general for any two sets; see Appendix A.)

3.1.7. Definition. If G and H are graphs with vertex set V, then the **symmetric difference** $G \triangle H$ is the graph with vertex set V whose edges are all those edges appearing in exactly one of G and H. We also use this notation for sets of edges; in particular, if M and M' are matchings, then $M \triangle M' = (M - M') \cup (M' - M)$.

3.1.8. Example. In the graph below, M is the matching with five solid edges, M' is the one with six bold edges, and the dashed edges belong to neither M nor M'. The two matchings have one common edge e; it is not in their symmetric difference. The edges of $M \triangle M'$ form a cycle of length 6 and a path of length 3. ∎



3.1.9. Lemma. Every component of the symmetric difference of two matchings is a path or an even cycle.

Proof: Let M and M' be matchings, and let $F = M \triangle M'$. Since M and M' are matchings, every vertex has at most one incident edge from each of them. Thus F has at most two edges at each vertex. Since $\Delta(F) \le 2$, every component of F is a path or a cycle. Furthermore, every path or cycle in F alternates between edges of $M - M'$ and edges of $M' - M$. Thus each cycle has even length, with an equal number of edges from M and from M'. ∎

3.1.10. Theorem. (Berge [1957]) A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.

Proof: We prove the contrapositive of each direction; G has a matching larger than M if and only if G has an M-augmenting path. We have observed that an M-augmenting path can be used to produce a matching larger than M.

For the converse, let M' be a matching in G larger than M; we construct an M-augmenting path. Let $F = M \triangle M'$. By Lemma 3.1.9, F consists of paths and even cycles; the cycles have the same number of edges from M and M'. Since $|M'| > |M|$, F must have a component with more edges of M' than of M. Such a component can only be a path that starts and ends with an edge of M'; thus it is an M-augmenting path in G. ■

HALL'S MATCHING CONDITION

When we are filling jobs with applicants, there may be many more applicants than jobs; successfully filling the jobs will not use all applicants. To model this problem, we consider an X,Y-bigraph (bipartite graph with bipartition X,Y—Definition 1.2.17), and we seek a matching that saturates X.

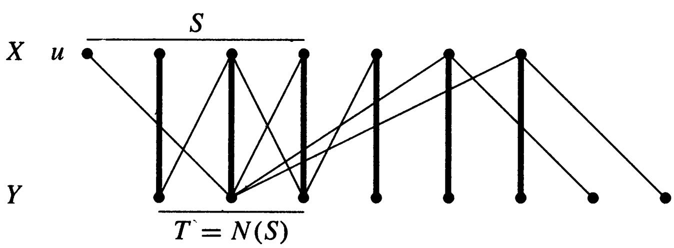
If a matching M saturates X, then for every $S \subseteq X$ there must be at least $|S|$ vertices that have neighbors in S, because the vertices matched to S must be chosen from that set. We use $N_G(S)$ or simply $N(S)$ to denote the set of vertices having a neighbor in S. Thus $|N(S)| \geq |S|$ is a necessary condition.

The condition "For all $S \subseteq X$, $|N(S)| \geq |S|$" is **Hall's Condition**. Hall proved that this obvious necessary condition is also sufficient (TONCAS).

3.1.11. Theorem. (Hall's Theorem—P. Hall [1935]) An X,Y-bigraph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof: *Necessity.* The $|S|$ vertices matched to S must lie in $N(S)$.

Sufficiency. To prove that Hall's Condition is sufficient, we prove the contrapositive. If M is a maximum matching in G and M does not saturate X, then we obtain a set $S \subseteq X$ such that $|N(S)| < |S|$. Let $u \in X$ be a vertex unsaturated by M. Among all the vertices reachable from u by M-alternating paths in G, let S consist of those in X, and let T consist of those in Y (see figure below with M in bold). Note that $u \in S$.



We claim that M matches T with $S - \{u\}$. The M-alternating paths from u reach Y along edges not in M and return to X along edges in M. Hence every vertex of $S - \{u\}$ is reached by an edge in M from a vertex in T. Since there is no M-augmenting path, every vertex of T is saturated; thus an M-alternating path reaching $y \in T$ extends via M to a vertex of S. Hence these edges of M yield a bijection from T to $S - \{u\}$, and we have $|T| = |S - \{u\}|$.

The matching between T and $S - \{u\}$ yields $T \subseteq N(S)$. In fact, $T = N(S)$. Suppose that $y \in Y - T$ has a neighbor $v \in S$. The edge vy cannot be in M, since u is unsaturated and the rest of S is matched to T by M. Thus adding vy to an M-alternating path reaching v yields an M-alternating path to y. This contradicts $y \notin T$, and hence vy cannot exist.

With $T = N(S)$, we have proved that $|N(S)| = |T| = |S| - 1 < |S|$ for this choice of S. This completes the proof of the contrapositive. ■

One can also prove sufficiency by assuming Hall's Condition, supposing that no matching saturates X, and obtaining a contradiction. As we have seen, lack of a matching saturating X yields a violation of Hall's Condition. Contradicting the hypothesis usually means that the contrapositive of the desired implication has been proved. Thus we have stated the proof in that language.

3.1.12. Remark. Theorem 3.1.11 implies that whenever an X, Y-bigraph has no matching saturating X, we can verify this by exhibiting a subset of X with too few neighbors.

Note also that the statement and proof permit multiple edges. ■

Many proofs of Hall's Theorem have been published; see Mirsky [1971, p38] and Jacobs [1969] for summaries. A proof by M. Hall [1948] leads to a lower bound on the number of matchings that saturate X, as a function of the vertex degrees. We consider algorithmic aspects in Section 3.2.

When the sets of the bipartition have the same size, Hall's Theorem is the **Marriage Theorem**, proved originally by Frobenius [1917]. The name arises from the setting of the compatibility relation between a set of n men and a set of n women. If every man is compatible with k women and every woman is compatible with k men, then a perfect matching must exist. Again multiple edges are allowed, which enlarges the scope of applications (see Theorem 3.3.9 and Theorem 7.1.7, for example).

3.1.13. Corollary. For $k > 0$, every k-regular bipartite graph has a perfect matching.

Proof: Let G be a k-regular X, Y-bigraph. Counting the edges by endpoints in X and by endpoints in Y shows that $k\,|X| = k\,|Y|$, so $|X| = |Y|$. Hence it suffices to verify Hall's Condition; a matching that saturates X will also saturate Y and be a perfect matching.

Consider $S \subseteq X$. Let m be the number of edges from S to $N(S)$. Since G is k-regular, $m = k\,|S|$. These m edges are incident to $N(S)$, so $m \le k\,|N(S)|$. Hence $k\,|S| \le k\,|N(S)|$, which yields $|N(S)| \ge |S|$ when $k > 0$. Having chosen $S \subseteq X$ arbitrarily, we have established Hall's condition. ■

One can also use contradiction here. Assuming that G has no perfect matching yields a set $S \subseteq X$ such that $|N(S)| < |S|$. The argument obtaining a contradiction amounts to a rewording of the direct proof given above.

MIN-MAX THEOREMS

When a graph G does not have a perfect matching, Theorem 3.1.10 allows us to prove that M is a maximum matching by proving that G has no M-augmenting path. Exploring all M-alternating paths to eliminate the possibility of augmentation could take a long time.

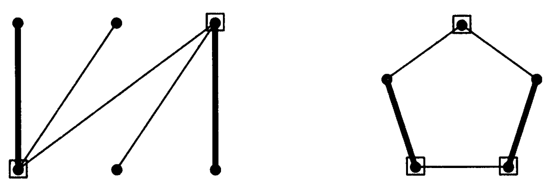
We faced a similar situation when proving that a graph is not bipartite. Instead of checking all possible bipartitions, we can exhibit an odd cycle. Here again, instead of exploring all M-alternating paths, we would prefer to exhibit an explicit structure in G that forbids a matching larger than M.

3.1.14. Definition. A **vertex cover** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in Q *cover* $E(G)$.

In a graph that represents a road network (with straight roads and no isolated vertices), we can interpret the problem of finding a minimum vertex cover as the problem of placing the minimum number of policemen to guard the entire road network. Thus "cover" means "watch" in this context.

Since no vertex can cover two edges of a matching, the size of every vertex cover is at least the size of every matching. Therefore, obtaining a matching and a vertex cover of the same size PROVES that each is optimal. Such proofs exist for bipartite graphs, but not for all graphs.

3.1.15. Example. *Matchings and vertex covers.* In the graph on the left below we mark a vertex cover of size 2 and show a matching of size 2 in bold. The vertex cover of size 2 prohibits matchings with more than 2 edges, and the matching of size 2 prohibits vertex covers with fewer than 2 vertices. As illustrated on the right, the optimal values differ by 1 for an odd cycle. The difference can be arbitrarily large (Exercise 3.3.10). ■



3.1.16. Theorem. (König [1931], Egerváry [1931]) If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

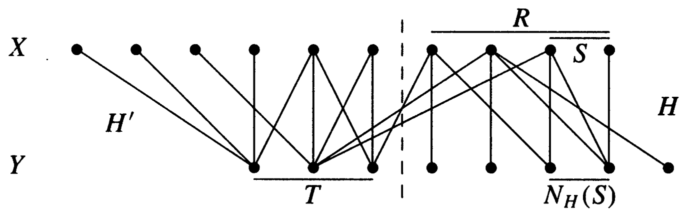
Proof: Let G be an X,Y-bigraph. Since distinct vertices must be used to cover the edges of a matching, $|Q| \geq |M|$ whenever Q is a vertex cover and M is a matching in G. Given a smallest vertex cover Q of G, we construct a matching of size $|Q|$ to prove that equality can always be achieved.

Partition Q by letting $R = Q \cap X$ and $T = Q \cap Y$. Let H and H' be the subgraphs of G induced by $R \cup (Y - T)$ and $T \cup (X - R)$, respectively. We use

Hall's Theorem to show that H has a matching that saturates R into $Y - T$ and H' has a matching that saturates T. Since H and H' are disjoint, the two matchings together form a matching of size $|Q|$ in G.

Since $R \cup T$ is a vertex cover, G has no edge from $Y - T$ to $X - R$. For each $S \subseteq R$, we consider $N_H(S)$, which is contained in $Y - T$. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for S in Q to obtain a smaller vertex cover, since $N_H(S)$ covers all edges incident to S that are not covered by T.

The minimality of Q thus yields Hall's Condition in H, and hence H has a matching that saturates R. Applying the same argument to H' yields the matching that saturates T. ∎



As graph theory continues to develop, new proofs of fundamental results like the König–Egerváry Theorem appear; see Rizzo [2000].

3.1.17. Remark. A **min-max relation** is a theorem stating equality between the answers to a minimization problem and a maximization problem over a class of instances. The König–Egerváry Theorem is such a relation for vertex covering and matching in bipartite graphs.

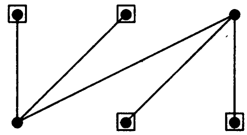
For the discussions in this text, we think of a **dual pair** of optimization problems as a maximization problem **M** and a minimization problem **N**, defined on the same instances (such as graphs), such that for every candidate solution M to **M** and every candidate solution N to **N**, the value of M is less than or equal to the value of N. Often the "value" is cardinality, as above where **M** is maximum matching and **N** is minimum vertex cover.

When **M** and **N** are dual problems, obtaining candidate solutions M and N that have the same value PROVES that M and N are optimal solutions for that instance. We will see many pairs of dual problems in this book. A min-max relation states that, on some class of instances, these short proofs of optimality exist. These theorems are desirable because they save work! Our next objective is another such theorem for independent sets in bipartite graphs. ∎

INDEPENDENT SETS AND COVERS

We now turn from matchings to independent sets. The **independence number** of a graph is the maximum size of an independent set of vertices.

3.1.18. Example. The independence number of a bipartite graph does *not* always equal the size of a partite set. In the graph below, both partite sets have size 3, but we have marked an independent set of size 4. ■



No vertex covers two edges of a matching. Similarly, no edge contains two vertices of an independent set. This yields another dual covering problem.

3.1.19. Definition. An **edge cover** of G is a set L of edges such that every vertex of G is incident to some edge of L.

We say that the vertices of G are *covered* by the edges of L. In Example 3.1.18, the four edges incident to the marked vertices form an edge cover; the remaining two vertices are covered "for free".

Only graphs without isolated vertices have edge covers. A perfect matching forms an edge cover with $n(G)/2$ edges. In general, we can obtain an edge cover by adding edges to a maximum matching.

3.1.20. Definition. For the optimal sizes of the sets in the independence and covering problems we have defined, we use the notation below.

maximum size of independent set	$\alpha(G)$
maximum size of matching	$\alpha'(G)$
minimum size of vertex cover	$\beta(G)$
minimum size of edge cover	$\beta'(G)$

A graph may have many independent sets of maximum size (C_5 has five of them), but the independence number $\alpha(G)$ is a single integer ($\alpha(C_5) = 2$). The notation treats the numbers that answer these optimization problems as graph parameters, like the order, size, maximum degree, diameter, etc. Our use of $\alpha'(G)$ to count the edges in a maximum matching suggests a relationship with the parameter $\alpha(G)$ that counts the vertices in a maximum independent set. We explore this relationship in Section 7.1.

We use $\beta(G)$ for minimum vertex cover due to its interaction with maximum matching. The "prime" goes on $\beta'(G)$ rather than on $\beta(G)$ because $\beta(G)$ counts a set of vertices and $\beta'(G)$ counts a set of edges.

In this notation, the König–Egerváry Theorem states that $\alpha'(G) = \beta(G)$ for every bipartite graph G. We will prove that also $\alpha(G) = \beta'(G)$ for bipartite graphs without isolated vertices. Since no edge can cover two vertices of an independent set, the inequality $\beta'(G) \geq \alpha(G)$ is immediate. (When $S \subseteq V(G)$, we often use $\overline{S}$ to denote $V(G) - S$, the remaining vertices).

3.1.21. Lemma. In a graph G, $S \subseteq V(G)$ is an independent set if and only if $\overline{S}$ is a vertex cover, and hence $\alpha(G) + \beta(G) = n(G)$.

Proof: If S is an independent set, then every edge is incident to at least one vertex of $\overline{S}$. Conversely, if $\overline{S}$ covers all the edges, then there are no edges joining vertices of S. Hence every maximum independent set is the complement of a minimum vertex cover, and $\alpha(G) + \beta(G) = n(G)$. ■

The relationship between matchings and edge coverings is more subtle. Nevertheless, a similar formula holds.

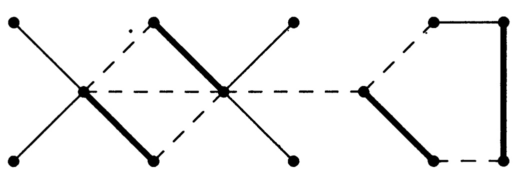
3.1.22. Theorem. (Gallai [1959]) If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.

Proof: From a maximum matching M, we will construct an edge cover of size $n(G) - |M|$. Since a smallest edge cover is no bigger than this cover, this will imply that $\beta'(G) \leq n(G) - \alpha'(G)$. Also, from a minimum edge cover L, we will construct a matching of size $n(G) - |L|$. Since a largest matching is no smaller than this matching, this will imply that $\alpha'(G) \geq n(G) - \beta'(G)$. These two inequalities complete the proof.

Let M be a maximum matching in G. We construct an edge cover of G by adding to M one edge incident to each unsaturated vertex. We have used one edge for each vertex, except that each edge of M takes care of two vertices, so the total size of this edge cover is $n(G) - |M|$, as desired.

Now let L be a minimum edge cover. If both endpoints of an edge e belong to edges in L other than e, then $e \notin L$, since $L - \{e\}$ is also an edge cover. Hence each component formed by edges of L has at most one vertex of degree exceeding 1 and is a star (a tree with at most one non-leaf). Let k be the number of these components. Since L has one edge for each non-central vertex in each star, we have $|L| = n(G) - k$. We form a matching M of size $k = n(G) - |L|$ by choosing one edge from each star in L. ■

3.1.23. Example. The graph below has 13 vertices. A matching of size 4 appears in bold, and adding the solid edges yields an edge cover of size 9. The dashed edges are not needed in the cover. The edge cover consists of four stars; from each we extract one edge (bold) to form the matching. ■



3.1.24. Corollary. (König [1916]) If G is a bipartite graph with no isolated vertices, then $\alpha(G) = \beta'(G)$.

Proof: By Lemma 3.1.21 and Theorem 3.1.22, $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$. Subtracting the König–Egerváry relation $\alpha'(G) = \beta(G)$ completes the proof. ■

DOMINATING SETS (optional)

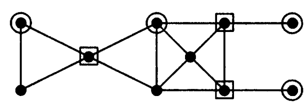
The edges covered by one vertex in a vertex cover are the edges incident to it; they form a star. The vertex cover problem can be described as covering the edge set with the fewest stars. Sometimes we instead want to cover the vertex set with fewest stars. This is equivalent to our next graph parameter.

3.1.25. Example. A company wants to establish transmission towers in a remote region. The towers are located at inhabited buildings, and each inhabited building must be reachable. If a transitter at x can reach y, then also one at y can reach x. Given the pairs that can reach each other, how many transmitters are needed to cover all the buildings?

A similar problem comes from recreational mathematics: How many queens are needed to attack all squares on a chessboard? (Exercise 56). ■

3.1.26. Definition. In a graph G, a set $S \subseteq V(G)$ is a **dominating set** if every vertex not in S has a neighbor in S. The **domination number** $\gamma(G)$ is the minimum size of a dominating set in G.

3.1.27. Example. The graph G below has a minimal dominating set of size 4 (circles) and a minimum dominating set of size 3 (squares): $\gamma(G) = 3$. ■



Berge [1962] introduced the notion of domination. Ore [1962] coined this terminology, and the notation $\gamma(G)$ appeared in an early survey (Cockayne–Hedetniemi [1977]). An entire book (Haynes–Hedetniemi–Slater [1998]) is devoted to domination and its variations.

3.1.28. Example. Covering the vertex set with stars may not require as many stars as covering the edge set. When a graph G has no isolated vertices, every vertex cover *is* a dominating set, so $\gamma(G) \le \beta(G)$. The difference can be large; $\gamma(K_n) = 1$, but $\beta(K_n) = n - 1$. ■

When studying domination as an extremal problem, we try to obtain bounds in terms of other graph parameters, such as the order and the minimum degree. A vertex of degree k dominates itself and k other vertices; thus every dominating set in a k-regular graph G has size at least $n(G)/(k+1)$. For every graph with minimum degree k, a greedy algorithm produces a dominating set not too much bigger than this.

3.1.29. Definition. The **closed neighborhood** $N[v]$ of a vertex v in a graph is $N(v) \cup \{v\}$; it is the set of vertices *dominated by* v.

3.1.30. Theorem. (Arnautov [1974], Payan [1975]) Every n-vertex graph with minimum degree k has a dominating set of size at most $n\frac{1+\ln(k+1)}{k+1}$.

Proof: (Alon [1990]) Let G be a graph with minimum degree k. Given $S \subseteq V(G)$, let U be the set of vertices not dominated by S. We claim that some vertex y outside S dominates at least $|U|(k+1)/n$ vertices of U. Each vertex in U has at least k neighbors, so $\sum_{v\in U}|N[v]| \ge |U|(k+1)$. Each vertex of G is counted at most n times by these $|U|$ sets, so some vertex y appears at least $|U|(k+1)/n$ times and satisfies the claim.

We iteratively select a vertex that dominates the most of the remaining undominated vertices. We have proved that when r undominated vertices remain, after the next selection at most $r(1-(k+1)/n)$ undominated vertices remain. Hence after $n\frac{\ln(k+1)}{k+1}$ steps the number of undominated vertices is at most

$$n(1-\tfrac{k+1}{n})^{n\ln(k+1)/(k+1)} < ne^{-\ln(k+1)} = \tfrac{n}{k+1}$$

The selected vertices and these remaining undominated vertices together form a dominating set of size at most $n\frac{1+\ln(k+1)}{k+1}$. ■

3.1.31. Remark. This bound is also proved by probabilistic methods in Theorem 8.5.10. Caro–Yuster–West [2000] showed that for large k the total domination number satisfies a bound asymptotic to this. Alon [1990] used probabilistic methods to show that this bound is asymptotically sharp when k is large.

Exact bounds remain of interest for small k. Among connected n-vertex graphs, $\delta(G) \ge 2$ implies $\gamma(G) \le 2n/5$ (McCuaig–Shepherd [1989], with seven small exceptions), and $\delta(G) \ge 3$ implies $\gamma(G) \le 3n/8$ (Reed [1996]). Exercise 53 requests constructions achieving these bounds. ■

Many variations on the concept of domination are studied. In Example 3.1.25, for example, one might want the transmitters to be able to communicate with each other, which requires that they induce a connected subgraph.

3.1.32. Definition. A dominating set S in G is
a **connected dominating set** if $G[S]$ is connected,
an **independent dominating set** if $G[S]$ is independent, and
a **total dominating set** if $G[S]$ has no isolated vertex.

Each variation adds a constraint, so dominating sets of these types are at least as large as $\gamma(G)$. Exercises 54–60 explore these variations. Studying independent dominating sets amounts to studying maximal independent sets. This leads to a nice result about claw-free graphs.

3.1.33. Lemma. A set of vertices in a graph is an independent dominating set if and only if it is a maximal independent set.

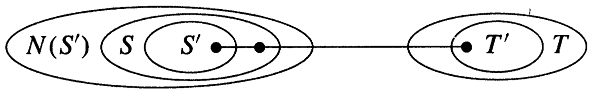
Proof: Among independent sets, S is maximal if and only if every vertex outside S has a neighbor in S, which is the condition for S to be a dominating set. ■

3.1.34. Theorem. (Bollobás–Cockayne [1979]) Every claw-free graph has an independent dominating set of size $\gamma(G)$.

Proof: Let S be a minimum dominating set in a claw-free graph G. Let S' be a maximal independent subset of S. Let $T = V(G) - N(S')$. Let T' be a maximal independent subset of S.

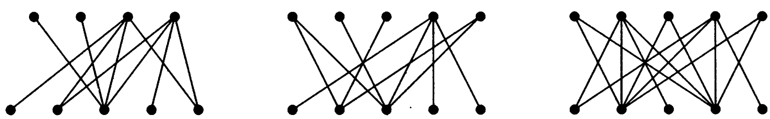
Since T' contains no neighbor of S', $S' \cup T'$ is independent. Since S' is maximal in S, we have $S \subseteq N(S')$. Since T' is maximal in T, T' dominates T. Hence $S' \cup T'$ is a dominating set.

It remains to show that $|S' \cup T'| \le \gamma(G)$. Since S' is maximal in S, T' is independent, and G is claw-free, each vertex of $S - S'$ has at most one neighbor in T'. Since S is dominating, each vertex of T' has at least one neighbor in $S - S'$. Hence $|T'| \le |S - S'|$, which yields $|S' \cup T'| \le |S| = \gamma(G)$. ■



EXERCISES

3.1.1. (−) Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover). Explain why this proves that the matching is optimal.



3.1.2. (−) Determine the minimum size of a maximal matching in the cycle C_n.

3.1.3. (−) Let S be the set of vertices saturated by a matching M in a graph G. Prove that some maximum matching also saturates all of S. Must the statement be true for every maximum matching?

3.1.4. (−) For each of α, α', β, β', characterize the simple graphs for which the value of the parameter is 1.

3.1.5. (−) Prove that $\alpha(G) \ge \frac{n(G)}{\Delta(G)+1}$ for every graph G.

3.1.6. (−) Let T be a tree with n vertices, and let k be the maximum size of an independent set in T. Determine $\alpha'(T)$ in terms of n and k.

3.1.7. (−) Use Corollary 3.1.24 to prove that a graph G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G with no isolated vertices.

• • • • •

3.1.8. (!) Prove or disprove: Every tree has at most one perfect matching.

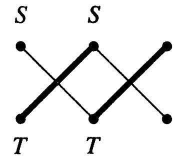
3.1.9. (!) Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

3.1.10. Let M and N be matchings in a graph G, with $|M| > |N|$. Prove that there exist matchings M' and N' in G such that $|M'| = |M| - 1$, $|N'| = |N| + 1$, and M', N' have the same union and intersection (as edge sets) as M, N.

3.1.11. Let C and C' be cycles in a graph G. Prove that $C \triangle C'$ decomposes into cycles.

3.1.12. Let C and C' be cycles of length k in a graph with girth k. Prove that $C \triangle C'$ is a single cycle if and only if $C \cap C'$ is a single path. (Jiang [2001])

3.1.13. Let M and M' be matchings in an X, Y-bigraph G. Suppose that M saturates $S \subseteq X$ and that M' saturates $T \subseteq Y$. Prove that G has a matching that saturates $S \cup T$. For example, below we show M as bold edges and M' as thin edges; we can saturate $S \cup T$ by using one edge from each.



3.1.14. Let G be the Petersen graph. In Example 7.1.9, analysis by cases is used to show that if M is a perfect matching in G, then $G - M = C_5 + C_5$. Assume this.

a) Prove that every edge of G lies in four 5-cycles, and count the 5-cycles in G.

b) Determine the number of perfect matchings in G.

3.1.15. a) Prove that for every perfect matching M in Q_k and every coordinate $i \in [k]$, there are an even number of edges in M whose endpoints differ in coordinate i.

b) Use part (a) to count the perfect matchings in Q_3.

3.1.16. For $k \ge 2$, prove that Q_k has at least $2^{(2^{k-2})}$ perfect matchings.

3.1.17. The *weight* of a vertex in Q_k is the number of 1s in its label. Prove that for every perfect matching in Q_k, the number of edges matching words of weight i to words of weight $i+1$ is $\binom{k-1}{i}$, for $0 \le i \le k-1$.

3.1.18. (!) Two people play a game on a graph G, alternately choosing distinct vertices. Player 1 start by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.

Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy. (Hint: For the second part, the first player should start with a vertex omitted by some maximum matching.)

3.1.19. (!) Let $\mathbf{A} = (A_1, \ldots, A_m)$ be a collection of subsets of a set Y. A **system of distinct representatives** (SDR) for $\mathbf{A}$ is a set of distinct elements $a_1, \ldots, a_m$ in Y such that $a_i \in A_i$. Prove that $\mathbf{A}$ has an SDR if and only if $|\cup_{i\in S} A_i| \ge |S|$ for every $S \subseteq \{1, \ldots, m\}$. (Hint: Transform this to a graph problem.)

3.1.20. The people in a club are planning their summer vacations. Trips $t_1, \ldots, t_n$ are available, but trip t_i has capacity n_i. Each person likes some of the trips and will travel on at most one. In terms of which people like which trips, derive a necessary and sufficient condition for being able to fill all trips (to capacity) with people who like them.

3.1.21. (!) Let G be an X, Y-bigraph such that $|N(S)| > |S|$ whenever $\emptyset \ne S \subset X$. Prove that every edge of G belongs to some matching that saturates X.

3.1.22. Prove that a bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V(G)$, and present an infinite class of examples to prove that this characterization does not hold for all graphs.

3.1.23. (+) *Alternative proof of Hall's Theorem.* Consider a bipartite graph G with bipartition X, Y, satisfying $|N(S)| \geq |S|$ for every $S \subseteq X$. Use induction on $|X|$ to prove that G has a matching that saturates X. (Hint: First consider the case where $|N(S)| > |S|$ for every proper subset S of X. When this does not hold, consider a minimal nonempty $T \subseteq X$ such that $|N(T)| = |T|$.) (M. Hall [1948], Halmos–Vaughan [1950])

3.1.24. (!) A **permutation matrix** P is a 0,1-matrix having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of k permutation matrices if and only if all row sums and column sums equal k.

3.1.25. (!) A **doubly stochastic matrix** Q is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix Q can be expressed $Q = c_1P_1 + \cdots + c_mP_m$, where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(Hint: Use induction on the number of nonzero entries in Q.) (Birkhoff [1946], von Neumann [1953])

3.1.26. (!) A deck of mn cards with m values and n suits consists of one card of each value in each suit. The cards are dealt into an n-by-m array.

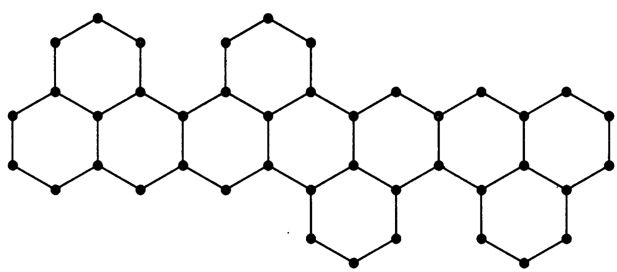
a) Prove that there is a set of m cards, one in each column, having distinct values.

b) Use part (a) to prove that by a sequence of exchanges of cards of the same value, the cards can be rearranged so that each column consists of n cards of distinct suits. (Enchev [1997])

3.1.27. (!) *Generalizing Tic-Tac-Toe.* A **positional game** consists of a set $X = x_1, \ldots, x_n$ of positions and a family $W_1, \ldots, W_m$ of winning sets of positions (Tic-Tac-Toe has nine positions and eight winning sets). Two players alternately choose positions; a player wins by collecting a winning set.

Suppose that each winning set has size at least a and each position appears in at most b winning sets (in Tic-Tac-Toe, $a = 3$ and $b = 4$). Prove that Player 2 can force a draw if $a \geq 2b$. (Hint: Form an X, Y-bigraph G, where $Y = \{w_1, \ldots, w_m\} \cup \{w'_1, \ldots, w'_m\}$, with edges x_iw_j and $x_iw'_j$ whenever $x_i \in W_j$. How can Player 2 use a matching in G? Comment: This result implies that Player 2 can force a draw in d-dimensional Tic-Tac-Toe when the sides are long enough.)

3.1.28. (!) Exhibit a perfect matching in the graph below or give a short proof that it has none. (Lovász–Plummer [1986, p7])



3.1.29. (!) Use the König–Egerváry Theorem to prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

3.1.30. (!) Determine the maximum number of edges in a simple bipartite graph that contains no matching with k edges and no star with l edges. (Isaak)

3.1.31. Use the König–Egerváry Theorem to prove Hall's Theorem.

3.1.32. (!) In an X,Y-bigraph G, the **deficiency** of a set S is $\mathrm{def}(S) = |S| - |N(S)|$; note that $\mathrm{def}(\emptyset) = 0$. Prove that $\alpha'(G) = |X| - \max_{S\subseteq X} \mathrm{def}(S)$. (Hint: Form a bipartite graph G' such that G' has a matching that saturates X if and only if G has a matching of the desired size, and prove that G' satisfies Hall's Condition.) (Ore [1955])

3.1.33. (!) Use Exercise 3.1.32 to prove the König–Egerváry Theorem. (Hint: Obtain a matching and a vertex cover of the same size from a set with maximum deficiency.)

3.1.34. (!) Let G be an X,Y-bigraph with no isolated vertices, and define *deficiency* as in Exercise 3.1.32. Prove that Hall's Condition holds for a matching saturating X if and only if each subset of Y has deficiency at most $|Y| - |X|$.

3.1.35. Let G be an X,Y-bigraph. Prove that G is $(k+1)K_2$-free if and only if each $S \subseteq X$ has a subset of size at most k with neighborhood $N(S)$. (Liu–Zhou [1997])

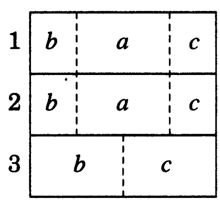
3.1.36. Let G be an X,Y-bigraph having a matching that saturates X. Letting $m = |X|$, prove that G has at most $\binom{m}{2}$ edges belonging to no matching of size m. Construct examples to show that this is best possible for every m.

3.1.37. (+) Let G be an X,Y-bigraph having a matching that saturates X.

a) Let S and T be subsets of X such that $|N(S)| = |S|$ and $|N(T)| = |T|$. Prove that $|N(S \cap T)| = |S \cap T|$.

b) Prove that X has some vertex x such that every edge incident to x belongs to some maximum matching. (Hint: Consider a minimal nonempty set $S \subseteq X$ such that $|N(S)| = |S|$, if any exists.)

3.1.38. (+) An island of area n has n married hunter/farmer couples. The Ministry of Hunting divides the island into n equal-sized hunting regions. The Ministry of Agriculture divides it into n equal-sized farming regions. The Ministry of Marriage requires that each couple receive two overlapping regions. By Exercise 3.1.25, this is always possible. Prove a stronger result: guarantee a pairing where each couple's two regions share area at least $4/(n+1)^2$ when n is odd and $4/[n(n+2)]$ when n is even. Prove also that no larger common area can be guaranteed; the example below achieves equality for $n = 3$. (Marcus–Ree [1959], Floyd [1990])



$$\begin{pmatrix} .5 & .25 & .25 \\ .5 & .25 & .25 \\ 0 & .5 & .5 \end{pmatrix}$$

3.1.39. Let G be a nontrivial simple graph. Prove that $\alpha(G) \le n(G) - e(G)/\Delta(G)$. Conclude that $\alpha(G) \le n(G)/2$ when G also is regular. (P. Kwok)

3.1.40. Let G be a bipartite graph. Prove that $\alpha(G) = n(G)/2$ if and only if G has a perfect matching.

3.1.41. A connected n-vertex graph has exactly one cycle if and only if it has exactly n edges (Exercise 2.1.30). Let C be the cycle in such a graph G. Assuming the result of Exercise 3.1.40 for trees, prove that $\alpha(G) \geq \lfloor n(G)/2 \rfloor$, with equality if and only if $G - V(C)$ has a perfect matching.

3.1.42. (!) An algorithm to greedily build a large independent set iteratively selects a vertex of minimum degree in the remaining graph and deletes it and its neighbors. Prove that this algorithm produces an independent set of size at least $\sum_{v \in V(G)} \frac{1}{d(v)+1}$ in a graph G. (Caro [1979], Wei [1981])

3.1.43. Let M be a maximal matching and L a minimal edge cover in a graph with no isolated vertices. Prove the statements below. (Norman–Rabin [1959], Gallai [1959])

a) M is a maximum matching if and only if M is contained in a minimum edge cover.

b) L is a minimum edge cover if and only if L contains a maximum matching.

3.1.44. (−) Let G be a simple graph in which the sum of the degrees of any k vertices is less than $n - k$. Prove that every maximal independent set in G has more than k vertices. (Meyer [1972])

3.1.45. An edge e of a graph G is **α-critical** if $\alpha(G - e) > \alpha(G)$. Suppose that xy and xz are α-critical edges in G. Prove that G has an induced subgraph that is an odd cycle containing xy and xz. (Hint: Let Y, Z be maximum independent sets in $G - xy$ and $G - xz$, respectively. Let $H = G[Y \triangle Z]$. Prove that every component of H has the same number of vertices from Y and from Z. Use this to prove that y and z belong to the same component of H.) (Berge [1970], with a difficult generalization in Markossian–Karapetian [1984])

3.1.46. (∗−) Characterize the graphs with domination number 1.

3.1.47. (∗−) Find the smallest tree where the domination number and the vertex cover number are not equal.

3.1.48. (∗−) Determine $\gamma(C_n)$ and $\gamma(P_n)$.

3.1.49. (∗) Let G be a graph without isolated vertices, and let S be a minimal dominating set in G. Prove that $\overline{S}$ is a dominating set. Conclude that $\gamma(G) \leq n(G)/2$. (Ore [1962])

3.1.50. (∗) Prove that $\gamma(G) \leq n - \beta'(G) \leq n/2$ when G is an n-vertex graph without isolated vertices. For $1 \leq k \leq n/2$, construct a connected n-vertex graph G with $\gamma(G) = k$.

3.1.51. (∗) Let G be an n-vertex graph.

a) Prove that $\lceil n/(1 + \Delta(G) \rceil \geq \gamma(G) \leq n - \Delta(G)$.

b) Prove that $(1 + \text{diam } G)/3 \leq \gamma(G) \leq n - \lfloor \text{diam } G/3 \rfloor$.

3.1.52. (∗) Prove that if the diameter of G is at least 3, then $\gamma(\overline{G}) \leq 2$.

3.1.53. (∗) For all $k \in \mathbb{N}$, construct a connected graph with $5k$ vertices and domination number $2k$. Construct a single 3-regular graph G such that $\gamma(G) = 3n(G)/8$.

3.1.54. (∗) Determine the domination number of the Petersen graph, and determine the minimum size of a total dominating set in the Petersen graph.

3.1.55. (∗) In the hypercube Q_4, determine the minimum sizes of a dominating set, an independent dominating set, a connected dominating set, and a total dominating set.

3.1.56. (∗) Find a way to place five queens on an eight-by-eight chessboard that attack all other squares. Show that the five queens cannot be placed so that also they do not attack each other. (Comment: Thus the independent domination number of the "queen's graph" exceeds its domination number; it is 7.)

3.1.57. (*) For all $n \in \mathbb{N}$, construct an n-vertex tree with domination number 2 in which the minimum size of an independent dominating set is $\lfloor n/2 \rfloor$.

3.1.58. (*) Prove that a $K_{1,r}$-free graph G has an independent dominating set of size at most $(r-2)\gamma(G) - (r-3)$. (Hint: Generalize the argument of Theorem 3.1.34.) (Bollobás–Cockayne [1979])

3.1.59. (*) In a graph G of order n, prove that the minimum size of a connected dominating set is n minus the maximum number of leaves in a spanning tree.

3.1.60. (*) For $k \le 5$, every graph G with $\delta(G) \le k$ has a connected dominating set of size at most $3n(G)/(k+1)$ (Kleitman–West [1991], Griggs–Wu [1992]). Prove that this is sharp using a graph formed from a cyclic arrangement of $3m$ pairwise-disjoint cliques by making each vertex adjacent to every vertex in the clique before it and the clique after it. Let the clique sizes be $\lceil k/2 \rceil$, $\lfloor k/2 \rfloor$, 1, $\lceil k/2 \rceil$, $\lfloor k/2 \rfloor$, 1, $\cdots$.

3.2. Algorithms and Applications

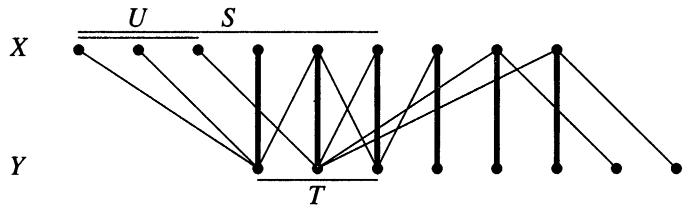
MAXIMUM BIPARTITE MATCHING

To find a maximum matching, we iteratively seek augmenting paths to enlarge the current matching. In a bipartite graph, if we don't find an augmenting path, we will find a vertex cover with the same size as the current matching, thereby proving that the current matching has maximum size. This yields both an algorithm to solve the maximum matching problem and an algorithmic proof of the König–Egerváry Theorem.

Given a matching M in an X,Y-bigraph G, we search for M-augmenting paths from each M-unsaturated vertex in X. We need only search from vertices in X, because every augmenting path has odd length and thus has ends in both X and Y. We will search from the unsaturated vertices in X simultaneously. Starting with a matching of size 0, $\alpha'(G)$ applications of the Augmenting Path Algorithm produce a maximum matching.

3.2.1. Algorithm. (Augmenting Path Algorithm).
Input: An X,Y-bigraph G, a matching M in G, and the set U of M-unsaturated vertices in X.
Idea: Explore M-alternating paths from U, letting $S \subseteq X$ and $T \subseteq Y$ be the sets of vertices reached. *Mark* vertices of S that have been explored for path extensions. As a vertex is reached, record the vertex from which it is reached.
Initialization: $S = U$ and $T = \emptyset$.
Iteration: If S has no unmarked vertex, stop and report $T \cup (X - S)$ as a minimum cover and M as a maximum matching. Otherwise, select an unmarked $x \in S$. To explore x, consider each $y \in N(x)$ such that $xy \notin M$. If y is unsaturated, terminate and report an M-augmenting path from U to y. Otherwise, y is matched to some $w \in X$ by M. In this case, include y in T (reached from x)

and include w in S (reached from y). After exploring all such edges incident to x, mark x and iterate. ■



When exploring x in the iterative step, we may reach a vertex $y \in T$ that we have reached previously. Recording x as the previous vertex on the path may change which M-augmenting path we report, but it won't change whether such a path exists.

3.2.2. Theorem. Repeatedly applying the Augmenting Path Algorithm to a bipartite graph produces a matching and a vertex cover of equal size.

Proof: We need only verify that the Augmenting Path Algorithm produces an M-augmenting path or a vertex cover of size $|M|$. If the algorithm produces an M-augmenting path, we are finished. Otherwise, it terminates by marking all vertices of S and claiming that $R = T \cup (X - S)$ is a vertex cover of size $|M|$. We must prove that R is a vertex cover and has size $|M|$.

To show that R is a vertex cover, it suffices to show that there is no edge joining S to $Y - T$. An M-alternating path from U enters X only on an edge of M. Hence every vertex x of $S - U$ is matched via M to a vertex of T, and there is no edge of M from S to $Y - T$. Also there is no such edge outside M. When the path reaches $x \in S$, it can continue along any edge not in M, and exploring x puts all other neighbors of x into T. Since the algorithm marks all of S before terminating, all edges from S go to T.

Now we study the size of R. The algorithm puts only saturated vertices in T; each $y \in T$ is matched via M to a vertex of S. Since $U \subseteq S$, also each vertex of $X - S$ is saturated, and the edges of M incident to $X - S$ cannot involve T. Hence they are different from the edges saturating T, and we find that M has at least $|T| + |X - S|$ edges. Since there is no matching larger than this vertex cover, we have $|M| = |T| + |X - S| = |R|$. ■

In addition to studying the correctness of algorithms, we are concerned about the time (number of computational steps) they use. We measure this as a function of the size of the input. For graph problems, we usually use the order $n(G)$ and/or size $e(G)$ to measure the input size.

3.2.3. Definition. The **running time** of an algorithm is the maximum number of computational steps used, expressed as a function of the size of the input. A **good algorithm** is one that has polynomial running time.

Running time is often expressed as "$O(f)$", where f is a function of the size of the input. Here $O(f)$ denotes the set of functions g such that $|g(x)|$ is bounded by a constant multiple of $|f(x)|$ when x is sufficiently large (that is, there exist c, a such that $|g(x)| \le c\,|f(x)|$ when $|x| \ge a$).

Many problems we study in Chapters 1-4 have good algorithms; other notions of complexity (Appendix B) need not trouble us yet. Since we don't know how long a particular operation may take on a particular computer, constant factors in running time have little meaning. Hence the "Big Oh" notation $O(f)$ is convenient. When f is a quadratic polynomial, we typically abuse notation by writing $O(n^2)$ instead of $O(f)$ to describe functions that grow at most quadratically in terms of n.

3.2.4. Remark. Let G be an X, Y-bigraph with n vertices and m edges. Since $\alpha'(G) \le n/2$, we find a maximum matching in G by applying Algorithm 3.2.1 at most $n/2$ times. Each application explores a vertex of X at most once, just before marking it; thus it considers each edge at most once. If the time for one edge exploration is bounded by a constant, then this algorithm to find a maximum matching runs in time $O(nm)$. Theorem 3.2.22 presents a faster algorithm, with running time $O(\sqrt{n}m)$. Section 3.3 discusses a good algorithm for maximum matching in general graphs. ∎

WEIGHTED BIPARTITE MATCHING

Our results on maximum matching generalize to weighted X, Y-bigraphs, where we seek a matching of maximum total weight. If our graph is not all of $K_{n,n}$, then we insert the missing edges and assign them weight 0. This does not affect the numbers we can obtain as the weight of a matching. Thus we assume that our graph is $K_{n,n}$.

Since we consider only nonnegative edge weights, some maximum weighted matching is a perfect matching; thus we seek a perfect matching. We solve both the maximum weighted matching problem and its dual.

3.2.5. Example. *Weighted bipartite matching and its dual.* A farming company owns n farms and n processing plants. Each farm can produce corn to the capacity of one plant. The profit that results from sending the output of farm i to plant j is $w_{i,j}$. Placing weight $w_{i,j}$ on edge $x_i y_j$ gives us a weighted bipartite graph with partite sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. The company wants to select edges forming a matching to maximize total profit.

The government claims that too much corn is being produced, so it will pay the company not to process corn. The government will pay u_i if the company agrees not to use farm i and v_j if it agrees not to use plant j. If $u_i + v_j < w_{i,j}$, then the company makes more by using the edge $x_i y_j$ than by taking the government payments for those vertices. In order to stop all production, the government must offer amounts such that $u_i + v_j \ge w_{i,j}$ for all i, j. The government wants to find such values to minimize $\sum u_i + \sum v_j$. ∎

3.2.6. Definition. A **transversal** of an n-by-n matrix consists of n positions, one in each row and each column. Finding a transversal with maximum sum is the **Assignment Problem**. This is the matrix formulation of the **maximum weighted matching** problem, where nonnegative weight $w_{i,j}$ is assigned to edge $x_i y_j$ of $K_{n,n}$ and we seek a perfect matching M to maximize the total weight $w(M)$.

With these weights, a **(weighted) cover** is a choice of labels $u_i, \dots, u_n$ and $v_j, \dots, v_n$ such that $u_i + v_j \geq w_{i,j}$ for all i, j. The **cost** $c(u, v)$ of a cover (u, v) is $\sum u_i + \sum v_j$. The **minimum weighted cover** problem is that of finding a cover of minimum cost.

Note that the problem of minimum weight perfect matching can be solved using maximum weight matching; simply replace each weight $w_{i,j}$ with $M - w_{i,j}$ for some large number M.

The next lemma shows that the weighted matching and weighted cover problems are dual problems.

3.2.7. Lemma. For a perfect matching M and cover (u, v) in a weighted bipartite graph G, $c(u, v) \geq w(M)$. Also, $c(u, v) = w(M)$ if and only if M consists of edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$. In this case, M and (u, v) are optimal.

Proof: Since M saturates each vertex, summing the constraints $u_i + v_j \geq w_{i,j}$ that arise from its edges yields $c(u, v) \geq w(M)$ for every cover (u, v). Furthermore, if $c(u, v) = w(M)$, then equality must hold in each of the n inequalities summed. Finally, since $c(u, v) \geq w(M)$ for every matching and every cover, $c(u, v) = w(M)$ implies that there is no matching with weight greater than $c(u, v)$ and no cover with cost less than $w(M)$. ■

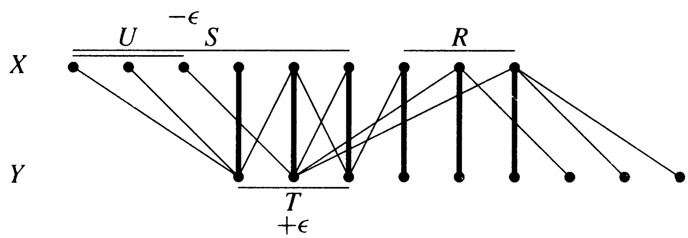
A matching and a cover have the same value only when the edges of the matching are covered with equality. This leads us to an algorithm.

3.2.8. Definition. The **equality subgraph** $G_{u,v}$ for a cover (u, v) is the spanning subgraph of $K_{n,n}$ having the edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$.

If $G_{u,v}$ has a perfect matching, then its weight is $\sum u_i + \sum v_j$, and by Lemma 3.2.7 we have the optimal solution. Otherwise, we find a matching M and a vertex cover Q of the same size in $G_{u,v}$ (by using the Augmenting Path Algorithm, for example). Let $R = Q \cap X$ and $T = Q \cap Y$. Our matching of size $|Q|$ consists of $|R|$ edges from R to $Y - T$ and $|T|$ edges from T to $X - R$, as shown below. To seek a larger matching in the equality subgraph, we change (u, v) to introduce an edge from $X - R$ to $Y - T$ while maintaining equality on all edges of M.

A cover requires $u_i + v_j \geq w_{i,j}$ for all i, j; the difference $u_i + v_j - w_{i,j}$ is the **excess** for i, j. Edges joining $X - R$ and $Y - T$ are not in $G_{u,v}$ and have positive excess. Let ϵ be the minimum excess on the edges from $X - R$ to $Y - T$. Reducing u_i by ϵ for all $x_i \in X - R$ maintains the cover condition for these edges while bringing at least one into the equality subgraph. To maintain the cover condition for the edges from $X - R$ to T, we also increase v_j by ϵ for $y_j \in T$.

We repeat the procedure with the new equality subgraph; eventually we obtain a cover whose equality subgraph has a perfect matching. The resulting algorithm was named the **Hungarian Algorithm** by Kuhn in honor of the work of König and Egerváry on which it is based.



3.2.9. Algorithm. (Hungarian Algorithm—Kuhn [1955], Munkres [1957]).
Input: A matrix of weights on the edges of $K_{n,n}$ with bipartition X, Y.
Idea: Iteratively adjusting the cover (u, v) until the equality subgraph $G_{u,v}$ has a perfect matching.
Initialization: Let (u, v) be a cover, such as $u_i = \max_j w_{i,j}$ and $v_j = 0$.
Iteration: Find a maximum matching M in $G_{u,v}$. If M is a perfect matching, stop and report M as a maximum weight matching. Otherwise, let Q be a vertex cover of size $|M|$ in $G_{u,v}$. Let $R = X \cap Q$ and $T = Y \cap Q$. Let

$$\epsilon = \min\{u_i + v_j - w_{i,j}\colon x_i \in X - R,\ y_j \in Y - T\}.$$

Decrease u_i by ϵ for $x_i \in X - R$, and increase v_j by ϵ for $y_j \in T$. Form the new equality subgraph and repeat. ■

We have presented the algorithm using bipartite graphs, but repeatedly drawing a changing equality subgraph is awkward. Therefore, we compute with matrices. The initial weights form a matrix A with $w_{i,j}$ in position i, j. We associate the vertices and the labels (u, v) with the rows and columns, which serve as X and Y, respectively. We subtract $w_{i,j}$ from $u_i + v_j$ to obtain the **excess matrix**: $c_{i,j} = u_i + v_j - w_{i,j}$. The edges of the equality subgraph correspond to 0s in the excess matrix.

3.2.10. Example. *Solving the Assignment Problem.* The first matrix below is the matrix of weights. The others display a cover (u, v) and the corresponding excess matrix. We underscore entries in the excess matrix to mark a maximum matching M of $G_{u,v}$, which appears as bold edges in the equality subgraphs drawn for the first two excess matrices. (Drawing the equality subgraphs is not necessary.) A matching in $G_{u,v}$ corresponds to a set of 0s in the excess matrix with no two in any row or column; call this a **partial transversal**.

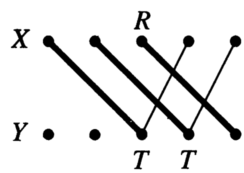
A set of rows and columns covering the 0s in the excess matrix is a **covering set**; this corresponds to a vertex cover in $G_{u,v}$. A covering set of size less than n yields progress toward a solution, since the next weighted cover costs less. We study the 0s in the excess matrix and find a partial transversal and a covering set of the same size. In a small matrix, we can do this by inspection.

We underscore the 0s of a partial transversal, and we use Rs and Ts to label the rows and columns of the covering set. At each iteration, we compute the minimum excess on the positions *not* in a covered row or column (in rows $X - R$ and columns $Y - T$). These uncovered positions have positive excess (the corresponding edges are not in the equality subgraph). The value ϵ defined in Algorithm 3.2.9 is the minimum of these excesses. We reduce the label u_i by ϵ on rows not in R and increase the label v_j by ϵ on columns in T.

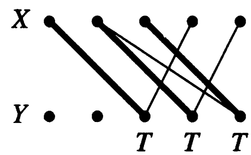
In the example below, the covering set used in the first iteration reduces the cost of the cover but does not augment the maximum matching in the equality subgraph. The second iteration produces a perfect matching. Using the last three columns as a covering set in the first iteration would augment the matching immediately.

The transversal of 0s after the final iteration identifies a perfect matching whose total weight equals the cost of the final cover. The corresponding edges have weights $5, 4, 6, 8, 8$ in the original data, which sum to 31. The labels $4, 5, 7, 4, 6$ and $0, 0, 2, 2, 1$ in the final cover satisfy each edge exactly and also sum to 31. The value of the optimal solution is unique, but the solution itself is not; this example has many maximum weight matchings and many minimum cost covers, but all have total weight 31. ■

$$\begin{pmatrix} 4 & 1 & 6 & 2 & 3 \\ 5 & 0 & 3 & 7 & 6 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 4 & 6 & 3 & 4 \\ 4 & 6 & 5 & 8 & 6 \end{pmatrix} \rightarrow \begin{array}{ccc} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & \\ \begin{matrix} 6 \\ 7 \\ 8 \\ 6 \\ 8 \end{matrix} & \begin{pmatrix} 2 & 5 & \underline{0} & 4 & 3 \\ 2 & 7 & 4 & \underline{0} & 1 \\ 6 & 5 & 4 & 3 & \underline{0} \\ 3 & 2 & 0 & 3 & 2 \\ 4 & 2 & 3 & 0 & 2 \end{pmatrix} & \begin{matrix} \\ \\ R \\ \\ \\ \end{matrix} \\ & \begin{matrix} & & T & T & \end{matrix} & \end{array}$$



$$\begin{array}{cc} & \begin{matrix} 0 & 0 & 1 & 1 & 0 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 8 \\ 5 \\ 7 \end{matrix} & \begin{pmatrix} 1 & 4 & \underline{0} & 4 & 2 \\ 1 & 6 & 4 & \underline{0} & 0 \\ 6 & 5 & 5 & 4 & \underline{0} \\ 2 & 1 & 0 & 3 & 1 \\ 3 & 1 & 3 & 0 & 1 \end{pmatrix} \\ & \begin{matrix} & & T & T & T \end{matrix} \end{array}$$



$$\rightarrow \begin{array}{cc} & \begin{matrix} 0 & 0 & 2 & 2 & 1 \end{matrix} \\ \begin{matrix} 4 \\ 5 \\ 7 \\ 4 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 3 & \underline{0} & 4 & 2 \\ \underline{0} & 5 & 4 & 0 & 0 \\ 5 & 4 & 5 & 4 & \underline{0} \\ 1 & \underline{0} & 0 & 3 & 1 \\ 2 & 0 & 3 & \underline{0} & 1 \end{pmatrix} \end{array}$$

3.2.11. Theorem. The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover.

Proof: The algorithm begins with a cover. It can terminate only when the equality subgraph has a perfect matching, which guarantees equal value for the current matching and cover. Suppose that (u, v) is the current cover and that the equality subgraph has no perfect matching. Let (u', v') denote the new lists of numbers assigned to the vertices. Because ϵ is the minimum of a nonempty finite set of positive numbers, $\epsilon > 0$.

We verify first that (u', v') is a cover. The change of labels on vertices of $X - R$ and T yields $u_i' + v_j' = u_i + v_j$ for edges $x_i y_j$ from $X - R$ to T or from R to $Y - T$. If $x_i \in R$ and $y_j \in T$, then $u_i' + v_j' = u_i + v_j + \epsilon$, and the weight remains covered. If $x_i \in X - R$ and $y_j \in Y - T$, then $u_i' + v_j'$ equals $u_i + v_j - \epsilon$, which by the choice of ϵ is at least $w_{i,j}$.

The algorithm terminates only when the equality subgraph has a perfect matching, so it suffices to show that it does terminate. Suppose that the weights $w_{i,j}$ are rational. Multiplying the weights by their least common denominator yields an equivalent problem with integer weights. We can now assume that the labels in the current cover also are integers. Thus each excess is also an integer, and at each iteration we reduce the cost of the cover by an integer amount. Since the cost starts at some value and is bounded below by the weight of a perfect matching, after finitely many iterations we have equality.

For real-valued weights in general, see Remark 3.2.12). ■

3.2.12.* Remark. When the weights are real numbers, the algorithm still works if we obtain vertex covers in the equality subgraph more carefully. We show that the algorithm terminates within n^2 iterations. Because the edges of M remain in the new equality subgraph, the size of the current matching never decreases. Since the size of the matching can increase at most n times, it suffices to show that it must increase within n iterations.

If we find the maximum matching M by iterating the Augmenting Path Algorithm, then the last iteration presents us with a vertex cover. We find it by exploring M-alternating paths from the set U of M-unsaturated vertices in X. With S and T denoting the sets of vertices reachable in X and T, we obtain the vertex cover $R \cup T$, where $R = X - S$.

Applying a step of the Hungarian Algorithm using the vertex cover $R \cup T$ maintains equality on M and all the edges in M-alternating paths from U. Edges from T to R disappear from the equality subgraph, but we don't care because they don't appear in M-alternating paths from U. Introducing an edge from S to $Y - T$ either creates an M-augmenting path or increases T while leaving U unchanged. Since we can increase T at most n times, we obtain a larger matching in the equality subgraph within n iterations. ■

3.2.13.* Remark. The maximum matching and vertex cover problems in bipartite graphs are special cases of the weighted problems. Given a bipartite graph G, form a weighted graph with weight 1 on the edges of G and weight 0 on the edges of $K_{n,n}$. The maximum weight of a matching is $\alpha'(G)$.

Given integer weights, the Hungarian algorithm always maintains integer labels in the weighted cover. Hence in this weighted cover problem we may restrict the values (labels) used to be integers. Further thought shows that these integers will always be 0 or 1.

The vertices receiving label 1 must cover the weight on the edges of G, so they form a vertex cover for G. Minimizing the sum of labels under the integer restriction is equivalent to finding the minimum number of vertices in a vertex cover for G. Hence the answer to the weighted cover problem is $\beta(G)$. ■

3.2.14.* Application. *Street Sweeping and the Transportation Problem.* A cleaning machine sweeping a curb must move in the same direction as traffic. This yields a digraph; a two-way street generates two oppositely directed edges, while a one-way street generates two edges in the same direction. We consider a simple version of the **Street Sweeping Problem**, discussed in more detail in Roberts [1978] as based on Tucker–Bodin [1976].

In New York City, parking is prohibited from some curbs each day to allow for street sweeping. For each day, this defines a **sweep subgraph** G of the full digraph H of curbs, consisting of those available for sweeping. Each $e \in E(H)$ has a **deadheading time** $t(e)$ needed to travel it without sweeping.

The question is how to sweep G while minimizing the total deadheading time spent without sweeping. This is a generalization of a directed version of the Chinese Postman Problem. If indegree equals outdegree at each vertex of G, then no deadheading is needed. Otherwise, we duplicate edges of G or add edges from H to obtain an Eulerian digraph G' containing G.

Let X be the set of vertices with excess indegree; let $\sigma(x) = d_G^-(x) - d_G^+(x)$ for $x \in X$. Let Y be the set with excess outdegree; let $\partial(y) = d_G^-(y) - d_G^+(y)$ for $y \in Y$. Note that $\sum_{x \in X} \sigma(x) = \sum_{y \in Y} \partial(y)$. To obtain G' from G, we must add $\sigma(x)$ edges with tails at $x \in X$ and $\partial(y)$ edges with heads at $y \in Y$. Since G' needs net outdegree 0 at each vertex, the additions form paths from X to Y. The cost $c(xy)$ of an x, y-path is the distance from x to y in the weighted digraph H, which can be found by Dijkstra's Algorithm.

This yields the **Transportation Problem**. Given supply $\sigma(x)$ for $x \in X$, demand $\partial(y)$ for $y \in Y$, cost $c(xy)$ per unit sent from x to y, and $\sum \sigma(x) = \sum \partial(y)$, we want to satisfy the demands at least total cost. A version of the problem was introduced by Kantorovich [1939]; the form above arose (with a constructive solution) in Hitchcock [1941] (see also Koopmans [1947]). The problem is discussed at length in Ford–Fulkerson [1962, p93–130].

When the supplies and demands are rational, the Assignment Problem can be applied. First scale up to obtain integer supplies and demands. Next define a matrix with $\sum \sigma(x)$ rows and columns. For each $x \in X$, create $\sigma(x)$ rows. For each $y \in Y$, create $\delta(y)$ columns. When row i and column j represent x and y, let $w_{i,j} = M - c(xy)$, where $M = \max_{x,y} c(xy)$. A maximum weight matching now yields a minimum cost solution to the Transportation Problem. A generalization of the Transportation Problem appears in Section 4.3. ■

STABLE MATCHINGS (optional)

Instead of optimizing total weight for a matching, we may try to optimize using preferences. Given n men and n women; we want to establish n "stable" marriages. If man x and woman a are paired with other partners, but x prefers a to his current partner and a prefers x to her current partner, then they might leave their current partners and switch to each other. In this situation we say that the unmatched pair (x, a) is an **unstable pair**.

3.2.15. Definition. A perfect matching is a **stable matching** if it yields no unstable unmatched pair.

3.2.16. Example. Given men x, y, z, w, women a, b, c, d, and preferences listed below, the matching $\{xa, yb, zd, wc\}$ is a stable matching. ■

Men $\{x, y, z, w\}$	Women $\{a, b, c, d\}$
$x : a > b > c > d$	$a : z > x > y > w$
$y : a > c > b > d$	$b : y > w > x > z$
$z : c > d > a > b$	$c : w > x > y > z$
$w : c > b > a > d$	$d : x > y > z > w$

In their paper "College admissions and the stability of marriage", Gale and Shapley proved that a stable matching always exists and can be found using a relatively simple algorithm. In the algorithm, men and women do not play symmetric roles; we will discuss this importance of this difference later. The algorithm below generates the matching of Example 3.2.16.

3.2.17. Algorithm. (Gale–Shapley Proposal Algorithm)
Input: Preference rankings by each of n men and n women.
Idea: Produce a stable matching using proposals by maintaining information about who has proposed to whom and who has rejected whom.
Iteration: Each man proposes to the highest woman on his preference list who has not previously rejected him. If each woman receives exactly one proposal, stop and use the resulting matching. Otherwise, every woman receiving more than one proposal rejects all of them except the one that is highest on her preference list. Every woman receiving a proposal says "maybe" to the most attractive proposal received. ■

3.2.18. Theorem. (Gale–Shapley [1962]) The Proposal Algorithm produces a stable matching.

Proof: The algorithm terminates (with some matching), because on each nonterminal iteration, the total length of the lists of potential mates for the men decreases. This can happen only n^2 times.

Key Observation: the sequence of proposals made by each man is nonincreasing in his preference list, and the sequence of men to whom a woman says "maybe" is nondecreasing in her preference list, culminating in the man assigned. This holds because men propose repeatedly to the same woman until rejected, and women say "maybe" to the same man until a better offer arrives.

If the result is not stable, then there is an unstable unmatched pair (x, a), with x matched to b and y matched to a. By the key observation, x never proposed to a during the algorithm, since a received a mate less desirable than x. The key observation also implies that x would not have proposed to b without earlier proposing to a. This contradiction confirms the stability of the result. ■

The asymmetry of the proposal algorithm suggests asking which sex is happier. When the first choices of the men are distinct, they all get their first choice, and the women are stuck with whomever proposed. When the algorithm runs with women proposing, every woman is at least as happy as when men do the proposing, and every man is at least as unhappy. In Example 3.2.16, running the algorithm with women proposing immediately yields the matching $\{xd, yb, ca, wc\}$, in which all women are matched to their first choices. In fact, among all stable matchings, every man is happiest in the one produced by the male-proposal algorithm, and every woman is happiest under the female-proposal algorithm (Exercise 11). Societal conventions thus favor men.

The algorithm is used in another setting. Each year, the graduates of medical schools submit preference lists of hospitals where they wish to be residents. The hospitals have their own preferences; we model a hospital with multiple openings as several hospitals with the same preference list. Chaos in the market for residents (then called interns) forced hospitals to devise and implement the algorithm ten years before the Gale–Shapley paper defined and solved the problem! The result was the National Resident Matching Program, a non-profit corporation established in 1952 to provide a uniform appointment date and matching procedure.

Who is happier with the outcome? Since the medical organizations ran the algorithm, it is not surprising that initially they did the proposing and were happier with the outcome. The distinction is even clearer in another setting; students applying for jobs have preferences, but the employers make the proposals, called "job offers". Unhappiness with the NRMP caused the system to be changed in 1998 to a student-proposing algorithm. In 1998 the system processed 35,823 applicants for 22,451 positions. Additional details about the system can be found at nrmp.aamc.org/nrmp/mainguid/ on the World Wide Web.

There may be stable matchings other than those found by the two versions of the proposal algorithm. To seek a "fair" stable matching, we could give each person a number of points with which to rate preferences. The weight for the pair xa is then the sum of the points that x gives to a and a gives to x. The Hungarian Algorithm would yield a matching of maximum total weight, but this might not be a stable matching (Exercise 10). Other approaches appear in the books Knuth [1976] and Gusfield–Irving [1989], which discuss stable marriages and related topics.

FASTER BIPARTITE MATCHING (optional)

We began this section with an algorithm for finding maximum matchings in bipartite graphs. The running time can be improved by seeking augmenting paths in a clever order; when short augmenting paths are available, we needn't explore many edges to find one. Using a Breadth-First Search simultaneously from all the unsaturated vertices of X, we can find many paths of the same length with one examination of the edge set. Hopcroft and Karp [1973] proved that subsequent augmentations must use longer paths, so the searches can be grouped in phases finding paths of the same lengths. They combined these ideas to show that few phases are needed, enabling maximum matchings in n-vertex bipartite graphs to be found in $O(n^{2.5})$ time.

3.2.19. Remark. If M is a matching of size r and M^* is a matching of size $s > r$, then there exist at least $s - r$ vertex-disjoint M-augmenting paths. At least this many such paths can be found in $M \triangle M^*$. ■

The next lemma implies that the sequence of path lengths in successive shortest augmentations is nondecreasing. Here we treat paths as sets of edges, and cardinality indicates number of edges.

3.2.20. Lemma. If P is a shortest M-augmenting path and P' is $M \triangle P$-augmenting, then $|P'| \geq |P| + 2|P \cap P'|$ (treating P as an edge set).

Proof: Note that $M \triangle P$ is the matching obtained by using P to augment M. Let N be the matching $(M \triangle P) \triangle P'$ obtained by using P' to augment $M \triangle P$. Since $|N| = |M| + 2$, Remark 3.2.19 guarantees that $M \triangle N$ contains two disjoint M-augmenting paths P_1 and P_2. Each of these is at least as long as P, since P is a shortest M-augmentating path.

Since N is obtained from M by switching the edges in P and then switching the edges in P', an edge belongs to exactly one of M and N if and only if it belongs to exactly one of P and P'. Therefore, $M \triangle N = P \triangle P'$. This yields $|P \triangle P'| \geq |P_1| + |P_2| \geq 2|P|$. Thus

$$2|P| \leq |P \triangle P'| = |P| + |P'| - 2|P \cap P'|.$$

We conclude that $|P'| \geq |P| + 2|P \cap P'|$. ■

3.2.21. Lemma. If $P_1, P_2, \cdots$ is a list of successive shortest augmentations, then the augmentations of the same length are vertex-disjoint paths.

Proof: We use the method of contradiction. Let P_k, P_l with $l > k$ be a closest pair in the list that have the same size but are not vertex-disjoint. By Lemma 3.2.20, the lengths of successive shortest augmenting paths are nondecreasing, so $P_k, \ldots, P_l$ all have the same length. Since P_k, P_l is a closest intersecting pair with the same length, the paths $P_{k+1}, \ldots, P_l$ are pairwise disjoint.

Let M' be the matching given by the augmentations $P_1, \ldots, P_k$. Since $P_{k+1}, \ldots, P_l$ are pairwise disjoint, P_l is an M'-augmenting path. By Lemma 3.2.20, $|P_l| \geq |P_k| + |P_l \cap P_k|$. Since $|P_l| = |P_k|$, there is no common edge.

On the other hand, there must be a common edge. Each vertex of P_k is saturated in M' using an edge of P_k, and every vertex of an M'-augmenting path P_l that is saturated in M' (such as a vertex common to P_l and P_k) must contribute its saturated edge to P_l.

The contradiction implies that there is no such pair P_k, P_l. ■

3.2.22. Theorem. (Hopcroft–Karp [1973]) The breadth-first phased maximum matching algorithm runs in $O(\sqrt{n}m)$ time on bipartite graphs with n vertices and m edges.

Proof: By Lemmas 3.2.20–3.2.21, searching simultaneously from all unsaturated vertices of X for shortest augmentations yields vertex-disjoint paths, after which all other augmenting paths are longer. Hence the augmentations of each length are found in one examination of the edge set, running in time $O(m)$. It suffices to prove that there are at most $2\left\lfloor\sqrt{n/2}\right\rfloor + 2$ phases.

List the augmenting paths as $P_1, \dots, P_s$ in order by length, with $s = \alpha'(G) \le n/2$. Since paths of the same length are vertex-disjoint, each P_{i+1} is an augmenting path for the matching M_i formed by using $P_1, \dots, P_i$. It suffices to prove the more general statement that whenever $P_1, \dots, P_s$ are successive shortest augmenting paths that build a maximum matching, the number of distinct lengths among these paths is at most $2\left\lfloor\sqrt{s}\right\rfloor + 2$.

Let $r = \left\lfloor s - \sqrt{s}\right\rfloor$. Because $|M_r| = r$ and the maximum matching has size s, Remark 3.2.19 yields at least $s - r$ vertex-disjoint M_r-augmenting paths. The shortest of these paths uses at most $\lfloor r/(s-r)\rfloor$ edges from M_r. Hence $|P_{r+1}| \le 2\lfloor r/(s-r)\rfloor + 1$. Since $\lfloor r/(s-r)\rfloor < \left\lfloor s/\left\lceil\sqrt{s}\right\rceil\right\rfloor \le \left\lfloor\sqrt{s}\right\rfloor$, the paths up to P_r provide all but the last $\left\lceil\sqrt{s}\right\rceil$ augmentations using length at most $2\left\lfloor\sqrt{s}\right\rfloor + 1$. There are at most $\left\lfloor\sqrt{s}\right\rfloor + 1$ distinct odd integers up to this value, and even if the last $\left\lceil\sqrt{s}\right\rceil$ paths have distinct lengths, they provide at most $\left\lfloor\sqrt{s}\right\rfloor + 1$ additional lengths, so altogether we use at most $2\left\lfloor\sqrt{s}\right\rfloor + 2$ distinct lengths. ■

Even and Tarjan [1975] extended this to solve in time $O(\sqrt{n}m)$ a more general problem that includes maximum bipartite matching.

EXERCISES

3.2.1. (−) Using nonnegative edge weights, construct a 4-vertex weighted graph in which the matching of maximum weight is not a matching of maximum size.

3.2.2. (−) Show how to use the Hungarian Algorithm to test for the existence of a perfect matching in a bipartite graph.

3.2.3. (∗−) Give an example of the stable matching problem with two men and two women in which there is more than one stable matching.

3.2.4. (∗−) Determine the stable matchings resulting from the Proposal Algorithm run with men proposing and with women proposing, given the preference lists below.

Men $\{u, v, w, x, y, z\}$	Women $\{a, b, c, d, e, f\}$
$u: a > b > d > c > f > e$	$a: z > x > y > u > v > w$
$v: a > b > c > f > e > d$	$b: y > z > w > x > v > u$
$w: c > b > d > a > f > e$	$c: v > x > w > y > u > z$
$x: c > a > d > b > e > f$	$d: w > y > u > x > z > v$
$y: c > d > a > b > f > e$	$e: u > v > x > w > y > z$
$z: d > e > f > c > b > a$	$f: u > w > x > v > z > y$

• • • • •

3.2.5. Find a transversal of maximum total sum (weight) in each matrix below. Prove that there is no larger weight transversal by exhibiting a solution to the dual problem. Explain why this proves that there is no larger transversal.

(a)	(b)	(c)
4 4 4 3 6	7 8 9 8 7	1 2 3 4 5
1 1 4 3 4	8 7 6 7 6	6 7 8 7 2
1 4 5 3 5	9 6 5 4 6	1 3 4 4 5
5 6 4 7 9	8 5 7 6 4	3 6 2 8 7
5 3 6 8 3	7 6 5 5 5	4 1 3 5 4

3.2.6. Find a minimum-weight transversal in the matrix below, and use duality to prove that the solution is optimal. (Hint: Use a transformation of the problem.)

$$\begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix}$$

3.2.7. *The Bus Driver Problem.* Let there be n bus drivers, n morning routes with durations $x_1, \ldots, x_n$, and n afternoon routes with durations $y_1, \ldots, y_n$. A driver is paid overtime when the morning route and afternoon route exceed total time t. The objective is to assign one morning run and one afternoon run to each driver to minimize the total amount of overtime. Express this as a weighted matching problem. Prove that giving the ith longest morning route and ith shortest afternoon route to the same driver, for each i, yields an optimal solution. (Hint: Do not use the Hungarian Algorithm; consider the special structure of the matrix.) (R.B. Potts)

3.2.8. Let the entries in matrix A have the form $w_{i,j} = a_i b_j$, where $a_1, \ldots, a_n$ are numbers associated with the rows and $b_1, \ldots, b_n$ are numbers associated with the columns. Determine the maximum weight of a transversal of A. What happens when $w_{i,j} = a_i + b_j$? (Hint: In each case, guess the general pattern by examining the solution when $n = 2$.)

3.2.9. (∗) A mathematics department offers k seminars in different topics to its n students. Each student will take one seminar; the ith seminar will have k_i students, where $\sum k_i = n$. Each student submits a preference list ranking the k seminars. An assignment of the students to seminars is *stable* if no two students can both obtain more preferable seminars by switching their assignments. Show how to find a stable assignment using weighted bipartite matching. (Isaak)

3.2.10. (∗) Consider n men and n women, each assigning $n - i$ points to the ith person in his or her preference list. Let the weight of a pair be the sum of the points assigned by those two people. Construct an example where no maximum weight matching is a stable matching.

3.2.11. (∗!) Prove that if man x is paired with woman a in some stable matching, then a does not reject x in the Gale–Shapley Proposal Algorithm with men proposing. Conclude that among all stable matchings, *every* man is happiest in the matching produced by this algorithm. (Hint: Consider the first occurrence of such a rejection.)

3.2.12. (∗) In the Stable Roommates Problem, each of $2n$ people has a preference ordering on the other $2n - 1$ people. A stable matching is a perfect matching such that no unmatched pair prefers each other to their current roommates. Prove that there is no stable matching when the preferences are those below. (Gale–Shapley [1962])

$$\begin{aligned} a &: b > c > d \\ b &: c > a > d \\ c &: a > b > d \\ d &: a > b > c \end{aligned}$$

3.2.13. (∗) In the stable roommates problem, suppose that each individual declares a top portion of the preference list as "acceptable". Define the *acceptability graph* to be the graph whose vertices are the people and whose edges are the pairs of people who rank each other as acceptable. Prove that all sets of rankings with acceptability graph G lead to a stable matching if and only if G is bipartite. (Abeledo–Isaak [1991]).

3.3. Matchings in General Graphs

When discussing perfect matchings in graphs, it is natural to consider more general spanning subgraphs.

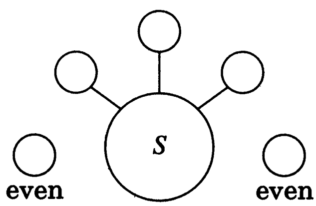
3.3.1. Definition. A **factor** of a graph G is a spanning subgraph of G. A **k-factor** is a spanning k-regular subgraph. An **odd component** of a graph is a component of odd order; the number of odd components of H is $o(H)$.

3.3.2. Remark. A 1-factor and a perfect matching are almost the same thing. The precise distinction is that "1-factor" is a spanning 1-regular subgraph of G, while "perfect matching" is the set of edges in such a subgraph.

A 3-regular graph that has a perfect matching decomposes into a 1-factor and a 2-factor. ∎

TUTTE'S 1-FACTOR THEOREM

Tutte found a necessary and sufficient condition for which graphs have 1-factors. If G has a 1-factor and we consider a set $S \subseteq V(G)$, then every odd component of $G - S$ has a vertex matched to something outside it, which can only belong to S. Since these vertices of S must be distinct, $o(G - S) \le |S|$.



The condition "For all $S \subseteq V(G)$, $o(G - S) \le |S|$" is **Tutte's Condition**. Tutte proved that this obvious necessary condition is also sufficient (TONCAS).

Many proofs are known, such as Exercise 13 and Exercise 27. We present the proof by Lovász using the ideas of symmetric difference and extremality.

3.3.3. Theorem. (Tutte [1947]) A graph G has a 1-factor if and only if $o(G - S) \le |S|$ for every $S \subseteq V(G)$.

Proof: (Lovász [1975]). *Necessity.* The odd components of $G - S$ must have vertices matched to distinct vertices of S.

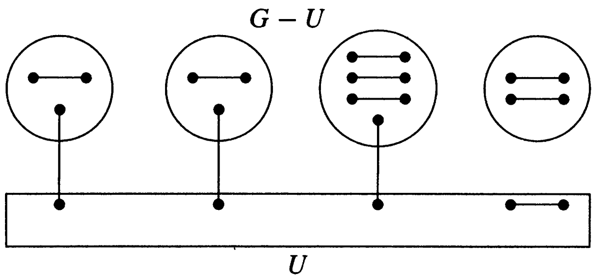
Sufficiency. When we add an edge joining two components of $G - S$, the number of odd components does not increase (odd and even together become one odd component, two components of the same parity become one even component). Hence Tutte's Condition is preserved by addition of edges: if $G' = G + e$ and $S \subseteq V(G)$, then $o(G' - S) \le o(G - S) \le |S|$. Also, if $G' = G + e$ has no 1-factor, then G has no 1-factor.

Therefore, the theorem holds unless there exists a simple graph G such that G satisfies Tutte's Condition, G has no 1-factor, and adding any missing edge to G yields a graph with a 1-factor. Let G be such a graph. We obtain a contradiction by showing that G actually does contain a 1-factor.

Let U be the set of vertices in G that have degree $n(G) - 1$.

Case 1: $G - U$ consists of disjoint complete graphs. In this case, the vertices in each component of $G - U$ can be paired in any way, with one extra in the odd components. Since $o(G - U) \le |U|$ and each vertex of U is adjacent to all of $G - U$, we can match the leftover vertices to vertices of U.

The remaining vertices are in U, which is a clique. To complete the 1-factor, we need only show that an even number of vertices remain in U. We have matched an even number, so it suffices to show that $n(G)$ is even. This follows by invoking Tutte's Condition for $S = \varnothing$, since a graph of odd order would have a component of odd order.

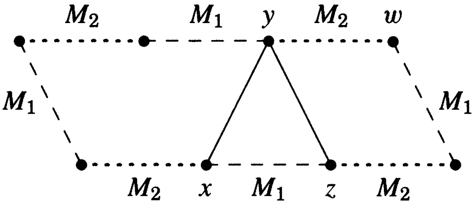


Case 2: $G - U$ is not a disjoint union of cliques. In this case, $G - U$ has two vertices at distance 2; these are nonadjacent vertices x, z with a common neighbor $y \notin U$ (Exercise 1.2.23a). Furthermore, $G - U$ has another vertex w not adjacent to y, since $y \notin U$. By the choice of G, adding an edge to G creates a 1-factor; let M_1 and M_2 be 1-factors in $G + xz$ and $G + yw$, respectively. It suffices to show that $M_1 \triangle M_2$ contains a 1-factor avoiding xz and yw, because this will be a 1-factor in G.

Let $F = M_1 \triangle M_2$. Since $xz \in M_1 - M_2$ and $yw \in M_2 - M_1$, both xz and yw are in F. Since every vertex of G has degree 1 in each of M_1 and M_2, every vertex of G has degree 0 or 2 in F. Hence the components of F are even cycles and isolated vertices (see Lemma 3.1.9). Let C be the cycle of F containing xz.

If C does not also contain yw, then the desired 1-factor consists of the edges of M_2 from C and all of M_1 not in C.

If C contains both yw and xz, as shown below, then to avoid them we use yx or yz. In the portion of C starting from y along yw, we use edges of M_1 to avoid using yw. When we reach $\{x, z\}$, we use zy if we arrive at z (as shown); otherwise, we use xy. In the remainder of C we use the edges of M_2. We have produced a 1-factor of C that does not use xz or yw. Combined with M_1 or M_2 outside C, we have a 1-factor of G. ■

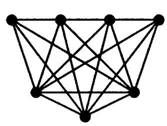


3.3.4. Remark. Like other characterization theorems (such as Theorem 1.2.18 and Theorem 3.1.11)), Theorem 3.3.3 yields short verifications both when the property holds *and* when it doesn't. We prove that G has a 1-factor exists by exhibiting one. When it doesn't exist, Theorem 3.3.3 guarantees that we can exhibit a set whose deletion leaves too many odd components. ■

3.3.5. Remark. For a graph G and any $S \subseteq V(G)$, counting the vertices modulo 2 shows that $|S| + o(G-S)$ has the same parity as $n(G)$. Thus also the difference $o(G - S) - |S|$ has the same parity as $n(G)$. We conclude that if $n(G)$ is even and G has no 1-factor, then $o(G - S)$ exceeds $|S|$ by at least 2 for some S. ■

For non-bipartite graphs (such as odd cycles), there may be a gap between $\alpha'(G)$ and $\beta(G)$ (see also Exercise 10). Nevertheless, another minimization problem yields a min-max relation for $\alpha'(T)$ in general graphs. This min-max relation generalizes Remark 3.3.5. The proof uses a graph transformation that involves a general graph operation.

3.3.6. Definition. The **join** of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$.

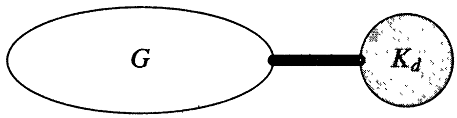
$P_4 \vee K_3$ 

3.3.7. Corollary. (Berge–Tutte Formula—Berge [1958]) The largest number of vertices saturated by a matching in G is $\min_{S \subseteq V(G)}\{n(G) - d(S)\}$, where $d(S) = o(G - S) - |S|$.

Proof: Given $S \subseteq V(G)$, at most $|S|$ edges can match vertices of S to vertices in odd components of $G - S$, so every matching has at least $o(G - S) - |S|$ unsaturated vertices. We want to achieve this bound.

Let $d = \max\{o(G - S) - |S| : S \subseteq V(G)\}$. The case $S = \emptyset$ yields $d \geq 0$. Let $G' = G \vee K_d$. Since $d(S)$ has the same parity as $n(G)$ for each S, we know that $n(G')$ is even. If G' satisfies Tutte's Condition, then we obtain a matching of the desired size in G from a perfect matching in G', because deleting the d added vertices eliminates edges that saturate at most d vertices of G.

The condition $o(G' - S') \leq |S'|$ holds for $S' = \emptyset$ because $n(G')$ is even. If S' is nonempty but does not contain all of K_d, then $G' - S'$ has only one component, and $1 \leq |S'|$. Finally, when $K_d \subseteq S'$, we let $S = S' - V(K_d)$. We have $G' - S' = G - S$, so $o(G' - S') = o(G - S) \leq |S| + d = |S'|$. We have verified that G' satisfies Tutte's Condition. ■



Corollary 3.3.7 guarantees that there is a short PROOF that a maximum matching indeed has maximum size by exhibiting a vertex set S whose deletion leaves the appropriate number of odd components.

Most applications of Tutte's Theorem involve showing that some other condition implies Tutte's Condition and hence guarantees a 1-factor. Some were proved by other means long before Tutte's Theorem was available.

3.3.8. Corollary. (Petersen [1891]) Every 3-regular graph with no cut-edge has a 1-factor.

Proof: Let G be a 3-regular graph with no cut-edge. We prove that G satisfies Tutte's Condition. Given $S \subseteq V(G)$, we count the edges between S and the odd components of $G - S$. Since G is 3-regular, each vertex of S is incident to at most three such edges. If each odd component H of $G - S$ is incident to at least three such edges, then $3o(G - S) \leq 3|S|$ and hence $o(G - S) \leq |S|$, as desired.

Let m be the number of edges from S to H. The sum of the vertex degrees in H is $3n(H) - m$. Since H is a graph, the sum of its vertex degrees must be even. Since $n(H)$ is odd, we conclude that m must also be odd. Since G has no cut-edge, m cannot equal 1. We conclude that there are at least three edges from S to H, as desired. ■

Proof by contradiction would also be natural here. Assuming $o(G - S) > |S|$ also leads to $o(G - S) \leq |S|$, so we rewrite the proof directly. Corollary 3.3.8 is best possible; the Petersen graph satisfies the hypothesis but does not have *two* edge-disjoint 1-factors (Petersen [1898]).

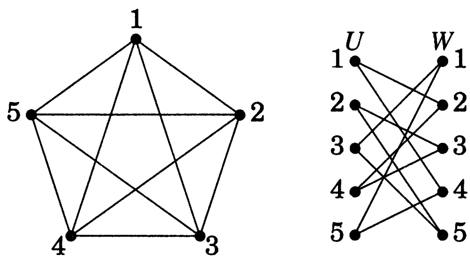
Petersen also proved a sufficient condition for 2-factors. A connected graph with even vertex degrees is Eulerian (Theorem 1.2.26) and decomposes into edge-disjoint cycles (Proposition 1.2.27). For regular graphs of even degree, the cycles in some decomposition can be grouped to form 2-factors.

3.3.9. Theorem. (Petersen [1891]) Every regular graph of even degree has a 2-factor.

Proof: Let G be a $2k$-regular graph with vertices $v_1, \ldots, v_n$. Every component of G is Eulerian, with some Eulerian circuit C. For each component, define a bipartite graph H with vertices $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ by putting $u_i \leftrightarrow w_j$ if v_j immediately follows v_i somewhere on C. Because C enters and exits each vertex k times, H is k-regular. (Actually, H is the split of the digraph obtained by orienting G in accordance with C—see Definition 1.4.20.)

Being a regular bipartite graph, H has a 1-factor M (Corollary 3.1.13). The edge incident to w_i in H corresponds to an edge entering v_i in C. The edge incident to u_i in H corresponds to an edge exiting v_i. Thus the 1-factor in H transforms into a 2-regular spanning subgraph of this component of G. Doing this for each component of G yields a 2-factor of G. ■

3.3.10. Example. *Construction of a 2-factor.* Consider the Eulerian circuit in $G = K_5$ that successively visits 1231425435. The corresponding bipartite graph H is on the right. For the 1-factor whose u, w-pairs are 12, 43, 25, 31, 54, the resulting 2-factor is the cycle $(1, 2, 5, 4, 3)$. The remaining edges form another 1-factor, which corresponds to the 2-factor $(1, 4, 2, 3, 5)$ that remains in G. ■



f-FACTORS OF GRAPHS (optional)

A factor is a spanning subgraph of G; we ask about existence of factors of special types. A k-factor is a k-regular factor; we have studied 1-factors and 2-factors. We can try to specify the degree at each vertex.

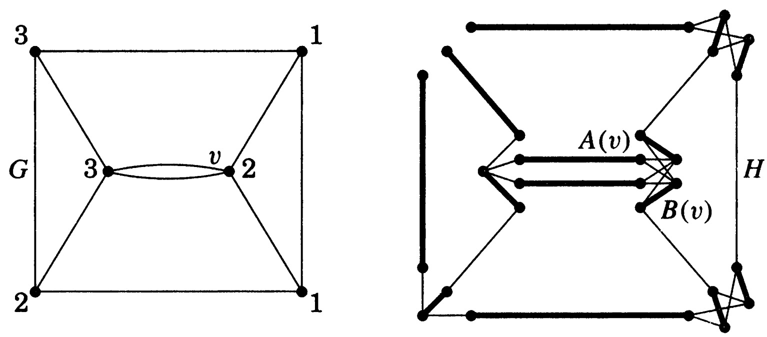
3.3.11. Definition. Given a function $f\colon V(G) \to \mathbb{N} \cup \{0\}$, an **$f$-factor** of a graph G is a subgraph H such that $d_H(v) = f(v)$ for all $v \in V(G)$.

Tutte [1952] proved a necessary and sufficient condition for a graph G to have an f-factor (see Exercise 29). He later reduced the problem to checking for a 1-factor in a related simple graph. We describe this reduction; it is a beautiful example of transforming a graph problem into a previously solved problem.

3.3.12. Example. *A graph transformation* (Tutte [1954a]). We assume that $f(w) \le d(w)$ for all w; otherwise G has too few edges at w to have an f-factor. We then construct a graph H that has a 1-factor if and only if G has an f-factor. Let $e(w) = d(w) - f(w)$; this is the *excess degree* at w and is nonnegative.

To construct H, replace each vertex v with a biclique $K_{d(v),e(v)}$ having partite sets $A(v)$ of size $d(v)$ and $B(v)$ of size $e(v)$. For each $vw \in E(G)$, add an edge joining one vertex of $A(v)$ to one vertex of $A(w)$. Each vertex of $A(v)$ participates in one such edge.

The figure below shows a graph G, vertex labels given by f, and the resulting simple graph H. The bold edges in H form a 1-factor that corresponds to an f-factor of G. In this example, the f-factor is not unique. ■



3.3.13. Theorem. A graph G has an f-factor if and only if the graph H constructed from G and f as in Example 3.3.12 has a 1-factor.

Proof: *Necessity.* If G has an f-factor, then the corresponding edges in H leave $e(v)$ vertices of $A(v)$ unmatched; match them arbitrarily to the vertices of $B(v)$ to obtain a 1-factor of H.

Sufficiency. From a 1-factor of H, deleting $B(v)$ and the vertices of $A(v)$ matched into $B(v)$ leaves $f(v)$ edges at v. Doing this for each v and merging the remaining $f(v)$ vertices of each $A(v)$ yields a subgraph of G with degree $f(v)$ at v. It is an f-factor of G. ■

Tutte's Condition for a 1-factor in the derived graph H of Example 3.3.12 transforms into a necessary and sufficient condition for an f-factor in G. Among the applications is a proof of the Erdős–Gallai [1960] characterization of degree sequences of simple graphs (Exercise 29).

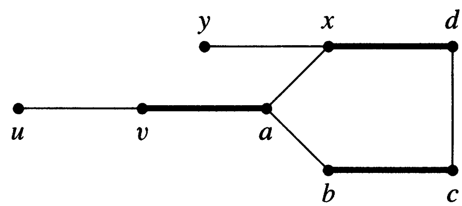
Given an algorithm to find a 1-factor, the correspondence in Theorem 3.3.13 provides an algorithmic test for an f-factor. Instead of just seeking a 1-factor (that is, a perfect matching), we next consider the more general problem of finding a maximum matching in a graph.

EDMONDS' BLOSSOM ALGORITHM (optional)

Berge's Theorem (Theorem 3.1.10) states that a matching M in G has maximum size if and only if G has no M-augmenting path. We can thus find a maximum matching using successive augmenting paths. Since we augment at most $n/2$ times, we obtain a good algorithm if the search for an augmenting path does not take too long. Edmonds [1965a] presented the first such algorithm in his famous paper "Paths, trees, and flowers".

In bipartite graphs, we can search quickly for augmenting paths (Algorithm 3.2.1) because we explore from each vertex at most once. An M-alternating path from u can reach a vertex x in the same partite set as u only along a saturated edge. Hence only once can we reach and explore x. This property fails in graphs with odd cycles, because M-alternating paths from an unsaturated vertex may reach x both along saturated and along unsaturated edges.

3.3.14. Example. In the graph below, with M indicated in bold, a search for shortest M-augmenting paths from u reaches x via the unsaturated edge ax. If we do not also consider a longer path reaching x via a saturated edge, then we miss the augmenting path u, v, a, b, c, d, x, y. ■



We describe Edmonds' solution to this difficulty. If an exploration of M-alternating paths from u reaches a vertex x by an unsaturated edge in one path and by a saturated edge in another path, then x belongs to an odd cycle. Alternating paths from u can diverge only when the next edge is unsaturated (leaving vertex a in Example 3.3.14); when the next edge is saturated there is only one choice for it. From the vertex where the paths diverge, the path reaching x on an unsaturated edge has odd length, and the path reaching it on a saturated edge has even length. Together, they form an odd cycle.

3.3.15. Definition. Let M be a matching in a graph G, and let u be an M-unsaturated vertex. A **flower** is the union of two M-alternating paths from u that reach a vertex x on steps of opposite parity (having not done so earlier). The **stem** of the flower is the maximal common initial path (of nonnegative even length). The **blossom** of the flower is the odd cycle obtained by deleting the stem.

In Example 3.3.14, the flower is the full graph except y, the stem is the path u, v, a, and the blossom is the 5-cycle. The horticultural terminology echoes the use of *tree* for the structures given by most search procedures.

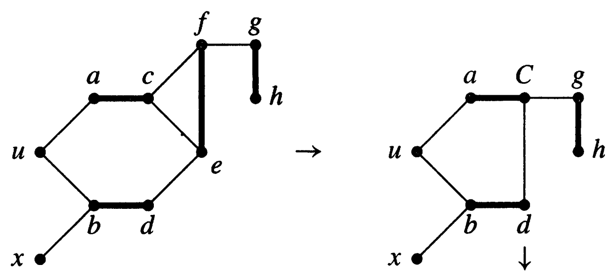
Blossoms do not impede our search. For each vertex z in a blossom, some M-alternating u, z-path reaches z on a saturated edge, found by traveling the proper direction around the blossom to reach z from the stem. We therefore can continue our search along any unsaturated edge from the blossom to a vertex not yet reached. Example 3.3.14 shows such an extension that immediately reaches an unsaturated vertex and completes an M-augmenting path.

Since each vertex of a blossom is saturated by an edge on these paths, no saturated edge emerges from a blossom (except the stem). The effect of these two observations is that we can view the entire blossom as a single "supervertex" reached along the saturated edge at the end of the stem. We search from all vertices of the supervertex blossom simultaneously along unsaturated edges.

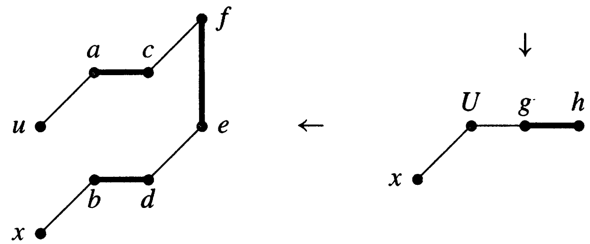
We implement this consolidation by contracting the edges of a blossom B when we find it. The result is a new saturated vertex b incident to the last (saturated) edge of the stem. Its other incident edges are the unsaturated edges joining vertices of B to vertices outside B. We explore from b in the usual way to extend our search. We may later find another blossom containing b; we then contract again. If we find an M-alternating path in the final graph from u to an unsaturated vertex x, then we can undo the contractions to obtain an M-augmenting path to x in the original graph.

Except for the treatment of blossoms, the approach is that of Algorithm 3.2.1 for exploring M-alternating paths. In the corresponding phrasing, T is the set of vertices of the current graph reached along unsaturated edges, and S is the set of vertices reached along saturated edges. The vertices that arise by contracting blossoms belong to S.

3.3.16. Example. Let M be the bold matching in the graph on the left below. We search from the unsaturated vertex u for an M-augmenting path. We first explore the unsaturated edges incident to u, reaching a and b. Since a and b are saturated, we immediately extend the paths along the edges ac and bd. Now $S = \{u, c, d\}$. If we next explore from c, then we find its neighbors e and f along unsaturated edges. Since $ef \in M$, we discover the blossom with vertex set $\{c, e, f\}$. We contract the blossom to obtain the new vertex C, changing S to $\{u, C, d\}$. This yields the graph on the right.



Suppose we now explore from the vertex $C \in S$. Unsaturated edges take us to g and to d. Since g is saturated by the edge gh, we place h in S. Since d is already in S, we have found another blossom. The paths reaching d are u, b, d and u, a, C, d. We contract the blossom, obtaining the new vertex U and the graph on the right below, with $S = \{U, h\}$. We next explore from h, finding nothing new (if we exhaust S without reaching an unsaturated vertex, then there is no M-augmenting path from u). Finally, we explore from U, reaching the unsaturated vertex x.



Having recorded the edge on which we reached each vertex, we can extract an M-augmenting u,x-path. We reached x from U, so we expand the blossom back into $\{u, a, C, d, b\}$ and find that x is reached from U along bx. The path in the blossom U that reaches b on a saturated edge ends with C, d, b. Since C is a blossom in the original graph, we expand C back into $\{c, f, e\}$. Note that d is reached from C by the unsaturated edge ed. The path from the "base" of C that reaches e along a saturated edge is c, f, e. Finally, c was reached from a and a from u, so we obtain the full M-augmenting path u, a, c, f, e, d, b, x. ■

We summarize the steps of the algorithm, glossing over the details of implementation, especially the treatment of contractions.

3.3.17. Algorithm. (Edmonds' Blossom Algorithm [1965a]—sketch).
Input: A graph G, a matching M in G, an M-unsaturated vertex u.
Idea: Explore M-alternating paths from u, recording for each vertex the vertex from which it was reached, and contracting blossoms when found. Maintain sets S and T analogous to those in Algorithm 3.2.1, with S consisting of u and the vertices reached along saturated edges. Reaching an unsaturated vertex yields an augmentation.
Initialization: $S = \{u\}$ and $T = \varnothing$.
Iteration: If S has no unmarked vertex, stop; there is no M-augmenting path from u. Otherwise, select an unmarked $v \in S$. To explore from v, successively consider each $y \in N(v)$ such that $y \notin T$.

If y is unsaturated by M, then trace back from y (expanding blossoms as needed) to report an M-augmenting u,y-path.

If $y \in S$, then a blossom has been found. Suspend the exploration of v and contract the blossom, replacing its vertices in S and T by a single new vertex in S. Continue the search from this vertex in the smaller graph.

Otherwise, y is matched to some w by M. Include y in T (reached from v), and include w in S (reached from y).

After exploring all such neighbors of v, mark v and iterate. ■

We cannot explore all unsaturated vertices simultaneously as in Algorithm 3.2.1, because the membership of vertices in blossoms depends on the choice of the initial unsaturated vertex. Nevertheless, if we find no M-augmenting path from u, then we can delete u from the graph and ignore it in the subsequent search for a maximum matching (Exercise 26).

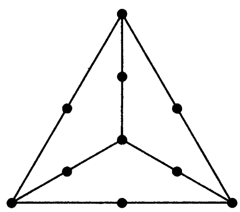
3.3.18. Remark. Edmonds' original algorithm runs in time $O(n^4)$. The implementation in Ahuja–Magnanti–Orlin [1993, p483–494] runs in time $O(n^3)$. This requires (1) appropriate data structures to represent the blossoms and to process contractions, and (2) careful analysis of the number of contractions that can be performed, the time spent exploring edges, and the time spent contracting and expanding blossoms.

The first algorithm solving the maximum matching problem in less than cubic time was the $O(n^{5/2})$ algorithm in Even–Kariv [1975]. The best algorithm now known runs in time $O(n^{1/2}m)$ for a graph with n vertices and m edges (this is faster than $O(n^{5/2})$ for sparse graphs). The algorithm is rather complicated and appears in Micali–Vazirani [1980], with a complete proof in Vazirani [1994].

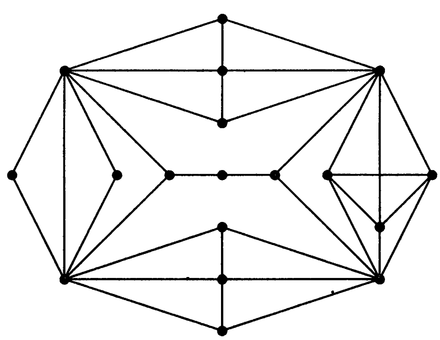
We have not discussed the weighted matching problem for general graphs. Edmonds [1965d] found an algorithm for this, which was implemented in time $O(n^3)$ by Gabow [1975] and by Lawler [1976]. Faster algorithms appear in Gabow [1990] and in Gabow–Tarjan [1989]. ■

EXERCISES

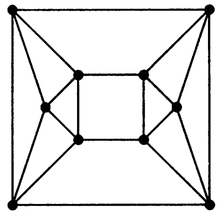
3.3.1. (−) Determine whether the graph below has a 1-factor.



3.3.2. (−) Exhibit a maximum matching in the graph below, and use a result in this section to give a short proof that it has no larger matching.



3.3.3. (−) In the graph drawn below, exhibit a k-factor for each k in $\{0, 1, 2, 3, 4\}$.



3.3.4. (−) Let G be a k-regular bipartite graph. Prove that G can be decomposed into r-factors if and only if r divides k.

3.3.5. (−) Given graphs G and H, determine the number of components and maximum degree of $G \vee H$ in terms of parameters of G and H.

• • • • •

3.3.6. (!) Prove that a tree T has a perfect matching if and only if $o(T - v) = 1$ for every $v \in V(T)$. (Chungphaisan)

3.3.7. (!) For each $k > 1$, construct a k-regular simple graph having no 1-factor.

3.3.8. Prove that if a graph G decomposes into 1-factors, then G has no cut-vertex. Draw a connected 3-regular simple graph that has a 1-factor and has a cut-vertex.

3.3.9. Prove that every graph G has a matching of size at least $n(G)/(1 + \Delta(G))$. (Hint: Apply induction on $e(G)$.) (Weinstein [1963])

3.3.10. (!) For every graph G, prove that $\beta(G) \leq 2\alpha'(G)$. For each $k \in \mathbb{N}$, construct a simple graph G with $\alpha'(G) = k$ and $\beta(G) = 2k$.

3.3.11. Let T be a set of vertices in a graph G. Prove that G has a matching that saturates T if and only if for all $S \subseteq V(G)$, the number of odd components of $G - S$ contained in $G[T]$ is at most $|S|$.

3.3.12. (!) *Extension of König–Egerváry Theorem to general graphs.* Given a graph G, let $S_1, \dots, S_k$ and T be subsets of $V(G)$ such that each S_i has odd size. These sets form a **generalized cover** of G if every edge of G has one endpoint in T or both endpoints in some S_i. The **weight** of a generalized cover is $|T| + \sum \lfloor |S_i| /2 \rfloor$. Let $\beta^*(G)$ be the minimum weight of a generalized cover. Prove that $\alpha'(G) = \beta^*(G)$. (Hint: Apply Corollary 3.3.7. Comment: every vertex cover is a generalized cover, and thus $\beta^*(G) \leq \beta(G)$.)

3.3.13. (+) *Tutte's Theorem from Hall's Theorem.* Let G be a graph such that $o(G - S) \leq |S|$ for all $S \subseteq V(G)$. Let T be a maximal vertex subset such that $o(G - T) = |T|$.

a) Prove that every component of $G - T$ is odd, and conclude that $T \neq \emptyset$.

b) Let C be a component of $G - T$. Prove that Tutte's Condition holds for every subgraph of C obtained by deleting one vertex. (Hint: Since $C - x$ has even order, a violation requires $o(C - x - S) \geq |S| + 2$.)

c) Let H be a bipartite graph with partite sets T and $\mathbf{C}$, where $\mathbf{C}$ is the set of components of $G - T$. For $t \in T$ and $C \in \mathbf{C}$, put $tC \in E(H)$ if and only if $N_G(t)$ contains a vertex of C. Prove that H satisfies Hall's Condition for a matching that saturates $\mathbf{C}$.

d) Use parts (a), (b), (c), and Hall's Theorem to prove Tutte's 1-factor Theorem by induction on $n(G)$. (Anderson [1971], Mader [1973])

3.3.14. For $k \in \mathbb{N}$, let G be a simple graph such that $\delta(G) \geq k$ and $n(G) \geq 2k$. Prove that $\alpha'(G) \geq k$. (Hint: Apply Corollary 3.3.7.) (Brandt [1994])

3.3.15. Let G be a 3-regular graph with at most two cut-edges. Prove that G has a 1-factor. (Petersen [1891])

3.3.16. (!) Let G be a k-regular graph of even order that remains connected when any $k-2$ edges are deleted. Prove that G has a 1-factor.

3.3.17. With G as in Exercise 3.3.16, use Remark 3.3.5 to prove that every edge of G belongs to some 1-factor. (Comment: This strengthens Exercise 3.3.16.) (Schönberger [1934] for $k = 3$, Berge [1973, p162])

3.3.18. (+) For each odd k greater than 1, construct a graph G with no 1-factor that is k-regular and remains connected when any $k-3$ edges are deleted. (Comment: Thus Exercise 3.3.16 is best possible.)

3.3.19. (!) Let G be a 3-regular simple graph with no cut-edge. Prove that G decomposes into copies of P_4. (Hint: Use Theorem 3.3.9.)

3.3.20. (!) Prove that a 3-regular simple graph has a 1-factor if and only if it decomposes into copies of P_4.

3.3.21. (+) Let G be a $2m$-regular graph, and let T be a tree with m edges Prove that if the diameter of T is less than the girth of G, then G decomposes into copies of T. (Hint: Use Theorem 3.3.9 to give an inductive proof of the stronger result that G has such a decomposition in which each vertex is used once as an image of each vertex of T.) (Häggkvist)

3.3.22. (!) Let G be an X, Y-bigraph. Let H be the graph obtained from G by adding one vertex to Y if $n(G)$ is odd and then adding edges to make Y a clique.

a) Prove that G has a matching of size $|X|$ if and only if H has a 1-factor.

b) Prove that if G satisfies Hall's Condition ($|N(S)| \geq |S|$ for all $S \subseteq X$), then H satisfies Tutte's Condition ($o(H-T) \leq |T|$ for all $T \subseteq V(H)$).

c) Use parts (a) and (b) to obtain Hall's Theorem from Tutte's Theorem.

3.3.23. Let G be a claw-free connected graph of even order.

a) Let T be a spanning tree of G generated by Breadth-First Search (Algorithm 2.3.8). Let x and y be vertices that have a common parent in T other than the root. Prove that x and y must be adjacent.

b) Use part (a) to prove that G has a 1-factor. (Comment: Without part (a), one can simply prove the stronger result that the last edge in a longest path belongs to a 1-factor.) (Sumner [1974a], Las Vergnas [1975])

3.3.24. (!) Let G be a simple graph of even order n having a set S of size k such that $o(G-S) > k$. Prove that G has at most $\binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$ edges, and that this bound is best possible. Use this to determine the maximum size of a simple n-vertex graph with no 1-factor. (Erdős–Gallai [1961])

3.3.25. A graph G is **factor-critical** if each subgraph $G-v$ obtained by deleting one vertex has a 1-factor. Prove that G is factor-critical if and only if $n(G)$ is odd and $o(G-S) \leq |S|$ for all nonempty $S \subseteq V(G)$. (Gallai [1963a])

3.3.26. (!) Let M be a matching in a graph G, and let u be an M-unsaturated vertex. Prove that if G has no M-augmenting path that starts at u, then u is unsaturated in some maximum matching in G.

3.3.27. $(*)$ Assuming that Algorithm 3.3.17 is correct, we develop an algorithmic proof of Tutte's Theorem (Theorem 3.3.3).

a) Let G be a graph with no perfect matching, and let M be a maximum matching in G. Let S and T be the sets generated when running Algorithm 3.3.17 from u. Prove that $|T| < |S| \le o(G - T)$.

b) Use part (a) to prove Theorem 3.3.3.

3.3.28. $(*)$ Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given $f\colon V(G) \to \mathbb{N}_0$, the graph G is **f-soluble** if there exists $w\colon E(G) \to \mathbb{N}_0$ such that $\sum_{uv \in E(G)} w(uv) = f(v)$ for every $v \in V(G)$.

a) Prove that G has an f-factor if and only if the graph H obtained from G by subdividing each edge twice and defining f to be 1 on the new vertices is f-soluble. (This reduces testing for an f-factor to testing f-solubility.)

b) Given G and an $f\colon V(G) \to \mathbb{N}_0$, construct a graph H (with proof) such that G is f-soluble if and only if H has a 1-factor. (Tutte [1954a])

3.3.29. $(*+)$ *Tutte's f-factor condition and graphic sequences.* Given $f\colon V(G) \to \mathbb{N}_0$, define $f(S) = \sum_{v \in S} f(v)$ for $S \subseteq V(G)$. For are disjoint subsets S, T of $V(G)$, let $q(S, T)$ denote the number of components Q of $G - S - T$ such that $e(Q, T) + f(V(Q))$ is odd, where $e(Q, T)$ is the number of edges from Q to T. Tutte [1952, 1954a] proved that G has an f-factor if and only if

$$q(S, T) + f(T) - \sum_{v \in T} d_{G-S}(v) \le f(S)$$

for all choices of disjoint subsets $S, T \subset V$.

a) **The Parity Lemma.** Let $\delta(S, T) = f(S) - f(T) + \sum_{v \in T} d_{G-S}(v) - q(S, T)$. Prove that $\delta(S, T)$ has the same parity as $f(V)$ for disjoint sets $S, T \subseteq V(G)$. (Hint: Use induction on $|T|$.)

b) Suppose that $G = K_n$ and $f(v_i) = d_i$, where $\sum d_i$ is even and $d_1 \ge \cdots \ge d_n$. Use the f-factor condition and part (a) to prove that G has an f-factor if and only if $\sum_{i=1}^{k} d_i \le (n - 1 - s)k + \sum_{i=n+1-s}^{n} d_i$ for all k, s with $k + s \le n$.

c) Conclude that $d_1, \ldots, d_n \ge 0$ are the vertex degrees of a simple graph if and only if $\sum d_i$ is even and $\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for $1 \le k \le n$. (Erdős–Gallai [1960])

Chapter 4

Connectivity and Paths

4.1. Cuts and Connectivity

A good communication network is hard to disrupt. We want the graph (or digraph) of possible transmissions to remain connected even when some vertices or edges fail. When communication links are expensive, we want to achieve these goals with few edges. Loops are irrelevant for connection, so in this chapter we assume that our graphs and digraphs **have no loops**, especially when considering degree conditions.

CONNECTIVITY

How many vertices must be deleted to disconnect a graph?

4.1.1. Definition. A **separating set** or **vertex cut** of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The **connectivity** of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is **k-connected** if its connectivity is at least k.

A graph other than a complete graph is k-connected if and only if every separating set has size at least k. We can view "k-connected" as a structural condition, while "connectivity k" is the solution of an optimization problem.

4.1.2. Example. *Connectivity of K_n and $K_{m,n}$.* Because a clique has no separating set, we need to adopt a convention for its connectivity. This explains the phrase "or has only one vertex" in Definition 4.1.1. We obtain $\kappa(K_n) = n - 1$, while $\kappa(G) \le n(G) - 2$ when G is not a complete graph. With this convention, most general results about connectivity remain valid on complete graphs.

Consider a bipartition X, Y of $K_{m,n}$. Every induced subgraph that has at least one vertex from X and from Y is connected. Hence every separating set of $K_{m,n}$ contains X or Y. Since X and Y themselves are separating sets (or leave only one vertex), we have $\kappa(K_{m,n}) = \min\{m, n\}$. The connectivity of $K_{3,3}$ is 3; the graph is 1-connected, 2-connected, and 3-connected, but not 4-connected. ■

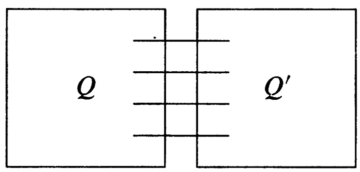
A graph with more than two vertices has connectivity 1 if and only if it is connected and has a cutvertex. A graph with more than one vertex has connectivity 0 if and only if it is disconnected. The 1-vertex graph K_1 is annoyingly inconsistent; it is connected, but for consistency in discussing connectivity we set $\kappa(K_1) = 0$.

4.1.3. Example. *The hypercube Q_k.* For $k \geq 2$, the neighbors of one vertex in Q_k form a separating set, so $\kappa(Q_k) \leq k$. To prove that $\kappa(Q_k) = k$, we show that every vertex cut has size at least k. We use induction on k.

Basis step: $k \in \{0, 1\}$. For $k \leq 1$, Q_k is a clique with $k + 1$ vertices and has connectivity k.

Induction step: $k \geq 2$). By the induction hypothesis, $\kappa(Q_{k-1}) = k - 1$. Consider the description of Q_k as two copies Q and Q' of Q_{k-1} plus a matching that joins corresponding vertices in Q and Q' (Example 1.3.8). Let S be a vertex cut in Q_k. If $Q - S$ is connected and $Q' - S$ is connected, then $Q_k - S$ is also connected unless S contains at least one endpoint of every matched pair. This requires $|S| \geq 2^{k-1}$, but $2^{k-1} \geq k$ for $k \geq 2$.

Hence we may assume that $Q - S$ is disconnected, which means that S has at least $k - 1$ vertices in Q, by the induction hypothesis. If S contains no vertices of Q', then $Q' - S$ is connected and all vertices of $Q - S$ have neighbors in $Q' - S$, so $Q_k - S$ is connected. Hence S must also contain a vertex of Q'. This yields $|S| \geq k$, as desired. ■

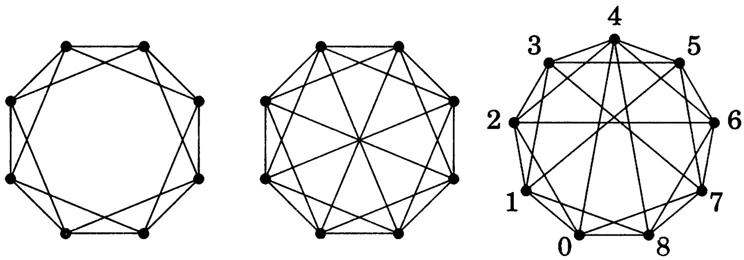


Deleting the neighbors of a vertex disconnects a graph (or leaves only one vertex), so $\kappa(G) \leq \delta(G)$. Equality need not hold; $2K_m$ has minimum degree $m - 1$ but connectivity 0.

Since connectivity k requires $\delta(G) \geq k$, it also requires at least $\lceil kn/2 \rceil$ edges. The k-dimensional cube achieves this bound, but only for the case $n = 2^k$. The bound is best possible whenever $k < n$, as shown by the next example.

4.1.4. Example. *Harary graphs.* Given $k < n$, place n vertices around a circle, equally spaced. If k is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle. If k is odd and n is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $(k-1)/2$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{k,n}$ is k-regular.

When k and n are both odd, index the vertices by the integers modulo n. Construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $i \leftrightarrow i + (n-1)/2$ for $0 \le i \le (n-1)/2$. The graphs $H_{4,8}$, $H_{5,8}$, and $H_{5,9}$ appear below. ■



4.1.5. Theorem. (Harary [1962a]) $\kappa(H_{k,n}) = k$, and hence the minimum number of edges in a k-connected graph on n vertices is $\lceil kn/2 \rceil$.

Proof: We prove only the even case $k = 2r$, leaving the odd case as Exercise 12. Let $G = H_{k,n}$. Since $\delta(G) = k$, it suffices to prove $\kappa(G) \ge k$. For $S \subseteq V(G)$ with $|S| < k$, we prove that $G - S$ is connected. Consider $u, v \in V(G) - S$. The original circular arrangement has a clockwise u,v-path and a counterclockwise u,v-path along the circle; let A and B be the sets of internal vertices on these two paths.

Since $|S| < k$, the pigeonhole principle implies that in one of $\{A, B\}$, S has fewer than $k/2$ vertices. Since in G each vertex has edges to the next $k/2$ vertices in a particular direction, deleting fewer than $k/2$ consecutive vertices cannot block travel in that direction. Thus we can find a u,v-path in $G - S$ via the set A or B in which S has fewer than $k/2$ vertices. ■

Harary's construction determines the degree conditions that *allow* a graph to be k-connected. Exercise 22 determines the degree conditions that *force* a simple graph to be k-connected. Since it depends on vertex deletions, connectivity is not affected by deleting extra copies of multiple edges. Hence we state degree conditions for k-connectedness only in the context of simple graphs.

4.1.6. Remark. A direct proof of $\kappa(G) \ge k$ considers a vertex cut S and proves that $|S| \ge k$, or it considers a set S with fewer than k vertices and proves that $G - S$ is connected. The indirect approach assumes a cut of size less than k and obtains a contradiction. The indirect proof may be easier to find, but the direct proof may be clearer to state.

Note also that if $k < n(G)$ and G has a vertex cut of size less than k, then G has a vertex cut of size $k-1$ (first delete the cut, then continue deleting vertices until $k-1$ are gone, retaining a vertex in each of two components).

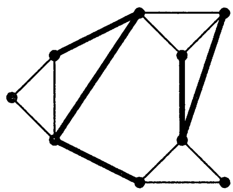
Finally, proving $\kappa(G) = k$ also requires presenting a vertex cut of size k; this is usually the easy part. ■

EDGE-CONNECTIVITY

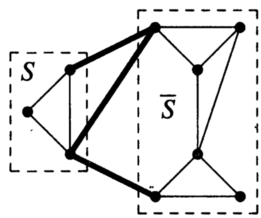
Perhaps our transmitters are secure and never fail, but our communication links are subject to noise or other disruptions. In this situation, we want to make it hard to disconnect our graph by deleting edges.

4.1.7. Definition. A **disconnecting set** of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. A graph is **k-edge-connected** if every disconnecting set has at least k edges. The **edge-connectivity** of G, written $\kappa'(G)$, is the minimum size of a disconnecting set (equivalently, the maximum k such that G is k-edge-connected).

Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one endpoint in S and the other in T. An **edge cut** is an edge set of the form $[S, \overline{S}]$, where S is a nonempty proper subset of $V(G)$ and $\overline{S}$ denotes $V(G) - S$.



disconnecting set



edge cut

4.1.8. Remark. *Disconnecting set vs. edge cut.* Every edge cut is a disconnecting set, since $G - [S, \overline{S}]$ has no path from S to $\overline{S}$. The converse is false, since a disconnecting set can have extra edges. Above we show a disconnecting set and an edge cut in bold; see also Exercise 13.

Nevertheless, *every minimal disconnecting set of edges is an edge cut* (when $n(G) > 1$). If $G - F$ has more than one component for some $F \subseteq E(G)$, then for some component H of $G - F$ we have deleted all edges with exactly one endpoint in H. Hence F contains the edge cut $[V(H), \overline{V(H)}]$, and F is not a minimal disconnecting set unless $F = [V(H), \overline{V(H)}]$. ■

The notation for edge-connectivity continues our convention of using a "prime" for an edge parameter analogous to a vertex parameter. Using the same base letter emphasizes the analogy and avoids the confusion of using many different letters – and running out of them.

Deleting one endpoint of each edge in an edge cut F deletes every edge of F. This suggests that $\kappa(G) \leq \kappa'(G)$. However, we must be careful not to delete the only vertex of a component of $G - F$ and thereby leave a connected subgraph.

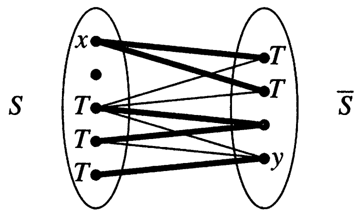
4.1.9. Theorem. (Whitney [1932a]) If G is a simple graph, then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof: The edges incident to a vertex v of minimum degree form an edge cut; hence $\kappa'(G) \leq \delta(G)$. It remains to show that $\kappa(G) \leq \kappa'(G)$.

We have observed that $\kappa(G) \le n(G) - 1$ (see Example 4.1.2). Consider a smallest edge cut $[S, \overline{S}]$. If every vertex of S is adjacent to every vertex of $\overline{S}$, then $|[S, \overline{S}]| = |S|\,|\overline{S}| \ge n(G) - 1 \ge \kappa(G)$, and the desired inequality holds.

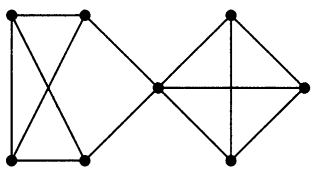
Otherwise, we choose $x \in S$ and $y \in \overline{S}$ with $x \nleftrightarrow y$. Let T consist of all neighbors of x in $\overline{S}$ and all vertices of $S - \{x\}$ with neighbors in $\overline{S}$. Every x,y-path passes through T, so T is a separating set. Also, picking the edges from x to $T \cap \overline{S}$ and one edge from each vertex of $T \cap S$ to $\overline{S}$ (shown bold below) yields $|T|$ distinct edges of $[S, \overline{S}]$. Thus $\kappa'(G) = |[S, \overline{S}]| \ge |T| \ge \kappa(G)$. ■



We have seen that $\kappa(G) = \delta(G)$ when G is a complete graph, a biclique, a hypercube, or a Harary graph. By Theorem 4.1.9, also $\kappa'(G) = \delta(G)$ for these graphs. Nevertheless, in many graphs the set of edges incident to a vertex of minimum degree is not a minimum edge cut. The situation $\kappa'(G) < \delta(G)$ is precisely the situation where no minimum edge cut isolates a vertex.

4.1.10. Example. *Possibility of* $\kappa < \kappa' < \delta$. For graph G below, $\kappa(G) = 1$, $\kappa'(G) = 2$, and $\delta(G) = 3$. Note that no minimum edge cut isolates a vertex.

Each inequality can be arbitarily weak. When $G = K_m + K_m$, we have $\kappa(G) = \kappa'(G) = 0$ but $\delta(G) = m - 1$. When G consists of two m-cliques sharing a single vertex, we have $\kappa'(G) = \delta(G) = m - 1$ but $\kappa(G) = 1$. ■

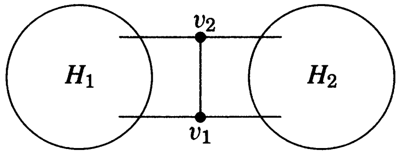


Various conditions force equalities among the parameters; for example, $\kappa'(G) = \delta(G)$ when G has diameter 2 (Exercise 25). For 3-regular graphs, connectivity and edge-connectivity are always equal.

4.1.11. Theorem. If G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$.

Proof: Let S be a minimum vertex cut ($|S| = \kappa(G)$). Since $\kappa(G) \le \kappa'(G)$ always, we need only provide an edge cut of size $|S|$. Let H_1, H_2 be two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2. Since G is 3-regular, v cannot have two neighbors in H_1 and two in H_2. For each $v \in S$, delete the edge from v to a member of $\{H_1, H_2\}$ where v has only one neighbor.

These $\kappa(G)$ edges break all paths from H_1 to H_2 except in the case below, where a path can enter S via v_1 and leave via v_2. In this case we delete the edge to H_1 for both v_1 and v_2 to break all paths from H_1 to H_2 through $\{v_1, v_2\}$. ■



When $\kappa'(G) < \delta(G)$, a minimum edge cut cannot isolate a vertex. In fact, whenever $|[S, \overline{S}]| < \delta(G)$, the set S (and also $\overline{S}$) must be much larger than a single vertex. This follows from a simple relationship between the size of the edge cut $[S, \overline{S}]$ and the size of the subgraph induced by S.

4.1.12. Proposition. If S is a set of vertices in a graph G, then

$$\left|[S, \overline{S}]\right| = [\textstyle\sum_{v \in S} d(v)] - 2e(G[S]).$$

Proof: An edge in $G[S]$ contributes twice to $\sum_{v \in S} d(v)$, while an edge in $[S, \overline{S}]$ contributes only once to the sum. Since this counts all contributions, we obtain $\sum_{v \in S} d(v) = \left|[S, \overline{S}]\right| + 2e(G[S])$. ■

4.1.13. Corollary. If G is a simple graph and $\left|[S, \overline{S}]\right| < \delta(G)$ for some nonempty proper subset S of $V(G)$, then $|S| > \delta(G)$.

Proof: By Proposition 4.1.12, we have $\delta(G) > \sum_{v \in S} d(v) - 2e(G[S])$. Using $d(v) \geq \delta(G)$ and $2e(G[S]) \leq |S|\,(|S| - 1)$ yields

$$\delta(G) > |S|\,\delta(G) - |S|\,(|S| - 1).$$

This inequality requires $|S| > 1$, so we can combine the terms involving $\delta(G)$ and cancel $|S| - 1$ to obtain $|S| > \delta(G)$. ■

As a set of edges, an edge cut may contain another edge cut. For example, $K_{1,2}$ has three edge cuts, but one of them contains the other two. The minimal non-empty edge cuts of a graph have useful structural properties.

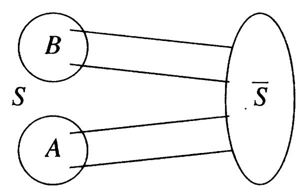
4.1.14. Definition. A **bond** is a minimal nonempty edge cut.

Here "minimal" means that no proper nonempty subset is also an edge cut. We characterize bonds in connected graphs.

4.1.15. Proposition. If G is a connected graph, then an edge cut F is a bond if and only if $G - F$ has exactly two components.

Proof: Let $F = [S, \overline{S}]$ be an edge cut. Suppose first that $G - F$ has exactly two components, and let F' be a proper subset of F. The graph $G - F'$ contains the two components of $G - F$ plus at least one edge between them, making it connected. Hence F is a minimal disconnecting set and is a bond.

For the converse, suppose that $G - F$ has more than two components. Since $G - F$ is the disjoint union of $G[S]$ and $G[\overline{S}]$, one of these has at least two components. Assume by symmetry that it is $G[S]$. We can thus write $S = A \cup B$, where no edges join A and B. Now the edge cuts $[A, \overline{A}]$ and $[B, \overline{B}]$ are proper subsets of F, so F is not a bond. ■

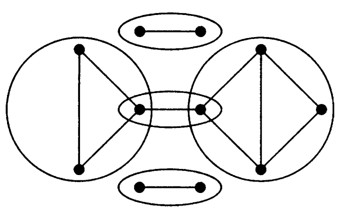


BLOCKS

A connected graph with no cut-vertex need not be 2-connected, since it can be K_1 or K_2. Connected subgraphs without cut-vertices provide a useful decomposition of a graph.

4.1.16. Definition. A **block** of a graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G *is* a block.

4.1.17. Example. *Blocks.* If H is a block of G, then H as a graph has no cut-vertex, but H may contain vertices that are cut-vertices of G. For example, the graph drawn below has five blocks; three copies of K_2, one of K_3, and one subgraph that is neither a cycle nor a clique. ■



4.1.18. Remark. *Properties of blocks.* An edge of a cycle cannot itself be a block, since it is in a larger subgraph with no cut-vertex. Hence an edge is a block if and only if it is a cut-edge; the blocks of a tree are its edges. If a block has more than two vertices, then it is 2-connected. The blocks of a loopless graph are its isolated vertices, its cut-edges, and its maximal 2-connected subgraphs. ■

4.1.19. Proposition. Two blocks in a graph share at most one vertex.

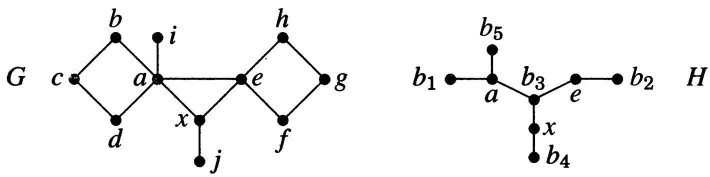
Proof: We use contradiction. Suppose that blocks B_1, B_2 have at least two common vertices. We show that $B_1 \cup B_2$ is a connected subgraph with no cut-vertex, which contradicts the maximality of B_1 and B_2.

When we delete one vertex from B_i, what remains is connected. Hence we retain a path in B_i from every vertex that remains to every vertex of $V(B_1) \cap V(B_2)$ that remains. Since the blocks have at least two common vertices, deleting a single vertex leaves a vertex in the intersection. We retain paths from all vertices to that vertex, so $B_1 \cup B_2$ cannot be disconnected by deleting one vertex. ∎

Every edge by itself is a subgraph with no cut-vertex and hence is in a block. We conclude that the blocks of a graph decompose the graph. Blocks in a graph behave somewhat like strong components of a digraph (Definition 1.4.12), but strong components share no vertices (Exercise 1.4.13a). Thus although blocks in a graph decompose the edge set, strong components in a digraph merely partition the vertex set and usually omit edges.

When two blocks of G share a vertex, it must be a cut-vertex of G. The interaction between blocks and cut-vertices is described by a special graph.

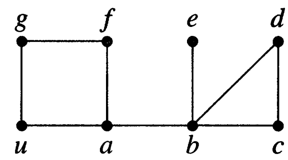
4.1.20.* Definition. The **block-cutpoint graph** of a graph G is a bipartite graph H in which one partite set consists of the cut-vertices of G, and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of H if and only if $v \in B_i$.



When G is connected, its block-cutpoint graph is a tree (Exercise 34) whose leaves are blocks of G. Thus a graph G that is not a single block has at least two blocks (**leaf blocks**) that each contain exactly one cut-vertex of G.

Blocks can be found using a technique for searching graphs. In **Depth-First Search** (DFS), we explore always from the most recently discovered vertex that has unexplored edges (also called **backtracking**). In contrast, Breadth-First Search (Algorithm 2.3.8) explores from the oldest vertex, so the difference between DFS and BFS is that in DFS we maintain the list of vertices to be searched as a Last-In First-Out "stack" rather than a queue.

4.1.21.* Example. *Depth-First Search.* In the graph below, one depth-first search from u finds the vertices in the order u, a, b, c, d, e, f, g. For both BFS and DFS, the order of discovery depends on the order of exploring edges from a searched vertex. ∎



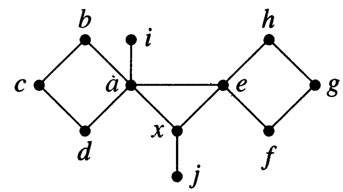
A breath-first or depth-first search from u generates a tree rooted at u; each time exploring a vertex x yields a new vertex v, we include the edge xv. This grows a tree that becomes a spanning tree of the component containing u. Applications of depth-first search rely on a fundamental property of the resulting spanning tree.

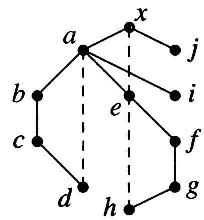
4.1.22.* Lemma. If T is a spanning tree of a connected graph G grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T.

Proof: Let vw be an edge of G, with v encountered before w in the depth-first search. Because vw is an edge, we cannot finish v before w is added to T. Hence w appears somewhere in the subtree formed before finishing v, and the path from w to u contains v. ■

4.1.23.* Algorithm. (Computing the blocks of a graph)
Input: A connected graph G. (The blocks of a graph are the blocks of its components, which can be found by depth-first search, so we may assume that G is connected.)
Idea: Build a depth-first search tree T of G, discarding portions of T as blocks are identified. Maintain one vertex called ACTIVE.
Initialization: Pick a root $x \in V(H)$; make x ACTIVE; set $T = \{x\}$.
Iteration: Let v denote the current active vertex.
1) If v has an unexplored incident edge vw, then
 1A) If $w \notin V(T)$, then add vw to T, mark vw explored, make w ACTIVE.
 1B) If $w \in V(T)$, then w is an ancestor of v; mark vw explored.
2) If v has no more unexplored incident edges, then
 2A) If $v \neq x$, and w is the parent of v, make w ACTIVE. If no vertex in the current subtree T' rooted at v has an explored edge to an ancestor above w, then $V(T') \cup \{w\}$ is the vertex set of a block; record this information and delete $V(T')$ from T.
 2B) If $v = x$, terminate. ■

4.1.24.* Example. *Finding blocks.* For the graph below, one depth-first traversal from x visits the other vertices in the order $a, b, c, d, e, f, g, h, i, j$. We find blocks in the order $\{a, b, c, d\}$, $\{e, f, g, h\}$, $\{a, i\}$, $\{x, a, e\}$, $\{x, j\}$. After finding each block, we delete the vertices other than the highest. Exercise 36 requests a proof of correctness. ■





EXERCISES

4.1.1. (−) Give a proof or a counterexample for each statement below.

a) Every graph with connectivity 4 is 2-connected.
b) Every 3-connected graph has connectivity 3.
c) Every k-connected graph is k-edge-connected.
d) Every k-edge-connected graph is k-connected.

4.1.2. (−) Give a counterexample to the following statement, add a hypothesis to correct it, and prove the corrected statement: If e is a cut-edge of G, then at least one vertex of e is a cut-vertex of G.

4.1.3. (−) Let G be an n-vertex simple graph other than K_n. Prove that if G is not k-connected, then G has a separating set of size $k-1$.

4.1.4. (−) Prove that a graph G is k-connected if and only if $G \vee K_r$ (Definition 3.3.6) is $k+r$-connected.

4.1.5. (−) Let G be a connected graph with at least three vertices. Form G' from G by adding an edge with endpoints x, y whenever $d_G(x, y) = 2$. Prove that G' is 2-connected.

4.1.6. (−) For a graph G with blocks $B_1, \ldots, B_k$, prove that $n(G) = (\sum_{i=1}^{k} n(B_i)) - k + 1$.

4.1.7. (−) Obtain a formula for the number of spanning trees of a connected graph in terms of the numbers of spanning trees of its blocks.

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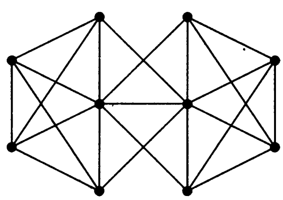
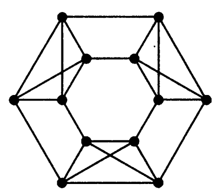
4.1.8. Determine $\kappa(G)$, $\kappa'(G)$, and $\delta(G)$ for each graph G drawn below.





4.1.9. For each choice of integers k, l, m with $0 < k \le l \le m$, construct a simple graph G with $\kappa(G) = k$, $\kappa'(G) = l$, and $\delta(G) = m$. (Chartrand–Harary [1968])

4.1.10. (!) Find (with proof) the smallest 3-regular simple graph having connectivity 1.

4.1.11. Prove that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \le 3$.

4.1.12. Let n, k be positive integers with n even, k odd, and $n > k > 1$. Let G be the k-regular simple graph formed by placing n vertices on a circle and making each vertex adjacent to the opposite vertex and to the $(k-1)/2$ nearest vertices in each direction. Prove that $\kappa(G) = k$. (Harary [1962a])

4.1.13. In $K_{m,n}$, let S consist of a vertices from one partite set and b from the other.

a) Compute $\left|[S, \overline{S}]\right|$ in terms of a, b, m, n.

b) Use part (a) to prove numerically that $\kappa'(K_{m,n}) = \min\{m, n\}$.

c) Prove that every set of seven edges in $K_{3,3}$ is a disconnecting set, but no set of seven edges is an edge cut.

4.1.14. (!) Let G be a connected graph in which for every edge e, there are cycles C_1 and C_2 containing e whose only common edge is e. Prove that G is 3-edge-connected. Use this to show that the Petersen graph is 3-edge-connected.

4.1.15. (!) Use Proposition 4.1.12 and Theorem 4.1.11 to prove that the Petersen graph is 3-connected.

4.1.16. Use Proposition 4.1.12 to prove that the Petersen graph has an edge cut of size m if and only if $3 \le m \le 12$. (Hint: Consider $\left|[S, \overline{S}]\right|$ for $1 \le |S| \le 5$.)

4.1.17. Prove that deleting an edge cut of size 3 in the Petersen graph isolates a vertex.

4.1.18. Let G be a triangle-free graph with minimum degree at least 3. Prove that if $n(G) \le 11$, then G is 3-edge-connected. Show that this inequality is sharp by finding a 3-regular bipartite graph with 12 vertices that is not 3-edge-connected. (Galvin)

4.1.19. Prove that $\kappa(G) = \delta(G)$ if G is simple and $\delta(G) \ge n(G) - 2$. Prove that this is best possible for each $n \ge 4$ by constructing a simple n-vertex graph with minimum degree $n-3$ and connectivity less than $n-3$.

4.1.20. (!) Let G be a simple n-vertex graph with $n/2 - 1 \le \delta(G) \le n-2$. Prove that G is k-connected for all k with $k \le 2\delta(G) + 2 - n$. Prove that this is best possible for all $\delta \ge n/2 - 1$ by constructing a simple n-vertex graph with minimum degree δ that is not k-connected for $k = 2\delta + 3 - n$. (Comment: Proposition 1.3.15 is the special case of this when $\delta(G) = (n-1)/2$.)

4.1.21. (+) Let G be a simple n-vertex graph with $n \ge k + l$ and $\delta(G) \ge \frac{n+l(k-2)}{l+1}$. Prove that if $G - S$ has more than l components, then $|S| \ge k$. Prove that the hypothesis on $\delta(G)$ is best possible for $n \ge k + l$ by constructing an appropriate n-vertex graph with minimum degree $\left\lfloor \frac{n+l(k-2)-1}{l+1} \right\rfloor$. (Comment: This generalizes Exercise 4.1.20.)

4.1.22. (!) *Sufficient condition for $k+1$-connected graphs.* (Bondy [1969])

a) Let G be a simple n-vertex graph with vertex degrees $d_1 \le \cdots \le d_n$. Prove that if $d_j \ge j + k$ whenever $j \le n - 1 - d_{n-k}$, then G is $k+1$-connected. (Comment: Exercise 1.3.64 is the special case of this when $k = 0$.)

b) Suppose that $0 \le j + k \le n$. Construct an n-vertex graph G such that $\kappa(G) \le k$ and G has j vertices of degree $j+k-1$, has $n-j-k$ vertices of degree $n-j-1$, and has k vertices of degree $n-1$. In what sense does this show that part (a) is best possible?

4.1.23. (!) Let G be an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor. (Sumner [1974b])

4.1.24. (!) *Degree conditions for $\kappa' = \delta$.* Let G be a simple n-vertex graph. Use Corollary 4.1.13 to prove the following statements.

a) If $\delta(G) \ge \lfloor n/2 \rfloor$, then $\kappa'(G) = \delta(G)$. Prove this best possible by constructing for each $n \ge 3$ a simple n-vertex graph with $\delta(G) = \lfloor n/2 \rfloor - 1$ and $\kappa'(G) < \delta(G)$.

b) If $d(x) + d(y) \geq n - 1$ whenever $x \not\leftrightarrow y$, then $\kappa'(G) = \delta(G)$. Prove that this is best possible by constructing for each $n \geq 4$ and $\delta(G) = m \leq n/2 - 1$ an n-vertex graph G with $\kappa'(G) < \delta(G) = m$ in which $d(x) + d(y) \geq n - 2$ whenever $x \not\leftrightarrow y$.

4.1.25. (!) $\kappa'(G) = \delta(G)$ *for diameter 2.* Let G be a simple graph with diameter 2, and let $[S, \overline{S}]$ be a minimum edge cut with $|S| \leq |\overline{S}|$.

a) Prove that every vertex of S has a neighbor in $\overline{S}$.

b) Use part (a) and Corollary 4.1.13 to prove that $\kappa'(G) = \delta(G)$. (Plesník [1975])

4.1.26. (!) Let F be a set of edges in G. Prove that F is an edge cut if and only if F contains an even number of edges from every cycle in G. For example, when $G = C_n$, every even subset of the edges is an edge cut, but no odd subset is an edge cut. (Hint: For sufficiency, the task is to show that the components of $G - F$ can be grouped into two nonempty collections so that every edge of F has an endpoint in each collection.)

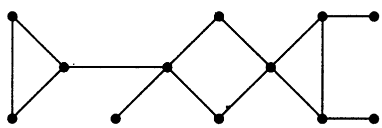
4.1.27. (!) Let $[S, \overline{S}]$ be an edge cut. Prove that there is a set of pairwise edge-disjoint bonds whose union (as edge sets) is $[S, \overline{S}]$. (Note: This is trivial if $[S, \overline{S}]$ is itself a bond.)

4.1.28. (!) Prove that the symmetric difference of two different edge cuts is an edge cut. (Hint: Draw a picture illustrating the two edge cuts and use it to guide the proof.)

4.1.29. (!) Let H be a spanning subgraph of a connected graph G. Prove that H is a spanning tree if and only if the subgraph $H^* = G - E(H)$ is a maximal subgraph that contains no bond. (Comment: See Section 8.2 for a more general context.)

4.1.30. (−) Let G be the simple graph with vertex set $\{1, \dots, 11\}$ defined by $i \leftrightarrow j$ if and only if i, j have a common factor bigger than 1. Determine the blocks of G.

4.1.31. A **cactus** is a connected graph in which every block is an edge or a cycle. Prove that the maximum number of edges in a simple n-vertex cactus is $\lfloor 3(n-1)/2 \rfloor$. (Hint: $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.)



4.1.32. Prove that every vertex of a graph has even degree if and only if every block is Eulerian.

4.1.33. Prove that a connected graph is k-edge-connected if and only if each of its blocks is k-edge-connected.

4.1.34. (!) *The block-cutpoint graph* (see Definition 4.1.20). Let H be the block-cutpoint graph of a graph G that has a cut-vertex. (Harary–Prins [1966])

a) Prove that H is a forest.

b) Prove that G has at least two blocks each of which contains exactly one cut-vertex of G.

c) Prove that a graph G with k components has exactly $k + \sum_{v \in V(G)} (b(v) - 1)$ blocks, where $b(v)$ is the number of blocks containing v.

d) Prove that every graph has fewer cut-vertices than blocks.

4.1.35. Let H and H' be two maximal k-connected subgraphs of a graph G. Prove that they have at most $k - 1$ common vertices. (Harary–Kodama [1964])

4.1.36. Prove that Algorithm 4.1.23 correctly computes blocks of graphs.

4.1.37. Develop an algorithm to compute the strong components of a digraph. Prove that it works. (Hint: Model the algorithm on Algorithm 4.1.23).

4.2. k-Connected Graphs

A communication network is fault-tolerant if it has alternative paths between vertices: the more disjoint paths, the better. In this section, we prove that this alternative measure of connection is essentially the same as k-connectedness. When $k = 1$, the definition already states that a graph G is 1-connected if and only if each pair of vertices is connected by a path. For larger k the equivalence is more subtle.

2-CONNECTED GRAPHS

We begin by characterizing 2-connected graphs.

4.2.1. Definition. Two paths from u to v are **internally disjoint** if they have no common internal vertex.

4.2.2. Theorem. (Whitney [1932a]) A graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G.

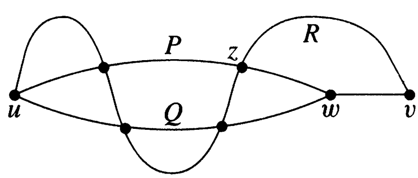
Proof: *Sufficiency.* When G has internally disjoint u, v-paths, deletion of one vertex cannot separate u from v. Since this condition is given for every pair u, v, deletion of one vertex cannot make any vertex unreachable from any other. We conclude that G is 2-connected.

Necessity. Suppose that G is 2-connected. We prove by induction on $d(u, v)$ that G has internally disjoint u, v-paths.

Basis step ($d(u, v) = 1$). When $d(u, v) = 1$, the graph $G - uv$ is connected, since $\kappa'(G) \ge \kappa(G) \ge 2$. A u, v-path in $G - uv$ is internally disjoint in G from the u, v-path formed by the edge uv itself.

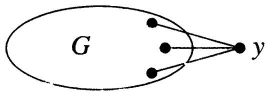
Induction step ($d(u, v) > 1$). Let $k = d(u, v)$. Let w be the vertex before v on a shortest u, v-path; we have $d(u, w) = k - 1$. By the induction hypothesis, G has internally disjoint u, w-paths P and Q. If $v \in V(P) \cup V(Q)$, then we find the desired paths in the cycle $P \cup Q$. Suppose not.

Since G is 2-connected, $G - w$ is connected and contains a u, v-path R. If R avoids P or Q, we are done, but R may share internal vertices with both P and Q. Let z be the last vertex of R (before v) belonging to $P \cup Q$. By symmetry, we may assume that $z \in P$. We combine the u, z-subpath of P with the z, v-subpath of R to obtain a u, v-path internally disjoint from $Q \cup wv$. ∎



4.2.3. Lemma. (Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.

Proof: We prove that a separating set S of G' must have size at least k. If $y \in S$, then $S - \{y\}$ separates G, so $|S| \geq k + 1$. If $y \notin S$ and $N(y) \subseteq S$, then $|S| \geq k$. Otherwise, y and $N(y) - S$ lie in a single component of $G' - S$. Thus again S must separate G and $|S| \geq k$. ■



4.2.4. Theorem. For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs).
A) G is connected and has no cut-vertex.
B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.
C) For all $x, y \in V(G)$, there is a cycle through x and y.
D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

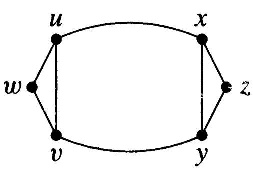
Proof: Theorem 4.2.2 proves A$\Leftrightarrow$B.

For B$\Leftrightarrow$C, note that cycles containing x and y correspond to pairs of internally disjoint x, y-paths.

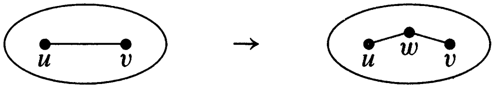
For D$\Rightarrow$C, the condition $\delta(G) \geq 1$ implies that vertices x and y are not isolated; we then apply the last part of D to edges incident to x and y. If there is only one such edge, then we use it and any edge incident to a third vertex.

To complete the proof, we assume that G satisfies the equivalent properties A and C and then derive D. Since G is connected, $\delta(G) \geq 1$. Now consider two edges uv and xy. Add to G the vertices w with neighborhood $\{u, v\}$ and z with neighborhood $\{x, y\}$. Since G is 2-connected, the Expansion Lemma (Lemma 4.2.3) implies that the resulting graph G' is 2-connected.

Hence condition C holds in G', so w and z lie on a cycle C in G'. Since w, z each have degree 2, C must contain the paths u, w, v and x, z, y but not the edges uv or xy. Replacing the paths u, w, v and x, z, y in C with the edges uv and xy yields the desired cycle through uv and xy in G. ■



4.2.5. Definition. In a graph G, **subdivision** of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.



4.2.6. Corollary. If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected.

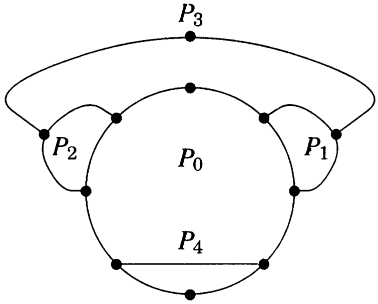
Proof: Let G' be formed from G by adding vertex w to subdivide uv. To show that G' is 2-connected, it suffices to find a cycle through arbitrary edges e, f of G' (by Theorem 4.2.4D).

Since G is 2-connected, any two edges of G lie on a common cycle (Theorem 4.2.4D). When our given edges e, f of G' lie in G, a cycle through them in G is also in G', unless it uses uv, in which case we modify the cycle. Here "modify the cycle" means "replace the edge uv with the u, v-path of length 2 through w".

When $e \in E(G)$ and $f \in \{uw, wv\}$, we modify a cycle passing through e and uv in G. When $\{e, f\} = \{uw, wv\}$, we modify a cycle through uv. ■

The class of 2-connected graphs has a characterization that expresses the construction of each such graph from a cycle and paths.

4.2.7. Definition. An **ear** of a graph G is a maximal path whose internal vertices have degree 2 in G. An **ear decomposition** of G is a decomposition $P_0, \dots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$.

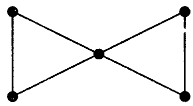


4.2.8. Theorem. (Whitney [1932a]) A graph is 2-connected if and only if it has an ear decomposition. Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

Proof: *Sufficiency.* Since cycles are 2-connected, it suffices to show that adding an ear preserves 2-connectedness. Let u, v be the endpoints of an ear P to be added to a 2-connected graph G. Adding an edge cannot reduce connectivity, so $G + uv$ is 2-connected. A succession of edge subdivisions converts $G + uv$ into the graph $G \cup P$ in which P is an ear; by Corollary 4.2.6, each subdivision preserves 2-connectedness.

Necessity. Given a 2-connected graph G, we build an ear decomposition of G from a cycle C in G. Let $G_0 = C$. Let G_i be a subgraph obtained by successively adding i ears. If $G_i \neq G$, then we can choose an edge uv of $G - E(G_i)$ and an edge $xy \in E(G_i)$. Because G is 2-connected, uv and xy lie on a common cycle C'. Let P be the path in C' that contains uv and exactly two vertices of G_i, one at each end of P. Now P can be added to G_i to obtain a larger subgraph G_{i+1} in which P is an ear. The process ends only by absorbing all of G. ■

Every 2-connected graph is 2-edge-connected, but the converse does not hold. Recall that the bowtie is the graph consisting of two triangles sharing one common vertex; it is 2-edge-connected but not 2-connected. Since more graphs are 2-edge-connected, decomposition of 2-edge-connected graphs needs a more general operation. The proof is like that of Theorem 4.2.8.



4.2.9. Definition. A **closed ear** in a graph G is a cycle C such that all vertices of C except one have degree 2 in G. A **closed-ear decomposition** of G is a decomposition $P_0, \dots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is either an (open) ear or a closed ear in G.

4.2.10. Theorem. A graph is 2-edge-connected if and only if it has a closed-ear decomposition, and every cycle in a 2-edge-connected graph is the initial cycle in some such decomposition.

Proof: *Sufficiency.* Cut-edges are the edges not on cycles (Theorem 1.2.14), so a connected graph is 2-edge-connected if and only if every edge lies on a cycle. The initial cycle is 2-edge-connected. When we add a closed ear, its edges form a cycle. When we add an open ear P to a connected graph G, a path in G connecting the endpoints of P completes a cycle containing all edges of P. In each case, the new graph also is connected. Thus adding an open or closed ear preserves 2-edge-connectedness.

Necessity. Given a 2-edge-connected graph G, let P_0 be a cycle in G. Consider a closed-ear decomposition $P_0, \dots, P_i$ of a subgraph G_i of G. When $G_i \neq G$, we find an ear to add. Since G is connected, there is an edge $uv \in E(G) - E(G_i)$ with $u \in V(G_i)$. Since G is 2-edge-connected, uv lies on a cycle C. Follow C until it returns to $V(G_i)$, forming up to this point a path or cycle P. Adding P to G_i yields a larger subgraph G_{i+1} in which P is an open or closed ear. The process ends only by absorbing all of G. ■

CONNECTIVITY OF DIGRAPHS

Our results about k-connected and k-edge-connected graphs will apply as well for digraphs, where we use analogous terminology.

4.2.11. Definition. A **separating set** or **vertex cut** of a digraph D is a set $S \subseteq V(D)$ such that $D - S$ is not strongly connected. A digraph is **k-connected** if every vertex cut has at least k vertices. The minimum size of a vertex cut is the **connectivity** $\kappa(D)$.

For vertex sets S, T in a digraph D, let $[S, T]$ denote the set of edges with tail in S and head in T. An **edge cut** is the set $[S, \overline{S}]$ for some $\emptyset \neq$

$S \subset V(D)$. A digraph is **k-edge-connected** if every edge cut has at least k edges. The minimum size of an edge cut is the **edge-connectivity** $\kappa'(D)$.

4.2.12. Remark. Because $|[S, \overline{S}]|$ is the number of edges leaving S, we can restate the definition of edge-connectivity as follows: A graph or digraph G is k-edge-connected if and only if for every nonempty proper vertex subset S, there are at least k edges in G leaving S.

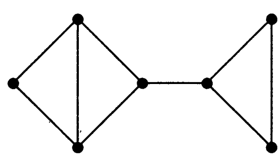
Note that $[S, T]$ is the set of edge from S to T. The meaning of this depends on whether we are discussing a graph or a digraph. In a graph, we take all edges that have endpoints in both sets. In a digraph, we take only the edges with tail in S and head in T. ∎

Strong digraphs are similar to 2-edge-connected graphs.

4.2.13. Proposition. Adding a (directed) ear to a strong digraph produces a larger strong digraph.

Proof: By Remark 4.2.12, a digraph is strong if and only if for every nonempty vertex subset there is a departing edge. If we add an open ear or closed ear P to a strong digraph D, then for every set S with $\emptyset \subset S \subset V(D)$ we already have an edge from S to $V(D) - S$. We need only consider sets that don't intersect $V(D)$ and sets that contain all of $V(D)$ but not all of $V(P)$. For every such set, there is an edge leaving it along P. ∎

When can the streets in a road network all be made one-way without making any location unreachable from some other location? In other words, when does a graph have a strong orientation? The graph below does not. The obvious necessary conditions are sufficient.



4.2.14. Theorem. (Robbins [1939]) A graph has a strong orientation if and only if it is 2-edge-connected.

Proof: *Necessity.* If a graph G is disconnected, then some vertices cannot reach others in any orientation. If G has a cut-edge xy oriented from x to y in an orientation D, then y cannot reach x in D. Hence G must be connected and have no cut-edge.

Sufficiency. When G is 2-edge-connected, it has a closed-ear decomposition. We orient the initial cycle consistently to obtain a strong digraph. As we add each new ear and direct it consistently, Proposition 4.2.13 guarantees that we still have a strong digraph. ∎

Robbins' Theorem generalizes for all k. When G has a k-edge-connected orientation, Remark 4.2.12 implies that G must be $2k$-edge-connected. Nash-Williams [1960] proved that this obvious necessary condition is also sufficient: a graph has a k-edge-connected orientation if and only if it is $2k$-edge-connected. This is easy when G is Eulerian (Exercise 21), but the general case is difficult (see Exercises 36–38). A thorough discussion of this and other orientation theorems appears in Frank [1993].

k-CONNECTED AND k-EDGE-CONNECTED GRAPHS

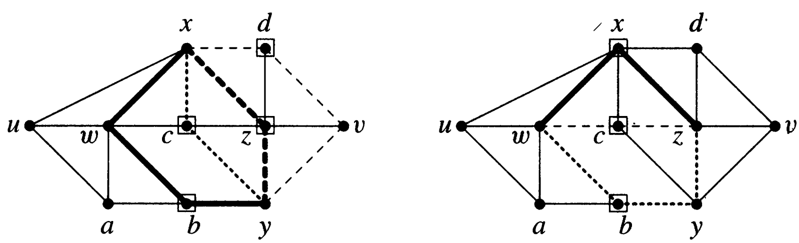
We have introduced two measures of good connection: invulnerability to deletions and multiplicity of alternative paths. Extending Whitney's Theorem, we show that these two notions are the same, for both vertex deletions and edge deletions, and for both graphs and digraphs.

We first discuss the "local" problem of x, y-paths for a fixed pair $x, y \in V(G)$. These definitions hold both for graphs and for digraphs.

4.2.15. Definition. Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-**separator** or x, y-**cut** if $G - S$ has no x, y-path. Let $\kappa(x, y)$ be the minimum size of an x, y-cut. Let $\lambda(x, y)$ be the maximum size of a set of pairwise internally disjoint x, y-paths. For $X, Y \subseteq V(G)$, an X, Y-**path** is a path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$.

An x, y-cut must contain an internal vertex of every x, y-path, and no vertex can cut two internally disjoint x, y-paths. Therefore, always $\kappa(x, y) \geq \lambda(x, y)$. Thus the problems of finding the smallest cut and the largest set of paths are dual problems, like the duality between matching and covering in Chapter 3.

4.2.16. Example. In the graph G below, the set $S = \{b, c, z, d\}$ is an x, y-cut of size 4; thus $\kappa(x, y) \leq 4$. As shown on the left, G has four pairwise internally disjoint x, y-paths; thus $\lambda(x, y) \geq 4$. Since $\kappa(x, y) \geq \lambda(x, y)$ always, we have $\kappa(x, y) = \lambda(x, y) = 4$.



Consider also the pair w, z. As shown on the right, $\kappa(w, z) = \lambda(w, z) = 3$, with $\{b, c, x\}$ being a minimum w, z-cut. The graph G is 3-connected; for every pair $u, v \in V(G)$, we can find three pairwise internally disjoint u, v-paths.

From the equality for internally disjoint paths, we will obtain an analogous equality for edge-disjoint paths. Although $\kappa(w, z) = 3$ above, it takes four edges to break all w, z-paths, and there are four pairwise edge-disjoint w, z-paths. ■

What we call Menger's Theorem states that the local equality $\kappa(x, y) = \lambda(x, y)$ always holds. The global statement for connectivity and analogous results for edge-connectivity and digraphs were observed by others. All are considered forms of Menger's Theorem. More than 15 proofs of Menger's Theorem have been published, some yielding stronger results, some incorrect. (A gap in Menger's original argument was later repaired by König.)

4.2.17. Theorem. (Menger [1927]) If x, y are vertices of a graph G and $xy \notin E(G)$, then the minimum size of an x, y-cut equals the maximum number of pairwise internally disjoint x, y-paths.

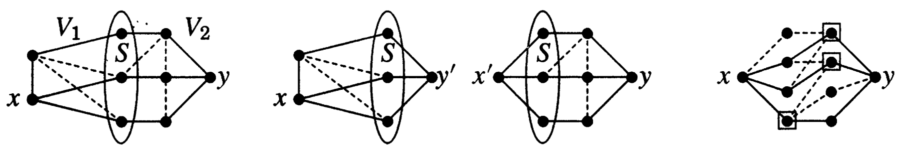
Proof: An x, y-cut must contain an internal vertex from each path in a set of pairwise internally disjoint x, y-paths. These vertices must be distinct, so $\kappa(x, y) \geq \lambda(x, y)$.

To prove equality, we use induction on $n(G)$. Basis step: $n(G) = 2$. Here $xy \notin E(G)$ yields $\kappa(x, y) = \lambda(x, y) = 0$. Induction step: $n(G) > 2$. Let $k = \kappa_G(x, y)$. We construct k pairwise internally disjoint x, y-paths. Note that since $N(x)$ and $N(y)$ are x, y-cuts, no minimum cut properly contains $N(x)$ or $N(y)$.

Case 1: G has a minimum x, y-cut S other than $N(x)$ or $N(y)$. To obtain the k desired paths, we combine x, S-paths and S, y-paths obtained from the induction hypothesis (as formed by solid edges shown below). Let V_1 be the set of vertices on x, S-paths, and let V_2 be the set of vertices on S, y-paths. We claim that $S = V_1 \cap V_2$. Since S is a minimal x, y-cut, every vertex of S lies on an x, y-path, and hence $S \subseteq V_1 \cap V_2$. If $v \in (V_1 \cap V_2)$, then following the x, v-portion of some x, S-path and then the v, y-portion of some S, y-path yields an x, y-path that avoids the x, y-cut S. This is impossible, so $S = V(G_1) \cap V(G_2)$. By the same argument, V_1 omits $N(y) - S$ and V_2 omits $N(x) - S$.

Form H_1 by adding to $G[V_1]$ a vertex y' with edges from S. Form H_2 by adding to $G[V_2]$ a vertex x' with edges to S. Every x, y-path in G starts with an x, S-path (contained in H_1), so every x, y'-cut in H_1 is an x, y-cut in G. Therefore, $\kappa_{H_1}(x, y') = k$, and similarly $\kappa_{H_2}(x', y) = k$.

Since V_1 omits $N(y) - S$ and V_2 omits $N(x) - S$, both H_1 and H_2 are smaller than G. Hence the induction hypothesis yields $\lambda_{H_1}(x, y') = k = \lambda_{H_2}(x', y)$. Since $V_1 \cap V_2 = S$, deleting y' from the k paths in H_1 and x' from the k paths in H_2 yields the desired x, S-paths and S, y-paths in G that combine to form k pairwise internally disjoint x, y-paths in G.



G H_1 H_2 G'

Case 1 Case 2

Case 2. Every minimum x, y-cut is $N(x)$ or $N(y)$. Again we construct the k desired paths. In this case, every vertex outside $\{x \cup N(x) \cup N(y) \cup y\}$ is in no minimum x, y-cut. If G has such a vertex v, then $\kappa_{G-v}(x, y) = k$, and applying the induction hypothesis to $G - v$ yields the desired x, y-paths in G. Also, if there exists $u \in N(x) \cap N(y)$, then u appears in every x, y-cut, and $\kappa_{G-u}(x, y) = k - 1$. Now applying the induction hypothesis to $G - v$ yields $k - 1$ paths to combine with the path x, v, y.

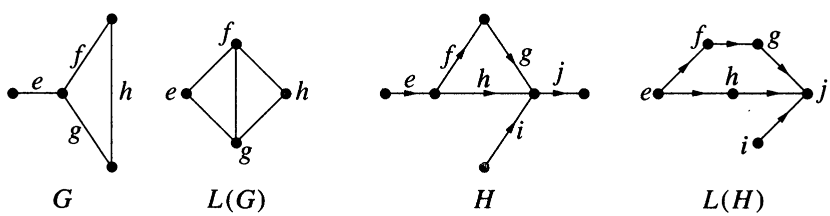
We may thus assume that $N(x)$ and $N(y)$ partition $V(G) - \{x, y\}$. Let G' be the bipartite graph with bipartition $N(x), N(y)$ and edge set $[N(x), N(y)]$. Every x, y-path in G uses some edge from $N(x)$ to $N(y)$, so the x, y-cuts in G are precisely the vertex covers of G'. Hence $\beta(G') = k$. By the König–Egerváry Theorem, G' has a matching of size k. These k edges yield k pairwise internally disjoint x, y-paths of length 3. ∎

Case 2 is needed in the proof because when $S = N(x)$, the induction hypothesis cannot be used to obtain the S, y-paths.

The statement of Theorem 4.2.17 makes sense also for digraphs. The proof of the digraph version is exactly the same; we only need to replace $N(x)$ and $N(y)$ with $N^+(x)$ and $N^-(y)$ throughout.

We next develop the analogue of Theorem 4.2.17 for edge-disjoint paths, which we prove by applying Theorem 4.2.17 to a transformed graph. The main part of the transformation is an operation that we will use again in Chapter 7.

4.2.18. Definition. The **line graph** of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. Substituting "digraph" for "graph" in this sentence yields the definition of **line digraph**. For graphs, e and f share a vertex; for digraphs, the head of e must be the tail of f.

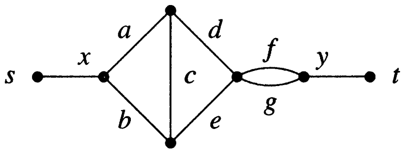


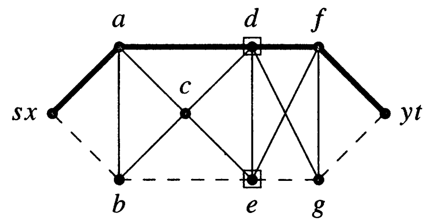
When disconnecting y from x by deleting edges, we use notation analogous to that of Definition 4.2.15: $\lambda'(x, y)$ is the maximum size of a set of pairwise edge-disjoint x, y-paths, and $\kappa'(x, y)$ is the minimum number of edges whose deletion makes y unreachable from x. Elias–Feinstein–Shannon [1956] and Ford–Fulkerson [1956] proved that always $\lambda'(x, y) = \kappa'(x, y)$ (using the methods of Section 4.3). We allow multiple edges and allow $xy \in E(G)$.

4.2.19. Theorem. If x and y are distinct vertices of a graph or digraph G, then the minimum size of an x, y-disconnecting set of edges equals the maximum number of pairwise edge-disjoint x, y-paths.

Proof: Modify G to obtain G' by adding two new vertices s, t and two new edges sx and yt. This does not change $\kappa'(x, y)$ or $\lambda'(x, y)$, and we can think of each path as starting from the edge sx and ending with the edge yt. A set of edges disconnects y from x in G if and only if the corresponding vertices of $L(G')$ form an sx, yt-cut. Similarly, edge-disjoint x, y-paths in G become internally disjoint sx, yt-paths in $L(G')$, and vice versa. Since $x \neq y$, we have no edge from sx to ty in $L(G')$. Applying Theorem 4.2.17 to $L(G')$ yields

$$\kappa'_G(x, y) = \kappa_{L(G')}(sx, yt) = \lambda_{L(G')}(sx, yt) = \lambda'_G(x, y). \qquad \blacksquare$$





The global version for k-connected graphs, observed first by Whitney [1932a], is also commonly called Menger's Theorem. The global versions for edges and digraphs appeared in Ford–Fulkerson [1956].

4.2.20. Lemma. Deletion of an edge reduces connectivity by at most 1.

Proof: We discuss only graphs; the argument for digraphs is similar (Exercise 7). Since every separating set of G is a separating set of $G - xy$, we have $\kappa(G - xy) \le \kappa(G)$. Equality holds unless $G - xy$ has a separating set S that is smaller than $\kappa(G)$ and hence is not a separating set of G. Since $G - S$ is connected, $G - xy - S$ has two components $G[X]$ and $G[Y]$, with $x \in X$ and $y \in Y$. In $G - S$, the only edge joining X and Y is xy.

If $|X| \ge 2$, then $S \cup \{x\}$ is a separating set of G, and $\kappa(G) \le \kappa(G - xy) + 1$. If $|Y| \ge 2$, then again the inequality holds. In the remaining case, $|S| = n(G) - 2$. Since we have assumed that $|S| < \kappa(G)$, $|S| = n(G) - 2$ implies that $\kappa(G) \ge n(G) - 1$, which holds only for a complete graph. Thus $\kappa(G - xy) = n(G) - 2 = \kappa(G) - 1$, as desired. $\blacksquare$

4.2.21. Theorem. The connectivity of G equals the maximum k such that $\lambda(x, y) \ge k$ for all $x, y \in V(G)$. The edge-connectivity of G equals the maximum k such that $\lambda'(x, y) \ge k$ for all $x, y \in V(G)$. Both statements hold for graphs and for digraphs.

Proof: Since $\kappa'(G) = \min_{x,y \in V(G)} \kappa'(x, y)$, Theorem 4.2.19 immediately yields the claim for edge-connectivity.

For connectivity, we have $\kappa(x, y) = \lambda(x, y)$ for $xy \notin E(G)$, and $\kappa(G)$ is the minimum of these values. We need only show that $\lambda(x, y)$ cannot be smaller than $\kappa(G)$ when $xy \in E(G)$. Certainly deletion of xy reduces $\lambda(x, y)$ by 1, since xy itself is an x, y-path and cannot contribute to any other x, y-path. With this, Theorem 4.2.17, and Lemma 4.2.20, we have

$$\lambda_G(x, y) = 1 + \lambda_{G-xy}(x, y) = 1 + \kappa_{G-xy}(x, y) \ge 1 + \kappa(G - xy) \ge \kappa(G). \qquad \blacksquare$$

APPLICATIONS OF MENGER'S THEOREM

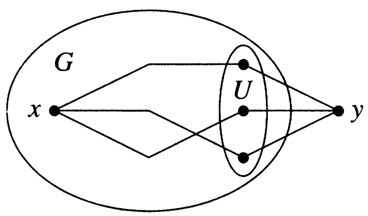
Dirac extended Menger's Theorem to other families of paths.

4.2.22. Definition. Given a vertex x and a set U of vertices, an x, U**-fan** is a set of paths from x to U such that any two of them share only the vertex x.

4.2.23. Theorem. (Fan Lemma, Dirac [1960]). A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an x, U-fan of size k.

Proof: *Necessity.* Given k-connected graph G, we construct G' from G by adding a new vertex y adjacent to all of U. The Expansion Lemma (Lemma 4.2.3) implies that G' also is k-connected, and then Menger's Theorem yields k pairwise internally disjoint x, y-paths in G'. Deleting y from these paths produces an x, U-fan of size k in G.

Sufficiency. Suppose that G satisfies the fan condition. For $v \in V(G)$ and $U = V(G) - \{v\}$, there is a v, U-fan of size k; thus $\delta(G) \geq k$. Given $w, z \in V(G)$, let $U = N(z)$. Since $|U| \geq k$, we have an w, U-fan of size k; extend each path by adding an edge to z. We obtain k pairwise internally disjoint w, z-paths, so $\lambda(w, z) \geq k$. This holds for all $w, z \in V(G)$, so G is k-connected. ■



The Fan Lemma generalizes considerably. Whenever X and Y are disjoint sets of vertices in a k-connected graph G and we specify integers at X and Y summing to k in each set, there are k pairwise internally disjoint X, Y-paths with the specified number ending at each point (Exercise 28). The Fan Lemma also yields the next result.

4.2.24.* Theorem. (Dirac [1960]) If G is a k-connected graph (with $k \geq 2$), and S is a set of k vertices in G, then G has a cycle including S in its vertex set.

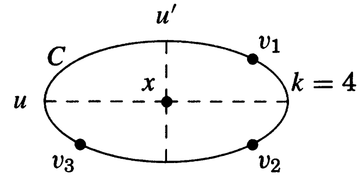
Proof: We use induction on k. Basis step ($k = 2$): Theorem 4.2.2 (or Theorem 4.2.21) implies that any two vertices are connected by two internally disjoint paths, which form a cycle containing them.

Induction step ($k > 2$): With G and S as specified, choose $x \in S$. Since G is also $k-1$-connected, the induction hypothesis implies that all of $S - \{x\}$ lies on a cycle C. Suppose first that $n(C) = k - 1$. Since G is $k-1$-connected, we have an $x, V(C)$-fan of size $k-1$, and the paths of the fan to two consecutive vertices of C enlarge the cycle to include x.

Hence we may assume that $n(C) \geq k$. Since G is k-connected, G has an

$x, V(C)$-fan of size k. We claim that again the fan has two paths forming a detour from C that includes x while keeping $S - \{x\}$. Let $v_1, \dots, v_{k-1}$ be the vertices of $S - \{x\}$ in order on C, and let V_i be the portion of $V(C)$ from v_i up to but not including v_{i+1} (here $v_k = v_1$).

The sets $V_1, \dots, V_{k-1}$ partition $V(C)$ into $k - 1$ disjoint sets. Since the $x, V(C)$-fan has k paths, two of them enter $V(C)$ in one of these sets, by the pigeonhole principle. Let u, u' be the vertices where these paths reach C. Replacing the u, u'-portion of C by the x, u-path and x, u'-path in the fan builds a new cycle that contains x and all of $S - \{x\}$. ■



Many applications of Menger's Theorem involve modeling a problem so that the desired objects correspond to paths in a graph or digraph, often by graph transformation arguments. For example, given sets $\mathbf{A} = A_1, \dots, A_m$ with union X, a **system of distinct representatives** (SDR) is a set of distinct elements $x_1, \dots, x_m$ such that $x_i \in A_i$. A necessary and sufficient condition for the existence of an SDR is that $\left|\bigcup_{i\in I} A_i\right| \ge |I|$ for all $I \subseteq [m]$. It is easy to prove this from Hall's Theorem by modeling $\mathbf{A}$ with an appropriate bipartite graph (Exercise 3.1.19). Indeed, Hall's Theorem was originally proved in the language of SDRs and is equivalent to Menger's Theorem (Exercise 23).

Ford and Fulkerson considered a more difficult problem. Let $\mathbf{A} = A_1, \dots, A_m$ and $\mathbf{B} = B_1, \dots, B_m$ be two families of sets. We may ask when there is a **common system of distinct representatives (CSDR)**, meaning a set of m elements that is both an SDR for $\mathbf{A}$ and an SDR for $\mathbf{B}$. They found a necessary and sufficient condition.

4.2.25.* Theorem. (Ford–Fulkerson [1958]) Families $\mathbf{A} = \{A_1, \dots, A_m\}$ and $\mathbf{B} = \{B_1, \dots, B_m\}$ have a common system of distinct representatives (CSDR) if and only if

$$\left|(\bigcup_{i\in I} A_i) \cap (\bigcup_{j\in J} B_j)\right| \ge |I| + |J| - m \quad \text{for each pair } I, J \subseteq [m].$$

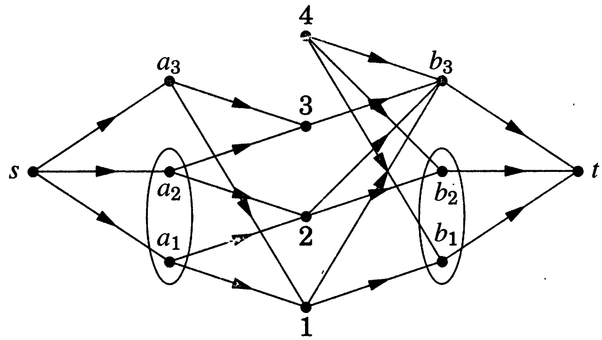
Proof: We create a digraph G with vertices $a_1, \dots, a_m$ and $b_1, \dots, b_m$, plus a vertex for each element in the sets and special vertices s, t. The edges are

$$\{sa_i\colon A_i \in \mathbf{A}\} \quad \{a_i x\colon x \in A_i\}$$
$$\{b_j t\colon B_j \in \mathbf{B}\} \quad \{x b_j\colon x \in B_j\}$$

Each s, t-path selects a member of the intersection of some A_i and some B_j. There is a CSDR if and only if there is a set of m pairwise internally disjoint s, t-paths. By Menger's Theorem, it suffices to show that the stated condition is equivalent to having no s, t-cut of size less than m. Given a set $R \subseteq V(G) - \{s, t\}$, let $I = \{a_i\} - R$ and $J = \{b_j\} - R$. The set R is an s, t-cut if and only if $(\bigcup_{i\in I} A_i) \cap (\bigcup_{j\in J} B_j) \subseteq R$. For an s, t-cut R, we thus have

$$|R| \geq \left|(\bigcup_{i\in I} A_i) \cap (\bigcup_{j\in J} B_j)\right| + (m - |I|) + (m - |J|).$$

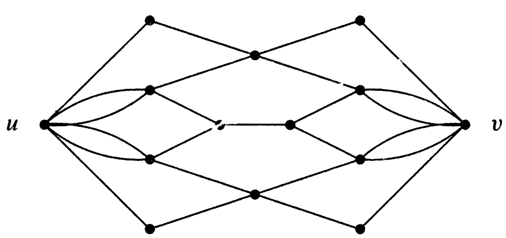
This lower bound is at least m for every s, t-cut if and only if the stated condition holds. ■



4.2.26.* Example. *Digraph for CSDR.* In the example above, the elements are $\{1, 2, 3, 4\}$, $\mathbf{A} = \{12, 23, 31\}$, and $\mathbf{B} = \{14, 24, 1234\}$. Suppose that $R \cap \{a_i\} = \{a_1, a_2\}$ and $R \cap \{b_j\} = \{b_1, b_2\}$. In the argument, we set $I = \{a_3\}$ and $J = \{b_3\}$, and we observe that R is an s, t-cut if and only if it also contains $\{1, 3\}$, which equals $(\bigcup_{i\in I} A_i) \cap (\bigcup_{j\in J} B_j)$. ■

EXERCISES

4.2.1. (−) Determine $\kappa(u, v)$ and $\kappa'(u, v)$ in the graph drawn below. (Hint: Use the dual problems to give short proofs of optimality.)



4.2.2. (−) Prove that if G is 2-edge-connected and G' is obtained from G by subdividing an edge of G, then G' is 2-edge-connected. Use this to prove that every graph having a closed-ear decomposition is 2-edge-connected. (Comment: This is an alternative proof of sufficiency for Theorem 4.2.10.)

4.2.3. ($-$) Let G be the digraph with vertex set $[12]$ in which $i \to j$ if and only if i divides j. Determine $\kappa(1, 12)$ and $\kappa'(1, 12)$.

4.2.4. ($-$) Prove or disprove: If P is a u, v-path in a 2-connected graph G, then there is a u, v-path Q that is internally disjoint from P.

4.2.5. ($-$) Let G be a simple graph, and let $H(G)$ be the graph with vertex set $V(G)$ such that $uv \in E(H)$ if and only if u, v appear on a common cycle in G. Characterize the graphs G such that H is a clique.

4.2.6. ($-$) Use results of this section to prove that a simple graph G is 2-connected if and only if G can be obtained from C_3 by a sequence of edge additions and edge subdivisions.

• • • • •

4.2.7. Let xy be an edge in a digraph G. Prove that $\kappa(G - xy) \geq \kappa(G) - 1$.

4.2.8. Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z-path through y. (Chein [1968])

4.2.9. Prove that a graph G with at least four vertices is 2-connected if and only if for every pair X, Y of disjoint vertex subsets with $|X|, |Y| \geq 2$, there exist two completely disjoint paths P_1, P_2 in G such that each has an endpoint in X and an endpoint in Y and no internal vertex in X or Y.

4.2.10. A **greedy ear decomposition** of a 2-connected graph is an ear decomposition that begins with a longest cycle and iteratively adds a longest ear from the remaining graph. Use a greedy ear decomposition to prove that every 2-connected claw-free graph G has $\lfloor n(G)/3 \rfloor$ pairwise-disjoint copies of P_3. (Kaneko–Kelmans–Nishimura [2000])

4.2.11. (!) For a connected graph G with at least three vertices, prove that the following statements are equivalent (use of Menger's Theorem is permitted).

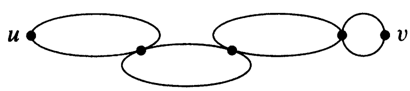
A) G is 2-edge-connected.
B) Every edge of G appears in a cycle.
C) G has a closed trail containing any specified pair of edges.
D) G has a closed trail containing any specified pair of vertices.

4.2.12. (!) Use Menger's Theorem to prove that $\kappa(G) = \kappa'(G)$ when G is 3-regular (Theorem 4.1.11).

4.2.13. (!) Let G be a 2-edge-connected graph. Define a relation R on $E(G)$ by $(e, f) \in R$ if $e = f$ or if $G - e - f$ is disconnected. (Lovász [1979, p277])

a) Prove that $(e, f) \in R$ if and only if e, f belong to the same cycles.
b) Prove that R is an equivalence relation on $E(G)$.
c) For each equivalence class F, prove that F is contained in a cycle.
d) For each equivalence class F, prove that $G - F$ has no cut-edge.

4.2.14. (!) A u, v-**necklace** is a list of cycles $C_1, \ldots, C_k$ such that $u \in C_1$, $v \in C_k$, consecutive cycles share one vertex, and nonconsecutive cycles are disjoint. Use induction on $d(u, v)$ to prove that a graph G is 2-edge-connected if and only if for all $u, v \in V(G)$ there is a u, v-necklace in G.



4.2.15. (+) Let v be a vertex of a 2-connected graph G. Prove that v has a neighbor u such that $G - u - v$ is connected. (Chartrand–Lesniak [1986, p51])

4.2.16. (+) Let G be a 2-connected graph. Prove that if T_1, T_2 are two spanning trees of G, then T_1 can be transformed into T_2 by a sequence of operations in which a leaf is removed and reattached using another edge of G.

4.2.17. Determine the smallest graph with connectivity 3 having a pair of nonadjacent vertices linked by four pairwise internally disjoint paths.

4.2.18. Let G be a graph without isolated vertices. Prove that if G has no even cycles, then every block of G is an edge or an odd cycle.

4.2.19. (!) *Membership in common cycles.*

a) Prove that two distinct edges lie in the same block of a graph if and only if they belong to a common cycle.

b) Given $e, f, g \in E(G)$, suppose that G has a cycle through e and f and a cycle through f and g. Prove that G also has a cycle through e and g. (Comment: This problem implies that for graphs without cut-edges, "belong to a common cycle" is an equivalence relation whose equivalence classes are the edge sets of blocks.)

4.2.20. Prove that the hypercube Q_k is k-connected by constructing k pairwise internally disjoint x, y-paths for each vertex pair $x, y \in V(Q_k)$.

4.2.21. (!) Let G be a $2k$-edge-connected graph with at most two vertices of odd degree. Prove that G has a k-edge-connected orientation. (Nash-Williams [1960])

4.2.22. (!) Suppose that $\kappa(G) = k$ and diam $G = d$. Prove that $n(G) \geq k(d-1)+2$ and $\alpha(G) \geq \lceil (1+d)/2 \rceil$. For each $k \geq 1$ and $d \geq 2$, construct a graph for which equality holds in both bounds.

4.2.23. (!) Use Menger's Theorem ($\kappa(x, y) = \lambda(x, y)$ when $xy \notin E(G)$) to prove the König–Egerváry Theorem ($\alpha'(G) = \beta(G)$ when G is bipartite).

4.2.24. (!) Let G be a k-connected graph, and let S, T be disjoint subsets of $V(G)$ with size at least k. Prove that G has k pairwise disjoint S, T-paths.

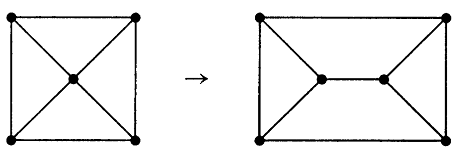
4.2.25. (*) Show that Theorem 4.2.24 is best possible by constructing for each k a k-connected graph having $k+1$ vertices that do not lie on a cycle.

4.2.26. For $k \geq 2$, prove that a graph with at least $k+1$ vertices is k-connected if and only if for every $T \subseteq S \subseteq V(G)$ with $|S| = k$ and $|T| = 2$, there is a cycle in G that contains T and avoids $S - T$. (Lick [1973])

4.2.27. A **vertex k-split** of a graph G is a graph H obtained from G by replacing one vertex $x \in V(G)$ by two adjacent vertices x_1, x_2 such that $d_H(x_i) \geq k$ and that $N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\}$.

a) Prove that every vertex k-split of a k-connected graph is k-connected.

b) Conclude that any graph obtained from a "wheel" $W_n = K_1 \vee C_{n-1}$ (Definition 3.3.6) by a sequence of edge additions and vertex 3-splits on vertices of degree at least 4 is 3-connected. (Comment: Tutte [1961b] proved also that every 3-connected graph arises in this way. The characterization does not extend easily for $k > 3$.)



4.2.28. (!) Let X and Y be disjoint sets of vertices in a k-connected graph G. Let $u(x)$ for $x \in X$ and $w(y)$ for $y \in Y$ be nonnegative integers such that $\sum_{x\in X} u(x) = \sum_{y\in Y} w(y) = k$. Prove that G has k pairwise internally disjoint X,Y-paths so that $u(x)$ of them start at x and $w(y)$ of them end at y, for $x \in X$ and $y \in Y$.

4.2.29. Given a graph G, let D be the digraph obtained by replacing each edge with two oppositely-directed edges having the same endpoints (thus D is the symmetric digraph with underlying graph G). Assume that for all $x, y \in V(D)$ both $\kappa'_D(x, y) = \lambda'_D(x, y)$ and $\kappa_D(x, y) = \lambda_D(x, y)$ hold, the latter applying only when $x \not\leftrightarrow y$. Use this hypothesis to prove that also $\kappa'_G(x, y) = \lambda'_G(x, y)$ and $\kappa_G(x, y) = \lambda_G(x, y)$, the latter for $x \not\leftrightarrow y$.

4.2.30. (!) Prove that applying the expansion operation of Example 1.3.26 to a 3-connected graph yields a 3-connected graph. Obtain the Petersen graph from K_4 by expansions. (Comment: Tutte [1966a] proved that a 3-regular graph is 3-connected if and only if it arises from K_4 by a sequence of these operations.)

4.2.31. Let G be a k-connected simple graph.

a) Let C and D be two cycles in G of maximum length. For $k = 2$ and $k = 3$, prove that C and D share at least k vertices. (Hint: If they don't, construct a longer cycle.)

b) For each $k \ge 2$, construct a k-connected graph that has distinct longest cycles with only k common vertices. (Hint: $K_{2,4}$ works for $k = 2$.)

4.2.32. *Graph splices.* Let G_1 and G_2 be disjoint k-connected graphs with $k \ge 2$. Choose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let B be a bipartite graph with partite sets $N_{G_1}(v_1)$ and $N_{G_2}(v_2)$ that has no isolated vertex and has a matching of size at least k. Prove that $(G_1 - v_1) \cup (G_2 - v_2) \cup B$ is k-connected.

4.2.33. ($*$) Prove Hall's Theorem from Theorem 4.2.25.

4.2.34. A k-connected graph G is **minimally k-connected** if for every $e \in E(G)$, the graph $G - e$ is not k-connected. Halin [1969] proved that $\delta(G) = k$ when G is minimally k-connected. Use ear decomposition to prove this for $k = 2$. Conclude that a minimally 2-connected graph G with at least 4 vertices has at most $2n(G) - 4$ edges, with equality only for $K_{2,n-2}$. (Dirac [1967])

4.2.35. Prove that if G is 2-connected, then $G - xy$ is 2-connected if and only if x and y lie on a cycle in $G - xy$. Conclude that a 2-connected graph is minimally 2-connected if and only if every cycle is an induced subgraph. (Dirac [1967], Plummer [1968])

4.2.36. (!) For $S \subseteq V(G)$, let $d(S) = \left|[S, \overline{S}]\right|$. Let X and Y be nonempty proper vertex subsets of G. Prove that $d(X \cap Y) + d(X \cup Y) \le d(X) + d(Y)$. (Hint: Draw a picture and consider contributions from various types of edges.)

4.2.37. (+) A k-edge-connected graph G is **minimally k-edge-connected** if for every $e \in E(G)$ the graph $G - e$ is not k-edge-connected. Prove that $\delta(G) = k$ when G is minimally k-edge-connected. (Hint: Consider a minimal set S such that $\left|[S, \overline{S}]\right| = k$. If $|S| \ne 1$, use $G - e$ for some $e \in E(G[S])$ to obtain another set T with $\left|[T, \overline{T}]\right| = k$ such that S, T contradict Exercise 4.2.36.) (Mader [1971]; see also Lovász [1979, p285])

4.2.38. Mader [1978] proved the following: "If z is a vertex of a graph G such that $d_G(z) \notin \{0, 1, 3\}$ and z is incident to no cut-edge, then z has neighbors x and y such that $\kappa_{G-xz-yz+xy}(u, v) = \kappa_G(u, v)$ for all $u, v \in V(G) - \{z\}$." Use Mader's Theorem and Exercise 4.2.37 to prove Nash-Williams' Orientation Theorem: every $2k$-edge-connected graph has a k-edge-connected orientation. (Comment: A weaker version of Mader's Theorem, given in Lovász [1979, p286–288], also yields Nash-Williams' Theorem in the same way.)

4.3. Network Flow Problems

Consider a network of pipes where valves allow flow in only one direction. Each pipe has a capacity per unit time. We model this with a vertex for each junction and a (directed) edge for each pipe, weighted by the capacity. We also assume that flow cannot accumulate at a junction. Given two locations s, t in the network, we may ask "what is the maximum flow (per unit time) from s to t?"

This question arises in many contexts. The network may represent roads with traffic capacities, or links in a computer network with data transmission capacities, or currents in an electrical network. There are applications in industrial settings and to combinatorial min-max theorems. The seminal book on the subject is Ford–Fulkerson [1962]. More recently, Ahuja–Magnanti–Orlin [1993] presents a thorough treatment of network flow problems.

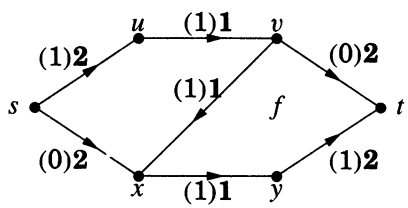
4.3.1. Definition. A **network** is a digraph with a nonnegative **capacity** $c(e)$ on each edge e and a distinguished **source vertex** s and **sink vertex** t. Vertices are also called **nodes**. A **flow** f assigns a value $f(e)$ to each edge e. We write $f^+(v)$ for the total flow on edges leaving v and $f^-(v)$ for the total flow on edges entering v. A flow is **feasible** if it satisfies the **capacity constraints** $0 \le f(e) \le c(e)$ for each edge and the **conservation constraints** $f^+(v) = f^-(v)$ for each node $v \notin \{s, t\}$.

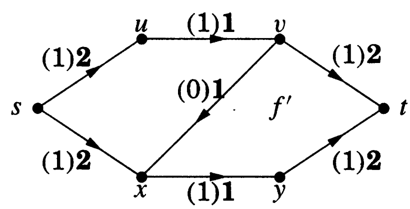
MAXIMUM NETWORK FLOW

We consider first the problem of maximizing the net flow into the sink.

4.3.2. Definition. The **value** $\mathrm{val}(f)$ of a flow f is the net flow $f^-(t) - f^+(t)$ into the sink. A **maximum flow** is a feasible flow of maximum value.

4.3.3. Example. The **zero flow** assigns flow 0 to each edge; this is feasible. In the network below we illustrate a nonzero feasible flow. Each capacities are shown in bold, flow values in parentheses. Our flow f assigns $f(sx) = f(vt) = 0$, and $f(e) = 1$ for every other edge e. This is a feasible flow of value 1.





A path from the source to the sink with excess capacity would allow us to increase flow. In this example, no path remains with excess capacity, but the flow f' with $f'(vx) = 0$ and $f'(e) = 1$ for $e \neq vx$ has value 2. The flow f is "maximal" in that no other feasible flow can be found by increasing the flow on some edges, but f is not a maximum flow.

We need a more general way to increase flow. In addition to traveling forward along edges with excess capacity, we allow traveling backward (against the arrow) along edges where the flow is nonzero. In this example, we can travel from s to x to v to t. Increasing the flow by 1 on sx and vt and decreasing it by one on vx changes f into f'. ■

4.3.4. Definition. When f is a feasible flow in a network N, an **f-augmenting path** is a source-to-sink path P in the underlying graph G such that for each $e \in E(P)$,

a) if P follows e in the forward direction, then $f(e) < c(e)$.

b) if P follows e in the backward direction, then $f(e) > 0$.

Let $\epsilon(e) = c(e) - f(e)$ when e is forward on P, and let $\epsilon(e) = f(e)$ when e is backward on P. The **tolerance** of P is $\min_{e \in E(P)} \epsilon(e)$.

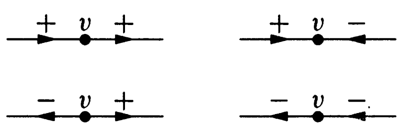
As in Example 4.3.3, an f-augmenting path leads to a flow with larger value. The definition of f-augmenting path ensures that the tolerance is positive; this amount is the increase in the flow value.

4.3.5. Lemma. If P is an f-augmenting path with tolerance z, then changing flow by $+z$ on edges followed forward by P and by $-z$ on edges followed backward by P produces a feasible flow f' with $\mathrm{val}(f') = \mathrm{val}(f) + z$.

Proof: The definition of tolerance ensures that $0 \leq f'(e) \leq c(e)$ for every edge e, so the capacity constraints hold. For the conservation constraints we need only check vertices of P, since flow elsewhere has not changed.

The edges of P incident to an internal vertex v of P occur in one of the four ways shown below. In each case, the change to the flow out of v is the same as the change to the flow into v, so the net flow out of v remains 0 in f'.

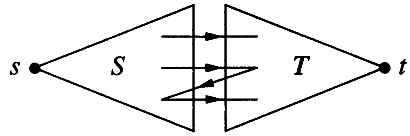
Finally, the net flow into the sink t increases by z. ■



The flow on backward edges did not disappear; it was redirected. In effect, the augmentation in Example 4.3.3 cuts the flow path and extends each portion to become a new flow path. We will soon describe an algorithm to find augmenting paths.

Meanwhile, we would like a quick way to know when our present flow is a maximum flow. In Example 4.3.3, the central edges seem to form a "bottleneck"; we only have capacity 2 from the left half of the network to the right half. This observation will give us a PROOF that the flow value can be no larger

4.3.6. Definition. In a network, a **source/sink cut** $[S, T]$ consists of the edges from a **source set** S to a **sink set** T, where S and T partition the set of nodes, with $s \in S$ and $t \in T$. The **capacity** of the cut $[S, T]$, written $\text{cap}(S, T)$, is the total of the capacities on the edges of $[S, T]$.



Keep in mind that in a digraph $[S, T]$ denotes the set of edges with tail in S and head in T. Thus the capacity of a cut $[S, T]$ is completely unaffected by edges from T to S.

Given a cut $[S, T]$, every s, t-path uses at least one edge of $[S, T]$, so intuition suggests that the value of a feasible flow should be bounded by $\text{cap}(S, T)$. To make this precise, we extend the notion of net flow to sets of nodes. Let $f^+(U)$ denote the total flow on edges leaving U, and let $f^-(U)$ be the total flow on edges entering U. The net flow out of U is then $f^+(U) - f^-(U)$.

4.3.7. Lemma. If U is a set of nodes in a network, then the net flow out of U is the sum of the net flows out of the nodes in U. In particular, if f is a feasible flow and $[S, T]$ is a source/sink cut, then the net flow out of S and net flow into T equal $\text{val}(f)$.

Proof: The stated claim is that

$$f^+(U) - f^-(U) = \textstyle\sum_{v \in U} [f^+(v) - f^-(v)].$$

We consider the contribution of the flow $f(xy)$ on an edge xy to each side of the formula. If $x, y \in U$, then $f(xy)$ is not counted on the left, but it contributes positively (via $f^+(x)$) and negatively (via $f^-(y)$) on the right. If $x, y \notin U$, then $f(xy)$ contributes to neither sum. If $xy \in [U, \overline{U}]$, then it contributes positively to each sum. If $xy \in [\overline{U}, U]$, then it contributes negatively to each sum. Summing over all edges yields the equality.

When $[S, T]$ is a source/sink cut and f is a feasible flow, net flow from nodes of S sums to $f^+(s) - f^-(s)$, and net flow from nodes of T sums to $f^+(t) - f^-(t)$, which equals $-\text{val}(f)$. Hence the net flow across any source/sink cut equals both the net flow out of s and the net flow into t. ■

4.3.8. Corollary. (Weak duality) If f is a feasible flow and $[S, T]$ is a source/sink cut, then $\text{val}(f) \le \text{cap}(S, T)$.

Proof: By the lemma, the value of f equals the net flow out of S. Thus

$$\text{val}(f) = f^+(S) - f^-(S) \le f^+(S),$$

since the flow into S is no less than 0. Since the capacity constraints require $f^+(S) \le \text{cap}(S, T)$, we obtain $\text{val}(f) \le \text{cap}(S, T)$. ■

Among source/sink cuts, one with minimum capacity yields the best bound on the value of a flow. This defines the **minimum cut** problem. The max flow and min cut problems on a network are dual optimization problems.† Given a flow with value α and a cut with value α, the duality inequality in Corollary 4.3.8 PROVES that the cut is a minimum cut and the flow is a maximum flow.

If every instance has solutions with the same value to both the max problem and the min problem ("strong duality"), then a short proof of optimality always exists. This does not hold for all dual pairs of problems (recall matching and covering in general graphs), but it holds for max flow and min cut.

The Ford–Fulkerson algorithm seeks an augmenting path to increase the flow value. If it does not find such a path, then it finds a cut with the same value (capacity) as this flow; by Corollary 4.3.8, both are optimal. If no infinite sequence of augmentations is possible, then the iteration leads to equality between the maximum flow value and the minimum cut capacity.

4.3.9. Algorithm. (Ford–Fulkerson labeling algorithm)
Input: A feasible flow f in a network.
Output: An f-augmenting path or a cut with capacity val(f).
Idea: Find the nodes reachable from s by paths with positive tolerance. Reaching t completes an f-augmenting path. During the search, R is the set of nodes labeled *Reached*, and S is the subset of R labeled *Searched*.
Initialization: $R = \{s\}$, $S = \emptyset$.
Iteration: Choose $v \in R - S$.

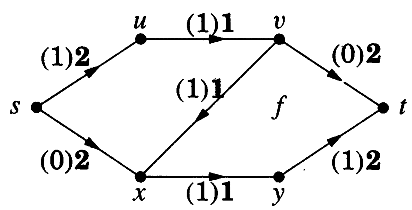
For each exiting edge vw with $f(vw) < c(vw)$ and $w \notin R$, add w to R.

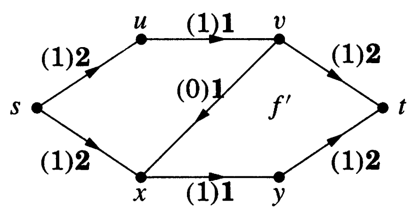
For each entering edge uv with $f(uv > 0)$ and $u \notin R$, add u to R.

Label each vertex added to R as "reached", and record v as the vertex reaching it. After exploring all edges at v, add v to S.

If the sink t has been reached (put in R), then trace the path reaching t to report an f-augmenting path and terminate. If $R = S$, then return the cut $[S, \overline{S}]$ and terminate. Otherwise, iterate. ■

4.3.10. Example. On the left below is the network of Example 4.3.3 with the flow f. We run the labeling algorithm. First we search from s and find excess capacity to u and x, labeling them reached. Now we have $u, v \in R - S$. There is no excess capacity on uv or xy, so searching from u reaches nothing, and also





†The precise notion of "dual problem" comes from linear programming. For our purposes, dual problems are a maximization problem and a minimization problem such that $a \leq b$ whenever a and b are the values of feasible solutions to the max problem and min problem, respectively. See Section 8.1 for further discussion.

searching from x does not reach y. However, there is nonzero flow on vx. Thus we label v from x. Now v is the only element of $R - S$, and searching from v reaches t. We labeled t from v, v from x, and x from s, so we have found the augmenting path s, x, v, t.

The tolerance on this path is 1, so the augmentation increases the flow value by 1. In the new flow f' shown on the right, every edge has unit flow except $f'(vx) = 0$. When we run the labeling algorithm again, we have excess capacity on su and sx and can label $\{u, x\}$, but from these nodes we can label no others. We terminate with $R = S = \{s, u, x\}$. The capacity of the resulting cut $[S, \overline{S}]$ is 2, which equals $\mathrm{val}(f')$ and proves that f' is a maximum flow. ∎

Repeated use of the labeling algorithm allows us to solve the maximum flow problem and prove the strong duality relationship.

4.3.11. Theorem. (Max-flow Min-cut Theorem—Ford and Fulkerson [1956]) In every network, the maximum value of a feasible flow equals the minimum capacity of a source/sink cut.

Proof: In the max-flow problem, the zero flow ($f(e) = 0$ for all e) is always a feasible flow and gives us a place to start. Given a feasible flow, we apply the labeling algorithm. It iteratively adds vertices to S (each vertex at most once) and terminates with $t \in R$ ("breakthrough") or with $S = R$.

In the breakthrough case, we have an f-augmenting path and increase the flow value. We then repeat the labeling algorithm. When the capacities are rational, each augmentation increases the flow by a multiple of $1/a$, where a is the least common multiple of the denominators, so after finitely many augmentations the capacity of some cut is reached. The labeling algorithm then terminates with $S = R$.

When terminating this way, we claim that $[S, T]$ is a source/sink cut with capacity $\mathrm{val}(f)$, where $T = \overline{S}$ and f is the present flow. It is a cut because $s \in S$ and $t \notin R = S$. Since applying the labeling algorithm to the flow f introduces no node of T into R, no edge from S to T has excess capacity, and no edge from T to S has nonzero flow in f. Hence $f^+(S) = \mathrm{cap}(S, T)$ and $f^-(S) = 0$.

Since the net flow out of any set containing the source but not the sink is $\mathrm{val}(f)$, we have proved

$$\mathrm{val}(f) = f^+(S) - f^-(S) = f^+(S) = \mathrm{cap}(S, T). \quad ∎$$

This proof of Theorem 4.3.11 requires rational capacities; otherwise, Algorithm 4.3.9 may yield augmenting paths forever! Ford and Fulkerson provided an example of this with only ten vertices (see Papadimitriou–Steiglitz [1982, p126-128]). Edmonds and Karp [1972] modified the labeling algorithm to use at most $(n^3 - n)/4$ augmentations in an n-vertex network and work for all real capacities. As in the bipartite matching problem (Theorem 3.2.22), this is done by searching always for shortest augmenting paths. Faster algorithms are now known; again we cite Ahuja–Magnanti–Orlin [1993] for a thorough discussion.

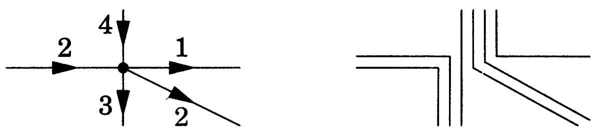
INTEGRAL FLOWS

In combinatorial applications, we typically have integer capacities and want a solution in which the flow on each edge is an integer.

4.3.12. Corollary. (Integrality Theorem) If all capacities in a network are integers, then there is a maximum flow assigning integral flow to each edge. Furthermore, some maximum flow can be partitioned into flows of unit value along paths from source to sink.

Proof: In the labeling algorithm of Ford and Fulkerson, the change in flow value when an augmenting path is found is always a flow value or the difference between a flow value and a capacity. When these are integers, the difference is also an integer. Starting with the zero flow, this implies that there is no first time when a noninteger flow appears.

The algorithm thus produces a maximum flow with integer flow on each edge. At each internal node, we now match units of entering flow to units of exiting flow. This forms s,t-paths and perhaps cycles. If a cycle arises, then we decrease flow on its edges by 1 to eliminate it without changing the flow value. This leaves $\mathrm{val}(f)$ paths from s to t, each corresponding to a unit of flow. ■



The integrality theorem yields paths of unit flow. In applications, we build networks where these units of flow have meaning.

The next two remarks show that the Max-flow Min-cut Theorem for networks with integer capacities is almost the same statement as Menger's Theorem for edge-disjoint paths in digraphs.

4.3.13. Remark. *Menger from Max-flow Min-cut.* When x,y are vertices in a digraph D, we can view D as a network with source x and sink y and capacity 1 on every edge. Capacity 1 ensures that units of flow from x to y correspond to pairwise edge-disjoint x,y-paths in D. Thus a flow of value k yields a set of k such paths.

Similarly, every source/sink partition S,T defines a set of edges whose deletion makes y unreachable from x: the set $[S,T]$. Since every capacity is 1, the size of this set is $\mathrm{cap}(S,T)$.

The paths and the edge cut we have obtained might not be optimal, but by the Max-flow Min-cut Theorem we have

$$\lambda'_D(x,y) \ge \max \mathrm{val}(f) = \min \mathrm{cap}(S,T) \ge \kappa'_D(x,y).$$

Since always $\kappa'(x,y) \ge \lambda'(x,y)$, equality now holds. ■

4.3.14. Remark. *Max-flow Min-cut from Menger.* To show that Menger's Theorem implies the Max-flow Min-cut Theorem for rational capacities, we take an arbitrary network and transform it into a digraph where we apply Menger's Theorem. By multiplying all capacities by the least common denominator, we may assume that the capacities are integers.

Given a network N with integer capacities, we form a digraph D by splitting each edge of capacity j into j edges with the same endpoints. For N, duality yields $\max \mathrm{val}(f) \le \min \mathrm{cap}(S, T)$. This time we want to use Menger's Theorem on D to obtain the reverse inquality, so in contrast to Remark 4.3.13 our desired computation is

$$\max \mathrm{val}(f) \ge \lambda'_D(s,t) = \kappa'_D(s,t) \ge \min \mathrm{cap}(S, T).$$

A set of $\lambda'(s,t)$ pairwise edge-disjoint s,t-paths in D collapses into a flow of value $\lambda'(s,t)$ in N, since the number of copies of each edge in D equals the capacity of the edge in N. Thus $\max \mathrm{val}(f) \ge \lambda'(s,t)$.

Now, let F be a set of $\kappa'(s,t)$ edges disconnecting t from s in D. If $e \in F$, then the minimality of F implies that $D - (F - e)$ has an s,t-path P through e. If some other copy e' of the edge $e = uv$ is not in F, then P can be rerouted along e' to obtain an s,t-path in $D - F$. Therefore, F contains all copies or no copies of each multiple edge in D. Hence $\kappa'(s,t)$ is the sum of the capacities on a set of edges that disconnects t from s in N. Letting S be the set of vertices reachable from s in $D - F$, we have $\mathrm{cap}(S, T) = \kappa'(s,t)$. The minimum cut has at most this capacity, so $\min \mathrm{cap}(S, T) \le \kappa'(s,t)$, and we have proved all the needed inequalities. ■

For combinatorial applications, Menger's Theorem may yield simpler proofs than the Max-flow Min-cut Theorem (compare Theorem 4.2.25 with ??). Nevertheless, our proof of Menger's Theorem in Section 4.2 is awkward to implement algorithmically. For large-scale computations, network flow and the Ford–Fulkerson labeling algorithm are more appropriate. Indeed, most algorithms that compute connectivity in graphs and digraphs use network flow methods (Stoer–Wagner [1994] presents a different approach).

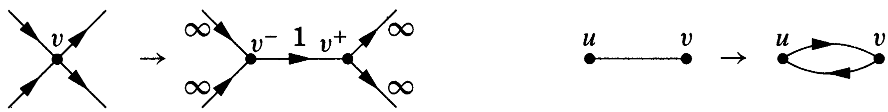
We present other network models for combinatorial problems. For example, the other local versions of Menger's Theorem can also be obtained directly.

4.3.15. Remark. *Other transformations.* For each version of Menger's Theorem, we encode the path problem using network flows with integer capacities.

To obtain a network model for the problem of internally disjoint paths in a digraph D, we must prevent two units of flow from passing through a vertex. This can be done by replacing each vertex v with two vertices v^-, v^+ that inherit the entering and exiting edges at v. By adding an edge of unit capacity from v^- to v^+, we obtain the effect of limiting flow through v to one unit. By putting very large capacity (essentially infinite) on the edges that were in D, we ensure that a minimum cut will count only edges of the form v^-v^+

To obtain a network model for the problem of edge-disjoint paths in a graph G, we must permit flow to pass either way in an edge. This can be done by replacing each edge uv with two directed edges uv and vu. When the network sends unit flow in both directions, in effect the edge is not being used at all.

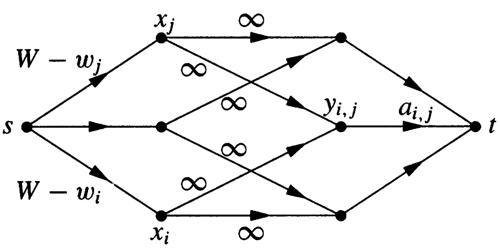
In each case, a flow in the network provides a set of paths, and a minimum cut leads to a separating set of vertices or edges. As in Remark 4.3.13, duality then gives us the desired equality in Menger's Theorem. To model the problem of internally disjoint paths in a graph, we need both of these transformations. Exercises 5–7 request the details of these proofs. ■



4.3.16. Application. *Baseball Elimination Problem* (Schwartz [1966]). At some time during the season, we may wonder whether team X can still win the championship. In other words, can winners be assigned for the remaining games so that no team ends with more victories than X? If so, then such an assignment exists with X winning all its remaining games, reaching W wins. We want to know whether winners can be chosen for other games so that no team obtains more than W wins. To test this, we create a network where units of flow correspond to the remaining games.

Let $X_1, \ldots, X_n$ be the other teams. Include nodes $x_1, \ldots, x_n$ for the n teams, nodes $y_{i,j}$ for the $\binom{n}{2}$ pairs of teams, and a source s and sink t. Put an edge from s to each team node and an edge from each pair node to t. Each pair node $y_{i,j}$ is entered by edges from x_i and x_j.

The capacities model the constraints. The capacity on edge $y_{i,j}t$ is $a_{i,j}$, the number of remaining games between X_i and X_j. Given that X_i has won w_i games already, the capacity on edge sx_i is $W - w_i$ to keep X in contention. The capacity on edges $x_i y_{i,j}$ and $x_j y_{i,j}$ is ∞ (the number of games x_i can win from x_j is constrained by the capacity on $y_{i,j}t$).



By the integrality theorem, a maximum flow breaks into flow units. Each unit corresponds to one game; the first edge specifies the winner, and the last edge specifies the pair. The network has a flow of value $\sum_{i,j} a_{i,j}$ if and only if *all* remaining games can be played with no team exceeding W wins; this is the condition for X remaining in contention.

By the Max-flow Min-cut Theorem, there is a flow of value $\sum a_{i,j}$ if and only if every cut has capacity at least $\sum a_{i,j}$. Let S, T be a cut with finite capacity, and let $Z = \{i\colon x_i \in T\}$. Since $c(x_i y_{i,j}) = \infty$, we cannot have $x_i \in S$ and $y_{i,j} \in T$; thus $y_{i,j} \in S$ whenever i or j is not in Z. To minimize capacity, we put $y_{i,j} \in T$ whenever $\{i, j\} \subseteq Z$. Now $\mathrm{cap}(S, T) = \sum_{i\in Z}(W - w_i) + \sum_{\{i,j\}\not\subseteq Z} a_{i,j}$. The condition that every cut have capacity at least $\sum a_{i,j}$ becomes

$$\sum_{i\in Z}(W - w_i) \ge \sum_{\{i,j\}\subseteq Z} a_{i,j} \qquad \text{for all } Z \subseteq [n].$$

Note that this condition is obviously necessary; it states that we need enough leeway in the total wins among teams indexed by Z in order to accommodate winners for all the games among these teams. We have proved TONCAS. ■

Combinatorial applications of network flow usually involve showing that the desired configuration exists if and only if a related network has a large enough flow. As in Application 4.3.16, the Max-flow Min-cut Theorem then yields a necessary and sufficient condition for its existence. Other examples include most of Exercises 5– and also Exercise 13 and Theorems 4.3.17–4.3.18.

SUPPLIES AND DEMANDS (optional)

Next we consider a more general network model. We allow multiple sources and sinks, and also we associate with each source x_i a **supply** $\sigma(x_i)$ and with each sink y_j a **demand** $\partial(y_j)$. To the capacity constraints for edges and conservation constraints for internal nodes, we add **transportation constraints** for the sources and sinks.

$$f^+(x_i) - f^-(x_i) \le \sigma(x_i) \text{ for each source } x_i$$
$$f^-(y_j) - f^+(y_j) \ge \partial(y_j) \text{ for each sink } y_j$$

The resulting configuration is a **transportation network**. With positive values for the demands, the zero flow is not feasible. We seek a feasible flow satisfying these additional constraints. The "supply/demand" terminology suggests the constraints; we must satisfy the demands at the sinks without exceeding the available supply at any source. This model is appropriate when a company has multiple distribution centers (sources) and retail outlets (sinks).

Let X and Y denote the sets of sources and sinks, respectively. Let $\sigma(A) = \sum_{v\in A}\sigma(v)$ and $\partial(B) = \sum_{v\in B}\partial(v)$ denote the total supply or demand at a set $A \subseteq X$ or $B \subseteq Y$. For a set F of edges, let $c(F) = \sum_{e\in F} c(e)$. Given a set T of vertices, the **net demand** $\partial(Y \cap T) - \sigma(X \cap T)$ must be satisfied by flow from the remaining vertices. Hence it is necessary that $c([\overline{T}, T])$ be at least this large. Satisfying this for every set T is also sufficient for a feasible flow (TONCAS).

4.3.17. Theorem. (Gale [1957]) In a transportation network N with sources X and sinks Y, a feasible flow exists if and only if

$$c([S, T]) \ge \partial(Y \cap T) - \sigma(X \cap T)$$

for every partition of the vertices of N into sets S and T.

Proof: We have already observed the necessity of the condition. For sufficiency, construct a new network N' by adding a supersource s and a supersink t, with an edge of capacity $\sigma(x_i)$ from s to each $x_i \in X$ and an edge of capacity $\partial(y_j)$ from each $y_j \in Y$ to t. The transportation network N has a feasible flow if and only if N' has a flow saturating each edge to t (a flow of value $\partial(Y)$).

By the Ford–Fulkerson Theorem, we know that N' has a flow of value $\partial(Y)$ if and only if $\text{cap}(S \cup s, T \cup t) \geq \partial(Y)$ for each partition S, T of $V(N)$. The cut $[S \cup s, T \cup t]$ in N' consists of $[S, T]$ from N, plus edges from s to T and edges from S to t in N'. Hence

$$\text{cap}(S \cup s, T \cup t) = c(S, T) + \sigma(T \cap X) + \partial(S \cap Y).$$

We now have $\text{cap}(S \cup s, T \cup t) \geq \partial(Y)$ if and only if

$$c(S, T) + \sigma(X \cap T) \geq \partial(Y) - \partial(Y \cap S) = \partial(Y \cap T),$$

which is the condition assumed. ■

For specific instances, the construction of N' is the key point, because we produce a feasible flow in N (when it exists) by running the Ford–Fulkerson algorithm on the network N'. When costs (per unit flow) are attached to the edges, we have the Min-cost Flow Problem, which generalizes the Transportation Problem of Application 3.2.14. Solution algorithms for the Min-cost Flow Problem appear in Ford–Fulkerson [1962] and in Ahuja–Magnanti–Orlin [1993].

We discuss several applications of Gale's condition. A pair of integer lists $p = (p_1, \dots, p_m)$ and $q = (q_1, \dots, q_n)$ is **bigraphic** (Exercise 1.4.31) if there is a simple X, Y-bigraph such that the vertices of X have degrees $p_1, \dots, p_m$ and the vertices of Y have degrees $q_1, \dots, q_n$. Clearly $\sum p_i = \sum q_j$ is necessary, but this condition is not sufficient. To test whether (p, q) is bigraphic, we create a network in which units of flows will correspond to edges in the desired graph. The result is a bipartite analogue of the Erdős–Gallai condition for graphic sequences (Exercise 3.3.28).

4.3.18. Theorem. (Gale [1957], Ryser [1957]) If p, q are lists of nonnegative integers with $p_1 \geq \cdots \geq p_m$ and $q_1 \geq \cdots \geq q_n$, then (p, q) is bigraphic if and only if $\sum_{i=1}^{m} \min\{p_i, k\} \geq \sum_{j=1}^{k} q_j$ for $1 \leq k \leq n$.

Proof: *Necessity.* Let G be a simple X, Y-bigraph realizing (p, q). Consider the edges incident to a set of k vertices in Y. Because G is simple, each $x_i \in X$ is incident to at most k of these edges, and also x_i is incident to at most p_i of these edges. Hence $\sum_{i=1}^{m} \min\{p_i, k\}$ is an upper bound on the number of edges incident to any k vertices of Y, such as those with degrees $q_1, \dots, q_k$.

Sufficiency. Given (p, q), create a network N with an edge of capacity 1 from x_i to y_j for each i, j, and let $\sigma(x_i) = p_i$ and $\partial(y_j) = q_j$. Unit capacity prevents multiple edges, and (p, q) is realizable if and only if N has a feasible flow.

It suffices to show that the stated condition on p and q implies the condition of Theorem 4.3.17. For $S \subseteq V(N)$, let $I(S) = \{i\colon x_i \in S\}$ and $J(S) = \{j\colon y_j \in S\}$. For a partition S, T of $V(N)$, we now have $\sigma(X \cap T) = \sum_{i \in I(T)} p_i$ and $\partial(Y \cap T) = \sum_{j \in J(T)} q_j$, and we have $c([S, T]) = |I(S)| \cdot |J(T)|$.

Letting $k = |J(T)|$, this last quantity becomes

$$c([S,T]) = |I(S)|\,k = \sum_{i\in I(S)} k \ge \sum_{i\in I(S)} \min\{p_i, k\}.$$

Also $\sum_{i\in I(T)} p_i \ge \sum_{i\in I(T)} \min\{p_i, k\}$, and $\sum_{j\in J(T)} q_j \le \sum_{j=1}^{k} q_j$. Combining these inequalities, the condition $\sum_{i=1}^{m} \min\{p_i, k\} \ge \sum_{j=1}^{k} q_j$ implies $c([S,T]) \ge \partial(Y\cap T) - \sigma(X\cap T)$. Since this holds for each partition S,T, the network has a feasible flow, which yields the desired bipartite graph. ∎

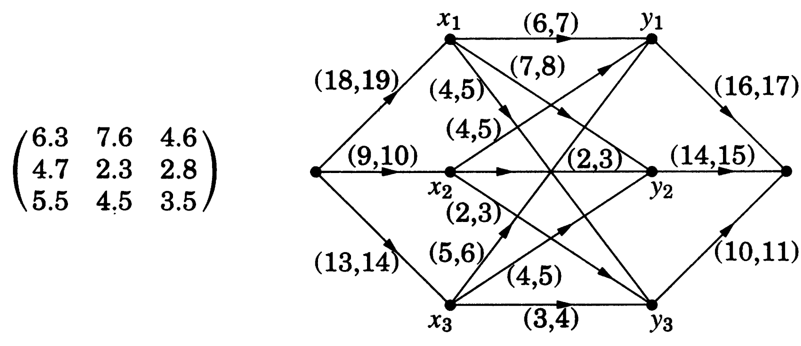
We can extend the maximum flow problem by imposing a nonnegative lower bound on the permitted flow in each edge. The capacity constraint remains as an upper bound, so we require $l(e) \le f(e) \le u(e)$ for the flow $f(e)$. We still impose conservation constraints on the internal nodes. If we have a feasible flow, then an easy modification of the Ford–Fulkerson labeling algorithm allows us to find a maximum (or minimum) feasible flow (Exercise 4). The difficulty is finding an initial feasible flow. First we present an application.

4.3.19. Application. *Matrix rounding* (Bacharach [1966]). We may want to round the entries of a data matrix up or down to integers. We also want to present integers for the row sums and column sums. The sum of each rounded row or column should be a rounding of the original sum. The resulting integer matrix, if it exists, is a **consistent rounding**.

We can represent the consistent rounding problem as a feasible flow problem. Establish vertices $x_1, \ldots, x_n$ for the rows and vertices $y_1, \ldots, y_n$ for the columns of the matrix. Add a source s and a sink t. Add edges sx_i, x_iy_j, y_jt for all values of i and j. If the matrix has entries $a_{i,j}$ with row-sums $r_1, \ldots, r_n$ and column-sums $s_1, \ldots, s_n$, set

$$\begin{array}{lll} l(sx_i) = \lfloor r_i \rfloor & l(x_iy_j) = \lfloor a_{i,j} \rfloor & l(y_jt) = \lfloor c_j \rfloor \\ u(sx_i) = \lceil r_i \rceil & u(x_iy_j) = \lceil a_{i,j} \rceil & u(y_jt) = \lceil c_j \rceil \end{array}$$

We test for a feasible flow by transforming again to an ordinary maximum flow problem. With these two transformations, we can use network flow to test for the existence of a consistent rounding. ∎



4.3.20. Solution. *Circulations and flows with lower bounds.* In a maximum flow problem with upper and lower bounds on edge capacities, the zero flow is not feasible, so the Ford–Fulkerson labeling algorithm has no place to start. We must first obtain a feasible flow, after which an easy modification of the labeling algorithm applies (Exercise 4).

The first step is to add an edge of infinite capacity from the sink to the source. The resulting network has a feasible flow with conservation at *every* node (called a **circulation**) if and only if the original network has a feasible flow. In a circulation problem, there is no source or sink.

Next, we convert a feasible circulation problem C into a maximum flow problem N by introducing supplies or demands at the nodes and adding a source and sink to satisfy the supplies and demands. Given the flow constraints $l(e) \le f(e) \le u(e)$, let $c(e) = u(e) - l(e)$ for each edge e. For each vertex v, let

$$\begin{aligned} l^-(v) &= \textstyle\sum_{e\in[V(C)-v,v]} l(e), \\ l^+(v) &= \textstyle\sum_{e\in[v,V(C)-v]} l(e), \\ b(v) &= l^-(v) - l^+(v). \end{aligned}$$

Since each $l(uv)$ contributes to $l^+(u)$ and $l^-(v)$, we have $\sum b(v) = 0$. A feasible circulation f must satisfy the flow constraints at each edge and satisfy $f^+(v) - f^-(v) = 0$ at each node. Letting $f'(e) = f(e) - l(e)$, we find that f is a feasible circulation in C if and only if f' satisfies $0 \le f'(e) \le c(e)$ on each edge and $f'^+(v) - f'^-(v) = b(v)$ at each vertex.

This transforms the feasible circulation problem into a flow problem with supplies and demands. If $b(v) \ge 0$, then v supplies flow $|b(v)|$ to the network; otherwise v demands $|b(v)|$. To restore conservation constraints, we add a source s with an edge of capacity $b(v)$ to each v with $b(v) \ge 0$, and we add a sink t with an edge of capacity $-b(v)$ from each v with $b(v) < 0$. This completes the construction of N.

Let α be the total capacity on the edges leaving s; since $\sum b(v) = 0$, the edges entering t also have total capacity α. Now C has a feasible circulation f if and only if N has a flow of value α (saturating all edges out of s or into t). ∎

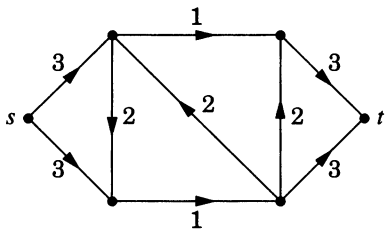
4.3.21. Corollary. A network D with conservation constraints at every node has a feasible circulation if and only if $\sum_{e\in[S,\overline{S}]} l(e) \le \sum_{e\in[\overline{S},S]} u(e)$ for every $S \subseteq V(D)$.

Proof: We can stop before the last step in the discussion of Solution 4.3.20 and interpret our problem with supplies and demands in the model of Theorem 4.3.17. Since $\sum b(v) = 0$, the only way to satisfy all the demands is to use up all the supply. Hence there is a circulation if and only if the supply/demand problem with supplies $\sigma(v) = b(v)$ for $\{v \in V(D) : b(v) \ge 0\}$ and demands $\partial(v) = -b(v)$ for $\{v \in V(D) : b(v) < 0\}$ has a solution.

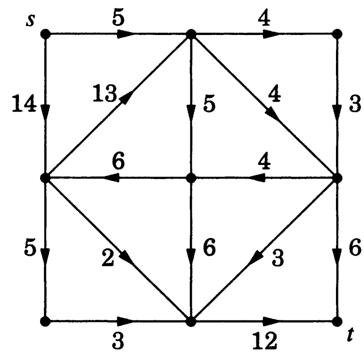
Theorem 4.3.17 characterizes when this problem has a solution. Translated back into the lower and upper bounds on flow in the original problem (Exercise 22), the criterion of Theorem 4.3.17 becomes $\sum_{e\in[S,\overline{S}]} l(e) \le \sum_{e\in[\overline{S},S]} u(e)$ for every $S \subseteq V(D)$. ∎

EXERCISES

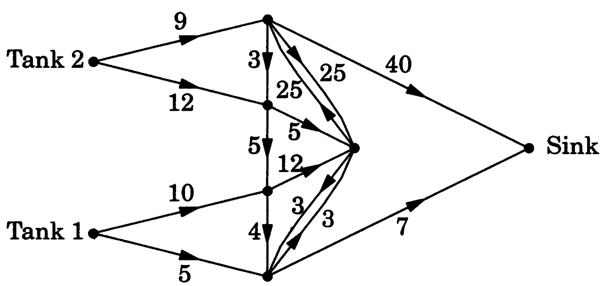
4.3.1. (−) In the network below, list all integer-valued feasible flows and select a flow of maximum value (this illustrates the advantage of duality over exhaustive search). Prove that this flow is a maximum flow by exhibiting a cut with the same value. Determine the number of source/sink cuts. (Comment: There is a nonzero flow with value 0.)



4.3.2. (−) In the network below, find a maximum flow from s to t. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.



4.3.3. (−) A kitchen sink draws water from two tanks according to the network of pipes with capacities per unit time shown below. Find the maximum flow. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.



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4.3.4. Let N be a network with edge capacity and node conservation constraints plus lower bound constraints $l(e)$ on the flow in edges, meaning that $f(e) \geq l(e)$ is required. If an initial feasible flow is given, how can the Ford–Fulkerson labeling algorithm be modified to search for a maximum feasible flow in this network?

4.3.5. (!) Use network flows to prove Menger's Theorem for internally-disjoint paths in digraphs: $\kappa(x, y) = \lambda(x, y)$ when xy is not an edge. (Hint: Use the first transformation suggested in Remark 4.3.15.)

4.3.6. (!) Use network flows to prove Menger's Theorem for edge-disjoint paths in graphs: $\kappa'(x, y) = \lambda'(x, y)$. (Hint: Use the second transformation suggested in Remark 4.3.15.)

4.3.7. (!) Use network flows to prove Menger's Theorem for nonadjacent vertices in graphs: $\kappa(x, y) = \lambda(x, y)$. (Hint: Use both transformations suggested in Remark 4.3.15.)

4.3.8. Let G be a directed graph with $x, y \in V(G)$. Suppose that capacities are specified *not* on the edges of G, but rather on the *vertices* (other than x, y); for each vertex there is a fixed limit on the total flow through it. There is no restriction on flows in edges. Show how to use ordinary network flow theory to determine the maximum value of a feasible flow from x to y in the vertex-capacitated graph G.

4.3.9. Use network flows to prove that a graph G is connected if and only if for every partition of $V(G)$ into two nonempty sets S, T, there is an edge with one endpoint in S and one endpoint in T. (Comment: Chapter 1 contains an easy direct proof of the conclusion, so this is an example of "using a sledgehammer to squash a bug".)

4.3.10. (!) Use network flows to prove the König–Egerváry Theorem ($\alpha'(G) = \beta(G)$ if G is bipartite).

4.3.11. Show that the Augmenting Path Algorithm for bipartite graphs (Algorithm 3.2.1) is a special case of the Ford–Fulkerson Labeling Algorithm.

4.3.12. Let $[S, \overline{S}]$ and $[T, \overline{T}]$ be source/sink cuts in a network N.

a) Prove that $\text{cap}(S \cup T, \overline{S \cup T}) + \text{cap}(S \cap T, \overline{S \cap T}) \le \text{cap}([S, \overline{S}]) + \text{cap}(T, \overline{T})$. (Hint: Draw a picture and consider contributions from various types of edges.)

b) Suppose that $[S, \overline{S}]$ and $[T, \overline{T}]$ are minimum cuts. Conclude from part (a) that $[S \cup T, \overline{S \cup T}]$ and $[S \cap T, \overline{S \cap T}]$ are also minimum cuts. Conclude also that no edge between $S - T$ and $T - S$ has positive capacity.

4.3.13. (!) Several companies send representatives to a conference; the ith company sends m_i representatives. The organizers of the conference conduct simultaneous networking groups; the jth group can accommodate up to n_j participants. The organizers want to schedule all the participants into groups, but the participants from the same company must be in different groups. The groups need not all be filled.

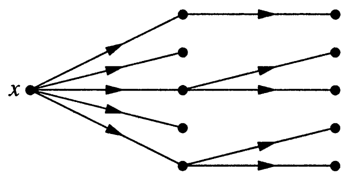
a) Show how to use network flows to test whether the constraints can be satisfied.

b) Let p be the number of companies, and let q be the number of groups, indexed so that $m_1 \ge \cdots \ge m_p$ and $n_1 \le \cdots \le n_q$. Prove that there exists an assignment of participants to groups that satisfies all the constraints if and only if, for all $0 \le k \le p$ and $0 \le l \le q$, it holds that $k(q - l) + \sum_{j=1}^{l} n_j \ge \sum_{i=1}^{k} m_i$.

4.3.14. In a large university with k academic departments, we must appoint an important committee. One professor will be chosen from each department. Some professors have joint appointments in two or more departments, but each must be the designated representative of at most one department. We must use equally many assistant professors, associate professors, and full professors among the chosen representatives (assume that k is divisible by 3). How can the committee be found? (Hint: Build a network in which units of flow correspond to professors chosen for the committee and capacities enforce the various constraints. Explain how to use the network to test whether such a committee exists and find it if it does.) (Hall [1956])

4.3.15. Let G be a weighted graph. Let the *value* of a spanning tree be the minimum weight of its edges. Let the *cap* from a edge cut $[S, \overline{S}]$ be the maximum weight of its edges. Prove that the maximum value of a spanning tree of G equals the minimum cap of an edge cut in G. (Ahuja–Magnanti–Orlin [1993, p538])

4.3.16. (+) Let x be a vertex of maximum outdegree in a tournament T. Prove that T has a spanning directed tree rooted at x such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2. (Hint: Create a network to model the desired paths to the non-successors of x, and show that every cut has enough capacity. Comment: This strengthens Proposition 1.4.30 about kings in tournaments; no vertex need be an intermediate vertex for more than two others.) (Lu [1996])



4.3.17. (∗−) Use the Gale–Ryser Theorem (Theorem 4.3.18) to determine whether there is a simple bipartite graph in which the vertices in one partite set have degrees (5, 4, 4, 2, 1) and the vertices in the other partite set also have degrees (5, 4, 4, 2, 1).

4.3.18. (∗−) Given list $r = (r_1, \dots, r_n)$ and $s = (s_1, \dots, s_n)$, obtain necessary and sufficient conditions for the existence of a digraph D with vertices $v_1, \dots, v_n$ such that each ordered pair occurs at most once as an edge and $d^+(v_i) = r_i$ and $d^-(v_i) = s_i$ for all i.

4.3.19. (∗−) Find a consistent rounding of the data in the matrix below. Is it unique? (Every entry must be 0 or 1.)

$$\begin{pmatrix} .55 & .6 & .6 \\ .55 & .65 & .7 \\ .6 & .65 & .7 \end{pmatrix}$$

4.3.20. (∗) Prove that every two-by-two matrix can be consistently rounded.

4.3.21. (∗) Suppose that every entry in an n-by-n matrix is strictly between $1/n$ and $1/(n-1)$. Describe all consistent roundings.

4.3.22. (∗) Complete the details of proving Corollary 4.3.21, proving the necessary and sufficient condition for a circulation in a network with lower and upper bounds.

4.3.23. (∗!) A $(k+l)$-regular graph G is **(k, l)-orientable** if it can be oriented so that each indegree is k or l.

a) Prove that G is (k, l)-orientable if and only if there is a partition X, Y of $V(G)$ such that for every $S \subseteq V(G)$,

$$(k-l)(|X \cap S| - |Y \cap S|) \le \left|[S, \overline{S}]\right|.$$

(Hint: Use Theorem 4.3.17.)

b) Conclude that if G is (k, l)-orientable and $k > l$, then G is also $(k-1, l+1)$-orientable. (Bondy–Murty [1976, p210–211])

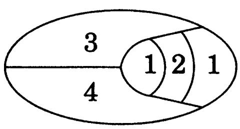
Chapter 5

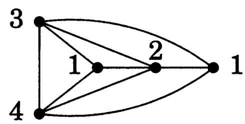
Coloring of Graphs

5.1. Vertex Coloring and Upper Bounds

The committee-scheduling example (Example 1.1.11) used graph coloring to model avoidance of conflicts. Similarly, in a university we want to assign time slots for final examinations so that two courses with a common student have different slots. The number of slots needed is the chromatic number of the graph in which two courses are adjacent if they have a common student.

Coloring the regions of a map with different colors on regions with common boundaries is another example; we return to it in Chapter 6. The map on the left below has five regions, and four colors suffice. The graph on the right models the "common boundary" relation and the corresponding coloring. Labeling of vertices is our context for coloring problems.





DEFINITIONS AND EXAMPLES

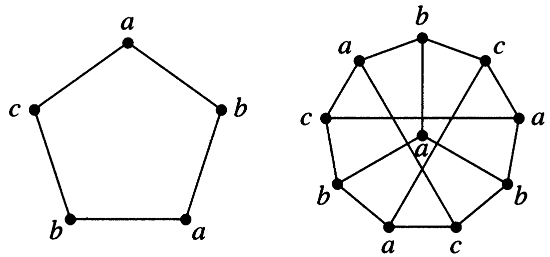
Graph coloring takes its name from the map-coloring application. We assign labels to vertices. When the numerical value of the labels is unimportant, we call them "colors" to indicate that they may be elements of any set.

5.1.1. Definition. A **k-coloring** of a graph G is a labeling $f\colon V(G) \to S$, where $|S| = k$ (often we use $S = [k]$). The labels are **colors**; the vertices of one color form a **color class**. A k-coloring is **proper** if adjacent vertices have different labels. A graph is **k-colorable** if it has a proper k-coloring. The **chromatic number** $\chi(G)$ is the least k such that G is k-colorable.

5.1.2. Remark. In a proper coloring, each color class is an independent set, so G is k-colorable if and only if $V(G)$ is the union of k independent sets. Thus "k-colorable" and "k-partite" have the same meaning. (The usage of the two terms is slightly different. Often "k-partite" is a structural hypothesis, while "k-colorable" is the result of an optimization problem.)

Graphs with loops are uncolorable; we cannot make the color of a vertex different from itself. Therefore, **in this chapter all graphs are loopless.** Also, multiple edges are irrelevant; extra copies don't affect colorings. Thus we usually think in terms of simple graphs when discussing colorings, and we will name edges by their endpoints. Most of the statements made without restriction to simple graphs remain valid when multiple edges are allowed. ∎

5.1.3. Example. Since a graph is 2-colorable if and only if it is bipartite, C_5 and the Petersen graph have chromatic number at least 3. Since they are 3-colorable, as shown below, they have chromatic number exactly 3. ∎



5.1.4. Definition. A graph G is **k-chromatic** if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an **optimal coloring**. If $\chi(H) < \chi(G) = k$ for every proper subgraph H of G, then G is **color-critical** or **k-critical**.

5.1.5. Example. *k-critical graphs for small k.* Properly coloring a graph needs at least two colors if and only if the graph has an edge. Thus K_2 is the only 2-critical graph (similarly, K_1 is the only 1-critical graph). Since 2-colorable is the same as bipartite, the characterization of bipartite graphs implies that the 3-critical graphs are the odd cycles.

We can test 2-colorability of a graph G by computing distances from a vertex x (in each component). Let $X = \{u \in V(G): d(u,x) \text{ is even}\}$, and let $Y = \{u \in V(G): d(u,x) \text{ is odd}\}$. The graph G is bipartite if and only if X, Y is a bipartition, meaning that $G[X]$ and $G[Y]$ are independent sets.

No good characterization of 4-critical graphs or test for 3-colorability is known. Appendix B discusses the computational ramifications. ∎

5.1.6. Definition. The **clique number** of a graph G, written $\omega(G)$, is the maximum size of a set of pairwise adjacent vertices (clique) in G.

We have used $\alpha(G)$ for the independence number of G; the usage of $\omega(G)$ is analogous. The letters α and ω are the first and last in the Greek alphabet.

This is consistent with viewing independent sets and cliques as the beginning and end of the "evolution" of a graph (see Section 8.5).

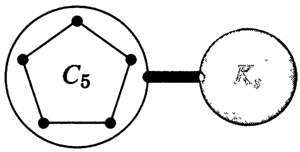
5.1.7. Proposition. For every graph G, $\chi(G) \ge \omega(G)$ and $\chi(G) \ge \frac{n(G)}{\alpha(G)}$.

Proof: The first bound holds because vertices of a clique require distinct colors. The second bound holds because each color class is an independent set and thus has at most $\alpha(G)$ vertices. ■

Both bounds in Proposition 5.1.7 are tight when G is a complete graph.

5.1.8. Example. *$\chi(G)$ may exceed $\omega(G)$.* For $r \ge 2$, let $G = C_{2r+1} \vee K_s$ (the join of C_{2r+1} and K_s—see Definition 3.3.6). Since C_{2r+1} has no triangle, $\omega(G) = s+2$.

Properly coloring the induced cycle requires at least three colors. The s-clique needs s colors. Since every vertex of the induced cycle is adjacent to every vertex of the clique, these s colors must differ from the first three, and $\chi(G) \ge s+3$. We conclude that $\chi(G) > \omega(G)$. ■

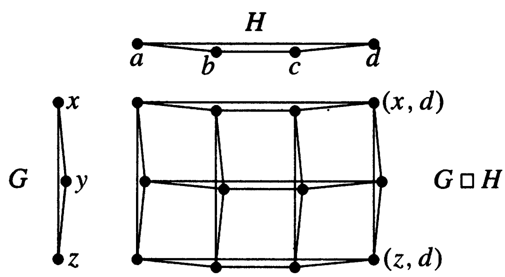


Exercises 23–30 discuss the chromatic number for special families of graphs. We can also ask how it behaves under graph operations. For the disjoint union, $\chi(G+H) = \max\{\chi(G), \chi(H)\}$. For the join, $\chi(G \vee H) = \chi(G) + \chi(H)$. Next we introduce another combining operation.

5.1.9. Definition. The **cartesian product** of G and H, written $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

5.1.10. Example. The cartesian product operation is symmetric; $G \square H \cong H \square G$. Below we show $C_3 \,\square\, C_4$. The hypercube is another familiar example: $Q_k = Q_{k-1} \,\square\, K_2$ when $k \ge 1$. The m-by-n **grid** is the cartesian product $P_m \,\square\, P_n$.

In general, $G \,\square\, H$ decomposes into copies of H for each vertex of G and copies of G for each vertex of H (Exercise 10). We use $\square$ instead of $\times$ to avoid confusion with other product operations, reserving $\times$ for the cartesian product of vertex sets. The symbol $\square$, due to Rödl, evokes the identity $K_2 \,\square\, K_2 = C_4$. ■

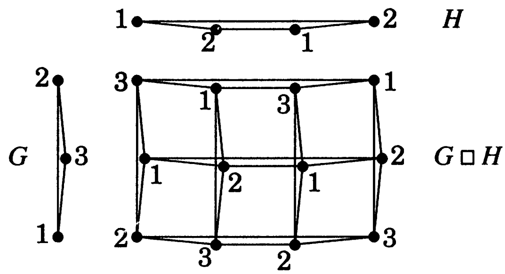


5.1.11. Proposition. (Vizing [1963], Aberth [1964]) $\chi(G\square H)=\max\{\chi(G),\chi(H)\}$.

Proof: The cartesian product $G \square H$ contains copies of G and H as subgraphs, so $\chi(G \square H) \geq \max\{\chi(G), \chi(H)\}$.

Let $k = \max\{\chi(G), \chi(H)\}$. To prove the upper bound, we produce a proper k-coloring of $G \square H$ using optimal colorings of G and H. Let g be a proper $\chi(G)$-coloring of G, and let h be a proper $\chi(H)$-coloring of H. Define a coloring f of $G \square H$ by letting $f(u, v)$ be the congruence class of $g(u) + h(v)$ modulo k. Thus f assigns colors to $V(G \square H)$ from a set of size k.

We claim that f properly colors $G \square H$. If (u, v) and (u', v') are adjacent in $G \square H$, then $g(u) + h(v)$ and $g(u') + h(v')$ agree in one summand and differ by between 1 and k in the other. Since the difference of the two sums is between 1 and k, they lie in different congruence classes modulo k. ■



The cartesian product allows us to compute chromatic numbers by computing independence numbers, because a graph G is m-colorable if and only if the cartesian product $G \square K_m$ has an independent set of size $n(G)$ (Exercise 31).

UPPER BOUNDS

Most upper bounds on the chromatic number come from algorithms that produce colorings. For example, assigning distinct colors to the vertices yields $\chi(G) \leq n(G)$. This bound is best possible, since $\chi(K_n) = n$, but it holds with equality only for complete graphs. We can improve a "best-possible" bound by obtaining another bound that is always at least as good. For example, $\chi(G) \leq n(G)$ uses nothing about the structure of G; we can do better by coloring the vertices in some order and always using the "least available" color.

5.1.12. Algorithm. (Greedy coloring)
The **greedy coloring** relative to a vertex ordering $v_1, \ldots, v_n$ of $V(G)$ is obtained by coloring vertices in the order $v_1, \ldots, v_n$, assigning to v_i the smallest-indexed color not already used on its lower-indexed neighbors. ■

5.1.13. Proposition. $\chi(G) \leq \Delta(G) + 1$.

Proof: In a vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbors, so the greedy coloring cannot be forced to use more than $\Delta(G) + 1$ colors. This proves constructively that $\chi(G) \leq \Delta(G) + 1$. ■

The bound $\Delta(G) + 1$ is the worst upper bound that greedy coloring could produce (although optimal for cliques and odd cycles). Choosing the vertex ordering carefully yields improvements. We can avoid the trouble caused by vertices of high degree by putting them at the beginning, where they won't have many earlier neighbors (see Exercise 36 for a better ordering).

5.1.14. Proposition. (Welsh–Powell [1967]) If a graph G has degree sequence $d_1 \geq \cdots \geq d_n$, then $\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}$.

Proof: We apply greedy coloring to the vertices in nonincreasing order of degree. When we color the ith vertex v_i, it has at most $\min\{d_i, i-1\}$ earlier neighbors, so at most this many colors appear on its earlier neighbors. Hence the color we assign to v_i is at most $1 + \min\{d_i, i-1\}$. This holds for each vertex, so we maximize over i to obtain the upper bound on the maximum color used. ■

The bound in Proposition 5.1.14 is always at most $1 + \Delta(G)$, so this is always at least as good as Proposition 5.1.13. It gives the optimal upper bound in Example 5.1.8, while $1 + \Delta(G)$ does not.

In Proposition 5.1.14, we use greedy coloring with a well-chosen ordering. In fact, every graph G has some vertex ordering for which the greedy algorithm uses only $\chi(G)$ colors (Exercise 33). Usually it is hard to find such an ordering.

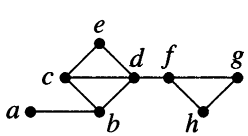
Our next example introduces a class of graphs where such an ordering is easy to find. The ordering produces a coloring that achieves equality in the bound $\chi(G) \geq \omega(G)$.

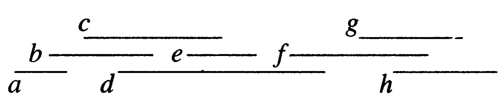
5.1.15. Example. *Register allocation and interval graphs.* A computer program stores the values of its variables in memory. For arithmetic computations, the values must be entered in easily accessed locations called *registers*. Registers are expensive, so we want to use them efficiently. If two variables are never used simultaneously, then we can allocate them to the same register. For each variable, we compute the first and last time when it is used. A variable is *active* during the interval between these times.

We define a graph whose vertices are the variables. Two vertices are adjacent if they are active at a common time. The number of registers needed is the chromatic number of this graph. The time when a variable is active is an interval, so we we obtain a special type of representation for the graph.

An **interval representation** of a graph is a family of intervals assigned to the vertices so that vertices are adjacent if and only if the corresponding intervals intersect. A graph having such a representation is an **interval graph**.

For the vertex ordering a, b, c, d, e, f, g, h of the interval graph below, greedy coloring assigns $1, 2, 1, 3, 2, 1, 2, 3$, respectively, which is optimal. Greedy colorings relative to orderings starting $a, d, \ldots$ use four colors. ■





5.1.16. Proposition. If G is an interval graph, then $\chi(G) = \omega(G)$.

Proof: Order the vertices according to the left endpoints of the intervals in an interval representation. Apply greedy coloring, and suppose that x receives k, the maximum color assigned. Since x does not receive a smaller color, the left endpoint a of its interval belongs also to intervals that already have colors 1 through $k-1$. These intervals all share the point a, so we have a k-clique consisting of x and neighbors of x with colors 1 through $k-1$. Hence $\omega(G) \geq k \geq \chi(G)$. Since $\chi(G) \geq \omega(G)$ always, this coloring is optimal. ∎

5.1.17.* Remark. The greedy coloring algorithm runs rapidly. It is "on-line" in the sense that it produces a proper coloring even if it sees only one new vertex at each step and must color it with no option to change earlier colors. For a random vertex ordering in a random graph (see Section 8.5), greedy coloring almost always uses only about twice as many colors as the minimum, although with a bad ordering it may use many colors on a tree (Exercise 34). ∎

We began with greedy coloring to underscore the constructive aspect of upper bounds on chromatic number. Other bounds follow from the properties of k-critical graphs but don't produce proper colorings: every k-chromatic graph *has* a k-critical subgraph, but we have no good algorithm for *finding* one. We derive the next bound using critical subgraphs; it can also be proved using greedy coloring (Exercise 36).

5.1.18. Lemma. If H is a k-critical graph, then $\delta(H) \geq k-1$.

Proof: Let x be a vertex of H. Because H is k-critical, $H-x$ is $k-1$-colorable. If $d_H(x) < k-1$, then the $k-1$ colors used on $H-x$ do not all appear on $N(x)$. We can assign x a color not used on $N(x)$ to obtain a proper $k-1$-coloring of H. This contradicts our hypothesis that $\chi(H) = k$. We conclude that $d_H(x) \geq k-1$ (for each $x \in V(H)$). ∎

5.1.19. Theorem. (Szekeres–Wilf [1968]) If G is a graph, then $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$.

Proof: Let $k = \chi(G)$, and let H' be a k-critical subgraph of G. Lemma 5.1.18 yields $\chi(G) - 1 = \chi(H') - 1 \leq \delta(H') \leq \max_{H \subseteq G} \delta(H)$. ∎

The next bound involves orientations (see also Exercises 43–45).

5.1.20. Example. If G is bipartite, then the orientation of G that directs every edge from one partite set to the other has no path (in the directed sense) of length more than 1. The next theorem thus implies that $\chi(G) \leq 2$.

Every orientation of an odd cycle must somewhere have two consecutive edges in the same direction. Thus each orientation has a path of length at least two, and the theorem confirms that an odd cycle is 3-chromatic. ∎

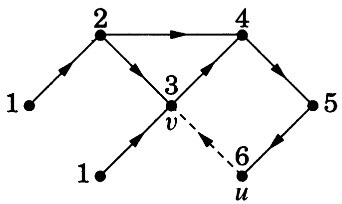
5.1.21. Theorem. Gallai–Roy–Vitaver Theorem (Gallai [1968], Roy [1967], Vitaver [1962]) If D is an orientation of G with longest path length $l(D)$, then $\chi(G) \leq 1 + l(D)$. Furthermore, equality holds for some orientation of G.

Proof: Let D be an orientation of G. Let D' be a maximal subdigraph of D that contains no cycle (in the example below, uv is the only edge of D not in D'). Note that D' includes all vertices of G. Color $V(G)$ by letting $f(v)$ be 1 plus the length of the longest path in D' that ends at v.

Let P be a path in D', and let u be the first vertex of P. Every path in D' ending at u has no other vertex on P, since D' is acyclic. Therefore, each path ending at u (including the longest such path) can be lengthened along P. This implies that f strictly increases along each path in D'.

The coloring f uses colors 1 through $1+l(D')$ on $V(D')$ (which is also $V(G)$). We claim that f is a proper coloring of G. For each $uv \in E(D)$, there is a path in D' between its endpoints (since uv is an edge of D' or its addition to D' creates a cycle). This implies that $f(u) \neq f(v)$, since f increases along paths of D'.

To prove the second statement, we construct an orientation D^* such that $l(D^*) \leq \chi(G) - 1$. Let f be an optimal coloring of G. For each edge uv in G, orient it from u to v in D^* if and only if $f(u) < f(v)$. Since f is a proper coloring, this defines an orientation. Since the labels used by f increase along each path in D^*, and there are only $\chi(G)$ labels in f, we have $l(D^*) \leq \chi(G) - 1$. ■



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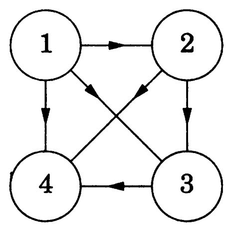


D^*

BROOKS' THEOREM

The bound $\chi(G) \leq 1 + \Delta(G)$ holds with equality for complete graphs and odd cycles. By choosing the vertex ordering more carefully, we can show that these are essentially the only such graphs. This implies, for example, that the Petersen graph is 3-colorable, without finding an explicit coloring. To avoid unimportant complications, we phrase the statement only for connected graphs. It extends to all graphs because the chromatic number of a graph is the maximum chromatic number of its components. Many proofs are known; we present a modification of the proof by Lovasz [1975].

5.1.22. Theorem. (Brooks [1941]) If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Proof: Let G be a connected graph, and let $k = \Delta(G)$. We may assume that $k \geq 3$, since G is a complete graph when $k \leq 1$, and G is an odd cycle or is bipartite when $k = 2$, in which case the bound holds.

Our aim is to order the vertices so that each has at most $k-1$ lower-indexed neighbors; greedy coloring for such an ordering yields the bound.

When G is not k-regular, we can choose a vertex of degree less than k as v_n. Since G is connected, we can grow a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices. Each vertex other than v_n in the resulting ordering $v_1, \dots, v_n$ has a higher-indexed neighbor along the path to v_n in the tree. Hence each vertex has at most $k-1$ lower-indexed neighbors, and the greedy coloring uses at most k colors.

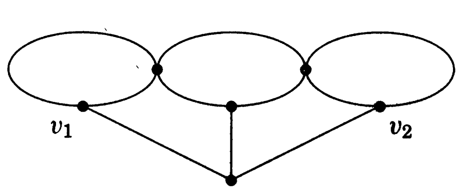
In the remaining case, G is k-regular. Suppose first that G has a cut-vertex x, and let G' be a subgraph consisting of a component of $G-x$ together with its edges to x. The degree of x in G' is less than k, so the method above provides a proper k-coloring of G'. By permuting the names of colors in the subgraphs resulting in this way from components of $G-x$, we can make the colorings agree on x to complete a proper k-coloring of G.

We may thus assume that G is 2-connected. In every vertex ordering, the last vertex has k earlier neighbors. The greedy coloring idea may still work if we arrange that two neighbors of v_n get the same color.

In particular, suppose that some vertex v_n has neighbors v_1, v_2 such that $v_1 \not\leftrightarrow v_2$ and $G - \{v_1, v_2\}$ is connected. In this case, we index the vertices of a spanning tree of $G - \{v_1, v_2\}$ using $3, \dots, n$ such that labels increase along paths to the root v_n. As before, each vertex before v_n has at most $k-1$ lower indexed neighbors. The greedy coloring also uses at most $k-1$ colors on neighbors of v_n, since v_1 and v_2 receive the same color.

Hence it suffices to show that every 2-connected k-regular graph with $k \geq 3$ has such a triple v_1, v_2, v_n. Choose a vertex x. If $\kappa(G-x) \geq 2$, let v_1 be x and let v_2 be a vertex with distance 2 from x. Such a vertex v_2 exists because G is regular and is not a complete graph; let v_n be a common neighbor of v_1 and v_2.

If $\kappa(G-x) = 1$, let $v_n = x$. Since G has no cut-vertex, x has a neighbor in every leaf block of $G-x$. Neighbors v_1, v_2 of x in two such blocks are nonadjacent. Also, $G - \{x, v_1, v_2\}$ is connected, since blocks have no cut-vertices. Since $k \geq 3$, vertex x has another neighbor, and $G - \{v_1, v_2\}$ is connected. ■



5.1.23.* Remark. The bound $\chi(G) \leq \Delta(G)$ can be improved when G has no large clique (Exercise 50). Brooks' Theorem implies that the complete graphs and odd cycles are the only $k-1$-regular k-critical graphs (Exercise 47). Gallai

[1963b] strengthened this by proving that in the subgraph of a k-critical graph induced by the vertices of degree $k-1$, every block is a clique or an odd cycle.

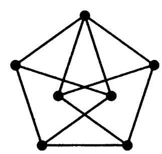
Brooks' Theorem states that $\chi(G) \le \Delta(G)$ whenever $3 \le \omega(G) \le \Delta(G)$. Borodin and Kostochka [1977] conjectured that $\omega(G) < \Delta(G)$ implies $\chi(G) < \Delta(G)$ if $\Delta(G) \ge 9$ (examples show that the condition $\Delta(G) \ge 9$ is needed). Reed [1999] proved that this is true when $\Delta(G) \ge 10^{14}$.

Reed [1998] also conjectured that the chromatic number is bounded by the average of the trivial upper and lower bounds; that is, $\chi(G) \le \lceil \frac{\Delta(G)+1+\omega(G)}{2} \rceil$. ∎

Because the idea of partitioning to satisfy constraints is so fundamental, there are many, many variations and generalizations of graph coloring. In Chapter 7 we consider coloring the edges of a graph. Sticking to vertices, we could allow color classes to induce subgraphs other than independent sets ("generalized coloring"—Exercises 49–53). We could restrict the colors allowed to be used on each vertex ("list coloring"—Section 8.4). We could ask questions involving numerical values of the colors (Exercise 54). We have only touched the tip of the iceberg on coloring problems.

EXERCISES

5.1.1. (−) Compute the clique number, the independence number, and the chromatic number of the graph below. Does either bound in Proposition 5.1.7 prove optimality for some proper coloring? Is the graph color-critical?



5.1.2. (−) Prove that the chromatic number of a graph equals the maximum of the chromatic numbers of its components.

5.1.3. (−) Let $G_1, \ldots, G_k$ be the blocks of a graph G. Prove that $\chi(G) = \max_i \chi(G_i)$.

5.1.4. (−) Exhibit a graph G with a vertex v so that $\chi(G-v) < \chi(G)$ and $\chi(\overline{G}-v) < \chi(\overline{G})$.

5.1.5. (−) Given graphs G and H, prove that $\chi(G+H) = \max\{\chi(G), \chi(H)\}$ and that $\chi(G \vee H) = \chi(G) + \chi(H)$.

5.1.6. (−) Suppose that $\chi(G) = \omega(G) + 1$, as in Example 5.1.8. Let $H_1 = G$ and $H_k = H_{k-1} \vee G$ for $k > 1$. Prove that $\chi(H)_k = \omega(H)_k + k$.

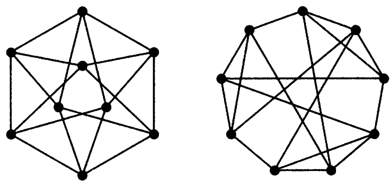
5.1.7. (−) Construct a graph G that is neither a clique nor an odd cycle but has a vertex ordering relative to which greedy coloring uses $\Delta(G) + 1$ colors.

5.1.8. (−) Prove that $\max_{H \subseteq G} \delta(H) \le \Delta(G)$ to explain why Theorem 5.1.19 is better than Proposition 5.1.13. Determine all graphs G such that $\max_{H \subseteq G} \delta(H) = \Delta(G)$.

5.1.9. (−) Draw the graph $K_{1,3} \,\square\, P_3$ and exhibit an optimal coloring of it. Draw $C_5 \,\square\, C_5$ and find a proper 3-coloring of it with color classes of sizes 9, 8, 8.

5.1.10. (−) Prove that $G \square H$ decomposes into $n(G)$ copies of H and $n(H)$ copies of G.

5.1.11. (−) Prove that each graph below is isomorphic to $C_3 \square C_3$.



5.1.12. (−) Prove or disprove: Every k-chromatic graph G has a proper k-coloring in which some color class has $\alpha(G)$ vertices.

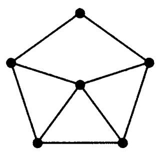
5.1.13. (−) Prove or disprove: If $G = F \cup H$, then $\chi(G) \le \chi(F) + \chi(H)$.

5.1.14. (−) Prove or disprove: For every graph G, $\chi(G) \le n(G) - \alpha(G) + 1$.

5.1.15. (−) Prove or disprove: If G is a connected graph, then $\chi(G) \le 1 + a(G)$, where $a(G)$ is the average of the vertex degrees in G.

5.1.16. (−) Use Theorem 5.1.21 to prove that every tournament has a spanning path. (Rédei [1934])

5.1.17. (−) Use Lemma 5.1.18 to prove that $\chi(G) \le 4$ for the graph G below.



5.1.18. (−) Determine the number of colors needed to label $V(K_n)$ such that each color class induces a subgraph with maximum degree at most k.

5.1.19. (−) Find the error in the false argument below for Brooks' Theorem (Theorem 5.1.22).

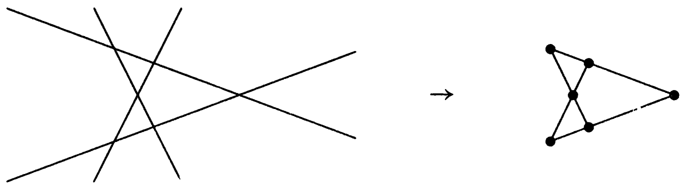
"We use induction on $n(G)$; the statement holds when $n(G) = 1$. For the induction step, suppose that G is not a complete graph or an odd cycle. Since $\kappa(G) \le \delta(G)$, the graph G has a separating set S of size at most $\Delta(G)$. Let $G_1, \dots, G_m$ be the components of $G - S$, and let $H_i = G[V(G_i) \cup S]$. By the induction hypothesis, each H_i is $\Delta(G)$-colorable. Permute the names of the colors used on these subgraphs to agree on S. This yields a proper $\Delta(G)$-coloring of G."

• • • • •

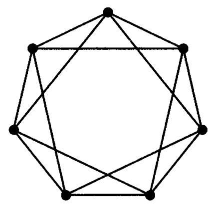
5.1.20. (!) Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \le 5$.

5.1.21. Suppose that every edge of a graph G appears in at most one cycle. Prove that every block of G is an edge, a cycle, or an isolated vertex. Use this to prove that $\chi(G) \le 3$.

5.1.22. (!) Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \le 3$. (Hint: This can be solved by using the Szekeres–Wilf Theorem or by using greedy colori: ; with an appropriate vertex ordering. Comment: The conclusion may fail when three lines are allowed to share a point.) (H. Sachs)



5.1.23. (!) Place n points on a circle, where $n \geq k(k+1)$. Let $G_{n,k}$ be the $2k$-regular graph obtained by joining each point to the k nearest points in each direction on the circle. For example, $G_{n,1} = C_n$, and $G_{7,2}$ appears below. Prove that $\chi(G_{n,k}) = k+1$ if $k+1$ divides n and $\chi(G_{n,k}) = k+2$ if $k+1$ does not divide n. Prove that the lower bound on n cannot be weakened, by proving that $\chi(G_{k(k+1)-1,k}) > k+2$ if $k \geq 2$.



5.1.24. (+) Let G be any 20-regular graph with 360 vertices formed in the following way. The vertices are evenly-spaced around a circle. Vertices separated by 1 or 2 degrees are nonadjacent. Vertices separated by 3, 4, 5 or 6 degrees are adjacent. No information is given about other adjacencies (except that G is 20-regular). Prove that $\chi(G) \leq 19$. (Hint: Color successive vertices in order around the circle.) (Pritikin)

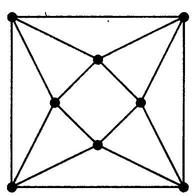
5.1.25. (+) Let G be the **unit-distance graph** in the plane; $V(G) = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1 (this is an infinite graph). Prove that $4 \leq \chi(G) \leq 7$. (Hint: For the upper bound, present an explicit coloring by regions, paying attention to the boundaries.) (Hadwiger [1945, 1961], Moser–Moser [1961])

5.1.26. Given finite sets $S_1, \ldots, S_m$, let $U = S_1 \times \cdots \times S_m$. Define a graph G with vertex set U by putting $u \leftrightarrow v$ if and only if u and v differ in every coordinate. Determine $\chi(G)$.

5.1.27. Let H be the complement of the graph in Exercise 5.1.26. Determine $\chi(H)$.

5.1.28. Consider a traffic signal controlled by two switches, each of which can be set in n positions. For each setting of the switches, the traffic signal shows one of its n possible colors. Whenever the setting of *both* switches changes, the color changes. Prove that the color shown is determined by the position of one of the switches. Interpret this in terms of the chromatic number of some graph. (Greenwell–Lovász [1974])

5.1.29. For the graph G below, compute $\chi(G)$ and find a $\chi(G)$-critical subgraph.



5.1.30. (+) Let $S = \binom{[n]}{2}$ denote the collection of 2-sets of the n-element set $[n]$. Define the graph G_n by $V(G_n) = S$ and $E(G_n) = \{(ij, jk)\colon 1 \le i < j < k \le n\}$ (disjoint pairs, for example, are nonadjacent). Prove that $\chi(G_n) = \lceil \lg n \rceil$. (Hint: Prove that G_n is r-colorable if and only if $[r]$ has at least n distinct subsets. Comment: G_n is called the **shift graph** of K_n.) (attributed to A. Hajnal)

5.1.31. (!) Prove that a graph G is m-colorable if and only if $\alpha(G \,\square\, K_m) \ge n(G)$. (Berge [1973, p379–80])

5.1.32. (!) Prove that a graph G is 2^k-colorable if and only if G is the union of k bipartite graphs. (Hint: This generalizes Theorem 1.2.23.)

5.1.33. (!) Prove that every graph G has a vertex ordering relative to which greedy coloring uses $\chi(G)$ colors.

5.1.34. (!) For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously. Comment: This result shows that the performance ratio of greedy coloring to optimal coloring can be as bad as $(\Delta(G)+1)/2$.) (Bean [1976])

5.1.35. Let G be a graph having no induced subgraph isomorphic to P_4. Prove that for every vertex ordering, greedy coloring produces an optimal coloring of G. (Hint: Suppose that the algorithm uses k colors for the ordering $v_1, \ldots, v_n$, and let i be the smallest integer such that G has a clique consisting of vertices assigned colors i through k in this coloring. Prove that $i = 1$. Comment: P_4-free graphs are also called **cographs**.)

5.1.36. Given a vertex ordering $\sigma = v_1, \ldots, v_n$ of a graph G, let $G_i = G[\{v_1, \ldots, v_i\}]$ and $f(\sigma) = 1 + \max_i d_{G_i}(v_i)$. Greedy coloring relative to σ yields $\chi(G) \le f(\sigma)$. Define σ^* by letting v_n be a minimum degree vertex of G and letting v_i for $i < n$ be a minimum degree vertex of $G - \{v_{i+1}, \ldots, v_n\}$. Show that $f(\sigma^*) = 1 + \max_{H \subseteq G} \delta(H)$, and thus that σ^* minimizes $f(\sigma)$. (Halin [1967], Matula [1968], Finck–Sachs [1969], Lick–White [1970])

5.1.37. Prove that $V(G)$ can be partitioned into $1 + \max_{H \subseteq G} \delta(H)/r$ classes such that every subgraph whose vertices lie in a single class has a vertex of degree less than r. (Hint: Consider ordering σ^* of Exercise 5.1.36. Comment: This generalizes Theorem 5.1.19. See also Chartrand–Kronk [1969] when $r = 2$.)

5.1.38. (!) Prove that $\chi(G) = \omega(G)$ when $\overline{G}$ is bipartite. (Hint: Phrase the claim in terms of $\overline{G}$ and apply results on bipartite graphs.)

5.1.39. (!) Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union of m complete graphs of order m, then $\chi(G) \le 1 + m\sqrt{m-1}$. (Comment: This bound is near tight, but the Erdős–Faber–Lovász Conjecture (see Erdős [1981]) asserts that $\chi(G) = m$ when the complete graphs are pairwise edge-disjoint.)

5.1.40. Prove that $\chi(G) \cdot \chi(\overline{G}) \ge n(G)$, use this to prove that $\chi(G) + \chi(\overline{G}) \ge 2\sqrt{n(G)}$, and provide a construction achieving these bounds whenever $\sqrt{n(G)}$ is an integer. (Nordhaus–Gaddum [1956], Finck [1968])

5.1.41. (!) Prove that $\chi(G) + \chi(\overline{G}) \le n(G) + 1$. (Hint: Use induction on $n(G)$.) (Nordhaus–Gaddum [1956])

5.1.42. (!) *Looseness of* $\chi(G) \ge n(G)/\alpha(G)$. Let G be an n-vertex graph, and let $c = (n+1)/\alpha(G)$. Use Exercise 5.1.41 to prove that $\chi(G) \cdot \chi(\overline{G}) \le (n+1)^2/4$, and use this to prove that $\chi(G) \le c(n+1)/4$. For each odd n, construct a graph such that $\chi(G) = c(n+1)/4$. (Nordhaus–Gaddum [1956], Finck [1968])

5.1.43. (!) *Paths and chromatic number in digraphs.*

a) Let $G = F \cup H$. Prove that $\chi(G) \le \chi(F)\chi(H)$.

b) Consider an orientation D of G and a function $f\colon V(G) \to \mathbb{R}$. Use part (a) and Theorem 5.1.21 to prove that if $\chi(G) > rs$, then D has a path $u_0 \to \cdots \to u_r$ with $f(u_0) \le \cdots \le f(u_r)$ or a path $v_0 \to \cdots \to v_s$ with $f(v_0) > \cdots > f(v_s)$.

c) Use part (b) to prove that every sequence of $rs + 1$ distinct real numbers has an increasing subsequence of size $r + 1$ or a decreasing subsequence of size $s + 1$. (Erdős–Szekeres [1935])

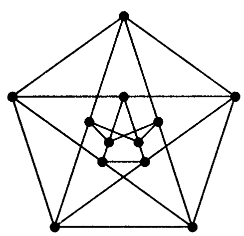
5.1.44. (!) *Minty's Theorem* (Minty [1962]). An **acyclic orientation** of a loopless graph is an orientation having no cycle. For each acyclic orientation D of G, let $r(D) = \max_C \lceil a/b \rceil$, where C is a cycle in G and a, b count the edges of C that are forward in D or backward in D, respectively. Fix a vertex $x \in V(G)$, and let W be a walk in G beginning at x. Let $g(W) = a - b \cdot r(D)$, where a is the number of steps along W that are forward edges in D and b is the number that are backward in D. For each $y \in V(G)$, let $g(y)$ be the maximum of $g(W)$ such that W is an x, y-walk (assume that G is connected).

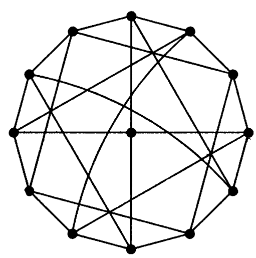
a) Prove that $g(y)$ is finite and thus well-defined, and use $g(y)$ to obtain a proper $1 + r(D)$-coloring of G. Thus G is $1 + r(D)$-colorable.

b) Prove that $\chi(G) = \min_{D \in \mathbf{D}}$, where $\mathbf{D}$ is the set of acyclic orientations of G.

5.1.45. (+) Use Minty's Theorem (Exercise 5.1.44) to prove Theorem 5.1.21. (Hint: Prove that $l(D)$ is maximized by some acyclic orientation of G.)

5.1.46. (+) Prove that the 4-regular triangle-free graphs below are 4-chromatic. (Hint: Consider the maximum independent sets. Comment: Chvátal [1970] showed that the graph on the left is the smallest triangle-free 4-regular 4-chromatic graph.)





5.1.47. (!) Prove that Brooks' Theorem is equivalent to the following statement: every $k - 1$-regular k-critical graph is a complete graph or an odd cycle.

5.1.48. Let G be a simple graph with n vertices and m edges and maximum degree at most 3. Suppose that no component of G is a complete graph on 4 vertices. Prove that G contains a bipartite subgraph with at least $m - n/3$ edges. (Hint: Apply Brooks' Theorem, and then show how to delete a few edges to change a proper 3-coloring of G into a proper 2-coloring of a large subgraph of G.)

5.1.49. (−) Prove that the Petersen graph can be 2-colored so that the subgraph induced by each color class consists of isolated edges and vertices.

5.1.50. (!) *Improvement of Brooks' Theorem.*

a) Given a graph G, let $k_1, \ldots, k_t$ be nonnegative integers with $\sum k_i \ge \Delta(G) - t + 1$. Prove that $V(G)$ can be partitioned into sets $V_1, \ldots, V_t$ so that for each i, the subgraph G_i induced by V_i has maximum degree at most k_i.. (Hint: Prove that the partition minimizing $\sum e(G_i)/k_i$ has the desired property.) (Lovász [1966])

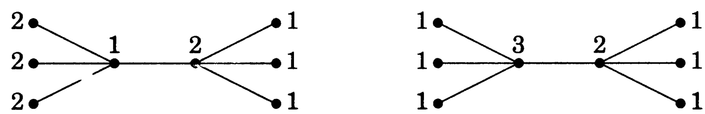
b) For $4 \le r \le \Delta(G) + 1$, use part (a) to prove that $\chi(G) \le \lceil \frac{r-1}{r}(\Delta(G) + 1) \rceil$ when G has no r-clique. (Borodin–Kostochka [1977], Catlin [1978], Lawrence [1978])

5.1.51. (!) Let G be an k-colorable graph, and let P be a set of vertices in G such that $d(x, y) \ge 4$ whenever $x, y \in P$. Prove that every coloring of P with colors from $[k + 1]$ extends to a proper $k + 1$ coloring of G. (Albertson–Moore [1999])

5.1.52. Prove that every graph G can be $\lceil(\Delta(G) + 1)/j\rceil$-colored so that each color class induces a subgraph having no j-edge-connected subgraph. For $j > 1$, prove that no smaller number of classes suffices when G is a j-regular j-edge-connected graph or is a complete graph with order congruent to 1 modulo j. (Comment: For $j = 1$, the restriction reduces to ordinary proper coloring.) (Matula [1973])

5.1.53. (+) Let $G_{n,k}$ be the $2k$-regular graph of Exercise 5.1.23. For $k \le 4$, determine the values of n such that $G_{n,k}$ can be 2-colored so that each color class induces a subgraph with maximum degree at most k. (Weaver–West [1994])

5.1.54. Let f be a proper coloring of a graph G in which the colors are natural numbers. The **color sum** is $\sum_{v \in V(G)} f(v)$. Minimizing the color sum may require using more than $\chi(G)$ colors. In the tree below, for example, the best proper 2-coloring has color sum 12, while there is a proper 3-coloring with color sum 11. Construct a sequence of trees in which the kth tree T_k use k colors in a proper coloring that minimizes the color sum. (Kubicka–Schwenk [1989])



5.1.55. (+) *Chromatic number is bounded by one plus longest odd cycle length.*

a) Let G be a 2-connected nonbipartite graph containing an even cycle C. Prove that there exist vertices x, y on C and an x, y-path P internally disjoint from C such that $d_C(x, y) \not\equiv d_P(x, y) \bmod 2$.

b) Let G be a simple graph with no odd cycle having length at least $2k + 1$. Prove that if $\delta(G) \ge 2k$, then G has a cycle of length at least $4k$. (Hint: Consider the neighbors of an endpoint of a maximal path.)

c) Let G be a 2-connected nonbipartite graph with no odd cycle longer than $2k - 1$. Prove that $\chi(G) \le 2k$. (Erdős–Hajnal [1966])

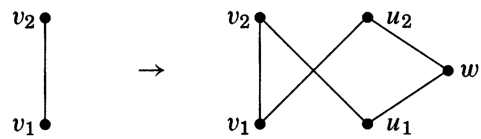
5.2. Structure of k-chromatic Graphs

We have observed that $\chi(H) \ge \omega(H)$ for all H. When equality holds in this bound for G and all its induced subgraphs (as for interval graphs), we say that G is **perfect**; we discuss such graphs in Sections 5.3 and 8.1. Our concern with the bound $\chi(G) \ge \omega(G)$ in this section is how *bad* it can be. Almost always $\chi(G)$ is much larger than $\omega(G)$, in a sense discussed precisely in Section 8.5. (The average values of $\omega(G)$, $\alpha(G)$, and $\chi(G)$ over all graphs with vertex set $[n]$ are very close to $2\lg n$, $2\lg n$, and $n/(2\lg n)$, respectively. Hence $\omega(G)$ is generally a bad lower bound on $\chi(G)$, and $n/\alpha(G)$ is generally a good lower bound.)

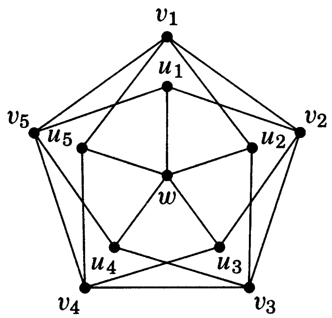
GRAPHS WITH LARGE CHROMATIC NUMBER

The bound $\chi(G) \ge \omega(G)$ can be tight, but it can also be very loose. There have been many constructions of graphs without triangles that have arbitrarily large chromatic number. We present one such construction here; others appear in Exercises 12–13.

5.2.1. Definition. From a simple graph G, **Mycielski's construction** produces a simple graph G' containing G. Beginning with G having vertex set $\{v_1, \ldots, v_n\}$, add vertices $U = \{u_1, \ldots, u_n\}$ and one more vertex w. Add edges to make u_i adjacent to all of $N_G(v_i)$, and finally let $N(w) = U$.



5.2.2. Example. From the 2-chromatic graph K_2, one iteration of Mycielski's construction yields the 3-chromatic graph C_5, as shown above. Below we apply the construction to C_5, producing the 4-chromatic **Grötzsch graph**. ■



5.2.3. Theorem. (Mycielski [1955]) From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph G'.

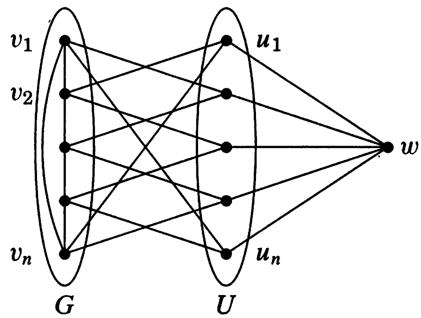
Proof: Let $V(G) = \{v_1, \ldots, v_n\}$, and let G' be the graph produced from it by Mycielski's construction. Let $u_1, \ldots, u_n$ be the copies of $v_1, \ldots, v_n$, with w the additional vertex. Let $U = \{u_1, \ldots, u_n\}$.

By construction, U is an independent set in G'. Hence the other vertices of any triangle containing u_i belong to $V(G)$ and are neighbors of v_i. This would complete a triangle in G, which can't exist. We conclude that G' is triangle-free.

A proper k-coloring f of G extends to a proper $k+1$-coloring of G' by setting $f(u_i) = f(v_i)$ and $f(w) = k+1$; hence $\chi(G') \le \chi(G) + 1$. We prove equality by showing that $\chi(G) < \chi(G')$. To prove this we consider any proper coloring of G' and obtain from it a proper coloring of G using fewer colors.

Let g be a proper k-coloring of G' By changing the names of colors, we may assume that $g(w) = k$. This restricts g to $\{1, \ldots, k-1\}$ on U. On $V(G)$, it may use all k colors. Let A be the set of vertices in G on which g uses color k; we change the colors used on A to obtain a proper $k-1$-coloring of G.

For each $v_i \in A$, we change the color of v_i to $g(u_i)$. Because all vertices of A have color k under g, no two edges of A are adjacent. Thus we need only check edges of the form $v_i v'$ with $v_i \in A$ and $v' \in V(G) - A$. If $v' \leftrightarrow v_i$, then by construction also $v' \leftrightarrow u_i$, which yields $g(v') \neq g(u_i)$. Since we change the color on v_i to $g(u_i)$, our change does not violate the edge $v_i v'$. We have shown that the modified coloring of $V(G)$ is a proper $k-1$-coloring of G. ■



If G is color-critical, then the graph G' resulting from Mycielski's construction is also color-critical (Exercise 9).

5.2.4.* Remark. Starting with $G_2 = K_2$, iterating Mycielski's construction produces a sequence $G_2, G_3, G_4, \ldots$ of graphs. The first three are K_2, C_5, and the Grötzsch graph. These are the smallest triangle-free 2-chromatic, 3-chromatic, and 4-chromatic graphs. The graphs then grow rapidly: $n(G_k) = 2n(G_{k-1}) + 1$. With $n(G_2) = 2$, this yields $n(G_k) = 3 \cdot 2^{k-2} - 1$ (exponential growth).

Let $f(k)$ be the minimum number of vertices in a triangle-free k-chromatic graph. Using probabilistic (non-constructive) methods, Erdős [1959] proved that $f(k) \leq ck^{2+\epsilon}$, where ϵ is any positive constant and c depends on ϵ but not on k. Using Ramsey numbers (Section 8.3), it is now known (non-constructively) that there are constants c_1, c_2 such that $c_1 k^2 \log k \leq f(k) \leq c_2 k^2 \log k$. Exercise 15 develops a quadratic lower bound.

Blanche Descartes† [1947, 1954] constructed color-critical graphs with girth 6 (Exercise 13). Using probabilistic methods, Erdős [1959] proved that graphs exist with chromatic number at least k and girth at least g (Theorem 8.5.11). Later, explicit constructions were found (Lovász [1968a], Nešetřil–Rödl [1979], Lubotzsky–Phillips–Sarnak [1988], Kriz [1989]).

By all these constructions, forbidding K_r from G does not place a bound on $\chi(G)$. Gyárfás [1975] and Sumner [1981]) conjectured that forbidding a fixed clique and a fixed forest as an *induced* subgraph does bound the chromatic number. Exercise 11 proves this when the forest is $2K_2$. (See also Kierstead–Penrice [1990, 1994], Kierstead [1992, 1997], Kierstead–Rödl [1996]) ■

†This pseudonym was used by W.T. Tutte and also by three others.

EXTREMAL PROBLEMS AND TURÁN'S THEOREM

Perhaps extremal questions can shed some light on the structure of k-chromatic graphs. For example, which are the smallest and largest k-chromatic graphs with n vertices?

5.2.5. Proposition. Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges. Equality holds for a complete graph plus isolated vertices.

Proof: An optimal coloring of a graph has an edge with endpoints of colors i and j for each pair i, j of colors. Otherwise, colors i and j could be combined into a single color class and use fewer colors. Since there are $\binom{k}{2}$ distinct pairs of colors, there must be $\binom{k}{2}$ distinct edges. ■

Exercise 6 asks for the minimum size among connected k-chromatic graphs with n vertices.

The maximization problem is more interesting (of course, it makes sense only when restricted to simple graphs). Given a proper k-coloring, we can continue to add edges without increasing the chromatic number as long as two vertices in different color classes are nonadjacent. Thus we may restrict our attention to graphs without such pairs.

5.2.6. Definition. A **complete multipartite graph** is a simple graph G whose vertices can be partitioned into sets so that $u \leftrightarrow v$ if and only if u and v belong to different sets of the partition. Equivalently, every component of $\overline{G}$ is a complete graph. When $k \geq 2$, we write $K_{n_1,\dots,n_k}$ for the complete k-partite graph with partite sets of sizes $n_1, \dots, n_k$ and complement $K_{n_1} + \cdots + K_{n_k}$.

We use this notation only for $k > 1$, since K_n denotes a complete graph. A complete k-partite graph is k-chromatic; the partite sets are the color classes in the only proper k-coloring. Also, since a vertex in a partite set of size t has degree $n(G) - t$, the edges can be counted using the degree-sum formula (Exercise 18). Which distribution of vertices to partite sets maximizes $e(G)$?

5.2.7. Example. *The Turán graph.* The **Turán graph** $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ in size by at most 1. By the pigeonhole principle (see Appendix A), some partite set has size at least $\lceil n/r \rceil$ and some has size at most $\lfloor n/r \rfloor$. Therefore, differing by at most 1 means that they all have size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

Let $a = \lfloor n/r \rfloor$. After putting a vertices in each partite set, $b = n - ra$ remain, so $T_{n,r}$ has b partite sets of size $a + 1$ and $r - b$ partite sets of size a. Thus the defining condition on $T_{n,r}$ specifies a single isomorphism class. ■

5.2.8. Lemma. Among simple r-partite (that is, r-colorable) graphs with n vertices, the Turán graph is the unique graph with the most edges.

Proof: As noted before Definition 5.2.6, we need only consider complete r-partite graphs. Given a complete r-partite graph with partite sets differing by more than 1 in size, we move a vertex v from the largest class (size i) to the smallest class (size j). The edges not involving v are the same as before, but v gains $i-1$ neighbors in its old class and loses j neighbors in its new class. Since $i-1>j$, the number of edges increases. Hence we maximize the number of edges only by equalizing the sizes as in $T_{n,r}$. ■

We used the idea of this local alteration previously in Theorem 1.3.19 and in Theorem 1.3.23; we are finding the largest r-partite subgraph of K_n.

What happens if we have more edges and thus force chromatic number at least $r+1$? We have seen that there are graphs with chromatic number $r+1$ that have no triangles. Nevertheless, if we go beyond the maximum number of edges in an r-colorable graph with n vertices, then we are forced not only to use $r+1$ colors but also to have K_{r+1} as a subgraph.

This famous result of Turán generalizes Theorem 1.3.23 and is viewed as the origin of extremal graph theory.

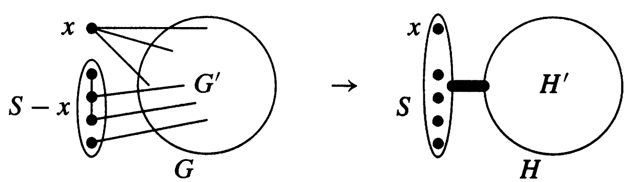
5.2.9. Theorem. (Turán [1941]) Among the n-vertex simple graphs with no $r+1$-clique, $T_{n,r}$ has the maximum number of edges.

Proof: The Turán graph $T_{n,r}$, like every r-colorable graph, has no $r+1$-clique, since each partite set contributes at most one vertex to each clique. If we can prove that the maximum is achieved by an r-partite graph, then Lemma 5.2.8 implies that the maximum is achieved by $T_{n,r}$. Thus it suffices to prove that if G has no $r+1$-clique, then there is an r-partite graph H with the same vertex set as G and at least as many edges.

We prove this by induction on r. When $r=1$, G and H have no edges. For the induction step, consider $r>1$. Let G be an n-vertex graph with no $r+1$-clique, and let $x\in V(G)$ be a vertex of degree $k=\Delta(G)$. Let G' be the subgraph of G induced by the neighbors of x. Since x is adjacent to every vertex in G' and G has no $r+1$-clique, the graph G' has no r-clique. We can thus apply the induction hypothesis to G'; this yields an $r-1$-partite graph H' with vertex set $N(x)$ such that $e(H')\ge e(G')$.

Let H be the graph formed from H' by joining all of $N(x)$ to all of $S=V(G)-N(x)$. Since S is an independent set, H is r-partite. We claim that $e(H)\ge e(G)$. By construction, $e(H)=e(H')+k(n-k)$. We also have $e(G)\le e(G')+\sum_{v\in S}d_G(v)$, since the sum counts each edge of G once for each endpoint it has outside $V(G')$. Since $\Delta(G)=k$, we have $d_G(v)\le k$ for each $v\in S$, and $|S|=n-k$, so $\sum_{v\in S}d_G(v)\le k(n-k)$. As desired, we have

$$e(G)\le e(G')+(n-k)k\le e(H')+k(n-k)=e(H)$$ ■

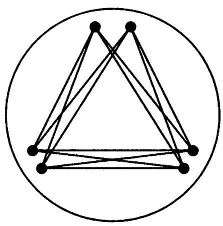


In fact, the Turán graph is the unique extremal graph (Exercise 21). Exercises 16–24 pertain to Turán's Theorem, including alternative proofs, the value of $e(T_{n,r})$, and applications. The argument used in Theorem 1.3.23 was simply one instance of the induction step in Theorem 5.2.9.

Turán's theorem applies to extremal problems when some condition forbids cliques of a given order; we describe a geometric application from Bondy–Murty [1976, p113–115].

5.2.10.* Example. *Distant pairs of points.* In a circular city of diameter 1, we might want to locate n police cars to maximize the number of pairs that are far apart, say separated by distance more than $d = 1/\sqrt{2}$. If six cars occupy equally spaced points on the circle, then the only pairs not more than d apart are the consecutive pairs around the outside: there are nine good pairs.

Instead, putting two cars each near the vertices of an equilateral triangle with side-length $\sqrt{3}/2$ yields three bad pairs and twelve good pairs. (This may not be the socially best criterion!) In general, with $\lceil n/3 \rceil$ or $\lfloor n/3 \rfloor$ cars near each vertex of this triangle, the good pairs correspond to edges of the tripartite Turán graph. We show next that this construction is best. ■

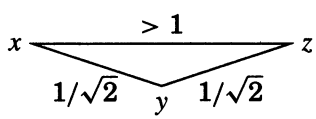


5.2.11.* Application. In a set of n points in the plane with no pair more than distance 1 apart, the maximum number of pairs separated by distance more than $1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$.

Proof: Draw a graph G on these points by making points adjacent when the distance between them exceeds $1/\sqrt{2}$. By Turán's Theorem and the construction in Example 5.2.10, it suffices to show that G has no K_4.

Among any four points, some three form an angle of at least 90°: if the four form a convex quadrilateral, then the interior angles sum to 360°, and if one point is inside the triangle formed by the others, then with them it forms three angles summing to 360°.

Suppose that G has a 4-vertex clique with points w, x, y, z, where $\angle xyz \geq 90°$. Since the lengths of xy and yz exceed $1/\sqrt{2}$, xz is longer than the hypotenuse of a right triangle with legs of length $1/\sqrt{2}$. Hence the distance between x and z exceeds 1, which contradicts the hypothesis. ■



Even without the full structural statement of Turán's Theorem, one can prove directly a rough bound on the number of edges in an n-vertex graph with no K_{r+1} (Exercise 16). Turning this around yields a sharp lower bound on the chromatic number of a graph in terms of the number of vertices and number of edges (Exercise 17).

COLOR-CRITICAL GRAPHS

The Turán graph solves a problem that is somehow opposite to understanding what forces chromatic number k. It considers the maximal graphs that *avoid* needing k colors instead of the minimal graphs that *do* need k colors.

Every k-chromatic graph has a k-critical subgraph, since we can continue discarding edges and isolated vertices without reducing the chromatic number until we reach a point where every such deletion reduces the chromatic number. Thus knowing the k-critical graphs could help us test for $k-1$-colorability. We begin with elementary properties of k-critical graphs.

5.2.12. Remark. A graph G with no isolated vertices is color-critical if and only if $\chi(G-e) < \chi(G)$ for every $e \in E(G)$. Hence when we prove that a connected graph is color-critical, we need only compare it with subgraphs obtained by deleting a single edge. ■

5.2.13. Proposition. Let G be a k-critical graph.

a) For $v \in V(G)$, there is a proper k-coloring of G in which the color on v appears nowhere else, and the other $k-1$ colors appear on $N(v)$.

b) For $e \in E(G)$, every proper $k-1$-coloring of $G-e$ gives the same color to the two endpoints of e.

Proof: (a) Given a proper $k-1$-coloring f of $G-v$, adding color k on v alone completes a proper k-coloring of G. The other colors must all appear on $N(v)$, since otherwise assigning a missing color to v would complete a proper $k-1$-coloring of G.

(b) If some proper $k-1$-coloring of $G-e$ gave distinct colors to the endpoints of e, then adding e would yield a proper $k-1$-coloring of G. ■

For any graph G, Proposition 5.2.13a holds for every $v \in V(G)$ such that $\chi(G-v) < \chi(G) = k$, and Proposition 5.2.13b holds for every $e \in E(G)$ such that $\chi(G-e) < \chi(G) = k$.

5.2.14. Example. The graph $C_5 \vee K_s$ of Example 5.1.8 is color-critical. In general, the join of two color-critical graphs is always color-critical. This is easy to prove using Remark 5.2.12 and Proposition 5.2.13 by considering cases for the deletion of an edge; the deleted edge e may belong to G or H or have an endpoint in each (Exercise 3). ■

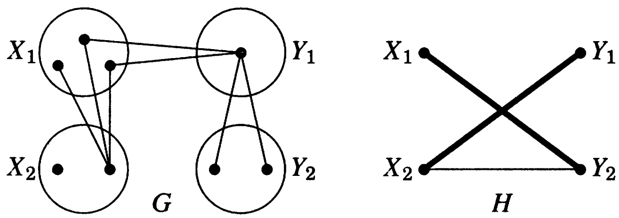
We proved in Lemma 5.1.18 that $\delta(G) \ge k-1$ when G is a k-critical graph. We can strengthen this to $\kappa'(G) \ge k-1$ by using the König–Egerváry Theorem.

5.2.15. Lemma. (Kainen) Let G be a graph with $\chi(G) > k$, and let X, Y be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges.

Proof: Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ be the partitions of X and Y formed by the color classes in proper k-colorings of $G[X]$ and $G[Y]$. If there is no edge between X_i and Y_j, then $X_i \cup Y_j$ is an independent set in G. We show that if $|[X, Y]| < k$, then we can combine color classes from $G[X]$ and $G[Y]$ in pairs to form a proper k-coloring of G.

Form a bipartite graph H with vertices $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$, putting $X_iY_j \in E(H)$ if in G there is *no edge* between the set X_i and the set Y_j. If $|[X, Y]| < k$, then H has more than $k(k-1)$ edges. Since m vertices can cover at most km edges in a subgraph of $K_{k,k}$, $E(H)$ cannot be covered by $k-1$ vertices. By the König–Egerváry Theorem, H therefore has a perfect matching M.

In G, we give color i to all of X_i and all of the set Y_j to which it is matched by M. Since there are no edges joining X_i and Y_j, doing this for all i produces a proper k-coloring of G, which contradicts the hypothesis that $\chi(G) > k$. Hence we conclude that $|[X, Y]| \geq k$. ■

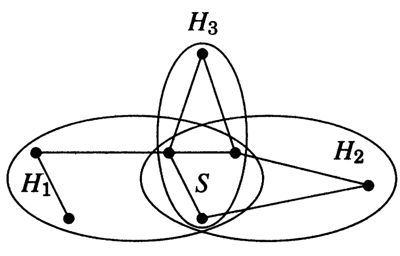


5.2.16. Theorem. (Dirac [1953]) Every k-critical graph is $k-1$-edge-connected.

Proof: Let G be a k-critical graph, and let $[X, Y]$ be a minimum edge cut. Since G is k-critical, $G[X]$ and $G[Y]$ are $k-1$-colorable. Applied with $k-1$ as the parameter, Lemma 5.2.15 then states that $|[X, Y]| \geq k-1$. ■

Although a k-critical graph must be $k-1$-edge-connected, it need not be $k-1$-connected; Exercise 32 shows how to construct k-critical graphs that have connectivity 2. Nevertheless, we can restrict the behavior of small vertex cutsets in k-critical graphs.

5.2.17. Definition. Let S be a set of vertices in a graph G. An **S-lobe** of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component of $G - S$.



For every $S \subseteq V(G)$, the graph G is the union of its S-lobes. We use this to prove a statement about vertex cutsets in k-critical graphs that will be useful in the next theorem. Exercise 33 strengthens the result when $|S| = 2$.

5.2.18. Proposition. If G is k-critical, then G has no cutset consisting of pairwise adjacent vertices. In particular, if G has a cutset $S = \{x, y\}$, then $x \not\leftrightarrow y$ and G has an S-lobe H such that $\chi(H + xy) = k$.

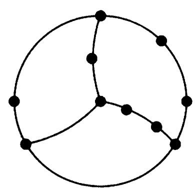
Proof: Let S be a cutset in a k-critical graph G. Let $H_1, \ldots, H_t$ be the S-lobes of G. Since each H_i is a proper subgraph of a k-critical graph, each H_i is $k-1$-colorable. If each H_i has a proper $k-1$-coloring giving distinct colors to the vertices of S, then the names of the colors in these $k-1$-colorings can be permuted to agree on S. The colorings then combine to form a $k-1$-coloring of G, which is impossible.

Hence some S-lobe H has no proper $k-1$-coloring with distinct colors on S. This implies that S is not a clique. If $S = \{x, y\}$, then every $k-1$-coloring of H assigns the same color to x and y, and hence $H + xy$ is not $k-1$-colorable. ■

FORCED SUBDIVISIONS

We need not have a k-clique to have chromatic number k, but perhaps we must have some weakened form of a k-clique.

5.2.19. Definition. An **H-subdivision** (or **subdivision of** H) is a graph obtained from a graph H by successive edge subdivisions (Definition 5.2.19). Equivalently, it is a graph obtained from H by replacing edges with pairwise internally disjoint paths.



a subdivision of K_4

5.2.20. Theorem. (Dirac [1952a]) Every graph with chromatic number at least 4 contains a K_4-subdivision.

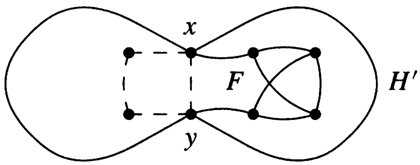
Proof: We use induction on $n(G)$.

Basis step: $n(G) = 4$. The graph G can only be K_4 itself.

Induction step: $n(G) > 4$. Since $\chi(G) \geq 4$, we may let H be a a 4-critical subgraph of G. By Proposition 5.2.18, H has no cut-vertex. If $\kappa(H) = 2$ and $S = \{x, y\}$ is a cutset of size 2, then by Proposition 5.2.18 $x \not\leftrightarrow y$ and H has an S-lobe H' such that $\chi(H' + xy) \geq 4$. Since $n(H') < n(G)$, we can apply the induction hypothesis to obtain a K_4-subdivision in H'.

This K_4-subdivision F appears also in G unless it contains xy (see figure below). In this case, we modify F to obtain a K_4-subdivision in G by replacing the edge xy with an x, y-path through another S-lobe of H. Such a path exists because the minimality of the cutset S implies that each vertex of S has a neighbor in each component of $H - S$.

Hence we may assume that H is 3-connected. Select a vertex $x \in V(G)$. Since $H - x$ is 2-connected, it has a cycle C of length at least 3. (Let x be the central vertex and C the outside cycle in the figure above.) Since H is 3-connected, the Fan Lemma (Theorem 4.2.23) yields an $x, V(C)$-fan of size 3 in H. These three paths, together with C, form a K_4-subdivision in H. ■



5.2.21.* Remark. Hajós [1961] conjectured that every k-chromatic graph contains a subdivision of K_k. For $k = 2$, the statement says that every 2-chromatic graph has a nontrivial path. For $k = 3$, it says that every 3-chromatic graph has a cycle. Theorem 5.2.20 proves it for $k = 4$, and it is open for $k \in \{5, 6\}$.

Hajós' Conjecture is false for $k \geq 7$ (Catlin [1979]—see Exercise 40). Hadwiger [1943] proposed a weaker conjecture: every k-chromatic graph has a subgraph that becomes K_k via edge contractions. This is weaker because a K_k-subdivision is a special subgraph of this type. For $k = 4$, Hadwiger's Conjecture is equivalent to Theorem 5.2.20. For $k = 5$, it is equivalent to the Four Color Theorem (Chapter 6). For $k = 6$, it was proved using the Four Color Theorem by Robertson, Seymour, and Thomas [1993]. For $k \geq 7$, it remains open. ■

Some results about k-critical graphs extend to the larger class of graphs with $\delta(G) \geq k - 1$. For example, every graph with minimum degree at least 3 has a K_4-subdivision (Exercise 38); this strengthens Theorem 5.2.20.

Dirac [1965] and Jung [1965] proved that sufficiently large chromatic number forces a K_k-subdivision in G. Mader improved this by weakening the hypothesis and generalizing the conclusion: for a simple graph F, every simple graph G with $\delta(G) \geq 2^{e(F)}$ contains a subdivision of F. The threshold $2^{e(F)}$ is larger than necessary but permits a short proof.

5.2.22.* Lemma. (Mader [1967], see Thomassen [1988]) If G is a simple graph with minimum degree at least $2k$, then G contains disjoint subgraphs G' and H such that 1) H is connected, 2) $\delta(G') \geq k$, and 3) each vertex of G' has a neighbor in H.

Proof: We may assume that G is connected. Let $G \cdot H'$ denote the graph obtained from G by contracting the edges of a connected subgraph H' and delete extra copies of multiple edges. In $G \cdot H'$, the set $V(H')$ becomes a single vertex. Consider all connected subgraphs H' of G such that $G \cdot H'$ has at least

$k(n(G) - n(H') + 1)$ edges. Since $\delta(G) \geq 2k$, every 1-vertex subgraph of G is such a subgraph. Since such subgraphs exist, we may choose H to be a maximal subgraph with this property.

Let S be the set of vertices outside H with neighbors in H, and let $G' = G[S]$. We need only show that $\delta(G') \geq k$. Each $x \in V(G')$ has a neighbor $y \in V(H)$. In $G \cdot (H \cup xy)$, the edges incident to x in G' collapse onto edges from $V(G')$ to H that appear in $G \cdot H$, and the edge xy contracts. Hence $e(G \cdot H) - e(G \cdot (H \cup xy)) = d_{G'}(x) + 1$. By the choice of H, this difference is more than k, and hence $\delta(G') \geq k$. ■

5.2.23.* Theorem. (Mader [1967], see Thomassen [1988]) If F and G are simple graphs with $e(F) = m$ and $\delta(F) \geq 1$, then $\delta(G) \geq 2^m$ implies that G contains a subdivision of F.

Proof: We use induction on m. The claim is trivial for $m \leq 1$. Consider $m \geq 2$. By Lemma 5.2.22, we may choose disjoint subgraphs H and G' in G such that H is connected, $\delta(G') \geq 2^{m-1}$, and every vertex of G' has a neighbor in H.

If F has an edge $e = xy$ such that $\delta(F - e) \geq 1$, then the induction hypothesis yields a subdivision J of $F - e$ in G'. A path through H can be added between the vertices of J representing x and y to complete a subdivision of F.

If $\delta(F - e) = 0$ for all $e \in E(F)$, then every edge of F is incident to a leaf. Now F is a forest of stars, and $\delta(G) \geq 2^m \geq 2m$ allows us to find F itself in G; we leave this claim to Exercise 42. ■

5.2.24.* Remark. The case when F is a complete graph remains of particular interest. Let $f(k)$ be the minimum d such that every graph with minimum degree at least d contains a K_k-subdivision. Theorem 5.2.23 yields $f(k) \leq 2^{\binom{k}{2}}$. Komlós–Szemerédi [1996] and Bollobás–Thomason [1998] proved that $f(k) < ck^2$ for some constant c (the latter shows $c \leq 256$). Since $K_{m,m-1}$ has no K_{2k}-subdivision when $m = k(k+1)/2$ (Exercise 41), we have $f(k) > k^2/8$.

Exercise 38 yields $f(4) = 3$. Furthermore, $f(5) = 6$. The icosahedron (Exercise 7.3.8) yields $f(5) \geq 6$, since this graph is 5-regular and has no K_5-subdivision. On the other hand, Mader [1998] proved Dirac's conjecture [1964] that every n-vertex graph with at least $3n - 5$ edges contains a K_5-subdivision. By the degree-sum formula, $\delta(G) \geq 6$ yields at least $3n$ edges; hence $f(5) \leq 6$.

Finally, we note that Scott [1997] proved a subdivision version of the Gyárfas–Sumner Conjecture (Remark 5.2.4) for each tree T and integer k: If G has with no k-clique but $\chi(G)$ is sufficiently large, then G contains a subdivision of T as an *induced* subgraph. ■

EXERCISES

5.2.1. (−) Let G be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs x, y of distinct vertices. Prove that G is a complete graph. (Comment: Lovász conjectured that the conclusion also holds when the condition is imposed only on pairs of adjacent vertices.)

5.2.2. (−) Prove that a simple graph is a complete multipartite graph if and only if it has no 3-vertex induced subgraph with one edge.

5.2.3. (−) The results below imply that there is no k-critical graph with $k+1$ vertices.

a) Let x and y be vertices in a k-critical graph G. Prove that $N(x) \subseteq N(y)$ is impossible. Conclude that no k-critical graph has $k+1$ vertices.

b) Prove that $\chi(G \vee H) = \chi(G) + \chi(H)$, and that $G \vee H$ is color-critical if and only if both G and H are color-critical. Conclude that $C_5 \vee K_{k-3}$, with $k+2$ vertices, is k-critical.

5.2.4. For $n \in \mathbb{N}$, let G be the graph with vertex set $\{v_0, \ldots, v_{3n}\}$ defined by $v_i \leftrightarrow v_j$ if and only if $|i-j| \le 2$ and $i+j$ is not divisible by 6.

a) Determine the blocks of G.

b) Prove that adding the edge v_0v_{3n} to G creates a 4-critical graph.

5.2.5. (−) Find a subdivision of K_4 in the Grötzsch graph (Example 5.2.2).

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5.2.6. Determine the minimum number of edges in a connected n-vertex graph with chromatic number k. (Hint: Consider a k-critical subgraph.) (Eršov–Kožuhin [1962]—see Bhasker–Samad–West [1994] for higher connectivity.)

5.2.7. (!) Given an optimal coloring of a k-chromatic graph, prove that for each color i there is a vertex with color i that is adjacent to vertices of the other $k-1$ colors.

5.2.8. Use properties of color-critical graphs to prove Proposition 5.1.14 again: $\chi(G) \le 1 + \max_i \min\{d_i, i-1\}$, where $d_1 \ge \cdots \ge d_n$ are the vertex degrees in G.

5.2.9. (!) Prove that if G is a color-critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical.

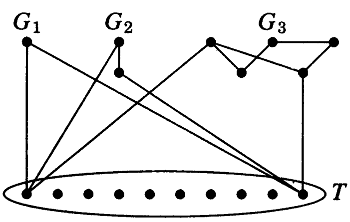
5.2.10. Given a graph G with vertex set $v_1, \ldots, v_n$, let G' be the graph generated from G by Mycielski's construction. Let H be a subgraph of G. Let G'' be the graph obtained from G' by adding the edges $\{u_iu_j \colon v_iv_j \in E(H)\}$. Prove that $\chi(G'') = \chi(G) + 1$ and that $\omega(G'') = \max\{\omega(G), \omega(H) + 1\}$. (Pritikin)

5.2.11. (!) Prove that if G has no induced $2K_2$, then $\chi(G) \le \binom{\omega(G)+1}{2}$. (Hint: Use a maximum clique to define a collection of $\binom{\omega(G)}{2} + \omega(G)$ independent sets that cover the vertices. Comment: This is a special case of the Gyárfás–Sumner Conjecture—Remark 5.2.4) (Wagon [1980])

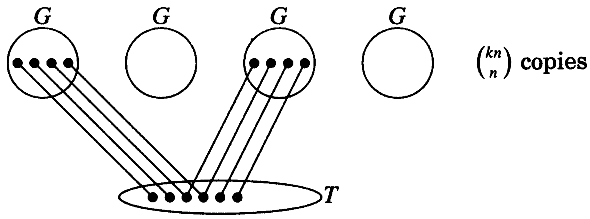
5.2.12. (!) Let $G_1 = K_1$. For $k > 1$, construct G_k as follows. To the disjoint union $G_1 + \cdots + G_{k-1}$, and add an independent set T of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1, \ldots, v_{k-1})$ in $V(G_1) \times \cdots \times V(G_{k-1})$, let one vertex of T have neighborhood $\{v_1, \ldots, v_{k-1}\}$. (In the sketch of G_4 below, neighbors are shown for only two elements of T.)

a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$. (Zykov [1949])

b) Prove that G_k is k-critical. (Schäuble [1969])



5.2.13. (+) Let G be a k-chromatic graph with girth 6 and order n. Construct G' as follows. Let T be an independent set of kn new vertices. Take $\binom{kn}{n}$ pairwise disjoint copies of G, one for each way to choose an n-set $S \subset T$. Add a matching between each copy of G and its corresponding n-set S. Prove that the resulting graph has chromatic number $k+1$ and girth 6. (Comment: Since C_6 is 2-chromatic with girth 6, the process can start and these graphs exist.) (Blanche Descartes [1947, 1954])



5.2.14. *Chromatic number and cycle lengths.*

a) Let v be a vertex in a graph G. Among all spanning trees of G, let T be one that maximizes $\sum_{u \in V(G)} d_T(u, v)$. Prove that every edge of G joins vertices belonging to a path in T starting at v.

b) Prove that if $\chi(G) > k$, then G has a cycle whose length is one more than a multiple of k. (Hint: Use the tree T of part (a) to define a k-coloring of G.) (Tuza)

5.2.15. (!) Prove that a triangle-free graph with n vertices is colorable with $2\sqrt{n}$ colors. (Comment: Thus every k-chromatic triangle-free graph has at least $k^2/4$ vertices.)

5.2.16. (!) Prove that every n-vertex simple graph with no $r+1$-clique has at most $(1 - 1/r)n^2/2$ edges. (Hint: This can be proved using Turán's Theorem or by induction on r without Turán's Theorem.)

5.2.17. (!) Let G be a simple n-vertex graph with m edges.

a) Prove that $\omega(G) \ge \lceil n^2/(n^2 - 2m) \rceil$ and that this bound is sharp. (Hint: Use Exercise 5.2.16. Comment: This also yields $\chi(G) \ge \lceil n^2/(n^2 - 2m) \rceil$.) (Myers–Liu [1972])

b) Prove that $\alpha(G) \ge \lceil n/(d+1) \rceil$, where d is the average vertex degree of G. (Hint: Use part (a).) (Erdős–Gallai [1961])

5.2.18. The Turán graph $T_{n,r}$ (Example 5.2.7) is the complete r-partite graph with b partite sets of size $a+1$ and $r-b$ partite sets of size a, where $a = \lfloor n/r \rfloor$ and $b = n - ra$.

a) Prove that $e(T_{n,r}) = (1 - 1/r)n^2/2 - b(r-b)/(2r)$.

b) Since $e(G)$ must be an integer, part (a) implies $e(T_{n,r}) \le \lfloor (1 - 1/r)n^2/2 \rfloor$. Determine the smallest r such that strict inequality occurs for some n. For this value of r, determine all n such that $e(T_{n,r}) < \lfloor (1 - 1/r)n^2/2 \rfloor$.

5.2.19. (+) Let $a = \lfloor n/r \rfloor$. Compare the Turán graph $T_{n,r}$ with the graph $\overline{K_a} + K_{n-a}$ to prove directly that $e(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$.

5.2.20. Given positive integers n and k, let $q = \lfloor n/k \rfloor$, $r = n - qk$, $s = \lfloor n/(k+1) \rfloor$, and $t = n - s(k+1)$. Prove that $\binom{q}{2}k + rq \ge \binom{s}{2}(k+1) + ts$. (Hint: Consider the complement of the Turán graph.) (Richter [1993])

5.2.21. Prove that among the n-vertex simple graphs with no $r+1$-clique, the Turán graph $T_{n,r}$ is the *unique* graph having the maximum number of edges. (Hint: Examine the proof of Theorem 5.2.9 more carefully.)

5.2.22. A circular city with diameter four miles will get 18 cellular-phone power stations. Each station has a transmission range of six miles. Prove that no matter where in the city the stations are placed, at least two will each be able to transmit to at least five others. (Adapted from Bondy–Murty [1976, p115])

5.2.23. (!) *Turán's proof of Turán's Theorem*, including uniqueness (Turán [1941]).

a) Prove that a maximal simple graph with no $r+1$-clique has an r-clique.

b) Prove that $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$.

c) Use parts (a) and (b) to prove Turán's Theorem by induction on n, including the characterization of graphs acheiving the bound.

5.2.24. (+) Let $t_r(n) = e(T_{n,r})$. Let G be a graph with n vertices that has $t_r(n) - k$ edges and at least one $r+1$-clique, where $k \geq 0$. Prove that G has at least $f_r(n) + 1 - k$ cliques of order $r+1$, where $f_r(n) = n - \lceil n/r \rceil - r$. (Hint: Prove that a graph with exactly one $r+1$-clique has at most $t_r(n) - f_r(n)$ edges.) (Erdős [1964], Moon [1965c])

5.2.25. *Partial analogue of Turán's Theorem for $K_{2,m}$.*

a) Prove that if G is simple and $\sum_{v \in V(G)} \binom{d(v)}{2} > (m-1)\binom{n}{2}$, then G contains $K_{2,m}$. (Hint: View $K_{2,m}$ as two vertices with m common neighbors.)

b) Prove that $\sum_{v \in V(G)} \binom{d(v)}{2} \geq e(2e/n - 1)$, where G has e edges.

c) Use parts (a) and (b) to prove that a graph with more than $\frac{1}{2}(m-1)^{1/2} n^{3/2} + n/4$ edges contains $K_{2,m}$.

d) Application: Given n points in the plane, prove that the distance is exactly 1 for at most $\frac{1}{\sqrt{2}} n^{3/2} + n/4$ pairs. (Bondy–Murty [1976, p111–112])

5.2.26. For $n \geq 4$, prove that every n-vertex graph with more than $\frac{1}{2} n\sqrt{n-1}$ edges has girth at most 4. (Hint: Use the methods of Exercise 5.2.25

5.2.27. (+) For $n \geq 6$, prove that the maximum number of edges in a simple m-vertex graph not having two edge-disjoint cycles is $n+3$. (Pósa)

5.2.28. (+) For $n \geq 6$, prove that the maximum number of edges in a simple n-vertex graph not having two disjoint cycles is $3n - 6$. (Pósa)

5.2.29. (!) Let G be a claw-free graph (no induced $K_{1,3}$).

a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles.

b) Prove that if G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one. (Niessen–Kind [2000])

5.2.30. (+) Prove that if G has a proper coloring g in which every color class has at least two vertices, then G has an optimal coloring f in which every color class has at least two vertices. (Hint: If f has a color class with only one vertex, use g to make an alteration in f. The proof can be given algorithmically or by induction on $\chi(G)$.) (Gallai [1963c])

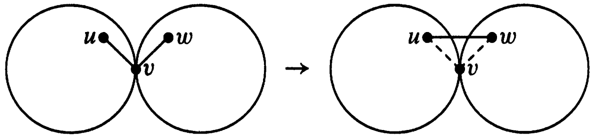
5.2.31. Let G be a connected k-chromatic graph that is not a complete graph or a cycle of length congruent to 3 modulo 6. Prove that every proper k-coloring of G has two vertices of the same color with a common neighbor. (Tomescu)

5.2.32. (!) *The Hajós construction* (Hajós [1961]).

a) Let G and H be k-critical graphs sharing only vertex v, with $vu \in E(G)$ and $vw \in E(H)$. Prove that $(G - vu) \cup (H - vw) \cup uw$ is k-critical.

b) For all $k \geq 3$, use part (a) to obtain a k-critical graph other than K_k.

c) For all $n \geq 4$ except $n = 5$, construct a 4-critical graph with n vertices.



5.2.33. Let G be a k-critical graph having a separating set $S = \{x, y\}$. By Proposition 5.2.18, $x \not\leftrightarrow y$. Prove that G has exactly two S-lobes and that they can be named G_1, G_2 such that $G_1 + xy$ is k-critical and $G_2 \cdot xy$ is k-critical (here $G_2 \cdot xy$ denotes the graph obtained from G_2 by adding xy and then contracting it).

5.2.34. (!) Let G be a 4-critical graph having a separating set S of size 4. Prove that $G[S]$ has at most four edges. (Pritikin)

5.2.35. (+) *Alternative proof that k-critical graphs are $k-1$-edge-connected.*

a) Let G be a k-critical graph, with $k \geq 3$. Prove that for every $e, f \in E(G)$ there is a $k-1$-critical subgraph of G containing e but not f. (Toft [1974])

b) Use part (a) and induction on k to prove Dirac's Theorem that every k-critical graph is $k-1$-edge-connected. (Toft [1974])

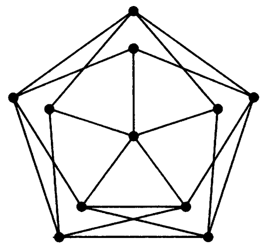
5.2.36. (+) Prove that if G is k-critical and every $k-1$-critical subgraph of G is isomorphic to K_{k-1}, then $G = K_k$ (if $k \geq 4$) (Hint: Use Toft's critical graph lemma—Exercise 5.2.35a.) (Stiebitz [1985])

5.2.37. A graph G is **vertex-color-critical** if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

a) Prove that every color-critical graph is vertex-color-critical.

b) Prove that every 3-chromatic vertex-color-critical graph is color-critical.

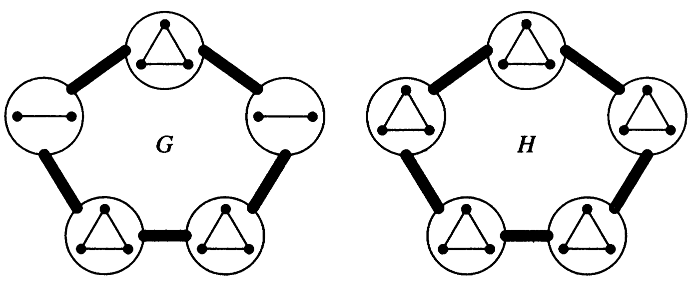
c) Prove that the graph below is vertex-color-critical but not color-critical. (Comment: This is *not* the Grötzsch graph.)



5.2.38. (!) Prove that every simple graph with minimum degree at least 3 contains a K_4-subdivision. (Hint: Prove a stronger result—every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision. The proof of Theorem 5.2.20 already shows that every 3-connected graph contains a K_4-subdivision.) (Dirac [1952a])

5.2.39. (!) Given that $\delta(G) \geq 3$ forces a K_4-subdivision in G, prove that the maximum number of edges in a simple n-vertex graph with no K_4-subdivision is $2n - 3$.

5.2.40. Thick edges below indicate that every vertex in one circle is adjacent to every vertex in the other. Prove that $\chi(G) = 7$ but G has no K_7-subdivision. Prove that $\chi(H) = 8$ but H has no K_8-subdivision. (Catlin [1979])



5.2.41. Let $m = k(k+1)/2$. Prove that $K_{m,m-1}$ has no K_{2k}-subdivision.

5.2.42. (+) Let F be a forest with m edges. Let G be a simple graph such that $\delta(G) \geq m$ and $n(G) \geq n(F)$. Prove that G contains F as a subgraph. (Hint: Delete one leaf from each nontrivial component of F to obtain F'. Let R be the set of neighbors of the deleted vertices. Map R onto an m-set $X \subseteq V(G)$ that minimizes $e(G[X])$. Extend X to a copy of F'. Use Hall's Theorem to show that X can be matched into the remaining vertices to complete a copy of F.) (Brandt [1994])

5.2.43. (+) Let G be a k-chromatic graph. It follows from Lemma 5.1.18 and Proposition 2.1.8 that G contains every k-vertex tree as a subgraph. Strengthen this to a labeled analogue: if f is a proper k-coloring of G and T is a tree with vertex set $\{w_1, \ldots, w_k\}$, then there is an adjacency-preserving map ϕ: $V(T) \to V(G)$ such that $f(\phi(w_i)) = i$ for all i. (Gyárfás–Szemerédi–Tuza [1980], Sumner [1981])

5.2.44. (+) Let G be a k-chromatic graph of girth at least 5. Prove that G contains every k-vertex tree as an induced subgraph. (Gyárfás–Szemerédi–Tuza [1980])

5.3. Enumerative Aspects

Sometimes we can shed light on a hard problem by considering a more general problem. No good algorithm to test existence of a proper k-coloring is known (see Appendix B), but still we can study the number of proper k-colorings (here we fix a particular set of k colors). The chromatic number $\chi(G)$ is the minimum k such that the count is positive; knowing the count for all k would tell us the chromatic number. Birkhoff [1912] introduced this counting problem as a possible way to attack the Four Color Problem (Section 6.3).

In this section, we will discuss properties of the counting function, classes where it is easy to compute, and further related topics.

COUNTING PROPER COLORINGS

We start by defining the counting problem as a function of k.

5.3.1. Definition. Given $k \in \mathbb{N}$ and a graph G, the value $\chi(G; k)$ is the number of proper colorings f: $V(G) \to [k]$. The set of available colors is $[k] = \{1, \ldots, k\}$; the k colors need not all be used in a coloring f. Changing the names of the colors that are used produces a different coloring.

5.3.2. Example. $\chi(\overline{K}_n; k)) = k^n$ *and* $\chi(K_n; k)) = k(k-1)\cdots(k-n+1)$.

When coloring the vertices of $\overline{K}_n$, we can use any of the k colors at each vertex no matter what colors we have used at other vertices. Each of the k^n functions from the vertex set to $[k]$ is a proper coloring, and hence $\chi(\overline{K}_n; k) = k^n$.

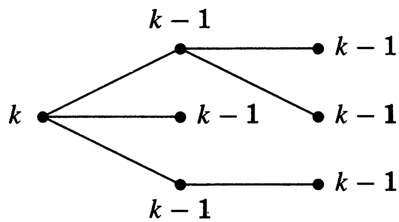
When we color the vertices of K_n, the colors chosen earlier cannot be used on the ith vertex. There remain $k-i+1$ choices for the color of the ith vertex no matter how the earlier colors were chosen. Hence $\chi(K_n; k) = k(k-1)\cdots(k-n+1)$.

We can also count this as $\binom{k}{n}n!$ by first choosing n distinct colors and then multiplying by $n!$ to count the ways to assign the chosen colors to the vertices. For example, $\chi(K_3; 3) = 6$ and $\chi(K_3; 4) = 24$.

The value of the product is 0 when $k < n$. This makes sense, since K_n has no proper k-colorings when $k < n$. ■

5.3.3. Proposition. If T is a tree with n vertices, then $\chi(T; k) = k(k-1)^{n-1}$.

Proof: Choose some vertex v of T as a root. We can color v in k ways. If we extend a proper coloring to new vertices as we grow the tree from v, at each step only the color of the parent is forbidden, and we have $k-1$ choices for the color of the new vertex. Furthermore, deleting a leaf shows inductively that every proper k-coloring arises in this way. Hence $\chi(T; k) = k(k-1)^{n-1}$. ■



Another way to count the colorings is to observe that the color classes of each proper coloring of G partition $V(G)$ into independent sets. Grouping the colorings according to this partition leads to a formula for $\chi(G; k)$ that is a polynomial in k of degree $n(G)$. Note that this holds for the answers in Example 5.3.2 and Proposition 5.3.3. Since every graph has this property, $\chi(G; k)$ as a function of k is called the **chromatic polynomial** of G.

5.3.4. Proposition. Let $x_{(r)} = x(x-1)\cdots(x-r+1)$. If $p_r(G)$ denotes the number of partitions of $V(G)$ into r nonempty independent sets, then $\chi(G; k) = \sum_{r=1}^{n(G)} p_r(G)k_{(r)}$, which is a polynomial in k of degree $n(G)$.

Proof: When r colors are actually used in a proper coloring, the color classes partition $V(G)$ into exactly r independent sets, which can happen in $p_r(G)$ ways. When k colors are available, there are exactly $k_{(r)}$ ways to choose colors and assign them to the classes. All the proper colorings arise in this way, so the formula for $\chi(G; k)$ is correct.

Since $k_{(r)}$ is a polynomial in k and $p_r(G)$ is a constant for each r, this formula implies that $\chi(G; k)$ is a polynomial function of k. When G has n vertices, there is exactly one partition of G into n independent sets and no partition using more sets, so the leading term is k^n. ■

5.3.5. Example. Always $p_n(G) = 1$, using independent sets of size 1. Also $p_1(G) = 0$ unless G has no edges, since only for $\overline{K}_n$ is the entire vertex set an independent set.

Consider $G = C_4$. There is exactly one partition into two independent sets: opposite vertices must be in the same independent set. When $r = 3$, we put two opposite vertices together and leave the other two in sets by themselves; we can do this in two ways. Thus $p_2 = 1$, $p_3 = 2$, $p_4 = 1$.

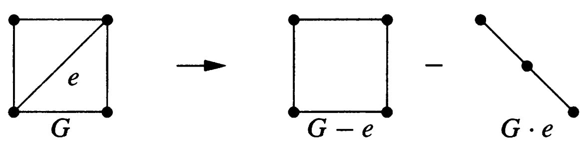
$$\begin{aligned}\chi(C_4; k) &= 1 \cdot k(k-1) + 2 \cdot k(k-1)(k-2) + 1 \cdot k(k-1)(k-2)(k-3) \\ &= k(k-1)(k^2 - 3k + 3).\end{aligned}$$ ■

Computing the chromatic polynomial in this way is not generally feasible, since there are too many partitions to consider. There is a recursive computation much like that used in Proposition 2.2.8 to count spanning trees. Again $G \cdot e$ denotes the graph obtained by contracting the edge e in G (Definition 2.2.7). Since the number of proper k-colorings is unaffected by multiple edges, **we discard multiple copies of edges that arise from the contraction**, keeping only one copy of each to form a simple graph.

5.3.6. Theorem. (Chromatic recurrence) If G is a simple graph and $e \in E(G)$, then $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$.

Proof: Every proper k-coloring of G is a proper k-coloring of $G - e$. A proper k-coloring of $G - e$ is a proper k-coloring of G if and only if it gives distinct colors to the endpoints u, v of e. Hence we can count the proper k-colorings of G by subtracting from $\chi(G - e; k)$ the number of proper k-colorings of $G - e$ that give u and v the same color.

Colorings of $G - e$ in which u and v have the same color correspond directly to proper k-colorings of $G \cdot e$, in which the color of the contracted vertex is the common color of u and v. Such a coloring properly colors all the edges of $G \cdot e$ if and only if it properly colors all the edges of G other than e. ■



5.3.7. Example. *Proper k-colorings of* C_4. Deleting an edge of C_4 produces P_4, while contracting an edge produces K_3. Since P_4 is a tree and K_3 is a complete graph, we have $\chi(P_4; k) = k(k-1)^3$ and $\chi(K_3; k) = k(k-1)(k-2)$. Using the chromatic recurrence, we obtain

$$\chi(C_4; k) = \chi(P_4; k) - \chi(K_3; k) = k(k-1)(k^2 - 3k + 3).$$ ■

Because both $G - e$ and $G \cdot e$ have fewer edges than G, we can use the chromatic recurrence inductively to compute $\chi(G; k)$. We need initial conditions for graphs with no edges, which we have already computed: $\chi(\overline{K}_n; k) = k^n$.

5.3.8. Theorem. (Whitney [1933c]) The chromatic polynomial $\chi(G;k)$ has degree $n(G)$, with integer coefficients alternating in sign and beginning $1, -e(G), \cdots$.

Proof: We use induction on $e(G)$. The claims hold trivially when $e(G) = 0$, where $\chi(\overline{K}_n;k) = k^n$. For the induction step, let G be an n-vertex graph with $e(G) \geq 1$. Each of $G-e$ and $G \cdot e$ has fewer edges than G, and $G \cdot e$ has $n-1$ vertices. By the induction hypothesis, there are nonnegative integers $\{a_i\}$ and $\{b_i\}$ such that $\chi(G-e;k) = \sum_{i=0}^{n}(-1)^i a_i k^{n-i}$ and $\chi(G \cdot e;k) = \sum_{i=0}^{n-1}(-1)^i b_i k^{n-1-i}$. By the chromatic recurrence,

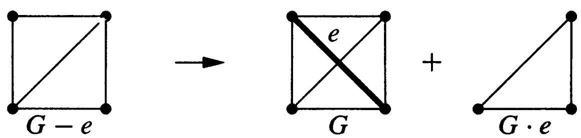
$$\begin{array}{rllllll}
\chi(G-e;k): & k^n & -\ [e(G)-1]k^{n-1} & +\ a_2k^{n-2} & -\cdots+ & (-1)^i & a_ik^{n-i}\cdots \\
-\ \chi(G\cdot e;k): & -(& k^{n-1} & -\ b_1k^{n-2} & +\cdots+ & (-1)^{i-1} & b_{i-1}k^{n-i}\cdots) \\
\hline
=\quad \chi(G;k): & k^n & -\ e(G)k^{n-1} & +\ (a_2+b_1)k^{n-2} & -\cdots+ & (-1)^i(a_i+b_{i-1})k^{n-i}\cdots &
\end{array}$$

Hence $\chi(G;k)$ is a polynomial with leading coefficient $a_0 = 1$ and next coefficient $-(a_1+b_0) = -e(G)$, and its coefficients alternate in sign. ■

5.3.9. Example. When adding an edge yields a graph whose chromatic polynomial is easy to compute, we can use the chromatic recurrence in a different way. Instead of $\chi(G;k) = \chi(G-e;k) - \chi(G \cdot e;k)$, we can write $\chi(G-e;k) = \chi(G;k) + \chi(G \cdot e;k)$. Thus we may be able to compute $\chi(G-e;k)$ using $\chi(G;k)$.

To compute $\chi(K_n - e;k)$, for example, we let G be K_n in this alternative formula and obtain

$$\chi(K_n - e;k) = \chi(K_n;k) + \chi(K_{n-1};k) = (k-n+2)^2\prod_{i=0}^{n-3}(k-i). \quad ■$$



We close our general discussion of $\chi(G;k)$ with an explicit formula. It has exponentially many terms, so its uses are primarily theoretical. The formula summarizes what happens if we iterate the chromatic recurrence until we dispose of all the edges.

5.3.10. Theorem. (Whitney [1932b]) Let $c(G)$ denote the number of components of a graph G. Given a set $S \subseteq E(G)$ of edges in G, let $G(S)$ denote the spanning subgraph of G with edge set S. Then the number $\chi(G;k)$ of proper k-colorings of G is given by

$$\chi(G;k) = \sum_{S \subseteq E(G)} (-1)^{|S|} k^{c(G(S))}$$

Proof: In applying the chromatic recurrence, contraction may produce multiple edges. We have observed that dropping these does not affect $\chi(G;k)$. We claim that deleting extra copies of edges also does not change the claimed formula.

Let e and e' be edges in G with the same endpoints. When $e' \in S$ and $e \notin S$, we have $c(G(S \cup \{e\})) = c(G(S))$, since both endpoints of e are in the same component of $G(S)$. However, $|S \cup \{e\}| = |S| + 1$. Thus the terms for S and $S \cup \{e\}$ in the sum cancel. Therefore, omitting all terms for sets of edges containing e' does not change the sum. This implies that we can keep or drop e' from the graph without changing the formula.

When computing the chromatic recurrence, we therefore obtain the same result if we do not discard multiple edges or loops and instead retain all edges for contraction or deletion. Iterating the recurrence now yields $2^{e(G)}$ terms as we dispose of all the edges; each in turn is deleted or contracted.

When all edges have been deleted or contracted, the graph that remains consists of isolated vertices. Let S be the set of edges that were contracted. The remaining vertices correspond to the components of $G(S)$; each such component becomes one vertex when the edges of S are contracted and the other edges are deleted. The $c(G(S))$ isolated vertices at the end yield a term with $k^{c(G(S))}$ colorings. Furthermore, the sign of the contribution changes for each contracted edge, so the contribution is positive if and only if $|S|$ is even.

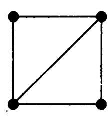
Thus the contribution when S is the set of contracted edges is $(-1)^{|S|}k^{c(G(S))}$, and this accounts for all terms in the sum. ■

5.3.11. Example. *A chromatic polynomial.* When G is a simple graph with n vertices, every spanning subgraph with 0, 1, or 2 edges has n, $n-1$, or $n-2$ components, respectively. When $|S| = 3$, the number of components is $n-2$ if and only if the three edges form a triangle; otherwise it is $n-3$.

For example, when G is a kite (four vertices, five edges) there are ten sets of three edges. For two of these, $G(S)$ consists of a triangle plus one isolated vertex. The other eight sets of three edges yield spanning subgraphs with one component. Both types of triples are counting negatively, since $|S| = 3$. All spanning subgraphs with four or five edges have only one component. Hence Theorem 5.3.10 yields

$$\chi(G; k) = k^4 - 5k^3 + 10k^2 - (2k^2 + 8k^1) + 5k - k = k^4 - 5k^3 + 8k^2 - 4k.$$

This agrees with $\chi(G; k) = k(k-1)(k-2)(k-2)$, computed by counting colorings directly or by using $\chi(G; k) = \chi(C_4; k) - \chi(P_3; k)$. ■



Whitney proved Theorem 5.3.10 using the inclusion-exclusion principle of elementary counting. Among the universe of k-colorings, the proper colorings are those not assigning the same color to the endpoints of any edge. Letting A_i be the set of k-colorings assigning the same color to the endpoints of edge e_i, we want to count the colorings that lie in none of $A_1, \ldots, A_m$ (see Exercise 17).

CHORDAL GRAPHS

Counting colorings is easy for cliques and trees (and the kite) because each such graph arises from K_1 by successively adding a vertex joined to a clique. The chromatic polynomial of such a graph is a product of linear factors.

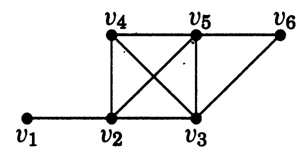
5.3.12. Definition. A vertex of G is **simplicial** if its neighborhood in G induces a clique. A **simplicial elimination ordering** is an ordering $v_n, \ldots, v_1$ for deletion of vertices so that each vertex v_i is a simplicial vertex of the remaining graph induced by $\{v_1, \ldots, v_i\}$. (These orderings are also called **perfect elimination orderings.**)

5.3.13. Example. *Chromatic polynomials from simplicial elimination orderings.* In a tree, a simplicial elimination ordering is a successive deletion of leaves. We have observed that $\chi(G; k) = k(k-1)^{n-1}$ when G is an n-vertex tree.

When $v_n, \ldots, v_1$ is a simplicial elimination ordering for G, the product rule of elementary combinatorics (Appendix A) allows us to count proper k-colorings of G. If we have colored $v_1, \ldots, v_i$, then when we add v_i there are $k-d(i)$ ways to color it, where $d(i) = |N(v_i) \cap \{v_1, \ldots, v_{i-1}\}|$. The factor $k - d(i)$ is independent of how previous color choices were made, because the neighbors of v_i that have been colored form a clique of size $d(i)$ and have distinct colors.

Deleting a simplicial vertex that starts a simplicial elimination ordering yields inductively that every proper k-coloring of G arises in this way. Thus we have expressed the chromatic polynomial as a product of linear factors.

In the graph below, $v_6, \ldots, v_1$ is a simplicial elimination ordering. When we form the graph in the order $v_1, \ldots, v_6$, the values $d(1), \ldots, d(6)$ are 0, 1, 1, 2, 3, 2, and the chromatic polynomial is $k(k-1)(k-1)(k-2)(k-3)(k-2)$. ■



5.3.14. Remark. It is important to note that some graphs without simplicial elimination orderings also have chromatic polynomials that can be expressed as a product of linear factors of the form $k - r_i$ with r_i a nonnegative integer. Exercise 19 presents an example. Thus the existence of a simplicial elimination ordering is a sufficient but not necessary condition for the chromatic polynomial to have this nice factorization property. ■

Trees, cliques, near-complete graphs ($K_n - e$), and interval graphs (Exercise 28) all have simplicial elimination orderings. When $n \geq 3$, the cycle C_n has no simplicial elimination ordering, because a cycle has no simplicial vertex to start the elimination. The existence of simplicial elimination orderings is equivalent to the absence of such cycles as induced subgraphs.

5.3.15. Definition. A **chord** of a cycle C is an edge not in C whose endpoints lie in C. A **chordless cycle** in G is a cycle of length at least 4 in G that has no chord (that is, the cycle is an induced subgraph). A graph G is **chordal** if it is simple and has no chordless cycle.

The motivation for the term "chord" is geometric. If a cycle is drawn with its vertices in order on a circle and its chords are drawn as line segments, then the chords of the cycle are chords of the circle.

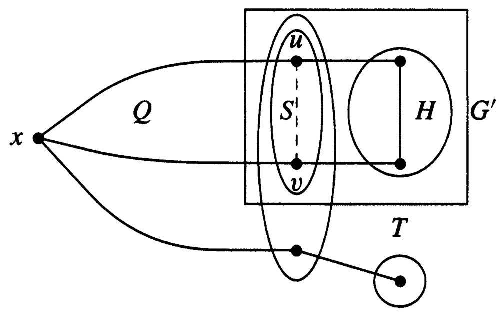
It is fairly easy to show that a graph with a simplicial elimination ordering cannot have a chordless cycle. Thus our characterization of these graphs is another TONCAS theorem. We separate the substantive part of the proof of sufficiency as a lemma that is useful on its own (see also Laskar–Shier [1983]).

5.3.16. Lemma. (Voloshin [1982], Farber–Jamison [1986]) For every vertex x in a chordal graph G, there is a simplicial vertex of G among the vertices farthest from x in G

Proof: We use induction on $n(G)$. Basis step ($n(G) = 1$): The one vertex in K_1 is simplicial.

Induction step ($n(G) \geq 2$): If x is adjacent to all other vertices, then we apply the induction hypothesis to the chordal graph $G - x$. Each simplicial vertex y of $G - x$ is also simplicial in G, since x is adjacent to all of $N(y) \cup \{y\}$.

Otherwise, let T be the set of vertices in G with maximum distance from x, and let H be a component of $G[T]$. Let S be the set of vertices in $G - T$ having neighbors in $V(H)$, and let Q be the component of $G - S$ containing x.

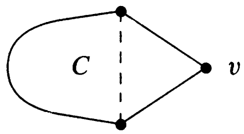


We claim that S is a clique. Each vertex of S has a neighbor in $V(H)$ and a neighbor in Q. For distinct vertices $u, v \in S$, the union of shortest u,v-paths through H and through Q is a cycle of length at least 4. Since there are no edges from $V(H)$ to $V(Q)$, this cycle has no chord other than uv. Since G has no chordless cycle, $u \leftrightarrow v$. Since $u, v \in S$ were chosen arbitrarily, S is a clique.

Now let $G' = G[S \cup V(H)]$; this omits x and thus is smaller than G. We apply the induction hypothesis to G' and a vertex $u \in S$. Since S is a clique, $S - \{u\} \subseteq N(u)$. Whether G' is a clique or not, it thus has a simplicial vertex z within $V(H)$. Since $N_G(z) \subseteq V(G')$, the vertex z is also simplicial in G, and z is a vertex with maximum distance from x, as desired. ∎

5.3.17. Theorem. (Dirac [1961]) A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

Proof: *Necessity.* Let G be a graph with a simplicial elimination ordering. Let C be a cycle in G of length at least 4. At the point when the elimination ordering first deletes a vertex of C, say v, the remaining neighbors of v form a clique. The clique includes the neighbors of v on C; the resulting edge joining them is a chord of C. Hence G has no chordless cycle.



Sufficiency. By Lemma 5.3.16, every chordal graph has a simplicial vertex. This yields a simplicial elimination ordering by induction on $n(G)$, since every induced subgraph of a chordal graph is a chordal graph. ■

Other properties of chordal graphs appear in Exercises 20–27.

A HINT OF PERFECT GRAPHS

In Proposition 5.1.16, we proved that $\chi(G) = \omega(G)$ when G is an interval graph. Furthermore, every induced subgraph of an interval graph is also an interval graph, since we can delete the interval representing v in an interval representation of G to obtain an interval representation of $G - v$. Thus $\chi(H) = \omega(H)$ holds for every induced subgraph H of an interval graph.

5.3.18. Definition. A graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph $H \subseteq G$. Equivalently, $\chi(G[A]) = \omega(G[A])$ for all $A \subseteq V(G)$.

The **clique cover number** $\theta(G)$ of a graph G is the minimum number of cliques in G needed to cover $V(G)$; note that $\theta(G) = \chi(\overline{G})$.

Since cliques and independent sets exchange roles under complementation, the statement of perfection for $\overline{G}$ is "$\alpha(H) = \theta(H)$ for every induced subgraph H of G". Lovász [1972a, 1972b] proved the **Perfect Graph Theorem** (PGT): G is perfect if and only if its complement $\overline{G}$ is perfect. We prove this in Theorem 8.1.6; here we merely illustrate perfect graphs.

5.3.19. Definition. A family of graphs **G** is **hereditary** if every induced subgraph of a graph in **G** is also a graph in **G**.

5.3.20. Remark. In order to prove that every graph in a hereditary class **G** is perfect, it suffices to verify that $\chi(G) = \omega(G)$ for every $G \in \mathbf{G}$. Doing so includes the proof of equality for the induced subgraphs of G. ■

5.3.21. Example. *Bipartite graphs and their line graphs.* Bipartite graphs form a hereditary class, and $\chi(G) = \omega(G)$ for every bipartite graph; hence bipartite graphs are perfect. When H is bipartite, the statement of perfection for $\overline{H}$ is Exercise 5.1.38 and follows from $\alpha(H) = \beta'(H)$ (Corollary 3.1.24). For bipartite graphs, the nontrivial $\alpha(G) = \theta(G) = \beta'(G)$ follows at once from the trivial $\chi(G) = \omega(G)$ by the PGT.

We briefly introduced line graphs in Definition 4.2.18 to prove the edge versions of Menger's Theorem; recall that the line graph $L(G)$ has a vertex for each edge of G, with $e, f \in V(L(G))$ adjacent in $L(G)$ if they have a common endpoint in G. Line graphs of bipartite graphs form a hereditary family, since deleting a vertex in the line graph represents deleting the corresponding edge in the original graph.

Therefore, proving that $\alpha(L(G)) = \theta(L(G))$ when G is bipartite will show that complements of line graphs are perfect. A clique in $L(G)$ (when G is bipartite) consists of edges in G with a common endpoint. Thus covering the vertices of $L(G)$ with cliques corresponds to selecting vertices in G to form a vertex cover. Independent sets in $L(G)$ are matchings in G. Thus perfection for complements of line graphs of bipartite graphs amounts to the König–Egerváry Theorem ($\alpha'(G) = \beta(G)$) for matchings and vertex covers in bipartite graphs.

From this the PGT yields also $\chi(L(G)) = \omega(L(G))$. A proper coloring of $L(G)$ is a partition of $E(G)$ into matchings, and $\omega(L(G)) = \Delta(G)$ (for bipartite G). Hence $\chi(L(G)) = \omega(L(G))$ means that the edges of a bipartite graph G can be partitioned into $\Delta(G)$ matchings. In Theorem 7.1.7, we prove directly this additional result of König [1916]. ■

Since every interval graph is a chordal graph (Exercise 28), proving that all chordal graphs are perfect strengthens Proposition 5.1.16. We explore other characterizations of interval graphs and chordal graphs in Section 8.1.

5.3.22. Theorem. (Berge [1960]) Chordal graphs are perfect.

Proof: Deleting vertices cannot create chordless cycles, so the family is hereditary. By Remark 5.3.20, we need only prove $\chi(G) = \omega(G)$ when G is chordal.

In Theorem 5.3.17, we proved that G has a simplicial elimination ordering. Let $v_1, \ldots, v_n$ be the reverse of such an ordering. For each i, the neighbors of v_i among $\{v_1, \ldots, v_{i-1}\}$ form a clique.

We apply greedy coloring with this ordering. If v_i receives color k, then colors $1, \ldots, k-1$ appear on earlier neighbors of v_i. Since they form a clique, with v_i we have a clique of size k. Thus we obtain a clique whose size equals the number of colors used. ■

The argument of Theorem 5.3.22 shows that greedy coloring relative to the reverse of a simplicial elimination ordering produces an optimal coloring. This generalizes Proposition 5.1.16 about interval graphs.

We present one more fundamental class of perfect graphs; it includes all bipartite graphs.

5.3.23.* Definition. A **transitive orientation** of a graph G is an orientation D such that whenever xy and yz are edges in D, also there is an edge xz in G that is oriented from x to z in D. A simple graph G is a **comparability graph** if it has a transitive orientation.

5.3.24.* Example. If G is an X, Y-bigraph, then directing every edge from X to Y yields a transitive orientation. Thus every bipartite graph is a comparability graph. Transitive orientations arise from order relations; $x \to y$ could mean "x contains y", which is a transitive relation. ■

5.3.25.* Proposition. (Berge [1960]) Comparability graphs are perfect.

Proof: Every induced subdigraph of a transitive digraph is transitive, so the class of comparability graphs is hereditary. Thus we need only show that each comparability graph G is $\omega(G)$-colorable.

Let F be a transitive orientation of G; note that F has no cycle. As shown in proving Theorem 5.1.21, the coloring of G that assigns to each vertex v the number of vertices in the longest path of F ending at v is a proper coloring. By transitivity, the vertices of a path in F form a clique in G. Thus we have $\chi(G) \le \omega(G)$. ■

COUNTING ACYCLIC ORIENTATIONS (optional)

Surprisingly, $\chi(G; k)$ has meaning when k is a negative integer. An **acyclic orientation** of a graph is an orientation having no cycle. Setting $k = -1$ in $\chi(G; k)$ enables us to count the acyclic orientations of G.

5.3.26. Example. Since C_4 has 4 edges, it has 16 orientations. Of these, 14 are acyclic. In Example 5.3.7, we proved that $\chi(C_4; k) = k(k-1)(k^2 - 3k + 3)$. Evaluated at $k = -1$, this equals $(-1)(-2)(7) = 14$. ■

5.3.27. Theorem. (Stanley [1973]) The value of $\chi(G; k)$ at $k = -1$ is $(-1)^{n(G)}$ times the number of acyclic orientations of G.

Proof: We use induction on $e(G)$. Let $a(G)$ be the number of acyclic orientations of G. When G has no edges, $a(G) = 1$ and $\chi(G; -1) = (-1)^{n(G)}$, so the claim holds. We will prove that $a(G) = a(G - e) + a(G \cdot e)$ for $e \in E(G)$. If so, then we apply the recurrence for a, the induction hypothesis for $a(G)$ in terms of $\chi(G; k)$, and the chromatic recurrence to compute

$$a(G) = (-1)^{n(G)}\chi(G - e; -1) + (-1)^{n(G)-1}\chi(G \cdot e; -1) = (-1)^{n(G)}\chi(G; -1).$$

Now we prove the recurrence for a. Every acyclic orientation of G contains an acyclic orientation of $G - e$. An acyclic orientation D of $G - e$ may extend to 0, 1, or 2 acyclic orientations of G by orienting the edge $e = uv$. When D has no u, v-path, we can choose $v \to u$. When D has no v, u-path, we can choose $u \to v$. Since D is acyclic, D cannot have both a u, v-path and a v, u-path, so the two choices for e cannot both be forbidden.

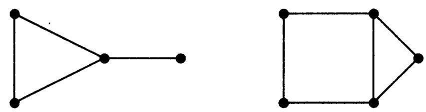
Hence every D extends in at least one way, and $a(G)$ equals $a(G-e)$ plus the number of orientations that extend in both ways. Those extending in both ways are the acyclic orientations of $G-e$ with no u,v-path *and* no v,u-path. There are exactly $a(G\cdot e)$ of these, since a u,v-path or a v,u-path in an orientation of $G-e$ becomes a cycle in $G\cdot e$. ■

The interpretation of $\chi(G;k)$ for general negative k (Exercise 32) is an instance of the phenomenon of "combinatorial reciprocity" (Stanley [1974]).

EXERCISES

Keep in mind that the notation $\chi(G;k)$ may be viewed as a polynomial or as the number of proper k-colorings of G.

5.3.1. (−) Compute the chromatic polynomials of the graphs below.



5.3.2. (−) Use the chromatic recurrence to obtain the chromatic polynomial of every tree with n vertices.

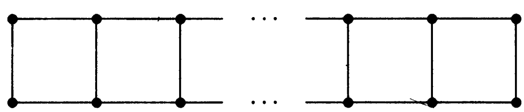
5.3.3. (−) Prove that $k^4-4k^3+3k^2$ is not a chromatic polynomial.

• • • • •

5.3.4. a) Prove that $\chi(C_n;k)=(k-1)^n+(-1)^n(k-1)$.

b) For $H=G\vee K_1$, prove that $\chi(F;k)=k\chi(G;k-1)$. From this and part (a), find the chromatic polynomial of the wheel $C_n\vee K_1$.

5.3.5. For $n\geq 1$, let $G_n=P_n\,\square\,K_2$; this is the graph with $2n$ vertices and $3n-2$ edges shown below. Prove that $\chi(G_n;k)=(k^2-3k+3)^{n-1}k(k-1)$.



5.3.6. (!) Let G be a graph with n vertices. Use Proposition 5.3.4 to give a non-inductive proof that the coefficient of k^{n-1} in $\chi(G;k)$ is $-e(G)$.

5.3.7. Prove that the chromatic polynomial of an n-vertex graph has no real root larger than $n-1$. (Hint: Use Proposition 5.3.4.)

5.3.8. (!) Prove that the number of proper k-colorings of a connected graph G is less than $k(k-1)^{n-1}$ if $k\geq 3$ and G is not a tree. What happens when $k=2$?

5.3.9. (!) Prove that $\chi(G;x+y)=\sum_{U\subseteq V(G)}\chi(G[U];x)\chi(G[\overline{U}];y)$. (Hint: Since both sides are polynomials, it suffices to prove equality when x and y are positive integers; do this by counting proper $x+y$-colorings in a different way.)

5.3.10. Let G be a connected graph with $\chi(G;k) = \sum_{i=0}^{n-1}(-1)^i a_i k^{n-i}$. For $1 \le i \le n$, prove that $a_i \ge \binom{n-1}{i}$. (Hint: Use the chromatic recurrence.)

5.3.11. (!) Prove that the sum of the coefficients of $\chi(G;k)$ is 0 unless G has no edges. (Hint: When a function is a polynomial, how can one obtain the sum of the coefficients?)

5.3.12. (+) *Coefficients of* $\chi(G;k)$.

a) Prove that the last nonzero term in the chromatic polynomial of G is the term whose exponent is the number of components of G.

b) Use part (a) to prove that if $p(k) = k^n - ak^{n-1} + \cdots \pm ck^r$ and $a > \binom{n-r+1}{2}$, then p is not a chromatic polynomial. (For example, this immediately implies that the polynomial in Exercise 5.3.3 is not a chromatic polynomial.)

5.3.13. Let G and H be graphs, possibly overlapping.

a) Prove that $\chi(G \cup H;k) = \frac{\chi(G;k)\chi(H;k)}{\chi(G \cap H;k)}$ when $G \cap H$ is a complete graph.

b) Consider two paths whose union is a cycle to show that the formula may fail when $G \cap H$ is not a complete graph.

c) Apply part (a) to conclude that the chromatic number of a graph is the maximum of the chromatic numbers of its blocks.

5.3.14. (!) Let P be the Petersen graph. By Brooks' Theorem, the Petersen graph is 3-colorable, and hence by the pigeonhole principle it has an independent set S of size 4.

a) Prove that $P - S = 3K_2$.

b) Using part (a) and symmetry, determine the number of vertex partitions of P into three independent sets.

c) In general, how can the number of partitions into the minimum number of independent sets be obtained from the chromatic polynomial of G?

5.3.15. Prove that a graph with chromatic number k has at most k^{n-k} vertex partitions into k independent sets, with equality achieved only by $K_k + (n-k)K_1$ (a k-clique plus $n-k$ isolated vertices). (Hint: Use induction on n and consider the deletion of a single vertex.) (Tomescu [1971])

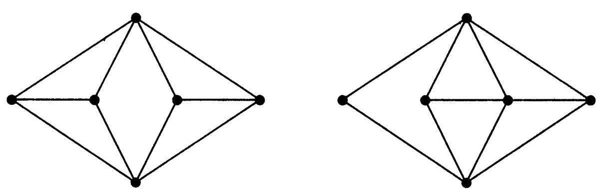
5.3.16. Let G be a simple graph with n vertices and m edges. Prove that G has at most $\frac{1}{3}\binom{m}{2}$ triangles. Conclude that the coefficient of k^{n-2} in $\chi(G;k)$ is positive, unless G has at most one edge. (Hint: Use Theorem 5.3.10.)

5.3.17. (∗) Use the inclusion-exclusion principle to prove Theorem 5.3.10 directly.

5.3.18. (!) Consider the chromatic polynomials of the graphs below.

a) Without computing them, give a short proof that they are equal.

b) Express this chromatic polynomial as the sum of the chromatic polynomials of two chordal graphs, and use this to give a one-line computation of it.



5.3.19. (−) Let G be the graph obtained from K_6 by subdividing one edge. Use the chromatic recurrence to Compute $\chi(G;k)$ as a product of linear factors (factors of the form $k - c_i$). Show that G is not a chordal graph. (Read [1975], Dmitriev [1980])

5.3.20. Let G be a chordal graph. Use a simplicial elimination ordering of G to prove the following statements.

a) G has at most n maximal cliques, with equality if and only if G has no edges. (Fulkerson–Gross [1965])

b) Every maximal clique of G containing no simplicial vertex of G is a separating set.

5.3.21. The **Szekeres–Wilf number** of a graph G is $1 + \max_{H \subseteq G} \delta(H)$. Prove that a graph G is chordal if and only if in every induced subgraph the Szekeres–Wilf number equals the clique number. (Voloshin [1982])

5.3.22. Let $k_r(G)$ be the number of r-cliques in a connected chordal graph G. Prove that $\sum_{r \ge 1} (-1)^{r-1} k_r(G) = 1$. (Hint: Use induction on $n(G)$. Note that the binomial formula (Appendix A) implies that $\sum_{j \ge 0} (-1)^j \binom{m}{j} = 0$ when $m \in \mathbb{N}$.)

5.3.23. Let S be the vertex set of a cycle in a chordal graph G. Prove that G has a cycle whose vertex set consists of all but one element of S. (Comment: When G has a spanning cycle and $S \subset V(G)$, Hendry conjectured that G also has a cycle whose vertex set consists of S plus one vertex.) (Hendry [1990])

5.3.24. Let e be a edge of a cycle C in a chordal graph. Prove that e forms a triangle with a third vertex of C.

5.3.25. Let Q be a maximal clique in a chordal graph G. Prove that if $G - Q$ is connected, then Q contains a simplicial vertex. (Voloshin–Gorgos [1982])

5.3.26. Exercise 5.3.13 establishes the formula $\chi(G \cup H; k) = \frac{\chi(G;k)\chi(H;k)}{\chi(G \cap H;k)}$ when $G \cap H$ is a complete graph.

a) Prove that the formula holds when $G \cup H$ is a chordal graph regardless of whether $G \cap H$ is a complete graph.

b) Prove that if x is a vertex in a chordal graph G, then

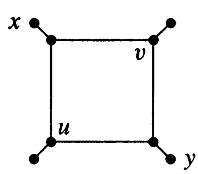
$$\chi(G; k) = \chi(G - x; k)k \frac{\chi(G[N(x)]; k - 1)}{\chi(G[N(x)]; k)}.$$

(Comment: Part (b) allows the chromatic polynomial of a chordal graph to be computed via an arbitrary elimination ordering. For example, eliminating the central vertex of P_5 yields $\chi(P_5; k) = [k(k-1)]^2 k \frac{(k-1)^2}{k^2} = k(k-1)^4$.) (Voloshin [1982])

5.3.27. (+) A **minimal vertex separator** in a graph G is a set $S \subseteq V(G)$ that for some pair x, y is a minimal set whose deletion separates x and y. Every minimal separating set is a minimal vertex separator, but u, v below show that the converse need not hold.

a) Prove that if every minimal vertex separator in G is a clique, then the same property holds in every induced subgraph of G.

b) Prove that a graph G is chordal if and only if every minimal vertex separator is a clique. (Dirac [1961])



5.3.28. (!) Let G be an interval graph. Prove that G is a chordal graph and that $\overline{G}$ is a comparability graph.

5.3.29. Determine the smallest imperfect graph G such that $\chi(G) = \omega(G)$.

5.3.30. An edge in an acyclic orientation of G is **dependent** if reversing it yields a cycle.

a) Prove that every acyclic orientation of a connected n-vertex graph has at least $n - 1$ independent edges.

b) Prove that if $\chi(G)$ is less than the girth of G, then G has an orientation with no dependent edges. (Hint: Use the technique in the proof of Theorem 5.1.21.)

5.3.31. $(*)$ The number $a(G)$ of acyclic orientations of G satisfies the recurrence $a(G) = a(G - e) + a(G \cdot e)$ (Theorem 5.3.27). The number of spanning trees of G appears to satisfy the same recurrence; does the number of acyclic orientations of G always equal the number of spanning trees? Why or why not?

5.3.32. $(*)$ Let D be an acyclic orientation of G, and let f be a coloring of $V(G)$ from the set $[k]$. We say that (D, f) is a **compatible pair** if $u \rightarrow v$ in D implies $f(u) \leq f(v)$. Let $\eta(G; k)$ be the number of compatible pairs. Prove that $\eta(G; k) = (-1)^{n(G)}\chi(G; k)$. (Stanley [1973])

Chapter 6

Planar Graphs

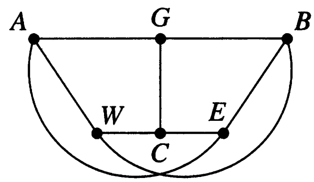
6.1. Embeddings and Euler's Formula

Topological graph theory, broadly conceived, is the study of graph layouts. Initial motivation involved the famous Four Color Problem: can the regions of every map on a globe be colored with four colors so that regions sharing a nontrivial boundary have different colors? Later motivation involves circuit layouts on silicon chips. Wire crossings cause problems in layouts, so we ask which circuits have layouts without crossings.

DRAWINGS IN THE PLANE

The following brain teaser appeared as early as Dudeney [1917].

6.1.1. Example. *Gas–water–electricity.* Three sworn enemies A, B, C live in houses in the woods. We must cut paths so that each has a path to each of three utilities, which by tradition are gas, water, and electricity. In order to avoid confrontations, we don't want any of the paths to cross. Can this be done? This asks whether $K_{3,3}$ can be drawn in the plane without edge crossings; we will give two proofs that it cannot. ■

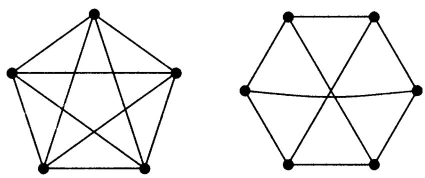


Arguments about drawings of graphs in the plane are based on the fact that every closed curve in the plane separates the plane into two regions (the inside and the outside). In elementary graph theory, we take this as an intuitive notion, but the full details in topology are quite difficult. Before discussing a way to make the arguments precise for graph theory, we show informally how this result is used to prove impossibility for planar drawings.

6.1.2. Proposition. K_5 and $K_{3,3}$ cannot be drawn without crossings.

Proof: Consider a drawing of K_5 or $K_{3,3}$ in the plane. Let C be a spanning cycle. If the drawing does not have crossing edges, then C is drawn as a closed curve. Chords of C must be drawn inside or outside this curve. Two chords conflict if their endpoints on C occur in alternating order. When two chords conflict, we can draw only one inside C and one outside C.

A 6-cycle in $K_{3,3}$ has three pairwise conflicting chords. We can put at most one inside and one outside, so it is not possible to complete the embedding. When C is a 5-cycle in K_5, at most two chords can go inside or outside. Since there are five chords, again it is not possible to complete the embeddings. Hence neither of these graphs is planar. ■



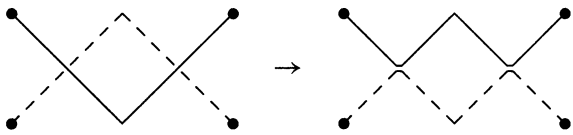
We need a precise notion of "drawing". We have used curves for edges. Using only curves formed from line segments avoids topological difficulties. These can approximate any curve well enough that the eye cannot tell the difference.

6.1.3. Definition. A **curve** is the image of a continuous map from $[0, 1]$ to $\mathbb{R}^2$. A **polygonal curve** is a curve composed of finitely many line segments. It is a **polygonal** u, v**-curve** when it starts at u and ends at v.

A **drawing** of a graph G is a function f defined on $V(G) \cup E(G)$ that assigns each vertex v a point $f(v)$ in the plane and assigns each edge with endpoints u, v a polygonal $f(u), f(v)$-curve. The images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a **crossing**.

It is common to use the same name for a graph G and a particular drawing of G, referring to the points and curves in the drawing as the vertices and edges of G. Since the endpoint relation between the points and curves is the same as the incidence relation between the vertices and edges, the drawing can be viewed as a member of the isomorphism class containing G.

By moving edges slightly, we can ensure that no three edges have a common internal point, that an edge contains no vertex except its endpoints, and that no two edges are tangent. If two edges cross more than once, then modifying them as shown below reduces the number of crossings; thus we also require that edges cross at most once. We consider only drawings with these properties.



6.1.4. Definition. A graph is **planar** if it has a drawing without crossings. Such a drawing is a **planar embedding** of G. A **plane graph** is a particular planar embedding of a planar graph.

A curve is **closed** if its first and last points are the same. It is **simple** if it has no repeated points except possibly first=last.

A planar embedding of a graph cuts the plane into pieces. These pieces are fundamental objects of study.

6.1.5. Definition. An **open set** in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every $p \in U$, all points within some small distance from p belong to U. A **region** is an open set U that contains a polygonal u, v-curve for every pair $u, v \in U$. The **faces** of a plane graph are the maximal regions of the plane that contain no point used in the embedding.

A finite plane graph G has one unbounded face (also called the **outer face**). The faces are pairwise disjoint. Points $p, q \in \mathbb{R}^2$ lying in no edge of G are in the same face if and only if there is a polygonal p, q-curve that crosses no edge.

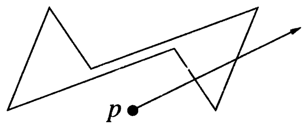
In a plane graph, every cycle is embedded as a simple closed curve. Some faces lie inside it, some outside. This again relies on the fact that a simple closed curve cuts the plane into two regions. As we have suggested, this is not too difficult for polygonal curves. We present some detail of this case in order to explain how to compute whether a point is in the inside or the outside. This proof appears in Tverberg [1980].

6.1.6.* Theorem. (Restricted Jordan Curve Theorem) A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary.

Proof: Because the list of segments is finite, nonintersecting segments cannot be arbitrarily close. Hence we can leave a face only by crossing C. As we follow C, the nearby points on our right are in a single face, and similarly for the points on the left. (There is a precise algebraic definition for "left" and "right" here.) If $x \notin C$ and $y \in C$, the segment xy first intersects C somewhere, approaching it from the right or the left. Hence every point not along C lies in the same face with at least one of the two sets we have described.

To prove that the points on the left and right lie in different faces, we consider rays in the plane. A ray emanating from a point p is "bad" if it contains an endpoint of a segment of C. Since C has finitely many segments, there are finitely many bad rays from p.

Since the list of segments is finite, each good ray from p crosses C finitely often. As the direction changes, the number of crossings changes only at a bad direction. Before and after such a direction, the parity of the number of crossings is the same. We say that p is an *even point* when every good ray from p crosses C an even number of times; otherwise p is an *odd point*.



Given points x and y in the same face of C, let P be a polygonal x, y-curve that avoids C. Since C has finitely many segments, the endpoints of segments on P can be adjusted slightly so that the rays along segments on P are good for their endpoints. A segment of P belongs to a ray from one end that contains the other; both points have good rays in the same direction. Since the segment does not intersect C, the two points have the same parity. Hence every two points in the same face have the same parity

Because the endpoints of a short segment intersecting C exactly once have opposite parity, there are two distinct faces. The even points and odd points form the outside face and the inside face, respectively. ■

DUAL GRAPHS

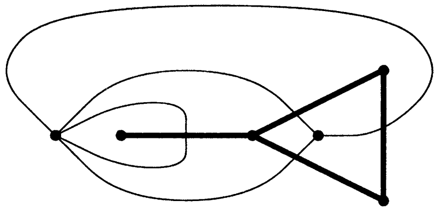
A map on the plane or the sphere can be viewed as a plane graph in which the faces are the territories, the vertices are places where boundaries meet, and the edges are the portions of the boundaries that join two vertices. We allow the full generality of loops and multiple edges. From any plane graph G, we can form a related plane graph called its "dual".

6.1.7. Definition. The **dual graph** G^* of a plane graph G is a plane graph whose vertices correspond to the faces of G. The edges of G^* correspond to the edges of G as follows: if e is an edge of G with face X on one side and face Y on the other side, then the endpoints of the dual edge $e^* \in E(G^*)$ are the vertices x, y of G^* that represent the faces X, Y of G. The order in the plane of the edges incident to $x \in V(G^*)$ is the order of the edges bounding the face X of G in a walk around its boundary.

6.1.8. Example. Every planar embedding of K_4 has four faces, and these pairwise share boundary edges. Hence the dual is another copy of K_4.

Every planar embedding of the cube Q_3 has eight vertices, 12 edges, and six faces. Opposite faces have no common boundary; the dual is a planar embedding of $K_{2,2,2}$, which has six vertices, 12 edges, and eight faces.

Taking the dual can introduce loops and multiple edges. For example, let G be the paw, drawn below in bold edges as a plane graph. Its dual graph G^* is drawn in solid edges. Since G has four vertices, four edges, and two faces, G^* has four faces, four edges, and two vertices. ■



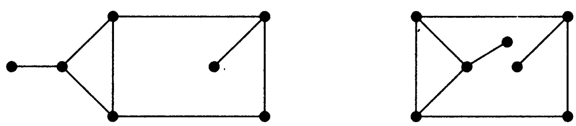
6.1.9. Remark. 1) Example 6.1.8 shows that a simple plane graph may have loops and multiple edges in its dual. A cut-edge of G becomes a loop in G^*, because the faces on both sides of it are the same. Multiple edges arise in the dual when distinct faces of G have more than one common boundary edge.

2) Some arguments require more careful geometric description of the dual. For each face X of G, we place the dual vertex x in the interior of X, so each face of G contains one vertex of G^*. For each edge e in the boundary of X, we draw a curve from x to a point on e; these do not cross. Each such curve meets another from the other side of e at the same point on e to form the edge of G^* that is dual to e. No other edges enter X. Hence G^* is a plane graph, and each edge of G^* in this layout crosses exactly one edge of G.

Such arguments lead to a proof that $(G^*)^*$ is isomorphic to G if and only if G is connected (Exercise 18). Mathematicians often use the word "dual" in a setting when performing an operation twice returns the original object. ■

6.1.10. Example. Two embeddings of a planar graph may have nonisomorphic duals. Each embedding shown below has three faces, so in each case the dual has three vertices. In the embedding on the right, the dual vertex corresponding to the outside face has degree 4. In the embedding on the left, no dual vertex has degree 4, so the duals are not isomorphic.

This does not happen with 3-connected graphs. Every 3-connected planar graph has essentially one embedding (see Exercise 8.2.45). ■



When a plane graph is connected, the boundary of each face is a closed walk. When the graph is not connected, there are faces whose boundary consists of more than one closed walk.

6.1.11. Definition. The **length** of a face in a plane graph G is the total length of the closed walk(s) in G bounding the face.

6.1.12. Example. A cut-edge belongs to the boundary of only one face, and it contributes twice to its length. Each graph in Example 6.1.10 has three faces. In the embedding on the left the lengths are 3, 6, 7; on the right they are 3, 4, 9. The sum of the lengths is 16 in each case, which is twice the number of edges. ■

6.1.13. Proposition. If $l(F_i)$ denotes the length of face F_i in a plane graph G, then $2e(G) = \sum l(F_i)$.

Proof: The face lengths are the degrees of the dual vertices. Since $e(G) = e(G^*)$, the statement $2e(G) = \sum l(F_i)$ is thus the same as the degree-sum formula $2e(G^*) = \sum d_{G^*}(x)$ for G^*. (Both sums count each edge twice.) ■

Proposition 6.1.13 illustrates that statements about a connected plane graph becomes statements about the dual graph when we interchange the roles of vertices and faces. Edges incident to a vertex become edges bounding a face, and vice versa, so the roles of face lengths and vertex degrees interchange.

We can also interpret coloring of G^* in terms of G. The edges of G^* represent shared boundaries between faces of G. Hence the chromatic number of G^* equals the number of colors needed to properly color the faces of G. Since the dual of the dual of a connected plane graph is the original graph, this means that four colors suffice to properly color the regions in every planar map if and only if every planar graph has chromatic number at most four.

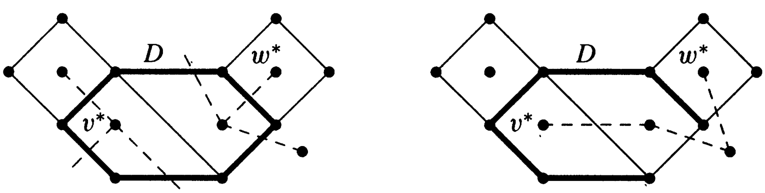
The Jordan Curve Theorem states that a simple closed curve cuts its interior from its exterior. In plane graphs, this duality between curve and cut becomes a duality between cycles and bonds.

6.1.14. Theorem. Edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.

Proof: Consider $D \subseteq E(G)$. Suppose first that D is the edge set of a cycle in G. The corresponding edge set $D^* \subseteq E(G^*)$ contains all dual edges joining faces inside D to faces outside D (the Jordan Curve Theorem implies that there is at least one of each). Thus D^* contains an edge cut.

If D contains a cycle and more, then D^* contains an edge cut and more. If D contains no cycle in G, then it encloses no region (see Exercise 24a). It remains possible to reach the unbounded face of G from every other without crossing D. Hence $G^* - D^*$ is connected, and D^* contains no edge cut.

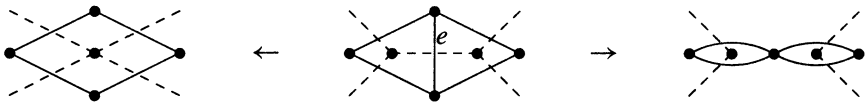
Thus D^* is a minimal edge cut if and only if D is a cycle. ■



The next remark yields an inductive proof of Theorem 6.1.14 (Exercise 19).

6.1.15. Remark. Deleting a non-cut edge of G has the effect of contracting an edge in G^*, as two faces of G merge into one. Contracting a non-loop edge of G has the effect of deleting an edge in G^*. Letting G be the central solid graph below, we have $G - e$ on the left and $G \cdot e$ on the right.

Note that to maintain this duality, we keep multiple edges and loops that arise from edge contraction in plane graphs. ■



Face boundaries allow us to characterize bipartite planar graphs. The characterization can also be proved by induction (Exercise 20).

6.1.16. Theorem. The following are equivalent for a plane graph G.

A) G is bipartite.

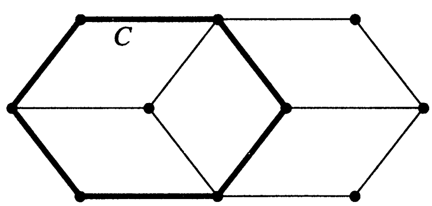
B) Every face of G has even length.

C) The dual graph G^* is Eulerian.

Proof: A $\Rightarrow$ B. A face boundary consists of closed walks. Every odd closed walk contains an odd cycle. Therefore, in a bipartite plane graph the contributions to the length of faces are all even.

B $\Rightarrow$ A. Let C be a cycle in G. Since G has no crossings, C is laid out as a simple closed curve; let F be the region enclosed by C. Every region of G is wholly within F or wholly outside F. If we sum the face lengths for the regions inside F, we obtain an even number, since each face length is even. This sum counts each edge of C once. It also counts each edge inside F twice, since each such edge belongs twice to faces in F. Hence the parity of the length of C is the same as the parity of the full sum, which is even.

B $\Leftrightarrow$ C. The dual graph G^* is connected, and its vertex degrees are the face lengths of G. ■



Many questions we consider for general planar graphs can be answered rather easily for a special class of planar graphs.

6.1.17. Definition. A graph is **outerplanar** if it has an embedding with every vertex on the boundary of the unbounded face. An **outerplane graph** is such an embedding of an outerplanar graph.

The graph in Example 6.1.10 is outerplanar, but another embedding is needed to demonstrate this.

6.1.18. Proposition. The boundary of the outer face a 2-connected outerplane graph is a spanning cycle.

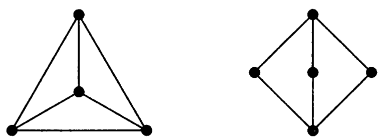
Proof: This boundary contains all the vertices. If it is not a cycle, then it passes through some vertex more than once. Such a vertex would be a cut-vertex. ■

6.1.19. Proposition. K_4 and $K_{2,3}$ are planar but not outerplanar.

Proof: The figure below shows that K_4 and $K_{2,3}$ are planar.

To show that they are not outerplanar, observe that they are 2-connected. Thus an outerplane embedding requires a spanning cycle. There is no spanning cycle in $K_{2,3}$, since it would be a cycle of length 5 in a bipartite graph.

There is a spanning cycle in K_4, but the endpoints of the remaining two edges alternate along it. Hence these chords conflict and cannot both be drawn inside. Drawing a chord outside separates a vertex from the outer face. ■



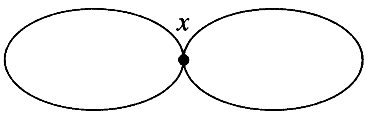
6.1.20. Proposition. Every simple outerplanar graph has a vertex of degree at most 2.

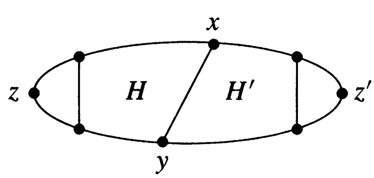
Proof: It suffices to prove the statement for connected graphs. We use induction on $n(G)$; when $n(G) \le 3$, every vertex has degree at most 2. For $n(G) \ge 4$, we prove the stronger statement that G has two nonadjacent vertices of degree at most 2.

Basis step ($n(G) = 4$): Since K_4 is not outerplanar, G has nonadjacent vertices, and two nonadjacent vertices have degree at most 2.

Induction step ($n(G) \ge 4$): If G has a cut-vertex x, then each $\{x\}$-lobe of G has a vertex of degree at most 2 other than x, and these are nonadjacent in G.

If G is 2-connected, then the outer face boundary is a cycle C. If C has no chords, then G is 2-regular. If xy is a chord of C, then the vertex sets of the two x, y-paths on C both induce outerplanar subgraphs. By the induction hypothesis, these subgraphs H, H' contain vertices z, z' of degree at most 2 that are not in $\{x, y\}$ (this includes the case where H or H' is K_3). Since no chord of C can be drawn outside C or cross xy, we have $z \not\leftrightarrow z'$. Thus z, z' is the desired pair of vertices. ■





EULER'S FORMULA

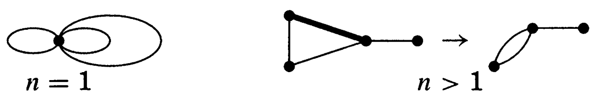
Euler's Formula ($n - e + f = 2$) is the basic counting tool relating vertices, edges, and faces in planar graphs.

6.1.21. Theorem. (Euler [1758]): If a connected plane graph G has exactly n vertices, e edges, and f faces, then $n - e + f = 2$.

Proof: We use induction on n. Basis step ($n = 1$): G is a "bouquet" of loops, each a closed curve in the embedding. If $e = 0$, then $f = 1$, and the formula holds. Each added loop passes through a face and cuts it into two faces (by the Jordan Curve Theorem). This augments the edge count and the face count each by 1. Thus the formula holds when $n = 1$ for any number of edges.

Induction step ($n > 1$): Since G is connected, we can find an edge that is not a loop. When we contract such an edge, we obtain a plane graph G' with n' vertices, e' edges, and f' faces. The contraction does not change the number of faces (we merely shortened boundaries), but it reduces the number of edges and vertices by 1, so $n' = n - 1$, $e' = e - 1$, and $f' = f$. Applying the induction hypothesis yields

$$n - e + f = n' + 1 - (e' + 1) + f' = n' - e' + f' = 2. \qquad \blacksquare$$



6.1.22. Remark. 1) By Euler's Formula, all planar embeddings of a connected graph G have the same number of faces. Although the dual may depend on the embedding chosen for G, the number of vertices in the dual does not.

2) Euler's Formula as stated fails for disconnected graphs. If a plane graph G has k components, then adding $k-1$ edges to G yields a connected plane graph without changing the number of faces. Hence Euler's Formula generalizes for plane graphs with k components as $n - e + f = k + 1$ (for example, consider a graph with n vertices and no edges). ■

Euler's Formula has many applications, particularly for simple plane graphs, where all faces have length at least 3.

6.1.23. Theorem. If G is a simple planar graph with at least three vertices, then $e(G) \le 3n(G) - 6$. If also G is triangle-free, then $e(G) \le 2n(G) - 4$.

Proof: It suffices to consider connected graphs; otherwise we could add edges. Euler's Formula will relate $n(G)$ and $e(G)$ if we can dispose of f.

Proposition 6.1.13 provides an inequality between e and f. Every face boundary in a simple graph contains at least three edges (if $n(G) \ge 3$). Letting $\{f_i\}$ be the list of face lengths, this yields $2e = \sum f_i \ge 3f$. Substituting into $n - e + f = 2$ yields $e \le 3n - 6$.

When G is triangle-free, the faces have length at least 4. In this case $2e = \sum f_i \geq 4f$, and we obtain $e \leq 2n - 4$. ■

6.1.24. Example. Nonplanarity of K_5 and $K_{3,3}$ follows immediately from Theorem 6.1.23. For K_5, we have $e = 10 > 9 = 3n - 6$. Since $K_{3,3}$ is triangle-free, we have $e = 9 > 8 = 2n - 4$. These graphs have too many edges to be planar. ■

6.1.25. Definition. A **maximal planar graph** is a simple planar graph that is not a spanning subgraph of another planar graph. A **triangulation** is a simple plane graph where every face boundary is a 3-cycle.

6.1.26. Proposition. For a simple n-vertex plane graph G, the following are equivalent.
A) G has $3n - 6$ edges.
B) G is a triangulation.
C) G is a maximal plane graph.

Proof: A $\Leftrightarrow$ B. For a simple n-vertex plane graph, the proof of Theorem 6.1.23 shows that having $3n - 6$ edges is equivalent to $2e = 3f$, which occurs if and only if every face is a 3-cycle.

B $\Leftrightarrow$ C. There is a face that is longer than a 3-cycle if and only if there is a way to add an edge to the drawing and obtain a larger simple plane graph. ■

6.1.27. Remark. A graph embeds in the plane if and only if it embeds on a sphere. Given an embedding on a sphere, we can puncture the sphere inside a face and project the embedding onto a plane tangent to the opposite point. This yields a planar embedding in which the punctured face on the sphere becomes the unbounded face in the plane. The process is reversible. ■

6.1.28. Application. *Regular polyhedra.* Informally, we think of a regular polyhedron as a solid whose boundary consists of regular polygons of the same length, with the same number of faces meeting at each vertex. When we expand the polyhedron out to a sphere and then lay out the drawing in the plane as in Remark 6.1.27, we obtain a regular plane graph with faces of the same length. Hence the dual also is a regular graph.

Let G be a plane graph with n vertices, e edges, and f faces. Suppose that G is regular of degree k and that all faces have length l. The degree-sum formula for G and for G^* yields $kn = 2e = lf$. By substituting for n and f in Euler's Formula, we obtain $e(\frac{2}{k} - 1 + \frac{2}{l}) = 2$. Since e and 2 are positive, the other factor must also be positive, which yields $(2/k) + (2/l) > 1$, and hence $2l + 2k > kl$. This inequality is equivalent to $(k - 2)(l - 2) < 4$.

Because the dual of a 2-regular graph is not simple, we require that $k, l \geq 3$. Now $(k - 2)(l - 2) < 4$ also requires $k, l \leq 5$. The only integer pairs satisfying these requirements for (k, l) are $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, and $(5, 3)$.

Once we specify k and l, there is only one way to lay out the plane graph when we start with any face. Hence there are only the five Platonic solids listed below, one for each pair (k, l) that satisfying the requirements. ■

k	l	$(k-2)(l-2)$	e	n	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	icosahedron

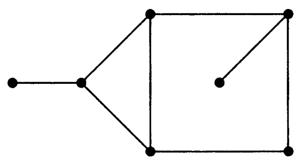
EXERCISES

6.1.1. (−) Prove or disprove:
a) Every subgraph of a planar graph is planar.
b) Every subgraph of a nonplanar graph is nonplanar.

6.1.2. (−) Show that the graphs formed by deleting one edge from K_5 and $K_{3,3}$ are planar.

6.1.3. (−) Determine all r, s such that $K_{r,s}$ is planar.

6.1.4. (−) Determine the number of isomorphism classes of planar graphs that can be obtained as planar duals of the graph below



6.1.5. (−) Prove that a plane graph has a cut-vertex if and only if its dual has a cut-vertex.

6.1.6. (−) Prove that a plane graph is 2-connected if and only if for every face, the bounding walk is a cycle.

6.1.7. (−) A **maximal outerplanar graph** is a simple outerplanar graph that is not a spanning subgraph of a larger simple outerplanar graph. Let G be a maximal outerplanar graph with at least three vertices. Prove that G is 2-connected.

6.1.8. (−) Prove that every simple planar graph has a vertex of degree at most 5.

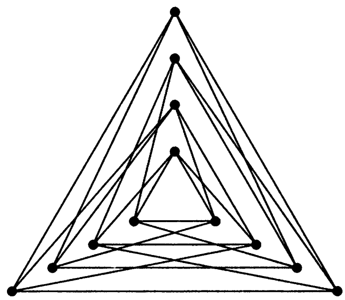
6.1.9. (−) Use Theorem 6.1.23 to prove that every simple planar graph with fewer than 12 vertices has a vertex of degree at most 4.

6.1.10. (−) Prove or disprove: There is no simple bipartite planar graph with minimum degree at least 4.

6.1.11. (−) Let G be a maximal planar graph. Prove that G^* is 2-edge-connected and 3-regular.

6.1.12. (−) Draw the five regular polyhedra as planar graphs. Show that the octahedron is the dual of the cube and the icosahedron is the dual of the dodecahedron.

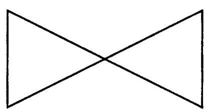
6.1.13. Find a planar embedding of the graph below.



6.1.14. Prove or disprove: For each $n \in \mathbb{N}$, there is a simple connected 4-regular planar graph with more than n vertices.

6.1.15. Construct a 3-regular planar graph of diameter 3 with 12 vertices. (Comment: T. Barcume proved that no such graph has more than 12 vertices.)

6.1.16. Let F be a figure drawn continuously in the plane without retracing any segment, ending at the start (this can be viewed as an Eulerian graph). Prove that F can be drawn without allowing the pencil point to cross what has already been drawn. For example, the figure below has two traversals; one crosses itself and the other does not.



6.1.17. Prove or disprove: If G is a 2-connected simple plane graph with minimum degree 3, then the dual graph G^* is simple.

6.1.18. Given a plane graph G, draw the dual graph G^* so that each dual edge intersects its corresponding edge in G and no other edge. Prove the following.

a) G^* is connected.
b) If G is connected, then each face of G^* contains exactly one vertex of G.
c) $(G^*)^* = G$ if and only if G is connected.

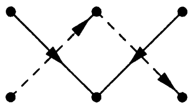
6.1.19. Let G be a plane graph. Use induction on $e(G)$ to prove Theorem 6.1.14: a set $D \subseteq E(G)$ is a cycle in G if and only if the corresponding set $D^* \subseteq E(G^*)$ is a bond in G^*. (Hint: Contract an edge of D and apply Remark 6.1.15.)

6.1.20. Prove by induction on the number of faces that a plane graph G is bipartite if and only if every face has even length.

6.1.21. (!) Prove that a set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*.

6.1.22. The **weak dual** of a plane graph G is the graph obtained from the dual G^* by deleting the vertex for the unbounded face of G. Prove that the weak dual of an outerplane graph is a forest.

6.1.23. (!) *Directed plane graphs.* Let G be a plane graph, and let D be an orientation of G. The **dual** D^* is an orientation of G^* such that when an edge of D is traversed from tail to head, the dual edge in D^* crosses it from right to left. For example, if the solid edges below are in D, then the dashed edges are in D^*.



Prove that if D is strongly connected, then D^* has no cycle, and $\delta^-(D^*) = \delta^+(D^*) = 0$. Conclude that if D is strongly connected, then D has a face on which the edges form a clockwise cycle and another face on which the edges form a counterclockwise cycle.

6.1.24. (!) *Alternative proof of Euler's Formula.*

a) Use polygonal curves (not Euler's Formula) to prove by induction on $n(G)$ that every planar embedding of a tree G has one face.

b) Prove Euler's Formula by induction on the number of cycles.

6.1.25. (!) Prove that every n-vertex plane graph isomorphic to its dual has $2n-2$ edges. For all $n \geq 4$, construct a simple n-vertex plane graph isomorphic to its dual.

6.1.26. Determine the maximum number of edges in a simple outerplane graph with n vertices, giving three proofs.

a) By induction on n.

b) By using Euler's Formula.

c) By adding a vertex in the unbounded face and using Theorem 6.1.23.

6.1.27. Let G be a connected 3-regular plane graph in which every vertex lies on one face of length 4, one face of length 6, and one face of length 8.

a) In terms of $n(G)$, determine the number of faces of each length.

b) Use Euler's Formula and part (a) to determine the number of faces of G.

6.1.28. Let C be a closed curve bounding a convex region in the plane. Suppose that m chords of C are drawn so that no three share a point and no two share an endpoint. Let p be the number of pairs of chords that cross. In terms of m and p, compute the number of segments and the number of regions formed inside C. (Alexanderson–Wetzel [1977])

6.1.29. Prove that the complement of a simple planar graph with at least 11 vertices is nonplanar. Construct a self-complementary simple planar graph with 8 vertices.

6.1.30. (!) Let G be an n-vertex simple planar graph with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges. Use this to prove that the Petersen graph is nonplanar.

6.1.31. Let G be the simple graph with vertex set $v_1, \ldots, v_n$ whose edges are $\{v_i v_j \colon |i-j| \leq 3\}$. Prove that G is a maximal planar graph.

6.1.32. Let G be a maximal planar graph. Prove that if S is a separating 3-set of G^*, then $G^* - S$ has two components. (Chappell)

6.1.33. (!) Let G be a triangulation, and let n_i be the number of vertices of degree i in G. Prove that $\sum (6-i)n_i = 12$.

6.1.34. Construct an infinite family of simple planar graphs with minimum degree 5 such that each has exactly 12 vertices of degree 5. (Hint: Modify the dodecahedron.)

6.1.35. (!) Prove that every simple planar graph with at least four vertices has at least four vertices with degree less than 6. For each even value of n with $n \geq 8$, construct an n-vertex simple planar graph G that has exactly four vertices with degree less than 6. (Grünbaum–Motzkin [1963])

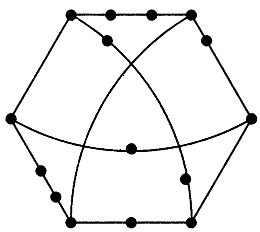
6.1.36. Let S be a set of n points in the plane such that for all $x, y \in S$, the distance in the plane between x and y is at least 1. Prove that there are at most $3n - 6$ pairs u, v in S such that the distance in the plane between u and v is exactly 1.

6.1.37. Given integers $k \geq 2$, $l \geq 1$, and kl even, construct a planar graph with exactly k faces in which every face has length l.

6.2. Characterization of Planar Graphs

Which graphs embed in the plane? We have proved that K_5 and $K_{3,3}$ do not. In fact, these are the crucial graphs and lead to a characterization of planar graphs known as Kuratowski's Theorem. Kasimir Kuratowski once asked Frank Harary about the origin of the notation for K_5 and $K_{3,3}$. Harary replied, "The K in K_5 stands for Kasimir, and the K in $K_{3,3}$ stands for Kuratowski!"

Recall that a subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths (Definition 5.2.19).



a subdivision of $K_{3,3}$

6.2.1. Proposition. If a graph G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is nonplanar.

Proof: Every subgraph of a planar graph is planar, so it suffices to show that subdivisions of K_5 and $K_{3,3}$ are nonplanar. Subdividing edges does not affect planarity; the curves in an embedding of a subdivision of G can be used to obtain an embedding of G, and vice versa. ■

By Proposition 6.2.1, avoiding subdivisions of K_5 and $K_{3,3}$ is a necessary condition for being a planar graph. Kuratowski proved TONCAS:

6.2.2. Theorem. (Kuratowski [1930]) A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. ■

Kuratowski's Theorem is our goal in the first half of this section, after which we will comment on other characterizations of planar graphs.

When G is planar, we can seek a planar embedding with additional properties. Wagner [1936], Fáry [1948], and Stein [1951] showed that every finite simple planar graph has an embedding in which all edges are straight line segments; this is known as **Fáry's Theorem** (Exercise 6). For 3-connected planar graphs, we will prove the stronger property that there exists an embedding in which every face is a convex polygon.

PREPARATION FOR KURATOWSKI'S THEOREM

We introduce short names for subgraphs that demonstrate nonplanarity.

6.2.3. Definition. A **Kuratowski subgraph** of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A **minimal nonplanar graph** is a nonplanar graph such that every proper subgraph is planar.

We will prove that a minimal nonplanar graph with no Kuratowski subgraph must be 3-connected. Showing that every 3-connected graph with no Kuratowski subgraph is planar then completes the proof of Kuratowski's Theorem.

6.2.4. Lemma. If F is the edge set of a face in a planar embedding of G, then G has an embedding with F being the edge set of the unbounded face.

Proof: Project the embedding onto the sphere, where the edge sets of regions remain the same and all regions are bounded, and then return to the plane by projecting from inside the face bounded by F. ■

6.2.5. Lemma. Every minimal nonplanar graph is 2-connected.

Proof: Let G be a minimal nonplanar graph. If G is disconnected, then we embed one component of G inside one face of an embedding of the rest.

If G has a cut-vertex v, let $G_1, \dots, G_k$ be the $\{v\}$-lobes of G. By the minimality of G, each G_i is planar. By Lemma 6.2.4, we can embed each G_i with v on the outside face. We squeeze each embedding to fit in an angle smaller than $360/k$ degrees at v, after which we combine the embeddings at v to obtain an embedding of G. ■

6.2.6. Lemma. Let $S = \{x, y\}$ be a separating 2-set of G. If G is nonplanar, then adding the edge xy to some S-lobe of G yields a nonplanar graph.

Proof: Let $G_1, \dots, G_k$ be the S-lobes of G, and let $H_i = G_i \cup xy$. If H_i is planar, then by Lemma 6.2.4 it has an embedding with xy on the outside face. For each $i > 1$, this allows H_i to be attached to an embedding of $\bigcup_{j=1}^{i-1} H_j$ by embedding H_i in a face that has xy on its boundary. Afterwards, deleting the edge xy if it is not in G yields a planar embedding of G. ■

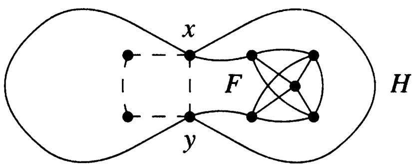
The next lemma allows us to restrict our attention to 3-connected graphs in order to prove Kuratowski's Theorem. The hypothesized graph doesn't exist, but if it did, it would be 3-connected.

6.2.7. Lemma. If G is a graph with fewest edges among all nonplanar graphs without Kuratowski subgraphs, then G is 3-connected.

Proof: Deleting an edge of G cannot create a Kuratowski subgraph in G. The hypothesis thus guarantees that deleting one edge produces a planar subgraph, and hence G is a minimal nonplanar graph. By Lemma 6.2.5, G is 2-connected.

Suppose that G has a separating 2-set $S = \{x, y\}$. Since G is nonplanar, the union of xy with some S-lobe is nonplanar (Lemma 6.2.6); let H be such a graph. Since H has fewer edges than G, the minimality of G forces H to have a Kuratowski subgraph F. All of F appears in G except possibly the edge xy.

Since S is a minimal vertex cut, both x and y have neighbors in every S-lobe. Thus we can replace xy in F with an x, y-path through another S-lobe to obtain a Kuratowski subgraph of G. This contradicts the hypothesis that G has no Kuratowski subgraph, so G has no separating 2-set. ■



CONVEX EMBEDDINGS

To complete the proof of Kuratowski's Theorem, it suffices to prove that 3-connected graphs without Kuratowski subgraphs are planar. We will use induction. In order to facilitate the proof of the induction step, it is helpful to prove a stronger statement.

6.2.8. Definition. A **convex embedding** of a graph is a planar embedding in which each face boundary is a convex polygon.

Tutte [1960, 1963] proved that every 3-connected planar graph has a convex embedding. This is best possible in terms of connectivity, since for $n \geq 4$ the 2-connected planar graph $K_{2,n}$ has no convex embedding. We follow Thomassen's approach to proving Kuratowski's Theorem by proving Tutte's stronger conclusion for 3-connected graphs without Kuratowski subgraphs. (Another proof of Tutte's result is based on ear decompositions—Kelmans [2000].)

We prove this theorem of Tutte by induction on $n(G)$. The paradigm for proving conditional statements by induction (Remark 1.3.25) tells us what lemmas we need. Our hypotheses are "3-connected" and "no Kuratowski subgraph"; our conclusion is "convex embedding". For a graph G satisfying the hypotheses, we need to find a smaller graph G' that satisfies *both* hypotheses in order to apply the induction hypothesis.

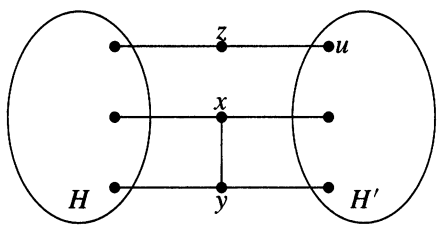
The first lemma allows us to obtain a smaller 3-connected graph G' by contracting some edge in G. The second shows that G' will also satisfy the hypothesis of having no Kuratowski subgraph. The proof will then be completed by obtaining a convex embedding of G from a convex embedding of G'.

6.2.9. Lemma. (Thomassen [1980]) Every 3-connected graph G with at least five vertices has an edge e such that $G \cdot e$ is 3-connected.

Proof: We use contradiction and extremality. Consider an edge e with endpoints x, y. If $G \cdot e$ is not 3-connected, then it has a separating 2-set S. Since G is 3-connected, S must include the vertex obtained by shrinking e. Let z denote the other vertex of S and call it the *mate* of the adjacent pair x, y. Note that $\{x, y, z\}$ is a separating 3-set in G.

Suppose that G has no edge whose contraction yields a 3-connected graph, so every adjacent pair has a mate. Among all the edges of G, choose $e = xy$ and their mate z so that the resulting disconnected graph $G - \{x, y, z\}$ has a component H with the largest order. Let H' be another component of $G - \{x, y, z\}$ (see the figure below). Since $\{x, y, z\}$ is a minimal separating set, each of x, y, z has a neighbor in each of H, H'. Let u be a neighbor of z in H', and let v be the mate of u, z.

By the definition of "mate", $G - \{z, u, v\}$ is disconnected. However, the subgraph of G induced by $V(H) \cup \{x, y\}$ is connected. Deleting v from this subgraph, if it occurs there, cannot disconnect it, since then $G - \{z, v\}$ would be disconnected. Therefore, $G_{V(H)\cup\{x,y\}} - v$ is contained in a component of $G - \{z, u, v\}$ that has more vertices than H, which contradicts the choice of x, y, z. ■



Next we need to show that edge contraction preserves the absence of Kuratowski subgraphs. We introduce a convenient term: the **branch vertices** in a subdivision H' of H are the vertices of degree at least 3 in H'.

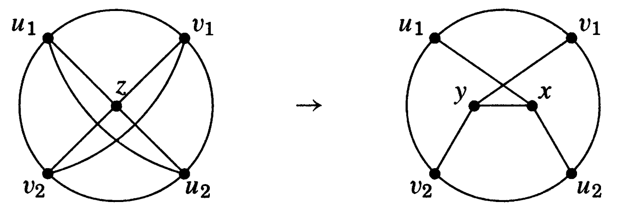
6.2.10. Lemma. If G has no Kuratowski subgraph, then also $G \cdot e$ has no Kuratowski subgraph.

Proof: We prove the contrapositive: If $G \cdot e$ contains a Kuratowski subgraph, then so does G. Let z be the vertex of $G \cdot e$ obtained by contracting $e = xy$. If z is not in H, then H itself is a Kuratowski subgraph of G. If $z \in V(H)$ but z is not a branch vertex of H, then we obtain a Kuratowski subgraph of G from H by replacing z with x or y or with the edge xy.

Similarly, if z is a branch vertex in H and at most one edge incident to z in

H is incident to x in G, then expanding z into xy lengthens that path, and y is the corresponding branch vertex for a Kuratowski subgraph in G.

In the remaining case (shown below), H is a subdivision of K_5 and z is a branch vertex, and the four edges incident to z in H consist of two incident to x and two incident to y in G. In this case, let u_1, u_2 be the branch vertices of H that are at the other ends of the paths leaving z on edges incident to x in G, and let v_1, v_2 be the branch vertices of H that are at the other ends of the paths leaving z on edges incident to y in G. By deleting the u_1, u_2-path and v_1, v_2-path from H, we obtain a subdivision of $K_{3,3}$ in G, in which y, u_1, u_2 are the branch vertices for one partite set and x, v_1, v_2 are the branch vertices of the other. ■



K_5 subdivision in G' $\qquad$ $K_{3,3}$ subdivision in G

Now we can prove Tutte's Theorem.

6.2.11. Theorem. (Tutte [1960, 1963]) If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then G has a convex embedding in the plane with no three vertices on a line.

Proof: (Thomassen [1980, 1981]) We use induction on $n(G)$.

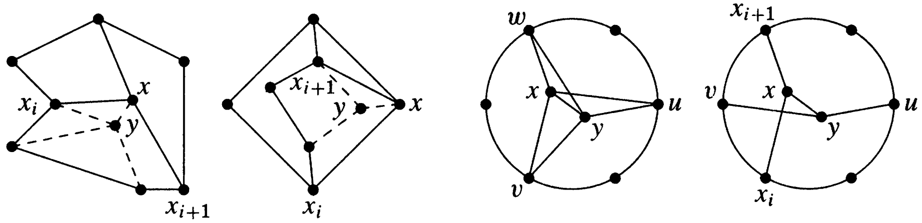
Basis step: $n(G) \leq 4$. The only 3-connected graph with at most four vertices is K_4, which has such an embedding.

Induction step: $n(G) \geq 5$. Let e be an edge such that $G \cdot e$ is 3-connected, as guaranteed by Lemma 6.2.9. Let z be the vertex obtained by contracting e. By Lemma 6.2.10, $G \cdot e$ has no Kuratowski subgraph. By the induction hypothesis, we obtain a convex embedding of $H = G \cdot e$ with no three vertices on a line.

In this embedding, the subgraph obtained by deleting the edges incident to z has a face containing z (perhaps unbounded). Since $H - z$ is 2-connected, the boundary of this face is a cycle C. All neighbors of z lie on C; they may be neighbors in G of x or y or both, where x and y are the original endpoints of e.

The convex embedding of H includes straight segments from z to all its neighbors. Let $x_1, \ldots, x_k$ be the neighbors of x in cyclic order on C. If all neighbors of y lie in the portion of C from x_i to x_{i+1}, then we obtain a convex embedding of G by putting x at z in H and putting y at a point close to z in the wedge formed by xx_i and xx_{i+1}, as shown in the diagrams for Case 0 below.

If this does not occur, then either 1) y shares three neighbors u, v, w with x, or 2) y has neighbors u, v that alternate on C with neighbors x_i, x_{i+1} of x. In Case 1, C together with xy and the edges from $\{x, y\}$ to $\{u, v, x\}$ form a subdivision of K_5. In Case 2, C together with the paths uyv, x_ixx_{i+1}, and xy form a subdivision of $K_{3,3}$. Since we are considering only graphs without Kuratowski subgraphs, in fact Case 0 must occur. ■



Case 0 Case 1 Case 2

Together, Lemma 6.2.7 and Theorem 6.2.11 imply Kuratowski's Theorem (Theorem 6.2.2). Fáry's Theorem can be obtained separately: if a graph has a planar embedding, then it has a straight-line planar embedding (Exercise 6).

For applications in computer science, we want more—a straight-line planar embedding in which the vertices are located at the integer points in a relatively small grid. Schnyder [1992] proved that every n-vertex planar graph has a straight-line embedding in which the vertices are located at integer points in the grid $[n-1] \times [n-1]$.

Many other characterizations of planar graphs have been proved; some are mentioned in the exercises. We describe two additional characterizations.

6.2.12.* Definition. A graph H is a **minor** of a graph G if a copy of H can be obtained from G by deleting and/or contracting edges of G.

For example, K_5 is a minor of the Petersen graph, although the Petersen graph does not contain a subdivision of K_5.

6.2.13.* Remark. Deletions and contractions can be performed in any order, as long as we keep track of which edge is which. Thus the minors of G can be described as "contractions of subgraphs of G".

If G contains a subdivision of H, say H', then H also is a minor of G, obtained by deleting the edges of G not in H' and then contracting edges incident to vertices of degree 2. If H has maximum degree at most 3, then H is a minor of G if and only if G contains a subdivision of H (Exercise 11).

Wagner [1937] proved that a graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G. Exercise 12 obtains this from Kuratowski's Theorem. ■

6.2.14.* Remark. Some characterizations are more closely related to actual embeddings. For example, when a 3-connected graph is drawn in the plane, deleting the vertex set of a facial cycle leaves a connected subgraph.

We say that a cycle in a graph is **nonseparating** if its vertex set is not a separating set. Kelmans [1980, 1981b] proved that a subdivision of a 3-connected graph is planar if and only if every edge e lies in exactly two nonseparating cycles. Kelmans [1993] surveys related material. ■

PLANARITY TESTING (optional)

Dirac and Schuster [1954] gave the first short proof of Kuratowski's Theorem. Appearing in Harary [1969, 109–112], Bondy–Murty [1976, p153–156], and Chartrand–Lesniak [1986, p96–98], it uses special subgraphs of a graph.

6.2.15. Definition. When H is a subgraph of G, an H**-fragment** of G is either
1) an edge not in H whose endpoints are in H, or
2) a component of $G - V(H)$ together with the edges (and vertices of attachment) that connect it to H.

Together with the subgraph H itself, the H-fragments form a decomposition of G. The H-fragments are the "pieces" that must be added to an embedding of H to obtain an embedding of G. Historically, the term "H-bridge" was used; we use "H-fragment" to avoid confusion with other uses of "bridge".

An H-fragment differs from a $V(H)$-lobe because the H-fragment omits the edges of H. Also, an H-fragment may be a single edge not in H but joining vertices of H, since H need not be an induced subgraph.

For the 3-connected case of Kuratowski's Theorem, Dirac and Schuster considered a minimal nonplanar 3-connected graph G with no Kuratowski subgraph. Deleting an edge e yields a planar 2-connected graph. After choosing a cycle C through the endpoints of e, we can add e to the embedding unless there is a C-fragment embedded inside C and another embedded outside C that "conflict" with e. As in the proof of Theorem 6.2.11, this produces a Kuratowski subgraph of G. Tutte used the idea of conflicting C-fragments to obtain another characterization of planar graphs.

6.2.16. Definition. Let C be a cycle in a graph G. Two C-fragments A, B **conflict** if they have three common vertices of attachment to C or if there are four vertices v_1, v_2, v_3, v_4 in cyclic order on C such that v_1, v_3 are vertices of attachment of A and v_2, v_4 are vertices of attachment of B. The **conflict graph** of C is a graph whose vertices are the C-fragments of G, with conflicting C-fragments adjacent.

Tutte [1958] proved that G is planar if and only if the conflict graph of each cycle in G is bipartite (Exercise 13). We used this idea in our first proof that K_5 and $K_{3,3}$ are nonplanar (Proposition 6.1.2); the conflict graph of a spanning cycle in $K_{3,3}$ is C_3, and the conflict graph of a spanning cycle in K_5 is C_5.

Nonplanar 3-connected graphs have Kuratowski subgraphs of a special type. Kelmans [1984a] conjectured this extension of Kuratowski's Theorem, and it was proved independently by Kelmans [1983, 1984b] and Thomassen [1984]: Every 3-connected nonplanar graph with at least six vertices contains a cycle with three pairwise crossing chords.

Characterizations of planarity lead us to ask whether we can test quickly whether a graph is planar. There are linear-time algorithms due to Hopcroft and Tarjan [1974] and to Booth and Luecker [1976], but these are very complicated (Gould [1988, p177–185] discusses the ideas used in the Hopcroft–Tarjan algorithm). A simpler earlier algorithm is not linear but runs in polynomial time. Due to Demoucron, Malgrange, and Pertuiset [1964], it uses H-fragments.

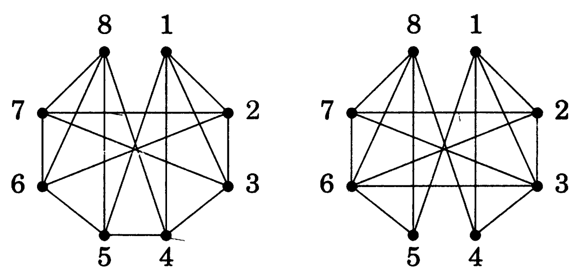
The idea is that if a planar embedding of H can be extended to a planar embedding of G, then in that extension every H-fragment of G appears inside a single face of H. We build increasingly larger plane subgraphs H of G that can be extended to an embedding of G if G is planar. We try to enlarge H by making small decisions that won't lead to trouble.

To enlarge H, we choose a face F that can accept an H-fragment B; the boundary of F must contain all vertices of attachment of B. Although we do not know the best way to embed B in F, a single path in B between vertices of attachment by itself has only one way to be added across F, so we add such a path. The details of choosing F and B appear below. Like the other algorithms mentioned, this algorithm produces an embedding if G is planar.

6.2.17. Algorithm. (Planarity Testing)
Input: A 2-connected graph. (Since G is planar if and only if each block of G is planar, and Algorithm 4.1.23 computes blocks, we may assume that G is a block with at least three vertices.)
Idea: Successively add paths from current fragments. Maintain the vertex sets forming face boundaries of the subgraph already embedded.
Initialization: G_0 is an arbitary cycle in G embedded in the plane, with two face boundaries consisting of its vertices.
Iteration: Having determined G_i, find G_{i+1} as follows.

1. Determine all G_i-fragments of the input block G.
2. For each G_i-fragment B, determine all faces of G_i that contain all vertices of attachment of B; call this set $F(B)$.
3. If $F(B)$ is empty for some B, return NONPLANAR. If $|F(B)| = 1$ for some B, select such a B. If $|F(B)| > 1$ for every B, select any B.
4. Choose a path P between two vertices of attachment of the selected B. Embed P across a face in $F(B)$. Call the resulting graph G_{i+1} and update the list of face boundaries.
5. If $G_{i+1} = G$, return PLANAR. Otherwise, augment i and return to Step 1.

6.2.18. Example. Consider the two graphs below (from Bondy–Murty [1976, p165–166]). Algorithm 6.2.17 produces a planar embedding of the graph on the left, but it terminates in Step 3 for the graph on the right. The cycle 12348765 has three pairwise crossing chords: 14, 27, 36. ■



6.2.19. Theorem. (Demoucron–Malgrange–Pertuiset [1964]) Algorithm 6.2.17 produces a planar embedding if G is planar.

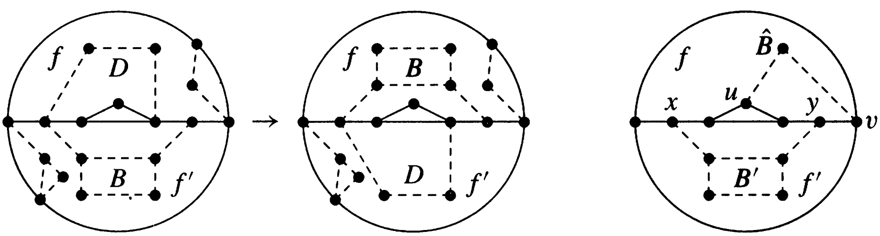
Proof: We may assume that G is 2-connected. A cycle appears as a simple closed curve in every planar embedding. Since we can reflect the plane, every embedding of a cycle in a planar graph G extends to an embedding of G.

Hence G_0 extends to a planar embedding of G if G is planar. It suffices to show that if the plane graph G_i is extendable to a planar embedding of G and the algorithm produces a plane graph G_{i+1} from G_i, then G_{i+1} also is extendable to a planar embedding of G. Note that every G_i-fragment has at least two vertices of attachment, since G is 2-connected,

If some G_i-fragment B has $|F(B)| = 1$, then there is only one face of G_i that can contain P in an extension of G_i to a planar embedding of G. The algorithm puts P in that face to obtain G_{i+1}, so in this case G_{i+1} is extendable.

Problems can arise only if $|F(B)| > 1$ for all B and we select the wrong face in which to embed a path P from the selected fragment. Suppose that (1) we embed P in face $f \in F(B)$, and (2) G_i can be extended to a planar embedding $\hat{G}$ of G in which P is inside face $f' \in F(B)$. We modify $\hat{G}$ to show that G_i can be extended to another embedding G' of G in which P is inside f. This shows that our choice causes no problem, and the constructed G_{i+1} is extendable.

Let C be the set of vertices in the boundaries of both f and f'; this includes the vertices of attachment of B. We draw G' by switching between f and f' all G_i-fragments that $\hat{G}$ places in f or f' and whose vertices of attachment lie in C. We show this on the left below, where edges of G not present in G_i are dashed.



$\hat{G}$ G'

The change switches B and produces the desired embedding G' unless some unswitched G_i-fragment $\hat{B}$ conflicts with a switched fragment. Since the switch is symmetric in f and f' and changes only their interiors, we may assume that $\hat{B}$ appears in f in $\hat{G}$. "Conflict" means that $\hat{G}$ has some B' in f', which we are trying to move to f, such that $\hat{B}$ and B' are adjacent in the conflict graph of f.

Let $\hat{A}$, A' denote the vertex sets where $\hat{B}$, B' attach to the boundary of f. Since $\hat{B}$ and B' conflict, $\hat{A}$, A' have three common vertices or four alternating vertices on the boundary of f. Since $A' \subseteq C$ but $\hat{A} \not\subseteq C$, the first possibility implies the second. Let x, u, y, v be the alternation, with $x, y \in A' \subseteq C$ and $u, v \in \hat{A}$. We may assume that $u \notin C$, as shown on the right above; if there is no such alternation, then $\hat{B}$, B' do not conflict or $\hat{B}$ can switch to f'.

Since $u \notin C$ and y is between u and v on f, no other face contains both u and v. Thus $\hat{B}$ fails to have its vertices of attachment contained in at least two faces, contradicting the hypothesis that $\left|F(\hat{B})\right| > 1$. ■

We can begin by checking that G has at most $3n - 6$ edges, maintain appropriate lists for the face boundaries, and perform the other operations via searches of linear size. Thus this algorithm runs in quadratic time. The proof of Kuratowski's Theorem by Klotz [1989] also gives a quadratic algorithm to test planarity, and it finds a Kuratowski subgraph when G is not planar.

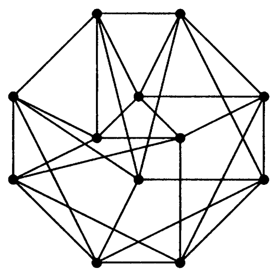
EXERCISES

6.2.1. (−) Prove that the complement of the 3-dimensional cube Q_3 is nonplanar.

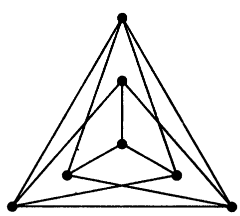
6.2.2. (−) Give three proofs that the Petersen graph is nonplanar.

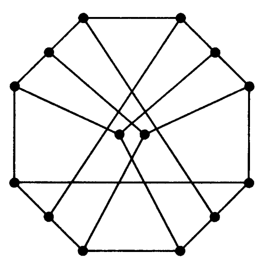
a) Using Kuratowski's Theorem.
b) Using Euler's Formula and the fact that the Petersen graph has girth 5.
c) Using the planarity-testing algorithm of Demoucron–Malgrange–Pertuiset.

6.2.3. (−) Find a convex embedding in the plane for the graph below.



6.2.4. (−) For each graph below, prove nonplanarity or provide a convex embedding.





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6.2.5. Determine the minimum number of edges that must be deleted from the Petersen graph to obtain a planar subgraph.

6.2.6. (!) *Fáry's Theorem.* Let R be a region in the plane bounded by a simple polygon with at most five sides (**simple polygon** means the edges are line segments that do not cross). Prove there is a point x inside R that "sees" all of R, meaning that the segment from x to any point of R does not cross the boundary of R. Use this to prove inductively that every simple planar graph has a straight-line embedding.

6.2.7. (!) Use Kuratowski's Theorem to prove that G is outerplanar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$. (Hint: To apply Kuratowski's Theorem, find an appropriate modification of G. This is much easier than trying to mimic a proof of Kuratowski's Theorem.)

6.2.8. (!) Prove that every 3-connected graph with at least six vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$. (Wagner [1937])

6.2.9. (+) For $n \geq 5$, prove that the maximum number of edges in a simple planar n-vertex graph not having two disjoint cycles is $2n-1$. (Comment: Compare with Exercise 5.2.28.) (Markus [1999])

6.2.10. (!) Let $f(n)$ be the maximum number of edges in a simple n-vertex graph containing no $K_{3,3}$-subdivision.

a) Given that $n-2$ is divisible by 3, construct a graph to show that $f(n) \geq 3n-5$.

b) Prove that $f(n) = 3n-5$ when $n-2$ is divisible by 3 and that otherwise $f(n) = 3n-6$. (Hint: Use induction on n, invoking Exercise 6.2.8 for the 3-connected case.) (Thomassen [1984])

(Comment: Mader [1998] proved the more difficult result that $3n-6$ is the maximum number of edges in an n-vertex simple graph with no K_5-subdivision.)

6.2.11. (!) Let H be a graph with maximum degree at most 3. Prove that a graph G contains a subdivision of H if and only if G contains a subgraph contractible to H.

6.2.12. (!) Wagner [1937] proved that the following condition is necessary and sufficient for a graph G to be planar: neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and contractions of edges.

a) Show that deletion and contraction of edges preserve planarity. Conclude from this that Wagner's condition is necessary.

b) Use Kuratowski's Theorem to prove that Wagner's condition is sufficient.

6.2.13. Prove that a graph G is planar if and only if for every cycle C in G, the conflict graph for C is bipartite. (Tutte [1958])

6.2.14. Let x and y be vertices of a planar graph G. Prove that G has a planar embedding with x and y on the same face unless $G-x-y$ has a cycle C with x and y in conflicting C-fragments in G. (Hint: Use Kuratowski's Theorem. Comment: Tutte proved this without Kuratowski's Theorem and used it to prove Kuratowski's Theorem.)

6.2.15. Let G be a 3-connected simple plane graph containing a cycle C. Prove that C is the boundary of a face in G if and only if G has exactly one C-fragment. (Comment: Tutte [1963] proved this to obtain Whitney's [1933b] result that 3-connected planar graphs have essentially only one planar embedding. See also Kelmans [1981a])

6.2.16. (+) Let G be an outerplanar graph with n vertices, and let P be a set of n points in the plane, no three of which lie on a line. The *extreme points* of P induce a convex polygon that contains the other points in its interior.

a) Let p_1, p_2 be consecutive extreme points of P. Prove that there is a point $p \in P - \{p_1, p_2\}$ such that 1) no point of P is inside $p_1 p_2 p$, and 2) some line l through p separates p_1 from p_2, meets P only at p, and has exactly $i-2$ points of P on the side of l containing p_2.

b) Prove that G has a straight-line embedding with its vertices mapped onto P. (Hint: Use part (a) to prove the stronger statement that if v_1, v_2 are two consecutive vertices of the unbounded face of a maximal outerplanar graph G, and p_1, p_2 are consecutive vertices of the convex hull of P, then G can be straight-line embedded on P such that $f(v_1) = p_1$ and $f(v_2) = p_2$.) (Gritzmann–Mohar–Pach–Pollack [1989])

6.3. Parameters of Planarity

Every property and parameter we have studied for general graphs can be studied for planar graphs. The problem of greatest historical interest is the maximum chromatic number of planar graphs. We will also study parameters that measure how far a graph is from being a planar graph.

COLORING OF PLANAR GRAPHS

Because every simple n-vertex planar graph has at most $3n-6$ edges, such a graph has a vertex of degree at most 5. This yields an inductive proof that planar graphs are 6-colorable (see Exercise 2). Heawood improved the bound.

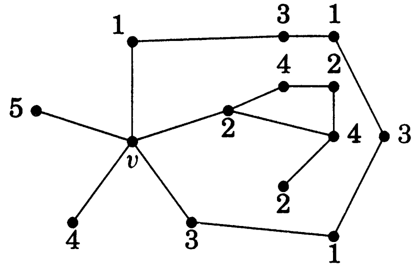
6.3.1. Theorem. (Five Color Theorem—Heawood [1890]) Every planar graph is 5-colorable.

Proof: We use induction on $n(G)$.

Basis step: $n(G) \leq 5$. All such graphs are 5-colorable.

Induction step: $n(G) > 5$. The edge bound (Theorem 6.1.23) implies that G has a vertex v of degree at most 5. By the induction hypothesis, $G - v$ is 5-colorable. Let $f\colon V(G-v) \to [5]$ be a proper 5-coloring of $G - v$. If G is not 5-colorable, then f assigns each color to some neighbor of v, and hence $d(v) = 5$. Let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in clockwise order around v. Name the colors so that $f(v_i) = i$.

Let $G_{i,j}$ denote the subgraph of $G - v$ induced by the vertices of colors i and j. Switching the two colors on any component of $G_{i,j}$ yields another proper 5-coloring of $G - v$. If the component of $G_{i,j}$ containing v_i does not contain v_j, then we can switch the colors on it to remove color i from $N(v)$. Now giving color i to v produces a proper 5-coloring of G. Thus G is 5-colorable unless, for each choice of i and j, the component of $G_{i,j}$ containing v_i also contains v_j. Let $P_{i,j}$ be a path in $G_{i,j}$ from v_i to v_j, illustrated below for $(i, j) = (1, 3)$.



Consider the cycle C completed with $P_{1,3}$ by v; this separates v_2 from v_4.

By the Jordan Curve Theorem, the path $P_{2,4}$ must cross C. Since G is planar, paths can cross only at shared vertices. The vertices of $P_{1,3}$ all have color 1 or 3, and the vertices of $P_{2,4}$ all have color 2 or 4, so they have no common vertex.

By this contradiction, G is 5-colorable. ■

Every planar graph is 5-colorable, but are five colors ever needed? The history of this infamous question is discussed in Aigner [1984, 1987], Ore [1967a], Saaty–Kainen [1977, 1986], Appel–Haken [1989], and Fritsch–Fritsch [1998]. The earliest known posing of the Four Color Problem is in a letter of October 23, 1852, from Augustus de Morgan to Sir William Hamilton. The question was asked by de Morgan's student Frederick Guthrie, who later attributed it to his brother Francis Guthrie. It was phrased in terms of map coloring.

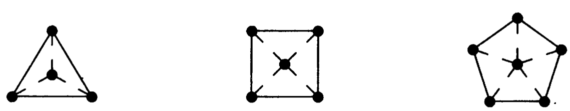
The problem's ease of statement and geometric subtleties invite fallacious proofs; some were published and remained unexposed for years. It does not suffice to forbid five pairwise-adjacent regions, since there are 5-chromatic graphs not containing K_5 (recall Mycielski's construction, for example).

Cayley announced the problem to the London Mathematical Society in 1878, and Kempe [1879] published a "solution". In 1890, Heawood published a refutation. Nevertheless, Kempe's idea of alternating paths, used by Heawood to prove the Five Color Theorem, led eventually to a proof by Appel and Haken [1976, 1977, 1986] (working with Koch). A path on which the colors alternate between two specified colors is a **Kempe chain**.

In proving the Five Color Theorem inductively, we argued that a minimal counterexample contains a vertex of degree at most 5 and that a planar graph with such a vertex cannot be a minimal counterexample. This suggests an approach to the Four Color Problem; we seek an unavoidable set of graphs that can't be present! We need only consider triangulations, since every simple planar graph is contained in a triangulation.

6.3.2. Definition. A **configuration** in a planar triangulation is a separating cycle C (the **ring**) together with the portion of the graph inside C. For the Four Color Problem, a set of configurations is **unavoidable** if a minimal counterexample must contain a member of it. A configuration is **reducible** if a planar graph containing it cannot be a minimal counterexample.

6.3.3. Example. *An unavoidable set.* We have remarked that $\delta(G) \leq 5$ for every simple planar graph. In a triangulation, every vertex has degree at least 3. Thus the set of three configurations below is unavoidable.



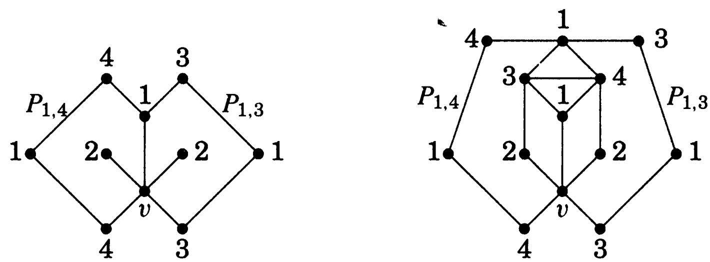
The edges from the ring to the interior are drawn with dashes because a configuration (in a triangulation) is completely determined if we state the degrees of the vertices adjacent to the ring and delete the ring (Exercise 7). Thus these configurations are written as "• 3", "• 4", and "• 5", respectively. ■

When we say that a configuration cannot be in a minimal counterexample, we mean that if it appears in a triangulation G, then it can be replaced to obtain a triangulation G' with fewer vertices such that every 4-coloring of G' can be manipulated to obtain a 4-coloring of G.

6.3.4. Remark. *Kempe's proof.* Let us try to prove the Four Color Theorem by induction using the unavoidable set $\{\bullet 3, \bullet 4, \bullet 5\}$. The approach is similar to Theorem 6.3.1. We can extend a 4-coloring of $G - v$ to complete a 4-coloring of G unless all four colors appear on $N(v)$. Thus "$\bullet$ 3" is reducible. If $d(v) = 4$, then the Kempe-chain argument works as in Theorem 6.3.1, and "$\bullet$ 4" is reducible.

Now consider "$\bullet$ 5". When $d(v) = 5$, the restriction to triangulations implies that the repeated color on $N(v)$ in the proper 4-coloring of $G - v$ appears on nonconsecutive neighbors of v. Let v_1, v_2, v_3, v_4, v_5 again be the neighbors of v in clockwise order. In the 4-coloring f of $G - v$, we may assume by symmetry that $f(v_5) = 2$ and that $f(v_i) = i$ for $1 \le i \le 4$.

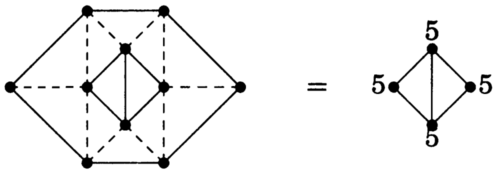
Define $G_{i,j}$ and $P_{i,j}$ as in Theorem 6.3.1. We can eliminate color 1 from $N(v)$ unless the chains $P_{1,3}$ and $P_{1,4}$ exist from v_1 to v_3 and v_4, respectively, as shown on the left below. The component H of $G_{2,4}$ containing v_2 is separated from v_4 and v_5 by the cycle completed by v with $P_{1,3}$. Also, the component H' of $G_{2,3}$ containing v_5 is separated from v_2 and v_3 by the cycle completed by v with $P_{1,4}$. We can eliminate color 2 from $N(v)$ by switching colors 2 and 4 in H and colors 2 and 3 in H'. Right? This was the final case in Kempe's proof.



The problem is that $P_{1,3}$ and $P_{1,4}$ can intertwine, intersecting at a vertex with color 1 as shown on the right above. We can make the switch in H or in H', but making them both creates a pair of adjacent vertices with color 2. ■

Because of this difficulty, we have not shown that "$\bullet$ 5" is reducible, and we must consider larger configurations. Heesch [1969] contributed the idea of seeking configurations with small ring size instead of few vertices inside. It is not hard to show that every configuration having ring size 3 or 4 is reducible (Exercise 9). This is equivalent to showing that no minimal 5-chromatic triangulation has a separating cycle of length at most 4.

6.3.5.* Example. Birkhoff [1913] pushed the idea farther. He proved that every configuration with ring size 5 that has more than one vertex inside is reducible. He also proved that the configuration with ring size 6 below, called the **Birkhoff diamond**, is reducible.



Proving that the Birkhoff diamond is reducible takes a full page of detailed analysis. One approach is to try to show that all proper 4-colorings of the ring extend to the interior. Although some cases can be combined, and some do extend, in some cases it is necessary to use Kempe chains to show that the coloring can be changed into one that extends. ■

The intricate analysis of this first nontrivial example suggests that we have barely begun. The detail remaining is enormous. From 1913 to 1950, additional reducible configurations were found, enough to prove that all planar graphs with at most 36 vertices are 4-colorable. This was slow progress. In the 1960s, Heesch focused attention on the size of the ring, gave heuristics for finding reducible configurations, and developed methods for generating unavoidable sets.

The first proof used configurations with ring size up to 14. A ring of size 13 has 66430 distinguishable 4-colorings. Reducibility requires showing that each leads to a 4-coloring of the full graph. Kempe-chain arguments and partial collapsing of the configuration may be needed, so reducibility proofs are not easy.

Appel and Haken, working with Koch, improved upon the heuristics of Heesch and others to restrict computer searches to "promising" configurations. Using 1000 hours of computer time on three computers in 1976, they found an unavoidable set of 1936 reducible configurations, all with ring size at most 14.

6.3.6. Theorem. (Four Color Theorem—Appel–Haken–Koch [1977]) Every planar graph is 4-colorable. ■

By 1983, refinements led to an unavoidable set of 1258 reducible configurations. The proof was revisited by Robertson, Sanders, Seymour, and Thomas [1996], using the same approach. They reduced the rules used for producing unavoidable sets to a set of 32 rules. Their simplifications yielded an unavoidable set of 633 reducible configurations. They made their computer code available on the Internet; in 1997, it would prove the Four Color Theorem on a desktop workstation in about three hours.

6.3.7.* Remark. *Discharging.* To generate unavoidable sets, we replace the problem case (vertex of degree 5) by larger configurations involving a vertex of degree 5; this can be viewed as a more detailed case analysis for the hard case. Systematic rules are needed to maintain a reasonably small exhaustive set.

In a triangulation, $\sum d(v) = 2e(G) = 6n - 12$. We rewrite this as $12 = \sum(6-d(v))$ and think of $6-d(v)$ as a **charge** on vertex v. Because 12 is positive, some vertices must have positive charge (degree 5). The rules for replacing bad cases involve moving the charge around; they are called **discharging rules**. Since positive charge must remain somewhere, we obtain new unavoidable sets. The next proposition describes the effect of the simplest discharging rule. ■

6.3.8.* Proposition. Every planar triangulation with minimum degree 5 contains a configuration in the set below.

5 •—• 5   5 •—• 6

Proof: Start with charge defined by $6 - d(v)$. The first discharging rule takes the charge from each vertex of positive charge (degree 5) and distributes that charge equally among its neighbors.

A vertex of degree 5 or 6 now having positive charge must have a neighbor of degree 5. A vertex of degree 7 now having positive charge must have at least six neighbors of degree 5. Since G is a triangulation, this requires adjacent vertices of degree 5. No vertex of degree 8 or more can acquire positive charge from this discharging rule.

The total charge in the graph remains 12, so some vertex v has positive charge. For each case of $d(v)$, one of the specified configurations occurs. ■

Discharging methods are now being applied to attack other problems using computer-assisted analysis by cases.

The proof of the Four Color Theorem met with considerable uproar. Some objected in principle to the use of a computer. Others complained that the proof was too long to be verified. Others worried about computer error. A few errors were found in the original algorithms, but these were fixed (Appel–Haken [1986]). Those who have checked calculations by hand recognize that the probability of human error in a mathematical proof is much higher than the probability of computer error when the algorithm has been proved correct.

CROSSING NUMBER

In the remainder of this section, we consider parameters that measure a graph's deviation from planarity. One natural parameter is the number of planar graphs needed to form the graph; Exercises 16–20 consider this.

6.3.9. Definition. The **thickness** of a graph G is the minimum number of planar graphs in a decomposition of G into planar graphs.

6.3.10. Proposition. A simple graph G with n vertices and m edges has thickness at least $m/(3n-6)$. If G has no triangles, then it has thickness at least $m/(2n-4)$.

Proof: By Theorem 6.1.23, the denominator is the maximum size of each planar subgraph. The pigeonhole principle then yields the inequality. ■

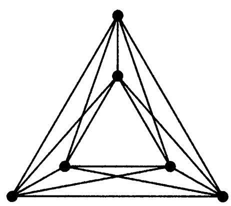
Sometimes we simply must draw a graph in the plane, even if it is not a planar graph. For example, a circuit laid out on a chip corresponds to a drawing of a graph. Since wire crossings lessen performance and cause potential problems, we try to minimize the number of crossings. We discuss the resulting parameter in the remainder of this subsection.

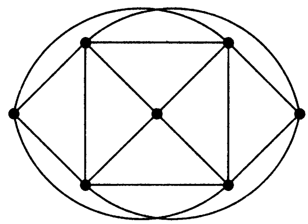
6.3.11. Definition. The **crossing number** $\nu(G)$ of a graph G is the minimum number of crossings in a drawing of G in the plane.

6.3.12. Example. *$\nu(K_6) = 3$ and $\nu(K_{3,2,2}) = 2$.* We can determine the crossing number of some small graphs by considering maximal planar subgraphs. Consider a drawing of G in the plane. If H is a maximal plane subgraph of this drawing, then every edge of G not in H crosses some edge of H, so the drawing has at least $e(G) - e(H)$ crossings. If G has n vertices, then $e(H) \le 3n - 6$. If also G has no triangles, then $e(H) \le 2n - 4$.

Since K_6 has 15 edges, and planar 6-vertex graphs have at most 12 edges, we have $\nu(K_6) \ge 3$. The drawing on the left below proves equality.

Since $K_{3,2,2}$ has 16 edges, and planar graphs with seven vertices have at most 15 edges, $\nu(K_{3,2,2}) \ge 1$. The best drawing we find has two crossings, as shown on the right below. To improve the lower bound, observe that $K_{3,2,2}$ contains $K_{3,4}$. Because $K_{3,4}$ is triangle-free, its planar subgraphs have at most $2 \cdot 7 - 4 = 10$ edges, and hence $\nu(K_{3,4}) \ge 2$. Every drawing of $K_{3,2,2}$ contains a drawing of $K_{3,4}$, so $\nu(K_{3,2,2}) \ge \nu(K_{3,4}) \ge 2$. ■





6.3.13. Proposition. Let G be an n-vertex graph with m edges. If k is the maximum number of edges in a planar subgraph of G, then $\nu(G) \ge m - k$. Furthermore, $\nu(G) \ge \frac{m^2}{2k} - \frac{m}{2}$.

Proof: Given a drawing of G in the plane, let H be a maximal subgraph of G whose edges do not cross in this drawing. Every edge not in H crosses at least one edge in H; otherwise, it could be added to H. Since H has at most k edges, we have at least $m - k$ crossings between edges of H and edges of $G - E(H)$.

After discarding $E(H)$, we have at least $m - k$ edges remaining. The same argument yields at least $(m - k) - k$ crossings in the drawing of the remaining graph. Iterating the argument yields at least $\sum_{i=1}^{t}(m - ik)$ crossings, where $t = \lfloor m/k \rfloor$. The value of the sum is $mt - kt(t + 1)/2$.

We now write $m = tk + r$, where $0 \le r \le k - 1$. We substitute $t = (m - r)/k$ in the value of the sum and simplify to obtain $\nu(G) \ge \frac{m^2}{2k} - \frac{m}{2} + \frac{r(k-r)}{2k}$. ■

The first bound $m - k$ in Proposition 6.3.13 is useful when G has few edges: the crossing number of a simple graph G is at least $e(G) - 3n + 6$, and when G is bipartite it is at least $e(G) - 2n + 4$. Iterating the argument improves the bound when $e(G)$ is larger, but for dense graphs this lower bound is weak.

Consider K_n, for example. Lacking an exact answer, we hope at least to determine the leading term in a polynomial expression for $\nu(K_n)$. To indicate a polynomial of degree k in n with leading term an^k, we often write $an^k + O(n^{k-1})$. This is consistent with the definition of "Big Oh" notation in Definition 3.2.3.

Proposition 6.3.13 yields $\nu(K_n) \ge \frac{1}{24}n^3 + O(n^2)$, but actually $\nu(K_n)$ grows like a polynomial of degree 4. The crossing number cannot exceed $\binom{n}{4}$, since we can place the vertices on the circumference of a circle and draw chords. For K_n, each set of four vertices contributes exactly one crossing. Actually, this is the worst possible straight-line drawing of K_n, since in every straight-line drawing, each set of four vertices contributes at most one crossing, depending on whether one vertex is inside the triangle formed by the other three. How many crossings can be saved by a better drawing?

6.3.14. Theorem. (R. Guy [1972]) $\frac{1}{80}n^4 + O(n^3) \le \nu(K_n) \le \frac{1}{64}n^4 + O(n^3)$.

Proof: A counting argument yields a recursive lower bound. A drawing of K_n with fewest crossings contains n drawings of K_{n-1}, each obtained by deleting one vertex. Each subdrawing has at least $\nu(K_{n-1}$ crossings. The total count is at least $n\nu(K_{n-1})$, but each crossing in the full drawing has been counted $(n-4)$ times. We conclude that $(n-4)\nu(K_n) \ge n\nu(K_{n-1})$.

From this inequality, we prove by induction on n that $\nu(K_n) \ge \frac{1}{5}\binom{n}{4}$ when $n \ge 5$. Basis step: $n = 5$. The crossing number of K_5 is 1. Induction step: $n > 5$. Using the induction hypothesis, we compute

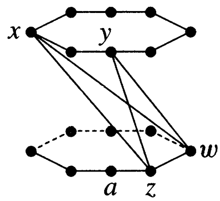
$$\nu(K_n) \ge \frac{n}{n-4}\nu(K_{n-1}) \ge \frac{n}{n-4}\frac{1}{5}\frac{(n-1)(n-2)(n-3)(n-4)}{24} = \frac{1}{5}\binom{n}{4}.$$

The denominator of the quartic term in the lower bound can be improved from 120 to 80 by considering copies of $K_{6,n-6}$, which has crossing number $6\left\lfloor\frac{n-6}{2}\right\rfloor\left\lfloor\frac{n-7}{2}\right\rfloor$ (Exercise 26b).

A better drawing lowers the upper bound from $\binom{n}{4}$ to $\frac{1}{64}n^4+O(n^3)$. Consider $n = 2k$. Drawing K_n in the plane is equivalent to drawing it on a sphere or on the surface of a can. Place k vertices on the top rim of the can and k vertices on the bottom rim, drawing chords on the top and bottom for those k-cliques.

The edges from top to bottom fall into k natural classes. The "class number" is the circular separation between the top and bottom endpoints, ranging from $\left\lceil\frac{-k+1}{2}\right\rceil$ to $\left\lceil\frac{k-1}{2}\right\rceil$. We draw these edges to wind around the can as little as possible in passing from top to bottom, so edges in the same class don't cross. We now twist the can to make the class displacements run from 1 to k. This makes them easier to count but doesn't change the pairs of edges that cross.

Crossings on the side of the can involve two vertices on the top and two on the bottom. For top vertices x, y and bottom vertices z, w, where xz has smaller positive displacement than xw, we have a crossing for x, y, z, w if and only if the displacements to y, z, w are distinct positive values in increasing order. (For example, this holds for x, y, z, w in the illustration, but not for x, y, z, a; the edge ya winds around the can.) Hence there are $k\binom{k}{3}$ crossings on the side of the twisted can, and $\nu(K_n) \le 2\binom{k}{4} + k\binom{k}{3} = \frac{1}{64}n^4 + O(n^3)$. ■



6.3.15. Example. $\nu(K_{m,n})$. The most naive drawing puts the vertices of one partite set on one side of a channel and the vertices of the other partite set on the other side, with all edges drawn straight across. This has $\binom{n}{2}\binom{m}{2}$ crossings, but it is easy to reduce this by a factor of 4. Place the vertices of $K_{m,n}$ along two perpendicular axes. Put $\lceil n/2 \rceil$ vertices along the positive y-axis and $\lfloor n/2 \rfloor$ along the negative y-axis; similarly split the m vertices along the positive and negative x-axis. Adding up the four types of crossings generated when we join every vertex on the x-axis to every vertex on the y-axis yields $\nu(K_{m,n}) \le \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ (Zarankiewicz [1954]).

This bound is conjectured to be optimal (Guy [1969] tells the history). Kleitman [1970] proved it for $\min\{n, m\} \le 6$. Aided by a computer search, Woodall [1993] extended this so that the smallest unknown cases are $K_{7,11}$ and $K_{9,9}$. From Kleitman's result, Guy [1970] proved that $\nu(K_{m,n}) \ge \frac{m(m-1)}{5} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$, which is not far from the upper bound (Exercise 26). ■

Another general lower bound for crossing number, conjectured in Erdős-Guy [1973], has an appealing geometric application. Our proof is inductive, generalizing the lower bound argument in Theorem 6.3.14. There is an elegant probabilistic proof in Exercise 8.5.11and stronger results in Pach–Tóth [1997].

6.3.16.* Theorem. (Ajtai-Chvátal-Newborn-Szemerédi [1982],Leighton[1983]) Let G be a simple graph. If $e(G) \ge 4n(G)$, then $\nu(G) \ge \frac{1}{64}e(G)^3/n(G)^2$.

Proof: Let $m = e(G)$ and $n = n(G)$. We use induction on n.

Basis step: $m \le 5n$ (this includes all simple graphs with at most 11 vertices). Note that $(\alpha - 3) \ge \frac{1}{64}\alpha^3$ when $4 \le \alpha \le 5$. Letting $m = \alpha n$ for $4 \le \alpha \le 5$, we obtain $\nu(G) \ge m - 3n \ge \frac{1}{64}m^3/n^2$, as desired.

Induction step: $n > 11$. Given an optimal drawing of G, each crossing appears in $n-4$ of the drawings obtained by deleting a single vertex. By the induction hypothesis, $\nu(G - v) \ge \frac{1}{64}\frac{(m-d(v))^3}{(n-1)^2}$. Thus $(n-4)\nu(G) \ge \sum_{v \in V(G)} \frac{1}{64}\frac{(m-d(v))^3}{(n-1)^2}$.

By convexity, the lower bound is always at least what results when the vertex degrees are all replaced by the average degree. In other words, $\sum(m - d(v))^3 \ge n(m - 2m/n)^3$. Also $(n-1)^2(n-4) \le (n-2)^3$. Thus

$$\nu(G) \ge \frac{1}{64}n\frac{(n-2)^3m^3}{n^3(n-1)^2(n-4)} \ge \frac{1}{64}\frac{m^3}{n^2}\,. \quad ■$$

6.3.17.* Example. *Achieving the bound.* The order of magnitude in Theorem 6.3.16 is best possible. Consider $G = \frac{n}{2m} K_{2m/n}$, where $2m$ is a multiple of n. The total number of vertices is n, and the total number of edges is asymptotic to $\frac{n^2}{2m}\frac{1}{2}(\frac{2m}{n})^2 = m$. Since $\nu(K_r) \le \frac{1}{64}r^4$, we have $\nu(G) \le \frac{n^2}{2m}\frac{1}{64}(\frac{2m}{n})^4 = \frac{1}{8}\frac{m^3}{n^2}$. This is within a constant factor of the lower bound from Theorem 6.3.16. ■

We apply Theorem 6.3.16 to a problem in combinatorial geometry. Erdős [1946] asked how many unit distances can occur among a set of n points in the plane. If the points occur in a unit grid, then the graph of unit distances is the cartesian product of two paths, and this produces about $n - O(\sqrt{n})$ edges. By taking all the points of a refined grid that lie within an appropriate distance from the origin, Erdős obtained about $n^{1+c/\log\log n}$ unit distances. This growth rate is superlinear, but it is slower than $n^{1+\epsilon}$ for each positive ϵ.

Erdős also proved an upper bound of $O(n^{3/2})$. Since two circles of radius 1 intersect in at most two points, the graph G of unit distances cannot contain $K_{2,3}$. Thus each pair of points has at most two common neighbors. Since each vertex v is a common neighbor for its $\binom{d(v)}{2}$ pairs of neighbors, $\sum\binom{d(v)}{2} \le 2\binom{n}{2}$. Since $2e(G)/n$ is the average vertex degree, convexity yields $\sum\binom{d(v)}{2} \ge n\binom{2e(G)/n}{2}$. Together, these inequalities yield the desired bound (Exercise 5.2.25 considers the edge-maximization problem in general when a biclique is forbidden).

Using number-theoretic arguments about incidences between lines and points in a point set, Spencer–Szemerédi–Trotter [1984] improved the upper bound to $O(n^{4/3})$. Székely applied Theorem 6.3.16 to give an elegant and short graph-theoretic proof of this bound.

6.3.18.* Theorem. (Spencer–Szemerédi–Trotter [1984]) There are at most $4n^{4/3}$ pairs of points at distance 1 among a set of n points in the plane.

Proof: (Székely [1997]) By moving points or pairs of points without reducing the number of pairs at distance 1, we can ensure that each point is involved in such a pair and that no two points have distance 1 only from each other. If any point now is involved in only one unit distance pair, we can rotate it around its mate until it is distance 1 from another point. This reduces the problem to the case that every point is involved in at least two such pairs.

Let P be an optimal n-point configuration, with q unit distance pairs. We obtain a graph from P, not by using the unit distance pairs as edges, but rather by drawing a unit circle around each point. If a point in P is at distance 1 from k other points in P, then these points partition the circle into k arcs. Altogether we obtain $2q$ arcs. These are the edges of a loopless graph G.

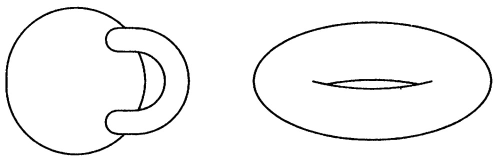
Since two points can appear on two (but not three) unit circles, G may have edges of multiplicity 2 but no larger multiplicity. We delete one copy of each duplicated edge to obtain a simple graph G' with at least q edges. We may assume that $q \ge 4n$; otherwise the bound already holds.

Because these arcs lie on n circles, they cannot produce many crossings; each pair of circles crosses at most twice. Thus our layout of G' has at most $2\binom{n}{2}$ crossings. By Theorem 6.3.16, G' has at least $\frac{1}{64}q^3/n^2$ crossings. Together, these inequalities yield $q \le 4n^{4/3}$. ■

SURFACES OF HIGHER GENUS (optional)

Instead of minimizing crossings in the plane, we could change the surface to avoid crossings. This is the effect of building overpasses and cloverleafs instead of installing traffic lights. The surface of the earth is a sphere, and for this discussion it is convenient to consider drawings on the sphere instead of in the plane. As observed in Remark 6.1.27, these settings are equivalent.

To avoid creating boundaries in the surface, we add an overpass by cutting two holes in the sphere and joining the edges of the holes by a tube. By stretching the tube and squeezing the rest of the sphere, we obtain a doughnut.



6.3.19. Definition. A **handle** is a tube joining two holes cut in a surface. The **torus** is the surface obtained by adding one handle to a sphere.

The torus is topologically the same as the sphere with one handle, in the sense that one surface can be continuously transformed into the other.†

A large graph may have many crossings and need more handles. For any graph, adding enough handles to a drawing on the sphere will eliminate all crossings and produce an embedding. When we add some number of handles, it doesn't matter how we do it, because a fundamental result of topology says that two surfaces obtained by adding the same number of handles to a sphere can be continuously deformed into each other.

6.3.20. Definition. The **genus** of a surface obtained by adding handles to a sphere is the number of handles added; we use S_γ for the surface of genus γ. The **genus** of a graph G is the minimum γ such that G embeds on S_γ. The graphs embeddable on the surfaces of genus 0, 1, 2 are the **planar**, **toroidal**, and **double-toroidal** graphs, respectively (the surface with two handles is the **double-torus**).

The theory of planar graphs extends in some ways to graphs embeddable on higher surfaces; we discuss this only briefly, for cultural interest. Drawings of large graphs on surfaces of large genus are hard to follow, even on the **pretzel** (S_3). Locally, the surface looks like a plane sheet of paper. To draw the graph we want to lay the entire surface flat; to do this we must cut the surface. If we keep track of how the edges should be pasted back together to get the surface, we can describe the surface on a flat piece of paper. Consider first the torus.

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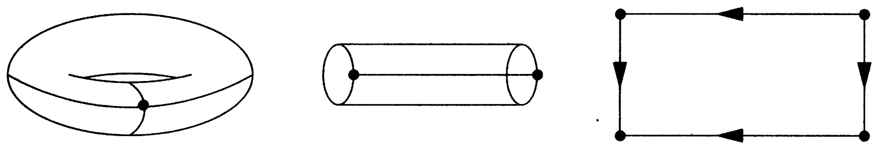
†This is the source of the joke that a topologist is a person who can't tell the difference between a doughnut and a coffee cup.

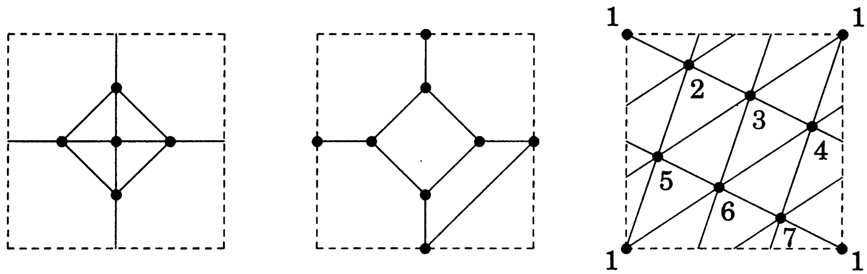
6.3.21. Example. *Combinatorial description of the torus.*



Cutting the closed tube once turns it into a cylinder, and then slitting the length of the cylinder allows us to lay it flat as a rectangle. Labeling the edges of the rectangle indicates how to paste it back together. The two sides of a cut labeled with the same letter are "identified".

Keeping track of the identifications is important because edges of an embedding on a surface may cross such a cut. When the edge reaches one border of the rectangle, it is reaching one side of the imagined cut. When it crosses the cut, it emerges from the identical point on the other copy of this border. The four "corners" of the rectangle correspond to the single point on the surface through which both cuts pass.

These ideas lead to nice toroidal embeddings of K_5, $K_{3,3}$, and K_7. ■



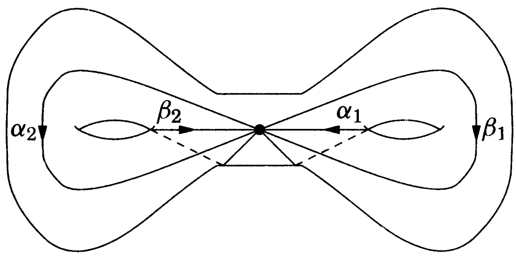
For surfaces of higher genus, there is some flexibility in making the cuts, but each way takes two cuts per handle before we can lay the surface flat. The usual representation comes from expressing the handles as "lobes" of the surface, with the cuts having a common point on the hub.

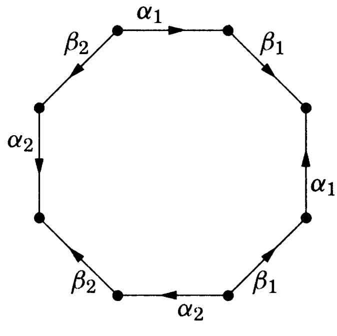
6.3.22. Example. *Laying the double torus flat.* Below is a polygonal representation for the double torus. Making the cuts is equivalent to adding loops at a single vertex until we have a one-face embedding of a bouquet of loops. In general, we make 2γ cuts through a single point to lay S_γ flat.

Keeping track of the borders from each cut leads to representing S_γ by a 4γ-gon in which a clockwise traversal of the boundary can be described by reading out the cuts as we traverse them. We record a cut using the notation of inverses when we traverse it in the opposite order.

Since we are following the boundary of a single face, with our left hand always on the wall, each edge will be followed once forward and once backward. For the example here, the traversal is $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}$.

Each surface S_γ has a layout of the form $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_\gamma\beta_\gamma\alpha_\gamma^{-1}\beta_\gamma^{-1}$. Other layouts result from other ways of making the cuts – different ways of embedding a bouquet of 2γ loops. For example, the double torus can also be represented by an octagon with boundary $\alpha\beta\gamma\delta\alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1}$. ■





6.3.23. Remark. *Euler's Formula for S_γ.* A **2-cell** is a region such that every closed curve in the interior can be continuously contracted to a point. A **2-cell embedding** is an embedding where every region is a 2-cell. Euler's Formula generalizes for 2-cell embeddings of connected graphs on S_γ (Exercise 35) as

$$n - e + f = 2 - 2\gamma.$$

For example, our embedding of K_7 on the torus ($\gamma = 1$) has 7 vertices, 21 edges, 14 faces, and $7 - 21 + 14 = 0$. The proof of Euler's Formula for S_γ is like the proof in the plane, except that the basis case of 1-vertex graphs needs more care. It requires showing that it takes 2γ cuts to lay the surface flat (that is, to obtain a 2-cell embedding of a graph with one vertex and one face). ■

6.3.24. Lemma. Every simple n-vertex graph embedded on S_γ has at most $3(n - 2 + 2\gamma)$ edges.

Proof: Exercise 35. ■

Note that K_7 satisfies Lemma 6.3:24 with equality on the torus ($\gamma = 1$), as every face in the toroidal embedding of K_7 is a 3-gon. Hence K_7 is a maximal toroidal graph. Rewriting $e \le 3(n-2+2\gamma)$ yields a lower bound on the number of handles we must add to obtain a surface on which G is embeddable; thus $\gamma(G) \ge 1 + (e - 3n)/6$.

Lemma 6.3.24 leads to an analogue of the Four Color Theorem for S_γ.

6.3.25. Theorem. (Heawood's Formula—Heawood [1890]) If G is embeddable on S_γ with $\gamma > 0$, then $\chi(G) \le \left\lfloor (7 + \sqrt{1 + 48\gamma})/2 \right\rfloor$.

Proof: Let $c = (7 + \sqrt{1 + 48\gamma})/2$. It suffices to prove that every simple graph embeddable on S_γ has a vertex of degree at most $c - 1$; the bound on $\chi(G)$ then follows by induction on $n(G)$. Since $\chi(G) \le c$ for all graphs with at most c vertices, so need only consider $n(G) > c$.

We use Lemma 6.3.24 to show that the average (and hence minimum) degree is at most $c - 1$. The second inequality below follows from $\gamma > 0$ and $n > c$.

Since c satisfies $c^2 - 7c + (12 - 12\gamma) = 0$, we have $c - 1 = 6 - (12 - 12\gamma)/c$, so the average degree satisfies the desired bound.

$$\frac{2e}{n} \le \frac{6(n - 2 + 2\gamma)}{n} \le 6 - \frac{12 - 12\gamma}{c} = c - 1. \qquad \blacksquare$$

The key inequality here fails when $\gamma = 0$. Thus the argument is invalid for planar graphs, even though the formula reduces to $\chi(G) \le 4$ when $\gamma = 0$. Proving that the Heawood bound is sharp involves embedding K_n on S_γ with $\gamma = \lceil (n-3)(n-4)/12 \rceil$. The proof breaks into cases by the congruence class of n modulo 12 (K_7 is the first example in the easy class). Completed in Ringel–Youngs [1968], it comprises the book *Map Color Theorem* (Ringel [1974]).

Having considered the coloring problem on S_γ, one naturally wonders which graphs embed on S_γ. Planar graphs have many characterizations, beginning with Kuratowski's Theorem (Theorem 6.2.2) and Wagner's Theorem (Exercise 6.2.12). On any surface, embeddability is preserved by deleting or contracting an edge. Thus every surface has a list of "minor-minimal" obstructions to embeddability. Wagner's Theorem states that the list for the plane is $\{K_{3,3}, K_5\}$; every nonplanar graph has one of these as a minor.

More than 800 minimal forbidden minors are known for the torus. For each surface, the list is finite; this follows from the much more general statement below (the *subdivision* relation in Kuratowski's Theorem leads to infinite lists).

6.3.26. Theorem. (The Graph Minor Theorem—Robertson–Seymour [1985]) In any infinite list of graphs, some graph is a minor of another. $\blacksquare$

This is perhaps the most difficult theorem known in graph theory. The complete proof takes well over 500 pages (without computer assistance) in a series of 20 papers stretching beyond the year 2000. It has many ramifications about structure of graphs and complexity of computation. The techniques involved in the proof have spawned new areas of graph theory. Some aspects of these techniques and their relation to the proof of the Graph Minor Theorem are presented in the final chapter of the text by Diestel [1997].

EXERCISES

6.3.1. (−) State a polynomial-time algorithm that takes an arbitrary planar graph as input and produces a proper 5-coloring of the graph.

6.3.2. (−) A graph G is k-**degenerate** if every subgraph of G has a vertex of degree at most k. Prove that every k-degenerate graph is $k+1$-colorable.

6.3.3. (−) Use the Four Color Theorem to prove that every outerplanar graph is 3-colorable.

6.3.4. (−) Determine the crossing numbers of $K_{2,2,2,2}$, $K_{4,4}$, and the Petersen graph.

• • • • •

6.3.5. Prove that every planar graph decomposes into two bipartite graphs. (Hedetniemi [1969], Mabry [1995])

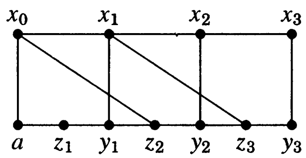
6.3.6. Without using the Four Color Theorem, prove that every planar graph with at most 12 vertices is 4-colorable. Use this to prove that every planar graph with at most 32 edges is 4-colorable.

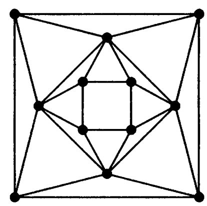
6.3.7. (!) Let H be a configuration in a planar triangulation (Definition 6.3.2). Let H' be obtained by labeling the neighbors of the ring vertices with their degrees and then deleting the ring vertices. Prove that H can be retrieved from H'.

6.3.8. Create a configuration with ring size 5 in a planar triangulation such that every internal vertex has degree at least five.

6.3.9. (+) Prove that every planar configuration having ring size at most four is reducible. (Hint: The ring is a separating cycle C. Prove that if smaller triangulations are 4-colorable, then the C-lobes of G have 4-colorings that agree on C.) (Birkhoff [1913])

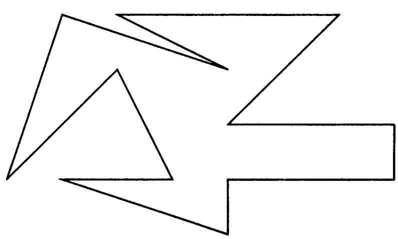
6.3.10. Grötzsch's Theorem [1959] (see Steinberg [1993], Thomassen [1994a]) states that a triangle-free planar graph G is 3-colorable. Hence $\alpha(G) \geq n(G)/3$. Tovey–Steinberg [1993] proved that $\alpha(G) > n(G)/3$ always. Prove that this is best possible by considering the family of graphs G_k defined as follows: G_1 is the 5-cycle, with vertices a, x_0, x_1, y_1, z_1 in order. For $k > 1$, G_k is obtained from G_{k-1} by adding the three vertices x_k, y_k, z_k and the five edges $x_{k-1}x_k, x_ky_k, y_kz_k, z_ky_{k-1}, z_kx_{k-2}$. The graph G_3 is shown on the left below. (Fraughnaugh [1985])





6.3.11. Define a sequence of plane graphs as follows. Let G_1 be C_4. For $n > 1$, obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to two consecutive vertices of the previous outside face. The graph G_3 is shown on the right above. Prove that if n is even, then every proper 4-coloring of G_n uses each color on exactly n vertices. (Albertson)

6.3.12. (!) Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior is visible to some guard. Construct a polygon that requires $\lfloor n/3 \rfloor$ guards. (Chvátal [1975], Fisk [1978])



6.3.13. An *art gallery with walls* is a polygon plus some nonintersecting chords called "walls" that join vertices. Each interior wall has a tiny opening called a "doorway". A guard in a doorway can see everything in the two neighboring rooms, but a guard not in a doorway cannot see past a wall. Determine the minimum number t such that for every walled art gallery with n vertices, it is possible to place t guards so that every interior point is visible to some guard. (Hutchinson [1995], Kündgen [1999])

6.3.14. (+) Prove that a maximal planar graph is 3-colorable if and only if it is Eulerian. (Hint: For sufficiency, use induction on $n(G)$. Choose an appropriate pair or triple of adjacent vertices to replace with appropriate edges.) (Heawood [1898])

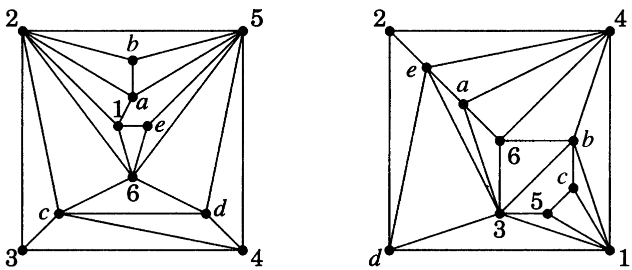
6.3.15. (!) Prove that the vertices of an outerplanar graph can be partitioned into two sets so that the subgraph induced by each set is a disjoint union of paths. (Hint: Define the partition using the parity of the distance from a fixed vertex.) (Akiyama–Era–Gervacio [1989], Goddard [1991])

6.3.16. (−) Prove that the 4-dimensional cube Q_4 is nonplanar. Decompose it into two isomorphic planar graphs; hence Q_4 has thickness 2.

6.3.17. Prove that K_n has thickness at least $\lfloor \frac{n+7}{6} \rfloor$. (Hint: $\lceil \frac{x}{r} \rceil = \lfloor \frac{x+r-1}{r} \rfloor$.) Show that equality holds for K_8 by finding a self-complementary planar graph with 8 vertices. (Comment: The thickness equals $\lfloor \frac{n+7}{6} \rfloor$ except that K_9 and K_{10} have thickness 3; Beineke–Harary [1965] for $n \not\equiv 4 \bmod 6$, and Alekseev–Gončakov [1976] for $n \equiv 4 \bmod 6$.)

6.3.18. Decompose K_9 into three pairwise-isomorphic planar graphs.

6.3.19. Prove that if G has thickness 2, then $\chi(G) \le 12$. Use the two graphs below to show that $\chi(G)$ may be as large as 9 when G has thickness 2. (Sulanke)



6.3.20. (!) When r is even and s is greater than $(r-2)^2/2$, prove that the thickness of $K_{r,s}$ is $r/2$. (Beineke–Harary–Moon [1964])

6.3.21. Determine $\nu(K_{1,2,2,2})$ and use it to compute $\nu(K_{2,2,2,2})$.

6.3.22. Prove that $K_{3,2,2}$ has no planar subgraph with 15 edges. Use this to give another proof that $\nu(K_{3,2,2}) \ge 2$.

6.3.23. Let M_n be the graph obtained from the cycle C_n by adding chords between vertices that are opposite (if n is even) or nearly opposite (if n is odd). The graph M_n is 3-regular if n is even, 4-regular if n is odd. Determine $\nu(M_n)$. (Guy–Harary [1967])

6.3.24. The graph P_n^k has vertex set $[n]$ and edge set $\{ij\colon |i-j| \le k\}$. Prove that P_n^3 is a maximal planar graph. Use a planar embedding of P_n^3 to prove that $\nu(P_n^4) = n-4$. (Harary–Kainen [1993])

6.3.25. For every positive integer k, construct a graph that embeds on the torus but requires at least k crossings when drawn in the plane. (Hint: A single easily described toroidal family suffices; use Proposition 6.3.13.)

6.3.26. (!) Use Kleitman's computation that $\nu(K_{6,n}) = 6\left\lfloor \frac{n-6}{2}\right\rfloor\left\lfloor\frac{n-7}{2}\right\rfloor$ to give counting arguments for the following lower bounds.

a) $\nu(K_{m,n}) \ge m\frac{m-1}{5}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. (Guy [1970])

b) $\nu(K_p) \ge \frac{1}{80}p^4 + O(p^3)$.

6.3.27. (!) It is conjectured that $\nu(K_{m,n}) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. Suppose that this conjecture holds for $K_{m,n}$ and that m is odd. Prove that the conjecture then holds also for $K_{m+1,n}$. (Kleitman [1970])

6.3.28. (!) Suppose that m and n are odd. Prove that in all drawings of $K_{m,n}$, the parity of the number of pairs of edges that cross is the same. (We consider only drawings where edges cross at most once and edges sharing an endpoint do not cross.) Conclude that $\nu(K_{m,m})$ is odd when $m-3$ and $n-3$ are divisible by 3 and even otherwise.

6.3.29. Suppose that n is odd. Prove that in all drawings of K_n, the parity of the number of pairs of edges that cross is the same. Conclude that $\nu(K_n)$ is even when n is congruent to 1 or 3 modulo 8 and is odd when n is congruent to 5 or 7 modulo 8.

6.3.30. (!) It is known that $\nu(C_m \square C_n) = (m-2)n$ if $m \le \min\{5, n\}$. Also $\nu(K_4 \square C_n) = 3n$.

a) Find drawings in the plane to establish the upper bounds.

b) Prove that $\nu(C_3 \square C_3) \ge 2$. (Hint: Find three subdivisions of $K_{3,3}$ that together use each edge exactly twice.)

6.3.31. Let $f(n) = \nu(K_{n,n,n})$.

a) Show that $3\nu(K_{n,n}) \le f(n) \le 3\binom{n}{2}^2$.

b) Show that $\nu(K_{3,2,2}) = 2$ and $\nu(K_{3,3,1}) = 3$. Show that $5 \le \nu(K_{3,3,2}) \le 7$ and $9 \le \nu(K_{3,3,3}) \le 15$.

c) Exercise 6.3.26a shows that the lower bound in part (a) is at least $(3/20)n^4 + O(n^3)$. Improve it by using a recurrence to show that $f(n) \ge n^3(n-1)/6$.

d) The upper bound in part (a) is $\frac{3}{4}n^4 + O(n^3)$. Improve it to $f(n) \le \frac{9}{16}n^4 + O(n^3)$. (Hint: One construction embeds the graph on a tetrahedron and generalizes to a construction for $K_{l,m,n}$; another uses K_n and generalizes to a construction for $K_{n,\dots,n}$.)

6.3.32. (*) Construct an embedding of a 3-regular nonbipartite simple graph on the torus so that every face has even length.

6.3.33. (*) Suppose that n is at least 9 and is not a prime or twice a prime. Construct a 6-regular toroidal graph with n vertices.

6.3.34. (*) An embedding of a graph on a surface is **regular** if its faces all have the same length. Construct regular embeddings of $K_{4,4}$, $K_{3,6}$, and $K_{3,3}$ on the torus.

6.3.35. (*) Prove Euler's Formula for genus γ: For every 2-cell embedding of a graph on the surface S_γ, the numbers of vertices, edges, and faces satisfy $n - e + f = 2 - 2\gamma$. Conclude that an n-vertex graph embeddable on S_γ has at most $3(n - 2 + 2\gamma)$ edges.

6.3.36. (*) Use Euler's Formula for S_γ to prove that $\gamma(K_{3,3,n}) \ge n - 2$, and determine the value exactly for $n \le 3$.

6.3.37. (*) For every positive integer k, use Euler's Formula for higher surfaces to prove that there exists a planar graph G such that $\gamma(G \square K_2) \ge k$.

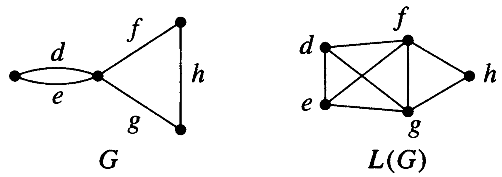
Chapter 7

Edges and Cycles

7.1. Line Graphs and Edge-coloring

Many questions about vertices have natural analogues for edges. Independent sets have no adjacent vertices; matchings have no "adjacent" edges. Vertex colorings partition vertices into independent sets; we can instead partition edges into matchings. These pairs of problems are related via line graphs (Definition 4.2.18). Here we repeat the definition, emphasizing our return to the context in which a graph may have multiple edges. We use "line graph" and $L(G)$ instead of "edge graph" because $E(G)$ already denotes the edge set.

7.1.1. Definition. The **line graph** of G, written $L(G)$, is the simple graph whose vertices are the edges of G, with $ef \in E(L(G))$ when e and f have a common endpoint in G.



Some questions about edges in a graph G can be phrased as questions about vertices in $L(G)$. When extended to all simple graphs, the vertex question may be more difficult. If we can solve it, then we can answer the original question about edges in G by applying the vertex result to $L(G)$.

In Chapter 1, we studied Eulerian circuits. An Eulerian circuit in G yields a spanning cycle in the line graph $L(G)$. (Exercise 7.2.10 shows that the converse need not hold!) In Section 7.2, we study spanning cycles for graphs in general. As discussed in Appendix B, this problem is computationally difficult.

In Chapter 3, we studied matchings. A matching in G becomes an independent set in $L(G)$. Thus $\alpha'(G) = \alpha(L(G))$, and the study of α' for graphs is the study of α for line graphs. Computing α is harder for general graphs than for line graphs. Section 3.1 considers this for bipartite graphs, and we describe the general case briefly in Appendix B.

In Chapter 4, we studied connectivity. Menger's Theorem gave a min-max relation for connectivity and internally disjoint paths in all graphs. By applying this theorem to an appropriate line graph, we proved the analogous min-max relation for edge-connectivity and edge-disjoint paths in all graphs.

In Chapter 5, we studied vertex coloring. Coloring edges so that each color class is a matching amounts to proper vertex coloring of the line graph. Thus edge-coloring is a special case of vertex coloring and therefore potentially easier. We discuss edge-coloring in this section. Our main result, when stated in terms of vertex coloring of line graphs, is an algorithm to compute $\chi(H)$ within 1 when H is the line graph of a simple graph.

Thus line graphs suggest the problems of edge-coloring and spanning cycles that are discussed in this chapter. We first study these separately. In Section 7.3, we study their connections to each other and to planar graphs.

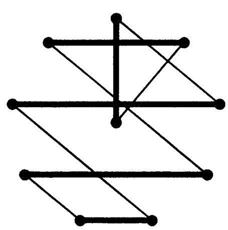
In applying algorithms for line graphs, we may need to know whether G is a line graph. There are good algorithms to check this; they use characterizations of line graphs, which we postpone to the end of this section.

EDGE-COLORINGS

In Example 1.1.11 that introduced vertex coloring, we needed to schedule Senate committees. Edge-coloring problems arise when the objects being scheduled are pairs of underlying elements.

7.1.2. Example. *Edge-coloring of K_{2n}.* In a league with $2n$ teams, we want to schedule games so that each pair of teams plays a game, but each team plays at most once a week. Since each team must play $2n-1$ others, the season lasts at least $2n-1$ weeks. The games of each week must form a matching. We can schedule the season in $2n-1$ weeks if and only if we can partition $E(K_{2n})$ into $2n-1$ matchings. Since K_{2n} is $2n-1$-regular, these must be perfect matchings.

The figure below describes the solution. Put one vertex in the center. Arrange the other $2n-1$ vertices cyclically, viewed as congruence classes modulo $2n-1$. As in Theorem 2.2.16, the *difference* between two congruence classes is 1 if they are consecutive, 2 if there is one class between them, and so on up to difference $n-1$. There are $2n-1$ edges with each difference i, for $1 \le i \le n-1$.



Each matching consists of one edge from each difference class plus one edge involving the center vertex. We show one such matching in bold. Rotating the picture (to obtain the solid matching) yields n new edges; again they are one of each length plus one to the center. The $2n - 1$ rotations of the figure yield the desired matchings, since these matchings take distinct edges from each difference class and distinct edges involving the center vertex. ■

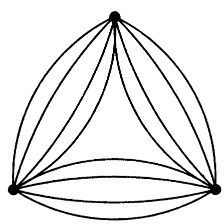
7.1.3. Definition. A **k-edge-coloring** of G is a labeling $f\colon E(G) \to S$, where $|S| = k$ (often we use $S = [k]$). The labels are **colors**; the edges of one color form a **color class**. A k-edge-coloring is **proper** if incident edges have different labels; that is, if each color class is a matching. A graph is **k-edge-colorable** if it has a proper k-edge-coloring. The **edge-chromatic number** $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.

Chromatic index is another name for $\chi'(G)$. Since edges sharing a vertex need different colors, $\chi'(G) \geq \Delta(G)$. Vizing [1964] and Gupta [1966] independently proved that $\Delta(G) + 1$ colors suffice when G is simple; this is our main objective. A clique in $L(G)$ is a set of pairwise-intersecting edges of G. When G is simple, such edges form a star or a triangle in G (Exercise 9). For the hereditary class of line graphs of simple graphs, Vizing's Theorem thus states that $\chi(H) \leq \omega(H) + 1$; thus line graphs are "almost" perfect.

In contrast to $\chi(G)$ in Chapter 5, multiple edges greatly affect $\chi'(G)$. A graph with a loop has no proper edge-coloring; the adjective "loopless" excludes loops but allows multiple edges.

7.1.4. Definition. In a graph G with multiple edges, we say that a vertex pair x, y is an edge of **multiplicity** m if there are m edges with endpoints x, y. We write $\mu(xy)$ for the multiplicity of the pair, and we write $\mu(G)$ for the maximum of the edge multiplicities in G.

7.1.5. Example. *The "Fat Triangle".* For loopless graphs with multiple edges, $\chi'(G)$ may exceed $\Delta(G)+1$. Shannon [1949] proved that the maximum of $\chi'(G)$ in terms of $\Delta(G)$ alone is $3\Delta(G)/2$ (see Theorem 7.1.13). Vizing and Gupta proved that $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum edge multiplicity. The graph below achieves both bounds. The edges are pairwise intersecting and hence require distinct colors. Thus $\chi'(G) = 3\Delta(G)/2 = \Delta(G) + \mu(G)$. ■



7.1.6. Remark. We have observed that always $\chi'(G) \geq \Delta(G)$. The upper bound $\chi'(G) \leq 2\Delta(G) - 1$ also follows easily. Color the edges in some order, always assigning the current edge the least-indexed color different from those already appearing on edges incident to it. Since no edge is incident to more than $2(\Delta(G)-1)$ other edges, this never uses more than $2\Delta(G)-1$ colors. The procedure is precisely greedy coloring for vertices of $L(G)$.

$$\chi'(G) = \chi(L(G)) \le \Delta(L(G)) + 1 \le 2\Delta(G) - 1. \qquad \blacksquare$$

For bipartite graphs, the results of Chapter 3 improve the upper bound of Remark 7.1.6, achieving the trivial lower bound even when multiple edges are allowed. Furthermore, there is a good algorithm to produce a proper $\Delta(G)$-edge-coloring in a bipartite graph G.

7.1.7. Theorem. (König [1916]) If G is bipartite, then $\chi'(G) = \Delta(G)$.

Proof: Corollary 3.1.13 states that every regular bipartite graph H has a 1-factor. By induction on $\Delta(H)$, this yields a proper $\Delta(H)$-edge-coloring. It therefore suffices to show that for every bipartite graph G with maximum degree k, there is a k-regular bipartite graph H containing G.

To construct such a graph, first add vertices to the smaller partite set of G, if necessary, to equalize the sizes. If the resulting G' is not regular, then each partite set has a vertex with degree less than k. Add an edge with these two vertices as endpoints. Continue adding such edges until the graph becomes k-regular; the resulting graph is H. ■

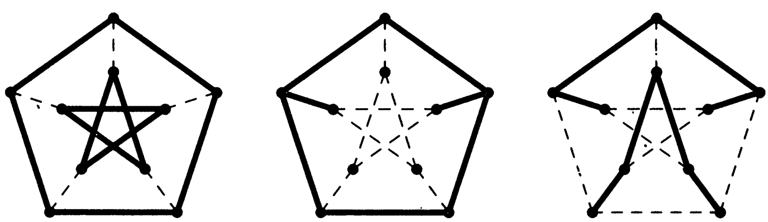
For a regular graph G, proper edge-coloring with $\Delta(G)$ colors is equivalent to decomposition into 1-factors.

7.1.8. Definition. A decomposition of a regular graph G into 1-factors is a **1-factorization** of G. A graph with a 1-factorization is **1-factorable**.

An odd cycle is not 1-factorable; $\chi'(C_{2m+1}) = 3 > \Delta(C_{2m+1})$. The Petersen graph also requires an extra color, but only one extra color.

7.1.9. Example. *The Petersen graph is 4-edge-chromatic* (Petersen [1898]). The Petersen graph is 3-regular; 3-edge-colorability requires a 1-factorization. Deleting a perfect matching leaves a 2-factor; all components are cycles. The 1-factorization can be completed only if these are all even cycles.

Thus it suffices to show that every 2-factor is isomorphic to $2C_5$. Consider the drawing consisting of two 5-cycles and a matching (the **cross edges**) between them. We consider cases by the number of cross edges used.



Every cycle uses an even number of cross edges, so a 2-factor H has an even number m of cross edges. If $m = 0$ (left figure), then $H = 2C_5$.

If $m = 2$ (central figure), then the two cross edges have nonadjacent endpoints on the inner cycle or the outer cycle. On the cycle where their endpoints are nonadjacent, the remaining three vertices force all five edges of that cycle into H, which violates the 2-factor requirement.

If $m = 4$ (right figure), then the cycle edges forced into H by the unused cross edges form a $2P_5$ whose only completion to a 2-factor in H is $2C_5$.

Note that since C_5 is 3-edge-colorable, the graph is 4-edge-colorable. ■

Now we consider all simple graphs. We make $\Delta(G) + 1$ colors available and build a proper edge-coloring, incorporating edges one by one until we have a proper $\Delta(G) + 1$-edge-coloring of G. The algorithm runs surprisingly quickly.

7.1.10. Theorem. (Vizing [1964, 1965], Gupta [1966]) If G is a simple graph, then $\chi'(G) \le \Delta(G) + 1$.

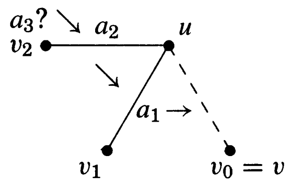
Proof: Let f be a proper $\Delta(G) + 1$-edge-coloring of a subgraph G' of G. If $G' \ne G$, then some edge uv is uncolored by f. After possibly recoloring some edges, we extend the coloring to include uv; call this an *augmentation*. After $e(G)$ augmentations, we obtain a proper $\Delta(G) + 1$-edge-coloring of G.

Since the number of colors exceeds $\Delta(G)$, every vertex has some color *not* appearing on its incident edges. Let a_0 be a color missing at u. We generate a list of neighbors of u and a corresponding list of colors. Begin with $v_0 = v$.

Let a_1 be a color missing at v_0. We may assume that a_1 appears at u on some edge uv_1; otherwise, we would use a_1 on uv_0.

Let a_2 be a color missing at v_1. We may assume that a_2 appears at u on some edge uv_2; otherwise, we would replace color a_1 with a_2 on uv_1 and then use a_1 on uv_0 to augment the coloring.

Having selected uv_{i-1} with color a_{i-1}, let a_i be a color missing at v_{i-1}. If a_i is missing at u, then we use a_i on uv_{i-1} and shift color a_j from uv_j to uv_{j-1} for $1 \le j \le i-1$ to complete the augmentation. We call this *downshifting from i*. If a_i appears at u (on some edge uv_i), then the process continues.



Since we have only $\Delta(G) + 1$ colors to choose from, the list of selected colors eventually repeats (or we complete the augmentation by downshifting). Let l be the smallest index such that a color missing at v_l is in the list $a_1, \dots, a_l$; let this color be a_k. Instead of extending the list, we use this repetition to perform the augmentation in one of several ways.

The color a_k missing at v_l is also missing at v_{k-1} and appears on uv_k. If a_0 does not appear at v_l, then we downshift from v_l and use color a_0 on uv_l to complete the augmentation. Hence we may assume that a_0 appears at v_l.

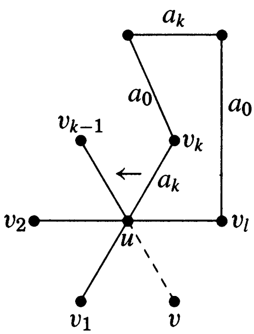
Let P be the maximal alternating path of edges colored a_0 and a_k that begins at v_l along color a_0. There is only one such path, because each vertex has at most one incident edge in each color (we ignore edges not yet colored). To complete the augmentation, we will interchange colors a_0 and a_k on P and downshift from an appropriate neighbor of u, depending on where P goes.

If P reaches v_k, then it arrives at v_k along an edge with color a_0, follows $v_k u$ in color a_k, and stops at u, which lacks color a_0. In this case, we downshift from v_k and switch colors on P (left picture below).

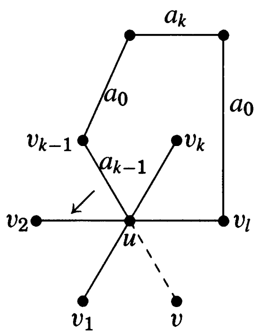
If P reaches v_{k-1}, then it reaches v_{k-1} on color a_0 and stops there, because a_k does not appear at v_{k-1}. In this case, we downshift from v_{k-1}, give color a_0 to uv_{k-1}, and switch colors on P (middle picture).

If P does not reach v_k or v_{k-1}, then it ends at some vertex outside $\{u, v_l, v_k, v_{k-1}\}$. In this case, we downshift from v_l, give color a_0 to uv_l, and switch colors on P (rightmost picture).

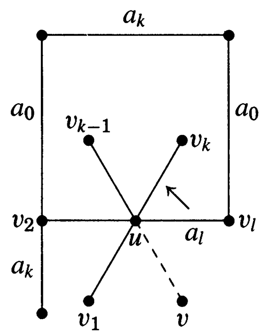
In each case, the changes described yield a proper $\Delta(G) + 1$-edge-coloring of $G' + uv$, so we have completed the desired augmentation. ■



P reaches v_k



P reaches v_{k-1}



otherwise

For simple graphs, we now have only two possibilities for χ'.

7.1.11. Definition. A simple graph G is **Class 1** if $\chi'(G) = \Delta(G)$. It is **Class 2** if $\chi'(G) = \Delta(G) + 1$.

Determining whether a graph is Class 1 or Class 2 is generally hard (Holyer [1981]; see Appendix B). Thus we seek conditions that forbid or guarantee $\Delta(G)$-edge-colorability. Examples of such conditions include Exercises 24–27.

7.1.12.* Remark. There is an obvious necessary condition for a graph to be Class 1 that is conjectured to be sufficient when $\Delta(G) > \frac{3}{10}n(G)$. Part (a) of Exercise 27 observes that a subgraph of G with odd order is an obstruction to $\Delta(G)$-edge-colorability if it has too many edges. A subgraph H of a simple graph G is an **overfull subgraph** if $n(H)$ is odd and $2e(H)/(n(H) - 1) > \Delta(G)$.

The **Overfull Conjecture** (Chetwynd–Hilton [1986]—see also Hilton [1989]) states that if $\Delta(G) > n(G)/3$, then a simple graph G is Class 1 if and only if G has no overfull subgraph. The Petersen graph with a vertex deleted shows that the condition is not sufficient when $\Delta(G) = n(G)/3$ (Exercise 28).

The Overfull Conjecture implies the **1-factorization Conjecture**: If $r \ge m$ (or $r \ge m - 1$ if m is even), then every r-regular simple graph of order $2m$ is Class 1. This also is sharp (Exercise 29).

The conclusions of the two conjectures hold when $\Delta(G)$ is large enough (Chetwynd–Hilton [1989], Niessen–Volkmann [1990], Perkovic–Reed [1997], Plantholt [2001]. ■

When G has multiple edges, $\chi'(G) \le \lfloor 3\Delta(G)/2 \rfloor$ (Shannon [1949]) and $\chi'(G) \le \Delta(G) + \mu(G)$ (Vizing [1964, 1965], Gupta [1966]) These bounds follow (Exercise 35) from that of Andersen [1977] and Goldberg [1977, 1984]:

$$\chi'(G) \le \max\{\Delta(G), \max_{\mathbf{P}} \left\lfloor \tfrac{1}{2}(d(x) + \mu(xy) + \mu(yz) + d(z)) \right\rfloor\}$$

where $\mathbf{P} = \{x, y, z \in V(G) \colon y \in N(x) \cap N(z)\}$. Proving this bound uses the methods of Theorem 7.1.10 plus counting arguments. To illustrate the use of counting arguments, we prove Shannon's Theorem from that of Vizing and Gupta.

7.1.13.* Theorem. (Shannon [1949]) If G is a graph, then $\chi'(G) \le \frac{3}{2}\Delta(G)$.

Proof: Let $k = \chi'(G)$, and assume $k \ge (3/2)\Delta(G)$. Let G' be a minimal subgraph of G with $\chi'(G') = k$. Since $k \le \Delta(G') + \mu(G')$ (Vizing–Gupta), we obtain $\mu(G') \ge \Delta(G)/2$. Let e with endpoints x, y be an edge with multiplicity $\mu(G')$.

Let f be a proper $k - 1$-edge-coloring of $G' - e$. In $G' - e$, both x and y have degree at most $\Delta(G) - 1$, so under f at least $(k - 1) - (\Delta(G) - 1)$ colors are missing at x, and similarly at y. No color is missing at both, since G' is not $k - 1$-edge-colorable. Accounting for the $\mu(G') - 1$ colors used on edges with endpoints x, y yields

$$2(k - \Delta(G)) + (\Delta(G)/2) - 1 \le 2(k - \Delta(G)) + \mu(G') - 1 \le k - 1,$$

and hence $k \le (3/2)\Delta(G)$. ■

Finally, there is a general conjecture analogous to the Overfull Conjecture.

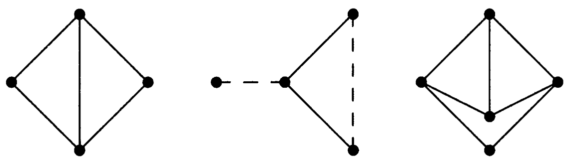
7.1.14.* Conjecture. (Goldberg [1973, 1984], Seymour [1979a])
If $\chi'(G) \ge \Delta(G) + 2$, then $\chi'(G) = \max_{H \subseteq G} \left\lceil \frac{e(H)}{\lfloor n(H)/2 \rfloor} \right\rceil$. ■

CHARACTERIZATION OF LINE GRAPHS (optional)

Characterizations of line graphs can lead to good algorithms to test whether a graph G is a line graph and, if so, to obtain H such that $L(H) = G$.

7.1.15. Example. To illustrate the ideas, we prove that the rightmost graph below is not the line graph of a simple graph. The kite G (two triangles with a common edge) is the line graph of the paw H (a claw plus an edge). By case analysis, we find that H is the only simple graph whose line graph is G, and the edges becoming the vertices of degree 2 in G must be the dashed edges.

The rightmost graph adds a vertex to G having only the vertices of degree 2 as neighbors. The result is not a line graph, because there is no way to add an edge to H that shares an endpoint with each dashed edge without sharing an endpoint with a solid edge. ■



$G = L(H)$ $\qquad$ H $\qquad$ not a line graph

Our first characterization encodes the process of taking the line graph. If $G = L(H)$ and H is simple, then each $v \in V(H)$ with $d(v) \geq 2$ generates a clique $Q(v)$ in G corresponding to edges incident to v. These cliques partition $E(G)$. Furthermore, each vertex $e \in V(G)$ belongs only to the cliques generated by the two endpoints of $e \in E(H)$.

For example, when G is the kite, we can partition $E(G)$ into three cliques (a triangle plus two edges), each vertex covered at most twice. These three cliques correspond to the vertices of degree at least 2 in the paw. The rightmost graph above does not have such a partition.

7.1.16. Theorem. (Krausz [1943]) For a simple graph G, there is a solution to $L(H) = G$ if and only if G decomposes into complete subgraphs, with each vertex of G appearing in at most two in the list.

Proof: We argued above that the condition is necessary. Note that when $G = L(H)$, the vertices of G that belong to only one of the cliques we have defined are those corresponding to edges of H that are incident to leaves.

For sufficiency, let $S_1, \ldots, S_k$ be the vertex sets of the specified complete subgraphs. We construct H such that $G = L(H)$. Isolated vertices of G become isolated edges of H, so we may assume that $\delta(G) \geq 1$. Let $v_1, \ldots, v_l$ be the vertices of G (if any) that appear in exactly one of $S_i, \ldots, S_1n$. Give H one vertex for each set in the list $\mathbf{A} = S_1, \ldots, S_k, \{v_1\}, \ldots, \{v_l\}\}$, and let vertices of H be adjacent if the corresponding sets intersect.

Each vertex of G appears in exactly two sets in $\mathbf{A}$, and no two vertices appear in the same two sets. Hence H is a simple graph with one edge for each vertex of G. If vertices are adjacent in G, then they appear together in some S_i, and the corresponding edges of H share the vertex for S_i. Hence $G = L(H)$. ■

Krausz's characterization does not directly yield an efficient test for line graphs, because there are too many possible decompositions to test. The next characterization tests substructures of fixed size and therefore yields a good algorithm. We say that each triangle T in G is odd or even as defined below.

T is **odd** if $|N(v) \cap V(T)|$ is odd for *some* $v \in V(G)$.
T is **even** if $|N(v) \cap V(T)|$ is even for *every* $v \in V(G)$.
An induced kite is a **double triangle**; it consists of two triangles sharing an edge, and the two vertices not in that edge are nonadjacent.

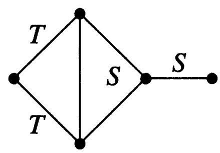
7.1.17. Theorem. (van Rooij and Wilf [1965]) For a simple graph G, there is a solution to $L(H) = G$ if and only if G is claw-free and no double triangle of G has two odd triangles.

Proof: *Necessity.* Suppose that $G = L(H)$. A vertex e in G with neighbors x, y, z corresponds to an edge e in H incident to edges x, y, z. Since e has only two endpoints in H, two of x, y, z are incident at one of them and hence are adjacent in G. This forbids the claw as an induced subgraph of G.

For the other condition, we saw in Example 7.1.15 that the vertices of a double triangle in G must correspond to the edges of a paw in H. In particular, the vertices of one of these triangles in G correspond to the edges of a triangle in H. This triangle must be even, because every edge in H incident to exactly one vertex of a triangle shares an endpoint with exactly two of its edges. Hence for each double triangle in G, at least one of its triangles is even.

Sufficiency. Suppose that G satisfies the specified conditions. We may assume that G is connected; otherwise, we apply the construction to each component. The case where G is claw-free and has a double triangle with both triangles even is very special; there are only three such graphs (Exercise 38). Here we consider only the general case, in which every double triangle of G has exactly one odd triangle.

By Theorem 7.1.16, it suffices to decompose G into complete subgraphs, using each vertex in at most two of them. Let $S_1, \ldots, S_k$ be the maximal complete subgraphs of G that are not even triangles, and let $T_1, \ldots, T_l$ be the edges that belong to one even triangle and no odd triangle. We claim that together these form the desired decomposition $\mathbf{B}$.



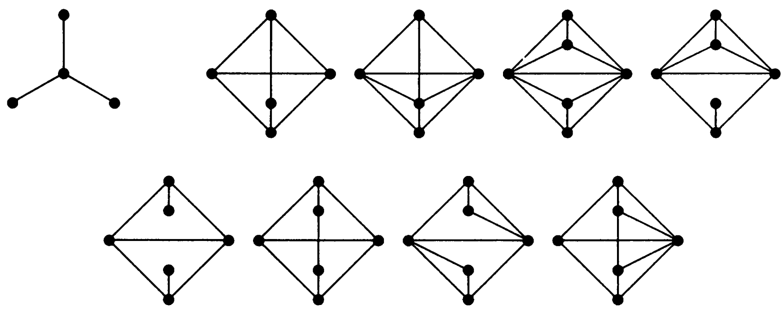
Every edge appears in a maximal complete subgraph, but every triangle in a complete subgraph with more than three vertices is odd. Hence each edge T_j in the list is not in any S_i. Also S_i and $S_{i'}$ share no edge, because G has no double triangles with both triangles odd. Hence the subgraphs in $\mathbf{B}$ are pairwise edge-disjoint.

If $e \in E(G)$, then e is in some S_i unless the only maximal clique containing e is an even triangle. In this case e is a T_j, since we have forbidden double triangles with both triangles even. Hence $\mathbf{B}$ is a decomposition.

It remains to show that each $v \in G$ appears in at most two of these subgraphs. Suppose that v belongs to $A, B, C \in \mathbf{B}$. Edge-disjointness implies that v has neighbors x, y, z with each belonging to only one of $\{A, B, C\}$. Since G has no induced claw, we may assume that $x \leftrightarrow y$. By edge-disjointness, the triangle vxy cannot belong to a member of **B**. Hence it must be an even triangle. Therefore, z must have exactly one other edge to vxy, say $z \leftrightarrow x$ and $z \not\leftrightarrow y$. But now the same argument shows zvx is an even triangle, and we have a double triangle with both triangles even. ■

Theorem 7.1.17 is close to a forbidden subgraph characterization.

7.1.18. Theorem. (Beineke [1968]) A simple graph G is the line graph of some simple graph if and only if G does not have any of the nine graphs below as an induced subgraph.



Proof: By Theorem 7.1.17, it suffices to show that the eight graphs listed other than $K_{1,3}$ are the vertex-minimal claw-free graphs containing a double triangle with both triangles odd. Each such graph has a double triangle and one or two additional vertices that make the triangles odd by having one or three neighbors in the triangles. The details of showing that this is the full list are requested in Exercise 40. ■

The characterizations in Theorems 7.1.17–7.1.18 yield algorithms to test whether G is a line graph that run in time polynomial in $n(G)$. In fact, there is such an algorithm that runs in linear time (Lehot [1974]) and produces a graph H such that $G = L(H)$ when G is a line graph. This graph H is unique if G has no component that is a triangle (Exercise 39).

EXERCISES

7.1.1. (−) For each graph G below, compute $\chi'(G)$ and draw $L(G)$.

7.1.2. (−) Give an explicit edge-coloring to prove that $\chi'(Q_k) = \Delta(Q_k)$

7.1.3. (−) Determine the edge-chromatic number of $C_n \square K_2$.

7.1.4. (−) Obtain an inequality for $\chi'(G)$ in terms of $e(G)$ and $\alpha'(G)$.

7.1.5. (−) Prove that the Petersen graph is the complement of $L(K_5)$.

7.1.6. (−) Determine the number of triangles in the line graph of the Petersen graph.

7.1.7. (−) Determine whether $\overline{P}_5$ is a line graph. If so, find H such that $L(H) = \overline{P}_5$.

7.1.8. (−) Prove that $L(K_{m,n}) \cong K_m \square K_n$.

• • • • •

7.1.9. Let G be a simple graph. Prove that vertices form a clique in $L(G)$ if and only if the corresponding edges in G have one common endpoint or form a triangle. (Comment: Thus $\omega(L(G)) = \Delta(G)$ unless $\Delta(G) = 2$ and some component of G is a triangle.)

7.1.10. Let G be a simple graph without isolated vertices. Prove that if $L(G)$ is connected and regular, then either G is regular or G is a bipartite graph in which vertices of the same partite set have the same degree. (Ray-Chaudhuri [1967])

7.1.11. (!) Let G be a simple graph.
a) Prove that the number of edges in $L(G)$ is $\sum_{v \in V(G)} \binom{d(v)}{2}$.
b) Prove that G is isomorphic to $L(G)$ if and only if G is 2-regular.

7.1.12. Let G be a connected simple graph. Use part (a) of Exercise 7.1.11 to determine when $e(L(G)) < e(G)$.

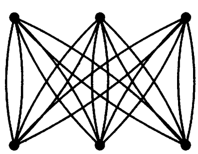
7.1.13. (+) Prove that the graph below is the only simple graph whose line graph is isomorphic to its complement. (Albertson)

7.1.14. (!) Let G be a k-edge-connected simple graph. Prove that $L(G)$ is k-connected and is $2k-2$-edge-connected. (Hint: For a minimum edge cut $[S, \overline{S}]$ in $L(G)$, describe what the cut corresponds to in G and count its edges in terms of the vertices of G.)

7.1.15. (!) Use Tutte's 1-factor Theorem to prove that every connected line graph of even order has a perfect matching. Conclude from this that the edges of a simple connected graph of even size can be partitioned into paths of length 2. (Comment: Exercise 3.3.22 shows that every connected claw-free graph has a perfect matching , but that stronger result is more difficult than this.) (Chartrand–Polimeni–Stewart [1973])

7.1.16. (∗) Let G be a simple graph. Prove that $\gamma(L(G)) \geq \gamma(G)$, where $\gamma(G)$ denotes the genus of G (Definition 6.3.20). (D. Greenwell)

7.1.17. Compute the number of proper 6-edge-colorings of the graph below.



7.1.18. (!) Give an explicit edge-coloring to prove that $\chi'(K_{r,s}) = \Delta(K_{r,s})$.

7.1.19. (!) Prove that for every simple bipartite graph G, there is a $\Delta(G)$-regular simple bipartite graph H that contains G.

7.1.20. (!) Let D be a digraph (loops allowed) such that $d^+(v) \le d$ and $d^-(v) \le d$ for all $v \in V(D)$. Prove that $E(D)$ can be colored using at most d colors so that the edges entering each vertex have distinct colors and the edges exiting each vertex have distinct colors. (Hint: Transform the digraph into another object where a known result applies.)

7.1.21. *Algorithmic proof of Theorem 7.1.7.* Let G be a bipartite graph with maximum degree k. Let f be a proper k-edge-coloring of a subgraph H of G. Let uv be an edge not in H. By using a path alternating in two colors, show that f can be altered and then extended to a proper k-edge-coloring of $H + uv$. Conclude that $\chi'(G) = \Delta(G)$.

7.1.22. Use Brooks' Theorem to an appropriate graph to prove that if G is a simple graph with $\Delta(G) = 3$, then G is 4-edge-colorable. (Comment: The result is a special case of Vizing's Theorem; do not use Vizing's Theorem to prove this.)

7.1.23. (+) Let $K(p, q)$ be the complete p-partite graph with q vertices in each partite set. Let $G[H]$ denote the composition operation, in which each vertex of G expands into a copy of H. Note that $K(p, q) = K(p, d)[\overline{K}_{q/d}]$ when d divides q.

a) Show that if G has a decomposition into copies of F, then $G[\overline{K}_m]$ has a decomposition into copies of $F[\overline{K}_m]$. Show also that the relation "G decomposes into spanning copies of F" is transitive.

b) Cliques of even order decompose into 1-factors. Cliques of odd order decompose into spanning cycles. Use these statements and part (a) to prove that $K(p, q)$ decomposes into 1-factors when pq is even. (Hartman [1997])

7.1.24. (!) Let G and H be nontrivial simple graphs. Use Vizing's Theorem to prove that $\chi'(H) = \Delta(H)$ implies $\chi'(G \square H) = \Delta(G \square H)$.

7.1.25. *Kotzig's Theorem for cartesian products of simple graphs.*

a) Use Vizing's Theorem to prove that $\chi'(G \square K_2) = \Delta(G \square K_2)$.

b) Let G_1, G_2 be edge-disjoint graphs with vertex set V, and let H_1, H_2 be edge-disjoint graphs with vertex set W. Prove that $(G_1 \cup G_2) \square (H_1 \cup H_2) = (G_1 \square H_2) \cup (G_2 \square H_1)$.

c) Use parts (a) and (b) to prove that $\chi'(G \square H) = \Delta(G \square H)$ if both G and H have 1-factors. (Comment: As a result, the product of the Petersen graph with itself is Class 1, which does not follow from Exercise 7.1.24. Here neither factor need be Class 1; there G need not have a 1-factor.) (Kotzig [1979], J. George [1991])

7.1.26. (!) Let G be a regular graph with a cut-vertex. Prove that $\chi'(G) > \Delta(G)$.

7.1.27. *Density conditions for $\chi'(G) > \Delta(G)$.*

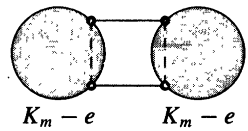
a) Prove that if $n(G) = 2m + 1$ and $e(G) > m \cdot \Delta(G)$, then $\chi'(G) > \Delta(G)$.

b) Prove that if G is obtained from a k-regular graph with $2m+1$ vertices by deleting fewer than $k/2$ edges, then $\chi'(G) > \Delta(G)$.

c) Prove that if G is obtained by subdividing an edge of a regular graph with $2m$ vertices and degree at least 2, then $\chi'(G) > \Delta(G)$.

7.1.28. ($*-$) Prove that the Petersen graph has no overfull subgraph.

7.1.29. Let G be the $m - 1$-regular connected graph formed from $2K_m$ by deleting an edge from each component and adding two edges between the components to restore regularity. Prove that G is not 1-factorable if m is odd and greater than 3. (Comment: This shows that the 1-factorization Conjecture (Remark 7.1.12) is sharp.)



7.1.30. (*!) *Overfull Conjecture* $\Rightarrow$ *1-factorization Conjecture* (Remark 7.1.12).

a) Prove that in a regular graph of even order, an induced subgraph is overfull if and only if the subgraph induced by the other vertices is overfull.

b) Let G be an k-regular graph of order $2m$ having an overfull subgraph. Prove that $k < m$ if m is odd and that $k < m - 1$ if m is even.

7.1.31. Given an edge-coloring of a graph G, let $c(v)$ denote the number of distinct colors appearing on edges incident to v. Among all k-edge-colorings of G, a coloring is **optimal** if it maximizes $\sum_{v \in V(G)} c(v)$.

a) Prove that if no component is an odd cycle, then G has a 2-edge-coloring where both colors appear at each vertex of degree at least 2. (Hint: Use Eulerian circuits.)

b) Let f be an optimal k-edge-coloring of G in which color a appears at least twice at $u \in V(G)$ and color b does not appear at u. Let H be the subgraph of G consisting of edges colored a or b. Prove that the component of H containing u is an odd cycle.

c) Let G be a bipartite graph. Conclude from part (b) that G is $\Delta(G)$-edge-colorable. (Comment: These ideas also lead to a proof of Vizing's Theorem.) (Fournier [1973])

7.1.32. Let G be a bipartite graph with minimum degree k. Prove that G has a k-edge-coloring in which at each vertex v, each color appears $\lceil d(v)/k \rceil$ or $\lfloor d(v)/k \rfloor$ times. (Hint: Use a graph transformation.) (Gupta [1966])

7.1.33. Use Vizing's Theorem to prove that every simple graph with maximum degree Δ has an "equitable" $\Delta + 1$-edge-coloring: a proper edge-coloring with each color used $\lceil e(G)/(\Delta + 1) \rceil$ or $\lfloor e(G)/(\Delta + 1) \rfloor$ times. (de Werra [1971], McDiarmid [1972])

7.1.34. Use Petersen's Theorem (every $2k$-regular graph has a 2-factor—Theorem 3.3.9) to prove that $\chi'(G) \le 3 \lceil \Delta(G)/2 \rceil$ when G is a loopless graph.

7.1.35. *Bounds on* $\chi'(G)$. Let $\mathbf{P} = \{x, y, z \in V(G)\colon y \in N(x) \cap N(z)$. Prove that the last bound below (Andersen [1977], Goldberg [1977, 1984]) implies the earlier bounds.

$\chi'(G) \le \lfloor 3\Delta(G)/2 \rfloor$. (Shannon [1949])

$\chi'(G) \le \Delta(G) + \mu(G)$. (Vizing [1964, 1965], Gupta [1966])

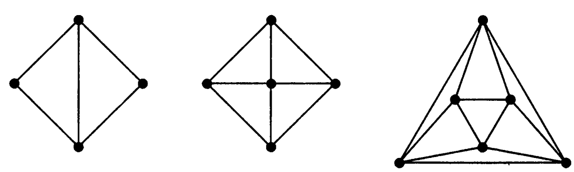
$\chi'(G) \le \max\{\Delta(G), \max_{\mathbf{P}} \left\lfloor \frac{1}{2}(d(x) + d(y) + d(z)) \right\rfloor\}$. (Ore [1967a])

$\chi'(G) \le \max\{\Delta(G), \max_{\mathbf{P}} \left\lfloor \frac{1}{2}(d(x) + \mu(xy) + \mu(yz) + d(z)) \right\rfloor\}$.

7.1.36. (+) For $n \ne 8$, prove that $L(K_n)$ is the only $2n - 4$-regular simple graph of order $\binom{n}{2}$ in which nonadjacent vertices have four common neighbors and adjacent vertices have $n - 2$ common neighbors. (Comment: When $n = 8$, three exceptional graphs satisfy the conditions.) (Chang [1959], Hoffman [1960])

7.1.37. (+) For n, m not both equalling 4, prove that $L(K_{m,n})$ is the only $(n+m-2)$-regular simple graph of order mn in which nonadjacent vertices have two common neighbors, $n\binom{m}{2}$ pairs of adjacent vertices have $m - 2$ common neighbors, and $m\binom{n}{2}$ pairs of adjacent vertices have $n - 2$ common neighbors. (Comment: When $n = m = 4$, there one exceptional graph—Shrikande [1959].) (Moon [1963], Hoffman [1964])

7.1.38. (*) Let G be a connected, simple, claw-free graph having a double triangle H with each triangle even. Prove that G is one of the three graphs below, and conclude that G is a line graph. (Comment: This completes the proof of Theorem 7.1.17.)



7.1.39. (*) A **Krausz decomposition** of a simple graph H is a partition of $E(H)$ into cliques such that each vertex of H appears in at most two of the cliques.

a) Prove that for a connected simple graph H, two Krausz decompositions of H that have a common clique are identical.

b) Find distinct Krausz decompositions for the graphs in Exercise 7.1.38.

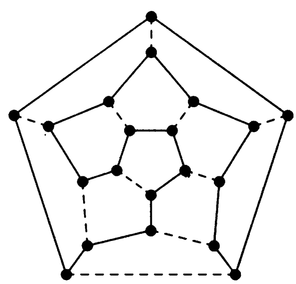
c) Prove that no other connected simple graph except K_3 has two distinct Krausz decompositions (use Exercise 7.1.38 and the proof of Theorem 7.1.17).

d) Conclude that $K_{1,3}$, K_3 is the only pair of nonisomorphic connected simple graphs with isomorphic line graphs. (Whitney [1932a])

7.1.40. (*) Complete the proof of Theorem 7.1.18 by proving that a simple graph with no induced claw has a double triangle with both triangles odd if and only if it contains an induced subgraph among the other eight graphs listed in the theorem statement.

7.2. Hamiltonian Cycles

Studied first by Kirkman [1856], Hamiltonian cycles are named for Sir William Hamilton, who described a game on the graph of the dodecahedron in which one player specifies a 5-vertex path and the other must extend it to a spanning cycle. The game was marketed as the "Traveller's Dodecahedron", a wooden version in which the vertices were named for 20 important cities.



7.2.1. Definition. A **Hamiltonian graph** is a graph with a spanning cycle, also called a **Hamiltonian cycle**.

Until the 1970s, interest in Hamiltonian cycles centered on their relationship to the Four Color Problem (Section 7.3). Later study was stimulated by practical applications and by the issue of complexity (Appendix B).

No easily testable characterization is known for Hamiltonian graphs; we will study necessary conditions and sufficient conditions. Loops and multiple edges are irrelevant; a graph is Hamiltonian if and only if the simple graph obtained by keeping one copy of each non-loop edge is Hamiltonian. Therefore, **in this section we restrict our attention to simple graphs**; this is relevant when discussing conditions involving vertex degrees.

For further material on Hamiltonian cycles, see Chvátal [1985a].

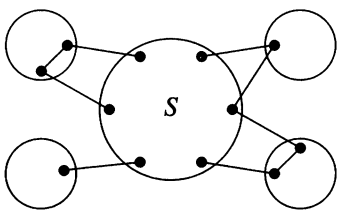
NECESSARY CONDITIONS

Every Hamiltonian graph is 2-connected, because deleting a vertex leaves a subgraph with a spanning path. Bipartite graphs suggest a way to strengthen this necessary condition.

7.2.2. Example. *Bipartite graphs.* A spanning cycle in a bipartite graph visits the two partite sets alternately, so there can be no such cycle unless the partite sets have the same size. Hence $K_{m,n}$ is Hamiltonian only if $m = n$. Alternatively, we can argue that the cycle returns to different vertices of one partite set after each visit to the other partite set. ∎

7.2.3. Proposition. If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, the graph $G - S$ has at most $|S|$ components.

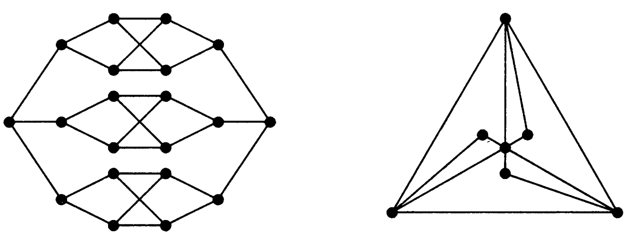
Proof: When leaving a component of $G - S$, a Hamiltonian cycle can go only to S, and the arrivals in S must use distinct vertices of S. Hence S must have at least as many vertices as $G - S$ has components. ∎



7.2.4. Definition. Let $c(H)$ denote the number of components of a graph H.

Thus the necessary condition is that $c(G - S) \le |S|$ for all $\emptyset \ne S \subseteq V$. This condition guarantees that G is 2-connected (deleting one vertex leaves at most one component), but it does not guarantee a Hamiltonian cycle.

7.2.5. Example. The graph on the left below is bipartite with partite sets of equal size. However, it fails the necessary condition of Proposition 7.2.3. Hence it is not Hamiltonian.



The graph on the right shows that the necessary condition is not sufficient. This graph satisfies the condition but has no spanning cycle. All edges incident to vertices of degree 2 must be used, but in this graph that requires three edges incident to the central vertex.

The Petersen graph is another non-Hamiltonian graph satisfying the condition. We proved in Example 7.1.9 that $2C_5$ is the only 2-factor of the Petersen graph, so it has no spanning cycle. ■

7.2.6.* Remark. Strengthening a necessary condition may yield a sufficient condition. Perhaps requiring $|S| \geq 2c(G-S)$ for every cutset S would guarantee a spanning cycle. A graph G is ***t*-tough** if $|S| \geq tc(G-S)$ for every cutset $S \subset V$. The **toughness** of G is the maximum t such that G is t-tough. For example, the toughness of the Petersen graph is 4/3 (Exercise 23).

By Proposition 7.2.3, spanning cycles require toughness at least 1. Chvátal [1974] conjectured that a sufficiently large toughness is sufficient. No value of toughness larger than 1 is necessary, since C_n itself is only 1-tough. For some years it was thought that toughness 2 would be sufficient. Enomoto–Jackson–Katerinis–Saito [1985] constructed non-Hamiltonian graphs with toughness $2-\epsilon$ for each $\epsilon > 0$. Finally, Bauer–Broersma–Veldman [2000] constructed non-Hamiltonian graphs with toughness approaching 9/4. Chvátal's conjecture that some value of toughness suffices remains open. ■

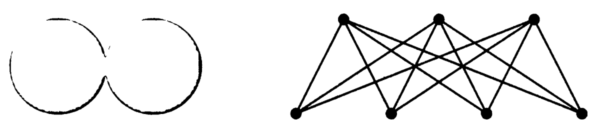
SUFFICIENT CONDITIONS

The number of edges needed to force an n-vertex graph to be Hamiltonian is quite large (Exercises 26–27). Under conditions that "spread out" the edges, we can reduce the number of edges while still guaranteeing Hamiltonian cycles. The simplest such condition is a lower bound on the minimum degree; $\delta(G) \geq n(G)/2$ suffices. We first note that no smaller minimum degree is sufficient.

7.2.7. Example. The graph consisting of cliques of orders $\lfloor (n+1)/2 \rfloor$ and $\lceil (n+1)/2 \rceil$ sharing a vertex has minimum degree $\lfloor (n-1)/2 \rfloor$ but is not Hamiltonian (not even 2-connected).

For odd order, another non-Hamiltonian graph with this minimum degree is the biclique with partite sets of sizes $(n-1)/2$ and $(n+1)/2$.

Proving that $\delta(G) \geq n(G)/2$ forces a spanning cycle thus shows that $\lfloor (n-1)/2 \rfloor$ is the largest value of the minimum degree among non-Hamiltonian graphs with n vertices. ■



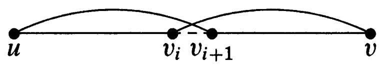
7.2.8. Theorem. (Dirac [1952b]). If G is a simple graph with at least three vertices and $\delta(G) \geq n(G)/2$, then G is Hamiltonian.

Proof: The condition $n(G) \geq 3$ is annoying but must be included, since K_2 is not Hamiltonian but satisfies $\delta(K_2) = n(K_2)/2$.

The proof uses contradiction and extremality. If there is a non-Hamiltonian graph satisfying the hypotheses, then adding edges cannot reduce the minimum degree. Thus we may restrict our attention to maximal non-Hamiltonian graphs with minimum degree at least $n/2$, where "maximal" means that adding any edge joining nonadjacent vertices creates a spanning cycle.

When $u \not\leftrightarrow v$ in G, the maximality of G implies that G has a spanning path $v_1, \ldots, v_n$ from $u = v_1$ to $v = v_n$, because every spanning cycle in $G+uv$ contains the new edge uv. To prove the theorem, it suffices to make a small change in this cycle to avoid using the edge uv; this will build a spanning cycle in G.

If a neighbor of u directly follows a neighbor of v on the path, such as $u \leftrightarrow v_{i+1}$ and $v \leftrightarrow v_i$, then $(u, v_{i+1}, v_{i+2}, \ldots, v, v_i, v_{i-1}, \ldots, v_2)$ is a spanning cycle.



To prove that such a cycle exists, we show that there is a common index in the sets S and T defined by $S = \{i\colon u \leftrightarrow v_{i+1}\}$ and $T = \{i\colon v \leftrightarrow v_i\}$. Summing the sizes of these sets yields

$$|S \cup T| + |S \cap T| = |S| + |T| = d(u) + d(v) \geq n.$$

Neither S nor T contains the index n. Thus $|S \cup T| < n$, and hence $|S \cap T| \geq 1$. We have established a contradiction by finding a spanning cycle in G; hence there is no (maximal) non-Hamiltonian graph satisfying the hypotheses. ■

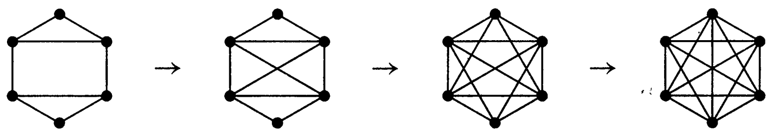
Ore observed that this argument uses $\delta(G) \geq n(G)/2$ only to show that $d(u)+d(v) \geq n$. Therefore, we can weaken the requirement of minimum degree $n/2$ to require only that $d(u) + d(v) \geq n$ whenever $u \not\leftrightarrow v$. We also did not need that G was a maximal non-Hamiltonian graph, only that $G + uv$ was Hamiltonian and thereby provided a spanning u, v-path.

7.2.9. Lemma. (Ore [1960]) Let G be a simple graph. If u, v are distinct nonadjacent vertices of G with $d(u) + d(v) \geq n(G)$, then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof: One direction is trivial, and the proof of the other direction is the same as for Theorem 7.2.8. ■

Bondy and Chvátal [1976] phrased the essence of Ore's argument in a much more general form that yields sufficient conditions for cycles of length l and other subgraphs. Here we discuss only the application to spanning cycles. Using Lemma 7.2.9 to add edges, we can test whether G is Hamiltonian by testing whether the larger graph is Hamiltonian.

7.2.10. Definition. The **(Hamiltonian) closure** of a graph G, denoted $C(G)$, is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains.



The graph above begins with vertices of degree 2, but its closure is K_6. Ore's Lemma yields the following theorem.

7.2.11. Theorem. (Bondy–Chvátal [1976]) A simple n-vertex graph is Hamiltonian if and only if its closure is Hamiltonian. ■

Fortunately, the closure does not depend on the order in which we choose to add edges when more than one is available.

7.2.12. Lemma. The closure of G is well-defined.

Proof: Let $e_1, \ldots, e_r$ and $f_1, \ldots, f_s$ be sequences of edges added in forming $C(G)$, the first yielding G_1 and the second G_2. If in either sequence nonadjacent vertices u and v acquire degree summing to at least $n(G)$, then the edge uv must be added before the sequence ends.

Thus f_1, being initially addable to G, must belong to G_1. Similarly, if $f_1, \ldots, f_{i-1} \in E(G_1)$, then f_i becomes addable to G_1 and therefore belongs to G_1. Hence neither sequence contains a first edge omitted by the other sequence, and we have $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$. ■

We now have a necessary and sufficient condition to test for Hamiltonian cycles in simple graphs. It doesn't help much, because it requires us to test whether another graph is Hamiltonian! Nevertheless, it does furnish a method for proving sufficient conditions. A condition that forces $C(G)$ to be Hamiltonian also forces a Hamiltonian cycle in G.

For example, the condition may imply $C(G) = K_n$. Chvátal used this method to prove the best possible degree sequence condition for Hamiltonian cycles. Some vertex degrees can be small if others are large enough.

7.2.13. Theorem. (Chvátal [1972]) Let G be a simple graph with vertex degrees $d_1 \le \cdots \le d_n$, where $n \ge 3$. If $i < n/2$ implies that $d_i > i$ or $d_{n-i} \ge n-i$ (**Chvátal's condition**), then G is Hamiltonian.

Proof: Adding edges to form the closure reduces no entry in the degree sequence. Also, G is Hamiltonian if and only if $C(G)$ is Hamiltonian. Thus it suffices to consider the case where $C(G) = G$, which we describe by saying that G is *closed*. In this case, we prove that Chvátal's condition implies that $G = K_n$.

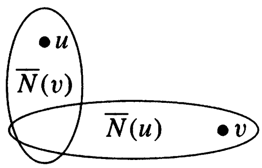
We prove the contrapositive; if G is a closed n-vertex graph that is not a complete graph, then we construct a value of i less than $n/2$ for which Chvátal's condition is violated. Violation means that at least i vertices have degree at most i and at least $n-i$ vertices have degree less than $n-i$.

With $G \neq K_n$, we choose among the pairs of nonadjacent vertices a pair u, v with maximum degree sum. Because G is closed, $u \not\leftrightarrow v$ implies that $d(u) + d(v) < n$. We choose the labels on u, v so that $d(u) \le d(v)$. Since $d(u) + d(v) < n$, we thus have $d(u) < n/2$. Let $i = d(u)$.

We need to find i vertices with degree at most i. Because we chose a nonadjacent pair with maximum degree sum, every vertex of $V - \{v\}$ that is not adjacent to v has degree at most $d(u)$, which equals i. There are $n - 1 - d(v)$ such vertices, and $d(u) + d(v) \le n - 1$ yields $n - 1 - d(v) \ge i$.

We also need $n - i$ vertices with degree less than $n - i$. Every vertex of $V - \{u\}$ that is not adjacent to u has degree at most $d(v)$, and we have $d(v) < n - d(u) = n - i$. There are $n - 1 - d(u)$ such vertices. Since $d(u) \le d(v)$, we can also add u itself to the set of vertices with degree at most $d(v)$. We thus obtain $n - i$ vertices with degree less than $n - i$.

We have proved that $d_i \le i$ and $d_{n-i} < n - i$ for this specially chosen i, which contradicts the hypothesis. ■

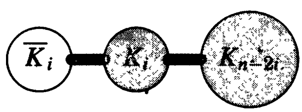


7.2.14. Example. *Non-Hamiltonian graphs with "large" vertex degrees.* Theorem 7.2.13 characterizes the degree sequences of simple graphs that force Hamiltonian cycles. If the degree sequence fails Chvátal's condition at i, then the largest we can make the terms in $d_1, \ldots, d_n$ is

$$\begin{array}{ll} d_j = i & \text{for } j \le i, \\ d_j = n - i - 1 & \text{for } i + 1 \le j \le n - i, \\ d_j = n - 1 & \text{for } j > n - i. \end{array}$$

Let G be a simple graph realizing this degree sequence (if it exists). The i vertices of degree $n - 1$ are adjacent to all others (the central clique in the figure). This already gives i neighbors to the i vertices of degree i, so they form an independent set and have no additional neighbors. With degree $n - i - 1$, each of the remaining $n - 2i$ vertices must be adjacent to all vertices except itself and the independent set. Thus these vertices form a clique. The only possible realization is $(\overline{K}_i + K_{n-2i}) \vee K_i$, shown below.

This graph is not Hamiltonian, because deleting the i vertices of degree $n - 1$ leaves a subgraph with $i + 1$ components. If a simple graph H is non-Hamiltonian and has vertex degrees $d'_1 \le \cdots \le d'_n$, then Chvátal's result implies that for some i the graph $(\overline{K}_i + K_{n-2i}) \vee K_i$ with vertex degrees $d_1 \le \cdots \le d_n$ satisfies $d_j \ge d'_j$ for all i. ■



7.2.15. Definition. A **Hamiltonian path** is a spanning path.

Every graph with a spanning cycle has a spanning path, but P_n shows that the converse is not true. We could make arguments like those above to prove sufficient conditions for Hamiltonian paths, but it is easier to use our previous work and prove the new theorem by invoking a theorem about cycles. To do this, we use a standard transformation.

7.2.16. Remark. A graph G has a spanning path if and only if the graph $G \vee K_1$ has a spanning cycle. ■

Remark 7.2.16 applies in several of the exercises. Here we use it to derive the analogue for paths of Chvátal's condition for spanning cycles.

7.2.17. Theorem. Let G be a simple graph with vertex degrees $d_1 \le \cdots \le d_n$. If $i < (n+1)/2$ implies ($d_i \ge i$ or $d_{n+1-i} \ge n-i$), then G has a spanning path.

Proof: Let $G' = G \vee K_1$, let $n' = n+1$, and let $d'_1, \ldots, d'_{n'}$ be the degree sequence of G'. Since a spanning cycle in $G \vee K_1$ becomes a spanning path in G when the extra vertex is deleted, it suffices to show that G' satisfies Chvátal's sufficient condition for Hamiltonian cycles.

Since the new vertex is adjacent to all of $V(G)$, we have $d'_{n'} = n$ and $d'_j = d_j + 1$ for $j < n'$. For $i < n'/2 = (n+1)/2$, the hypothesis on G yields

$$d'_i = d_i + 1 \ge i + 1 > i \quad \text{or} \quad d'_{n'-i} = d_{n+1-i} + 1 \ge n - i + 1 = n' - i.$$

This is precisely Chvátal's sufficient condition, so G' has a spanning cycle, and deleting the extra vertex leaves a spanning path in G. ■

7.2.18.* Remark. The degree requirements can be weakened under conditions such as regularity or high toughness. Every regular simple graph G with vertex degrees at least $n(G)/3$ is Hamiltonian (Jackson [1980]). Only the Petersen graph prevents lowering the threshold to $(n(G) - 1)/3$ (Zhu–Liu–Yu [1985], partly simplified in Bondy–Kouider [1988]; see also Exercise 13).

It may be possible to lower the degree condition further when connectivity is high. For example, Tutte [1971] conjectured that every 3-connected 3-regular bipartite graph is Hamiltonian. Horton [1982] found a counterexample with 96 vertices, and the smallest known counterexample has 50 vertices (Georges [1989]), but stronger conditions of this sort may suffice. ■

Our last sufficient condition for Hamiltonian cycles involves connectivity and independence, not degrees. The proof yields a good algorithm that constructs a Hamiltonian cycle or shows that the hypothesis is false.

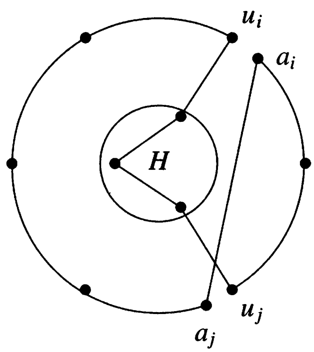
7.2.19. Theorem. (Chvátal–Erdős [1972]) If $\kappa(G) \ge \alpha(G)$, then G has a Hamiltonian cycle (unless $G = K_2$).

Proof: With $G \ne K_2$, the conditions require $\kappa(G) > 1$. Suppose that $\kappa(G) \ge \alpha(G)$. Let $k = \kappa(G)$, and let C be a longest cycle in G. Since $\delta(G) \ge \kappa(G)$, and every graph with $\delta(G) \geq 2$ has a cycle of length at least $\delta(G) + 1$ (Proposition 1.2.28), C has at least $k + 1$ vertices.

Let H be a component of $G - V(C)$. The cycle C has at least k vertices with edges to H; otherwise, deleting the vertices of C with edges to H contradicts $\kappa(G) = k$. Let $u_1, \ldots, u_k$ be k vertices of C with edges to H, in clockwise order.

For $i = 1, \ldots, k$, let a_i be the vertex immediately following u_i on C. If any two of these vertices are adjacent, say $a_i \leftrightarrow a_j$, then we construct a longer cycle by using $a_i a_j$, the portions of C from a_i to u_j and a_j to u_i, and a u_i, u_j-path through H (see illustration).

If a_i has a neighbor in H, then we can detour to H between u_i and a_i on C. Thus we also conclude that no a_i has a neighbor in H. Hence $\{a_1, \ldots, a_k\}$ plus a vertex of H forms an independent set of size $k + 1$. This contradiction implies that C is a Hamiltonian cycle. ■



7.2.20.* Remark. Most sufficient conditions for Hamiltonian cycles generalize to conditions for long cycles. The **circumference** of a graph is the length of its longest cycle. A weaker form of a sufficient condition for spanning cycles may force a long cycle. Dirac [1952b] proved the first such result: a 2-connected graph with minimum degree k has circumference at least $\min\{n, 2k\}$. Proposition 1.2.28 only guarantees a cycle of length at least $k + 1$. Most long-cycle results are more more difficult than the corresponding sufficient conditions for Hamiltonian cycles (see Lemma 8.4.36–Theorem 8.4.37). ■

CYCLES IN DIRECTED GRAPHS (optional)

The theory of cycles in digraphs is similar to that of cycles in graphs. For a digraph G, let $\delta^-(G) = \min d^-(v)$ and $\delta^+(G) = \min d^+(v)$. The arguments of Chapter 1 using maximal paths guarantee paths of length k and cycles of length $k + 1$, where $k = \max\{\delta^-(G), \delta^+(G)\}$.

Every complete graph is Hamiltonian, but orientations of complete graphs are more complicated. The necessary condition of 2-connectedness becomes a necessary condition of strong connectedness for spanning cycles in digraphs. For tournaments, this necessary condition is also sufficient (Exercise 45).

For arbitrary digraphs, we prove an analogue of Dirac's theorem (Theorem 7.2.8). Indeed, it yields Dirac's theorem as a special case (Exercise 49). Meyniel [1973] subtantially strengthened the theorem by weakening the hypothesis (Theorem 8.4.42).

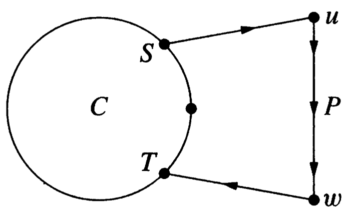
7.2.21. Definition. A digraph is **strict** if it has no loops and has at most one copy of each ordered pair as an edge.

7.2.22. Theorem. (Ghouilà-Houri [1960]) If D is a strict digraph, and $\min\{\delta^+(D), \delta^-(D)\} \geq n(D)/2$, then D is Hamiltonian.

Proof: Again we use contradiction and extremality. In an n-vertex counterexample D, let C be a longest cycle, with length l. As we have observed, $l > \max\{\delta^+, \delta^-\} \geq n/2$. Let P be a longest path in $D - V(C)$, beginning at u, ending at w, and having length $m \geq 0$. Now $l > n/2$ and $n \geq l + m + 1$ imply $m < n/2$.

Let S be the set of predecessors of u on C, and let T be the set of successors of w on C. By the maximality of P, every predecessor of u and successor of w lies in $V(C) \cup V(P)$. Thus S and T each have size at least $\min\{\delta^+, \delta^-\} - m$, which is at least $\geq n/2 - m$ and hence is positive. Thus S and T are nonempty.

The maximality of C guarantees that the distance along C from a vertex $u' \in S$ to a vertex $w' \in T$ must exceed $m + 1$. Otherwise, traveling along P instead of C from u' to w' yields a longer cycle. Hence we may assume that every vertex of S is followed on C by more than m vertices not in T.



If the distance between successive vertices of S along C is always at most $m + 1$, then there is no legal place to put a vertex of T. Since both S and T are nonempty, we may thus assume there is a vertex of S followed on C by at least $m + 1$ vertices not in S. These are forbidden from T, as is the immediate successor on C of all the other vertices of S.

Thus at least $|S| - 1 + m + 1 \geq n/2$ vertices of C are not in T. Together withthe vertices that are in T, this yields $|V(C)| \geq n - m$, which contradicts $l \leq n - m - 1$. The contradiction implies that C must be a spanning cycle. ■

EXERCISES

7.2.1. (−) For which values of r is $K_{r,r}$ Hamiltonian?

7.2.2. (−) Is the Grötzsch graph (Example 5.2.2) Hamiltonian?

7.2.3. (−) For $n > 1$, prove that $K_{n,n}$ has $(n-1)!n!/2$ Hamiltonian cycles.

7.2.4. (−) Prove that G has a Hamiltonian path only if for every $S \subseteq V(G)$, the number of components of $G - S$ is at most $|S| + 1$.

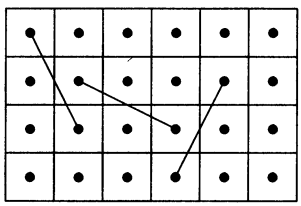
• • • • •

7.2.5. Prove that every 5-vertex path in the dodecahedron lies in a Hamiltonian cycle.

7.2.6. (!) Let G be a Hamiltonian bipartite graph, and choose $x, y \in V(G)$. Prove that $G - x - y$ has a perfect matching if and only if x and y are on opposite sides of the bipartition of G. Apply this to prove that deleting two unit squares from an 8 by 8 chessboard leaves a board that can be partitioned into 1 by 2 rectangles if and only if the two missing squares have opposite colors.

7.2.7. A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last? Give a method or prove impossible. (Ignore gravity.)

7.2.8. (!) On a chessboard, a **knight** can move from one square to another that differs by 1 in one coordinate and by 2 in the other coordinate, as shown below. Prove that no $4 \times n$ chessboard has a **knight's tour**: a traversal by knight's moves that visits each square once and returns to the start. (Hint: Find an appropriate set of vertices in the corresponding graph to violate the necessary condition.)



7.2.9. Construct an infinite family of non-Hamiltonian graphs satisfying the necessary condition of Proposition 7.2.3.

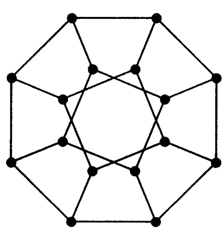
7.2.10. (!) *Hamiltonian vs. Eulerian.*

a) Find a 2-connected non–Eulerian graph whose line graph is Hamiltonian.

b) Prove that $L(G)$ is Hamiltonian if and only if G has a closed trail that contains at least one endpoint of each edge. (Harary and Nash-Williams [1965])

7.2.11. Construct a 3-regular 3-connected graph whose line graph is not Hamiltonian. (Hint: Replace each vertex in the Petersen graph with an appropriate graph and apply Exercise 7.2.10.)

7.2.12. Determine whether the graph below is Hamiltonian.

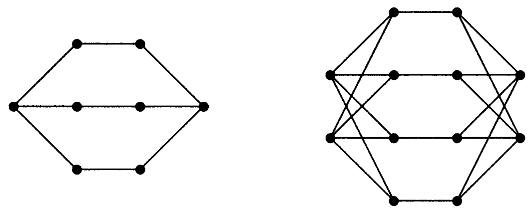


7.2.13. Let G be the 3-regular graph obtained from the Petersen graph by replacing one vertex with a triangle, matching the vertices of the triangle to the former neighbors of the deleted vertex. Prove that G is not Hamiltonian. (Comment: Except for this graph and the Petersen graph, every 2-connected, k-regular graph with at most $3k+3$ vertices is Hamiltonian.) (Hilbig [1986])

7.2.14. A graph G is **uniquely** k-edge-colorable if all proper k-edge-colorings of G induce the same partition of the edges. Prove that every uniquely 3-edge-colorable 3-regular graph is Hamiltonian. (Greenwell–Kronk [1973])

7.2.15. Place n points around a circle. Let G_n be the 4-regular graph obtained by joining each point to the nearest two points in each direction. If $n \geq 5$, prove that G_n is the union of two Hamiltonian cycles.

7.2.16. For $k \geq 3$, let G_k be the graph obtained from two disjoint copies of $K_{k,k-2}$ by adding a matching between the two "partite sets" of size k. Determine all values of k such that G_k is Hamiltonian.



7.2.17. (!) Prove that the cartesian product of two Hamiltonian graphs is Hamiltonian. Conclude that the k-dimensional cube Q_k is Hamiltonian for $k \geq 2$.

7.2.18. Prove that the cartesian product of two graphs with Hamiltonian paths fails to have a Hamiltonian cycle if and only if both graphs are bipartite and have odd order, in which case the product has a Hamiltonian path.

7.2.19. (+) For each odd natural number k, construct a $k-1$-connected k-regular simple bipartite graph that is not Hamiltonian.

7.2.20. (!) The kth **power** of a simple graph G is the simple graph G^k with vertex set $V(G)$ and edge set $\{uv\colon d_G(u, v) \leq k\}$.

a) Suppose that $G - x$ has at least three nontrivial components in each of which x has exactly one neighbor. Prove that G^2 is not Hamiltonian. (Hint: Consider the second graph in Example 7.2.5.)

b) Prove that the cube of each connected graph (with at least three vertices) is Hamiltonian. (Hint: Reduce this to the special case of trees, and prove it for trees by proving the stronger result that if xy is an edge of the tree T, then T^3 has a Hamiltonian cycle using the edge xy. Comment: Fleischner [1974] proved that the square of each 2-connected graph is Hamiltonian.)

7.2.21. Let $n = k(2l+1)$. Construct a non-Hamiltonian complete k-partite graph with n vertices and minimum degree $\frac{n}{2}\frac{k-1}{k}\frac{2l}{2l+1}$. (Snevily)

7.2.22. Let $\mathbf{G}(k, t)$ be the class of connected k-partite graphs in which each partite set has size t and each subgraph induced by two partite sets is a matching of size t. For $k \geq 4$ and $t \geq 4$, construct a graph in $\mathbf{G}(k, t)$ that is not Hamiltonian. (Hint: There is a graph in $\mathbf{G}(4, 4)$ with a 3-set whose deletion leaves four components; generalize this example. Comment: $\mathbf{G}(3, t) = \{C_{3t}\}$, and also every graph in $\mathbf{G}(k, 3)$ is Hamiltonian.) (Ayel [1982])

7.2.23. (∗) Prove that the Petersen graph has toughness 4/3.

7.2.24. (∗) Let $t(G)$ denote the toughness of G.

a) Prove that $t(G) \le \kappa(G)/2$. (Chvátal [1973])

b) Prove that equality holds in part (a) for claw-free graphs. (Hint: Consider a set S such that $|S| = t(G) \cdot c(G - S)$.) (Matthews–Sumner [1984])

7.2.25. (!) Let G be a simple graph that is not a forest and has girth at least 5. Prove that $\overline{G}$ is Hamiltonian. (Hint: Use Ore's condition.) (N. Graham)

7.2.26. (!) Prove that if G fails Chvátal's condition, then $\overline{G}$ has at least $n - 2$ edges. Conclude from this that the maximum number of edges in a simple non-Hamiltonian n-vertex graph is $\binom{n-1}{2} + 1$. (Ore [1961], Bondy [1972b])

7.2.27. Prove directly by induction on n that the maximum number of edges in a simple non-Hamiltonian n-vertex graph is $\binom{n-1}{2} + 1$.

7.2.28. *Generalization of the edge bound.*

a) Let $f(i) = 2i^2 - i + (n - i)(n - i - 1)$, and suppose that $n \ge 6k$. Prove that on the interval $k \le i \le n/2$, the maximum value of $f(i)$ is $f(k)$.

b) Let G be a simple graph with minimum degree k. Use part (a) and Chvátal's condition to prove that if G has at least $6k$ vertices and has more than $\binom{n(G)-k}{2} + k^2$ edges, then G is Hamiltonian. (Erdős [1962])

7.2.29. (!) Let G be a simple graph with vertex degrees $d_1 \le \cdots \le d_n$, and let $d_1' \le \cdots \le d_n'$ be the vertex degrees in $\overline{G}$. Prove that if $d_i \ge d_i'$ for all $i \le n/2$, then G has a Hamiltonian path. Conclude that every simple graph isomorphic to its complement has a Hamiltonian path. (Clapham [1974])

7.2.30. Obtain Lemma 7.2.9 (sufficiency of Ore's condition) from Theorem 7.2.13 (sufficiency of Chvátal's condition). (Bondy [1978])

7.2.31. (!) Prove or disprove: If G is a simple graph with at least three vertices, and G has at least $\alpha(G)$ vertices of degree $n(G) - 1$, then G is Hamiltonian.

7.2.32. (+) Suppose that n is even and G is a simple bipartite graph with partite sets X, Y of size $n/2$. Let the vertex degrees of G be $d_1, \ldots, d_n$. Let G' be the supergraph of G obtained by adding edges so that $G[Y] = K_{n/2}$.

a) Prove that G is Hamiltonian if and only if G' is Hamiltonian, and describe the relationship between the degree sequences of G and G'.

b) Suppose that $d_k > k$ or $d_{n/2} > n/2 - k$ whenever $k \le n/4$. Prove that G is Hamiltonian. (Hint: Assume that the degree sequence of G' fails Chvátal's condition for some $i < n/2$, and obtain a contradiction.) (Chvátal [1972])

7.2.33. (!) A graph is **Hamiltonian-connected** if for every pair of vertices u, v there is a Hamiltonian path from u to v. Prove that a simple graph G is Hamiltonian if $e(G) \ge \binom{n(G)-1}{2} + 2$ and Hamiltonian-connected if $e(G) \ge \binom{n(G)-1}{2} + 3$. (Proving the two together permits a simpler proof.) (Ore [1963])

7.2.34. *Necessary condition for Hamiltonian-connected.* (Moon [1965a])

a) Prove that every Hamiltonian-connected graph G with at least four vertices has at least $\lceil 3n(G)/2 \rceil$ edges.

b) Prove that the bound in part (a) is best possible by showing that $C_m \,\square\, K_2$ is Hamiltonian-connected if m is odd.

7.2.35. (!) *Sufficient condition for Hamiltonian-connected.* (Ore [1963])

a) Prove that a simple graph G is Hamiltonian-connected if $x \not\leftrightarrow y$ implies $d(x) +$

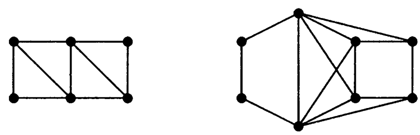
$d(y) > n(G)$. (Hint: Prove that appropriate graphs related to G are Hamiltonian by considering their closures.)

b) Prove that part (a) is sharp by constructing, for each even n greater than 2, a simple n-vertex graph with minimum degree $n/2$ that is not Hamiltonian-connected.

7.2.36. Las Vergnas' condition for a simple n-vertex graph is the existence of a vertex ordering $v_1, \ldots, v_n$ such that there is no nonadjacent pair v_i, v_j satisfying $i < j$, $d(v_i) \leq i$, $d(v_j) < j$, $d(v_i) + d(v_j) < n$, and $i + j \geq n$. Las Vergnas [1971] proved that this condition is sufficient for the existence of a spanning cycle.

a) Prove that Chvátal's condition (Theorem 7.2.13) implies Las Vergnas' condition, which means that Las Vergnas' theorem strengthens Chvátal's theorem.

b) Prove that each of the graphs below fails Chvátal's condition but has a complete graph as its Hamiltonian closure. Prove that the smaller graph satisfies Las Vergnas' condition but the larger one does not.



7.2.37. For $\emptyset \neq S \subset V(G)$, let $t(S) = \left|\overline{S} \cap N(S)\right| / \left|\overline{S}\right|$. Let $\theta(G) = \min t(S)$. Lu [1994] proved that if $\theta(G)n(G) \geq \alpha(G)$, then G is Hamiltonian. Prove that $\kappa(G) \geq \alpha(G)$ implies $\theta(G)n(G) \geq \alpha(G)$. (Comment: This shows that Lu's theorem implies the Chvátal–Erdős Theorem and is a stronger result.)

7.2.38. (!) *Long paths and cycles.* Let G be a connected simple graph with $\delta(G) = k \geq 2$ and $n(G) > 2k$.

a) Let P be a maximal path in G (not a subgraph of any longer path). If $n(P) \leq 2k$, prove that the induced subgraph $G[V(P)]$ has a spanning cycle (this cycle need not have its vertices in the same order as P).

b) Use part (a) to prove that G has a path with at least $2k + 1$ vertices. Give an example for each odd value of n to show that G need not have a cycle with more than $k + 1$ vertices.

7.2.39. Prove that if a simple graph G has degree sequence $d_1 \leq \cdots \leq d_n$ and $d_1 + d_2 < n$, then G has a path of length at least $d_1 + d_2 + 1$ unless G is the join of $n - (d_1 + 1)$ isolated vertices with a graph on $d_1 + 1$ vertices or $G = pK_{d_1} \vee K_1$ for some $p \geq 3$. (Ore [1967b])

7.2.40. (!) Dirac [1952b] proved that every 2-connected simple graph G has a cycle of length at least $\min\{n(G), 2\delta(G)\}$. Use this to prove that every $2k$-regular graph with $4k + 1$ vertices is Hamiltonian. (Nash-Williams)

7.2.41. Scott Smith conjectured that any two longest cycles in a k-connected graph have at least k common vertices. The approach below works for small k.

a) Suppose that G is a 4-regular graph with n vertices that is the union of two cycles (multiple edges may arise). Let G' be the 4-regular graph on $n + 2$ vertices obtained from G by subdividing two edges and adding a double edge between the two new vertices. Show that G' is also the union of two spanning cycles if $n \leq 5$.

b) Use part (a) to conclude that any pair of longest cycles in a k-connected graph intersect in at least k points if $k \leq 6$. (Smith, Burr)

7.2.42. (+) Let G be an Eulerian graph. Let V' be the set of Eulerian circuits of G, considering a circuit and its reversal to be the same. Let G' be the graph with vertex set V' such that two circuits are adjacent if and only if one arises from the other by reversing the edge order on a proper closed subcircuit. Prove that G' is Hamiltonian if $\Delta(G) \le 4$. (Hint: Use induction on the number of vertices of degree 4, proving that there is a Hamiltonian cycle through every edge of G'. Comment: The conclusion also holds without restriction on $\Delta(G)$.) (Xia [1982], Zhang–Guo [1986])

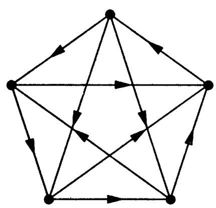
7.2.43. Prove that the Eulerian circuit graph G' of Exercise 7.2.42 is regular, and derive a formula for its vertex degree. Compare $\delta(G')$ and $n(G')$ when $n(G) = 2$ to show that the preceding problem cannot be solved by applying general results on Hamiltonicity of regular graphs with specified degree.

7.2.44. Prove that every tournament has a Hamiltonian path (a spanning directed path). (Hint: Use extremality). (Rédei [1934])

7.2.45. Let T be a strong tournament. For each $u \in V(T)$ and each k such that $3 \le k \le n$, prove that u belongs to a cycle of length k in T. (Hint: Use induction on k.) (Moon [1966])

7.2.46. Let G be a 7-vertex tournament in which every vertex has outdegree 3. Use Exercise 7.2.45 to prove that G has two vertex-disjoint cycles.

7.2.47. (+) Prove that every tournament has a Hamiltonian path that is not contained in a Hamiltonian cycle, except the cyclic tournament on three vertices and the tournament T_5 on five vertices drawn below. (Hint: Induction works, but some care is needed to prove the claim for six vertices. In all cases, find the desired configuration or $G = T_5$.) (Grünbaum, in Harary [1969, p211])



7.2.48. (*) Prove that Theorem 7.2.22 is best possible by showing that the strictness condition on the digraph cannot be weakened to allow loops. In particular, construct for each even n an n-vertex digraph D that is not Hamiltonian even though at most one copy of each ordered pair is an edge and $\min\{\delta^-(D), \delta^+(D)\} \ge n/2$.

7.2.49. (*) Obtain Theorem 7.2.8 (sufficiency of Dirac's condition in graphs) from Theorem 7.2.22 (sufficiency of Ghouilà-Houri's condition on digraphs). (Hint: Transform a simple graph G into a strict digraph by replacing each edge with a pair of directed edges in opposite directions.)

7.3. Planarity, Colorings, and Cycles

We return to the Four Color Problem to explore its historical relationship with the problems of edge-coloring and Hamiltonian cycles. We then consider ways in which the problem generalizes.

TAIT'S THEOREM

In 1878, Tait proved a theorem relating face-coloring and edge-coloring of plane graphs, and he used this in an approach to the Four Color Theorem. This stimulated interest in edge-coloring. We first define face-coloring precisely.

7.3.1. Definition. A **proper face-coloring** of a 2-edge-connected plane graph is an assignment of colors to its faces so that faces having a common edge in their boundaries have distinct colors.

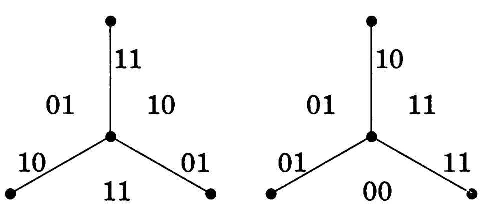
We often think of a face-coloring as a coloring of the dual graph. For this reason, we restrict our attention to face-colorings of 2-edge-connected graphs. When a plane graph has a cut-edge, its dual has a loop. We say that graphs with loops do not have proper colorings. In a plane graph with a cut-edge, a face shares a boundary with itself and is thus uncolorable.

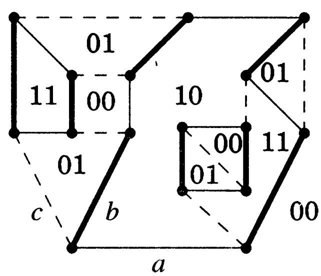
Since adding edges does not make ordinary coloring easier, to prove the Four Color Theorem it suffices to prove that all triangulations are 4-colorable. Equivalently, we could show that all duals of triangulations are 4-face-colorable. The dual G^* of a plane triangulation G is a 3-regular, 2-edge-connected plane graph (Exercise 6.1.11). Tait showed that for such graphs, proper 4-face-colorings are equivalent to proper 3-edge-colorings.

7.3.2. Theorem. (Tait [1878]) A simple 2-edge-connected 3-regular plane graph is 3-edge-colorable if and only if it is 4-face-colorable.

Proof: Let G be such a graph. Suppose first that G is 4-face-colorable; we obtain a 3-edge-coloring. Let the four colors be denoted by binary ordered pairs: $c_0 = 00$, $c_1 = 01$, $c_2 = 10$, $c_3 = 11$. Color $E(G)$ by assigning to the edge between faces with colors c_i and c_j the color obtained by adding c_i and c_j coordinatewise using addition modulo 2. (Thus $c_2 + c_3 = c_1$, for example.) We show that this is a proper 3-edge-coloring.

Because G is 2-edge-connected, each edge bounds two distinct faces. Hence the color 00 never occurs as a sum. We check that the edges at a vertex receive distinct colors. At vertex v the faces bordering the three incident edges must have distinct colors $\{c_i, c_j, c_k\}$, as illustrated below. If color 00 is not in this set, then the sum of any two of these is the third, and hence $\{c_i, c_j, c_k\}$ is also the set of colors on the edges. If $c_k = 00$, then c_i and c_j appear on two of the edges, and the third receives color $c_i + c_j$, which is the color not in $\{c_i, c_j, c_k\}$.



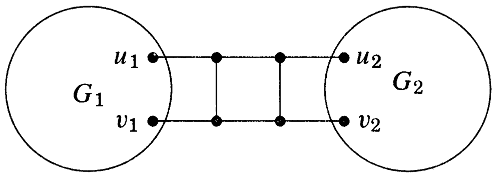


For the converse, suppose that G has a proper 3-edge-coloring using colors a, b, c (shown bold, solid, and dashed). Let E_a, E_b, E_c be the edge sets having the three colors, respectively. We construct a 4-face-coloring using the four colors defined above. Since G is 3-regular, each color appears at every vertex, and the union of any two of E_a, E_b, E_c is 2-regular, which makes it a union of disjoint cycles. Each face of this subgraph is a union of faces of the original graph. Let $H_1 = E_a \cup E_b$ and $H_2 = E_b \cup E_c$. To each face of G, assign the color whose ith coordinate ($i \in \{1, 2\}$) is the parity of the number of cycles in H_i that contain it (0 for even, 1 for odd).

We claim that this is a proper 4-face-coloring, as illustrated above. Faces F, F' sharing an edge e are distinct faces, since G is 2-edge-connected. Edge e belongs to a cycle C in at least one of H_1, H_2 (in both if e has color b). By the Jordan Curve Theorem, one of F, F' is inside C and the other is outside. All other cycles in H_1 and H_2 fail to separate F and F', leaving them on the same side. Hence if e has color a, c, or b, then the parity of the number of cycles containing F and F' is different in H_1, in H_2, or in *both*, respectively. Thus F and F' receive different colors in the face-coloring we have constructed. ■

Due to this theorem, a proper 3-edge-coloring of a 3-regular graph is called a **Tait coloring**. The problem of showing that every 2-edge-connected 3-regular planar graph is 3-edge-colorable reduces to showing that every 3-connected 3-regular planar graph is 3-edge-colorable.

7.3.3.* Lemma. If G is a 3-regular graph with edge-connectivity 2, then G has subgraphs G_1, G_2 and vertices $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ such that $u_1 \not\leftrightarrow v_1$, also $u_2 \not\leftrightarrow v_2$, and G consists of G_1, G_2 and a *ladder* of some length joining G_1, G_2 at u_1, v_1, u_2, v_2 as shown below.



Proof: If G has an edge cut of size 2 in which the two edges are incident, then the third edge incident to their common vertex is a cut-edge, contradicting $\kappa' = 2$. Hence we may assume that the four endpoints in our minimum edge cut xy, uv are distinct. If $x \not\leftrightarrow y$ and $u \not\leftrightarrow v$, then these are the four desired vertices and the ladder has only these two edges.

When $x \leftrightarrow y$, we extend the ladder (a similar argument applies when $u \leftrightarrow v$). Let w be the third neighbor of x and z the third neighbor of y. If $w = z$, then the third edge incident to this vertex is a cut-edge. Hence $w \neq z$ and the ladder extends. If $w \not\leftrightarrow z$, then we are finished in this direction; otherwise, we repeat the argument till we obtain a nonadjacent pair at the base of the ladder. ■

7.3.4.* Theorem. All 2-edge-connected 3-regular simple planar graphs are 3-edge-colorable if and only if all 3-connected 3-regular simple planar graphs are 3-edge-colorable.

Proof: The second family is contained in the first. Hence it suffices to show that 3-edge-colorability for all graphs in the smaller family implies it also for the larger family. We use induction on $n(G)$.

Basis step ($n(G) = 4$): The only 2-edge-connected 3-regular simple planar graph with at most 4 vertices is K_4, which is 3-edge-colorable.

Induction step ($n(G) > 4$): Since $\kappa(G) = \kappa'(G)$ when G is 3-regular (Theorem 4.1.11), we may restrict our attention to 3-regular graphs with edge-connectivity 2. Lemma 7.3.3 gives us a decomposition of G into G_1, G_2, and a ladder joining them. The *length* of the ladder is the distance from G_1 to G_2.

Both $G_1 + u_1v_1$ and $G_2 + u_2v_2$ are 2-edge-connected and 3-regular. By the induction hypothesis, they are 3-edge-colorable; let f_i be a proper 3-edge-coloring of $G_i + u_iv_i$. Permute names of colors so that $f_1(u_1v_1) = 1$ and so that $f_2(u_2v_2)$ is chosen from $\{1, 2\}$ to have the same parity as the length of the ladder.

Returning to G, color each G_i as in f_i. Beginning from the end of the ladder at G_1, color the rungs of the ladder with 3, and color the paths forming the sides of the ladder alternately with 1 and 2. The edges of the ladder at u_i and v_i now have the color $f_i(u_iv_i)$. Thus we have assembled a proper 3-edge-coloring of G. ■

Thus the Four Color Theorem reduces to finding Tait colorings of 3-edge-connected 3-regular planar graphs. The statement of their existence was known as **Tait's conjecture** and is equivalent to the Four Color Theorem.

GRINBERG'S THEOREM

Every Hamiltonian 3-regular graph has a Tait coloring (Exercise 1). Tait believed that this completed a proof of the Four Color Theorem, because he assumed that every 3-connected 3-regular planar graph is Hamiltonian. Not until 1946 was an explicit counterexample found, although the gap in the proof was noticed earlier. Later, Grinberg [1968] discovered a simple necessary condition that led to many 3-regular 3-connected non-Hamiltonian planar graphs, including the Grinberg graph of Exercise 16.

7.3.5. Theorem. (Grinberg [1968]) If G is a loopless plane graph having a Hamiltonian cycle C, and G has f_i' faces of length i inside C and f_i'' faces of length i outside C, then $\sum_i (i-2)(f_i' - f_i'') = 0$.

Proof: Considering the faces inside and outside C separately, we want to show that $\sum_i (i-2)f_i' = \sum_i (i-2)f_i''$. No changes on one side affect the sum on the other side. Furthermore, we can switch inside and outside by projecting the embedding onto a sphere and puncturing a face inside C.

Hence we need only show that $\sum_i (i-2)f_i'$ is constant. When there are no inside edges, the sum is $n-2$. With this as the basis step, we prove by induction on the number of inside edges that the sum is always $n-2$.

Suppose that $\sum_i (i-2)f_i' = n-2$ when there are k edges inside C. We can obtain any graph with $k+1$ edges inside C by adding an edge to such a graph.

The added edge cuts a face of some length r into two faces of lengths s and t. We have $s + t = r + 2$, because the new edge contributes to both new faces and each edge on the old face contributes to one new face.

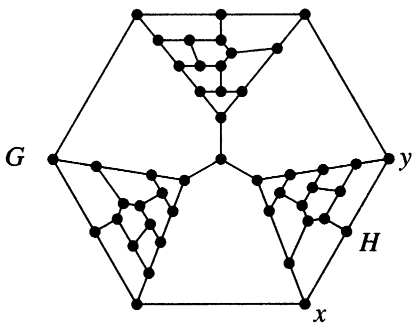
No other contribution to the sum changes. Since $(s - 2) + (t - 2) = (r - 2)$, the contribution from these faces also remains the same. By the induction hypothesis, the sum is $n - 2$. ∎

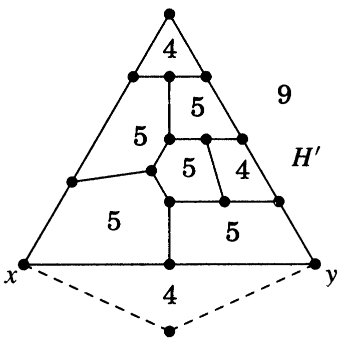
Being a necessary condition, Grinberg's condition can be used to show that graphs are *not* Hamiltonian. The arguments can often be simplified using modular arithmetic. Two numbers that are not congruent mod k are not equal.

We apply this to the first known non-Hamiltonian 3-connected 3-regular planar graph (Tutte [1946]). Tutte used an *ad hoc* argument to prove that this graph is not Hamiltonian. For many years it was the only known example (see Exercise 17 for the smallest now known).

7.3.6. Example. *Grinberg's condition and the Tutte graph.* The Tutte graph G appears on the left below. Let H denote each component obtained by deleting the central vertex and the three long edges. Since a Hamiltonian cycle must visit the central vertex of G, it must traverse one copy of H along a Hamiltonian path joining the other entrances to H, which we call x and y.

We therefore study a graph that has a Hamiltonian cycle if and only if H has a Hamiltonian x, y-path. Such a graph H' (on the right below) is obtained by adding an x, y-path of length two through a new vertex.





The plane graph H' has five 5-faces, three 4-faces, and one 9-face. Grinberg's condition becomes $2a_4 + 3a_5 + 7a_9 = 0$, where $a_i = f_i' - f_i''$. Since the unbounded face is always outside, the equation reduces mod 3 to $2a_4 \equiv 7 \bmod 3$. Since $f_4' + f_4'' = 3$, the possibilities for a_4 are $+3, +1, -1, -3$. The only choice satisfying $2a_4 \equiv 7 \bmod 3$ is $a_4 = -1$, which requires that two of the 4-faces lie outside the Hamiltonian cycle. However, the 4-faces having having a vertex of degree 2 cannot lie outside the cycle, since the edges incident to the vertex of degree 2 separate the face from the outside face.

We can reach a contradiction faster by subdividing one edge incident to each vertex of degree 2. This does not change the existence of a spanning cycle. The resulting graph has seven 5-faces, one 4-face, and one 11-face. The required equation becomes $2 \cdot (\pm 1) = 9 - 3a_5$, which has no solution since the left side is not a multiple of 3. ■

We have not presented a systematic procedure for proving the nonexistence of solutions to equations with integer variables. Our arguments involving divisibility are merely tricks to avoid listing cases, but such tricks often work.

High connectivity makes it harder to avoid spanning cycles. Tutte [1956] (extended by Thomassen [1983]) proved that every 4-connected planar graph is Hamiltonian. Barnette [1969] conjectured that every planar 3-connected 3-regular bipartite graph is Hamiltonian.

SNARKS (optional)

Another approach to the Four Color Theorem is to study which 3-regular graphs are 3-edge-colorable. In a discussion focusing on 3-regular graphs and graphs without cut-edges, it is convenient to have simple adjectives to describe these properties.

7.3.7. Definition. A **bridgeless graph** is a graph without cut-edges. A **cubic graph** is a graph that is regular of degree 3.

7.3.8. Conjecture. (3-edge-coloring Conjecture—Tutte [1967]) Every bridgeless cubic non-3-edge-colorable graph contains a subdivision of the Petersen graph.

Conjecture 7.3.8 has been proved! Like the Four Color Theorem, its computer-assisted proof uses discharging methods. The proof will appear in a series of five papers by Robertson, Sanders, Seymour, and Thomas [2001].

Since every subdivision of the Petersen graph is nonplanar, Conjecture 7.3.8 implies Tait's Conjecture and hence the Four Color Theorem. One natural approach to the conjecture, like the idea of reducibility for the Four Color Theorem, is to derive properties that a minimal counterexample must have. In this language, Theorem 7.3.4 says that a minimal counterexample must be 3-edge-connected. In the next lemma, we make this statement precise and obtain several other properties.

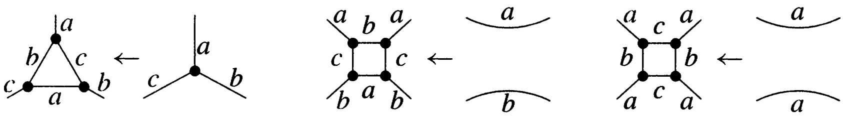
7.3.9. Definition. A **trivial edge cut** is an edge cut whose deletion isolates a single vertex. Other edge cuts are **nontrivial**.

7.3.10. Lemma. If a non-3-edge-colorable cubic graph G has connectivity 2 or girth less than 4 or a nontrivial 3-edge cut, then G contains a subdivision of a smaller non-3-edge-colorable cubic graph.

Proof: Suppose first that G has an edge cut of size 2. As discussed in Lemma 7.3.3, these edges have no common vertices. Deleting the edge cut and adding one edge to each piece yields cubic graphs $G_1 + u_1v_1$ and $G_2 + u_2v_2$. As argued in Theorem 7.3.4, at least one of these graphs is not 3-edge-colorable. Since the added edge can be replaced by a path through the other piece, G contains a subdivision of this smaller non-3-edge-colorable graph.

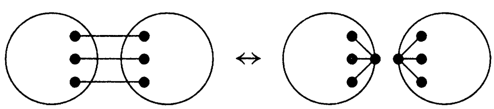
Next suppose that G contains a triangle. Let G' be the graph obtained from G by contracting the triangle to a single vertex. A proper 3-edge-coloring of G' could be expanded into a proper 3-edge-coloring of G as shown below. Also, G contains a subdivision of G', obtained by deleting one edge of the triangle.

Suppose that G contains a 4-cycle but no triangle. Let G' be the cubic graph obtained from G by deleting two opposite edges of the 4-cycle and replacing the resulting paths of length 3 with single edges. Since G has no triangle, the new edges are not loops. A proper 3-edge-coloring of G' yields a proper 3-edge-coloring of G via the two cases shown below. Also G contains a subdivision of G', so G' is the desired smaller graph.



Finally, suppose that G contains a nontrivial 3-edge cut $[S, \overline{S}]$. Since we may assume that G is 3-edge-connected, the three edges of the cut are pairwise disjoint. The two graphs obtained by contracting $G[S]$ or $G[\overline{S}]$ to a single vertex are also 3-regular. If both are 3-edge-colorable, then the colors can be renamed to agree on the edges of the cut, yielding a proper 3-edge-coloring of G. Thus at least one of these graphs is not 3-edge-colorable.

It remains only to show that G contains a subdivision of $G[S]$ (and similarly of $G[\overline{S}]$). Let a, b, c be the endpoints in $\overline{S}$ of the edges in the cut. Since G is 3-edge-connected, the cut is a bond, and $G[\overline{S}]$ is connected (Proposition 4.1.15). Thus $G[\overline{S}]$ contains an a,b-path P and a path from c to P. Adding these paths and the edges of the cut to $G[S]$ completes a subdivision of $G[S]$. ■



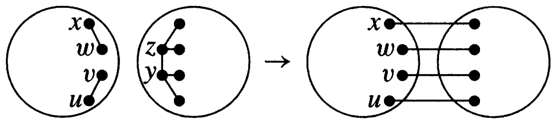
7.3.11. Definition. A **snark** is a 2-edge-connected 3-regular graph that is not 3-colorable, has girth at least 5, and has no non-trivial 3-edge cut. A **prime snark** is one that contains no subdivision of a smaller snark.

In this language, we have reduced Tutte's 3-edge-coloring Conjecture to the statement that the Petersen graph is the only prime snark. Again, we note that the conjecture has been proved (Robertson–Sanders–Seymour–Thomas [2001]).

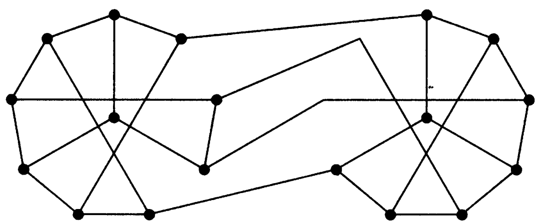
After the Petersen graph in 1898, by 1975 only three more snarks had been found: the 18-vertex Blanuša [1946] snark, the 210-vertex Descartes [1948] snark, and the 50-vertex Szekeres [1973] snark. This prompted Martin Gardner [1976] to invent the term "snark", evoking the rarity of the creature in Lewis Carroll's "The Hunting of the Snark".

Isaacs [1975] then showed that the earlier snarks arise from the Petersen graph via an operation that generates infinite families of snarks.

7.3.12. Definition. The **dot product** of cubic graphs G and H is the cubic graph formed from $G + H$ by deleting disjoint edges uv and wx from G, deleting adjacent vertices y and z from H, and adding edges from u and v to $N_H(y) - \{z\}$ and from w and x to $N_H(z) - \{y\}$.

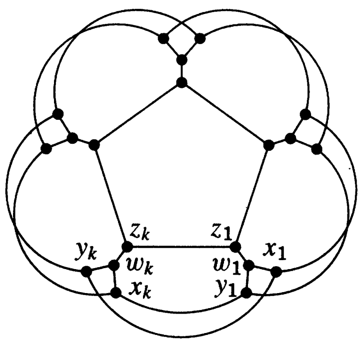


The dot product of two snarks is a snark (Exercise 23). Applying it to two copies of the Petersen graph yields the Blanuša snark shown below. This graph has a non-trivial 4-edge cut. Kochol [1996] introduced a more general operation that yields snarks with large girth and higher connectedness properties.



7.3.13. Example. *The flower snarks.* Isaacs also found an explicit infinite family of snarks (Exercise 21) that don't arise via the dot product. Independently discovered by Grinberg, they have $4k$ vertices, for odd $k \geq 5$.

Begin with three disjoint k-cycles. Let $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ be the three vertex sets, indexed cyclically. For each i add a vertex w_i with $N(w_i) = \{x_i, y_i, z_i\}$. The resulting graph G_k is 3-edge-colorable. Let H_k be the graph obtained by replacing the edges $x_k x_1$ and $y_k y_1$ with $x_k y_1$ and $y_k x_1$. If k is odd and $k \geq 5$, then H_k is a snark. If k is even, then H_k is 3-edge-colorable. The drawing of H_k in which $\{z_i\}$ is a central cycle suggests the name "flower snark". ■



FLOWS AND CYCLE COVERS (optional)

Tait's Theorem (Theorem 7.3.2) states that 3-edge-colorability and 4-face-colorability are equivalent for plane triangulations. When extending this beyond planar graphs, we need a concept that makes sense for all graphs and is equivalent to 4-face-coloring on plane graphs. Additional information about this topic (and about snarks) appears in the monograph by Zhang [1997].

7.3.14. Definition. A **flow** on a graph G is a pair (D, f) such that

1) D is an orientation of G,
2) f is a weight function on $E(G)$, and
3) each $v \in V(G)$ satisfies $\sum_{w \in N_D^+(v)} f(vw) = \sum_{u \in N_D^-(v)} f(uv)$.

A k-**flow** is an integer-valued flow such that $|f(e)| \le k-1$ for all $e \in E(G)$. A flow is **nowhere-zero** or **positive** if $f(e)$ is nonzero or positive, respectively, for all $e \in E(G)$.

The usage of "flow" here is somewhat different from that in Chapter 4. In both contexts, the word "flow" suggests the conservation constraints imposed at each vertex. The bound of $k-1$ on flow value evokes the notion of capacity.

We can alter the orientation to make all weights positive.

7.3.15. Proposition. For a graph G, the following are equivalent:

A) G has a positive k-flow.
B) G has a nowhere-zero k-flow.
C) G has a nowhere-zero k-flow for each orientation of G.

Proof: Simultaneously changing the orientation of an edge and the sign of its weight does not affect the conservation constraints. ■

Thus the existence of a nowhere-zero k-flow does not depend on the choice of the orientation. We can also take linear combinations of flows.

7.3.16. Proposition. If $(D, f_1), \dots, (D, f_r)$ are flows on G, and $g = \sum_{i=1}^r \alpha_i f_i$, then (D, g) is a flow on G.

Proof: For each $v \in V(G)$, the net flow out of v under each f_i is zero, and hence it is also zero under g. ■

7.3.17. Proposition. For a flow on G, the net flow out of any set $S \subseteq V(G)$ is zero. Thus a graph with a nowhere-zero flow has no cut-edge.

Proof: We sum the net flows out of vertices of S. Edges leaving S contribute with positive weight, edges entering S contribute with negative weight, and edges within S contribute positively at their tails and negatively at their heads. The net flow out of S is thus the sum of the net flows out of the vertices of S, which is zero.

This implies that the net flow across any edge cut is zero, so it cannot consist of a single edge with nonzero weight. ■

Thus we restrict our attention to graphs without cut-edges (bridgeless graphs). What distinguishes flows here from circulations in Section 4.3 is that we forbid zero as a weight. Nowhere-zero flows enable us to extend Tait's Theorem. We begin by interpreting Eulerian graphs in the context of nowhere-zero flows; connectedness is no longer important.

7.3.18. Definition. A graph is an **even graph** if every vertex has even degree.

7.3.19. Proposition. A graph has a nowhere-zero 2-flow if and only if it is an even graph.

Proof: Given a nowhere-zero 2-flow, we obtain a positive 2-flow. Since this assigns weight 1 to every edge, the orientation must have as many edges entering each vertex as leaving it. Thus each vertex degree is even.

Conversely, when each vertex degree is even, each component has an Eulerian circuit. Orienting the edges to follow such a circuit and assigning weight 1 to each edge yields a positive 2-flow. ∎

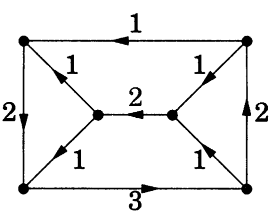
Nowhere-zero 3-flows are more subtle, even for 3-regular graphs.

7.3.20. Proposition. (Tutte [1949]) A cubic graph has a nowhere-zero 3-flow if and only if it is bipartite.

Proof: Let G be a cubic X, Y-bigraph. Every regular bipartite graph has a 1-factor. Orient the edges of a 1-factor from X to Y, and give them weight 2. Orient all other edges from Y to X, and give them weight 1. The flow in and out of every vertex is 2, so this is a nowhere-zero 3-flow.

Conversely, let G be a cubic graph with a nowhere-zero 3-flow. By Proposition 7.3.15, we may assume that the flow is 1 or 2 on each edge. Since the net flow is 0, there must be one edge with flow 2 and two edges with flow 1 at each vertex. Thus the edges with flow 2 form a matching. The X be the set of tails and Y the set of heads of these edges. Since the net flow is 0 at each vertex, each edge with flow 2 points from X to Y, and each edge with flow 1 points from Y to X. Thus X, Y is a bipartition of G. ∎

7.3.21. Example. Since the Petersen graph is cubic and not bipartite, it has no nowhere-zero 3-flow. We will see that it also has no nowhere-zero 4-flow. Below we show a nowhere-zero 4-flow in the 3-regular simple graph $C_3 \square K_2$. ∎

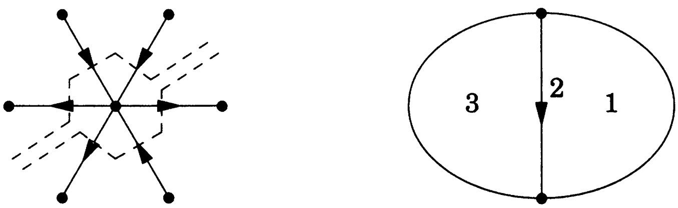


To understand the duality between flows and colorings, we characterize the plane graphs with nowhere-zero k-flows.

7.3.22. Theorem. (Tutte [1954b]) A plane bridgeless graph is k-face-colorable if and only if it has a nowhere-zero k-flow.

Proof: (Younger [1983], refined by Seymour) Let f be a flow on a plane graph G. We define a function g on the set of faces by letting $g(F)$ be the net flow accumulated by traveling from face F out to the unbounded face. Each time we cross an edge e we count $+f(e)$ if e is directed toward our right, $-f(e)$ if e is directed toward our left. The value assigned to the outside face is 0.

The function g is well-defined; that is, $g(F)$ is independent of our route to the outside face. We can change a route into any other by a succession of changes where we go the "other way" around some vertex v (shown on the left below). The change increases or decreases our accumulation for this portion by the net flow out of v, which is 0. Note that the difference between the values on faces with a common edge e is $\pm f(e)$.



Conversely, given a function g defined on the faces, we can invert the process to obtain a flow (shown on the right above). As we stand on face F and look and face F' across edge e, we let $f(e) = g(F) - g(F')$ if e is directed toward our right, $f(e) = g(F') - g(F)$ if e is directed toward our left.

Thus flows correspond to face-colorings. The face-coloring is proper if and only if the flow is nowhere-zero. If the flow is a nowhere-zero k-flow, then reducing the labels in the coloring to congruence classes in $\{0, \ldots, k-1\}$ produces a proper k-coloring. Conversely, a proper k-face-coloring using these colors produces a nowhere-zero k-flow. ∎

The correspondence between face-labelings and flows in Theorem 7.3.22 is valid when the labels come from any abelian group. Applied using the group of binary ordered pairs under addition ((0, 0) is the identity), the statement proved by this argument is precisely Tait's Theorem itself.

Since we can study flows on all graphs, we can consider the flow problem as a general dual notion to vertex coloring. "Nowhere-zero" is the analogue of "proper". Since every nowhere-zero k-flow *is* a nowhere-zero $k+1$-flow, the natural problem is to minimize k such that G has a nowhere-zero k-flow. This minimum is the **flow number** of G, by analogy with "chromatic number". Since we say "G is k-colorable" when G has a proper k-coloring, the natural analogue would be to say "G is k-**flowable**" instead of "G has a nowhere-zero k-flow". This language is not yet common, so we will use it sparingly.

By Tait's Theorem, Theorem 7.3.22 states that a cubic bridgeless planar graph is 3-edge-colorable if and only if it has a nowhere-zero 4-flow. We want to extend this correspondence by dropping the condition on planarity. A simple observation about parity will be useful.

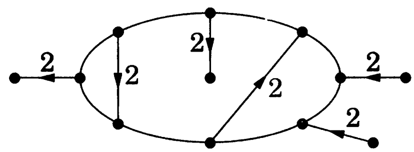
7.3.23. Lemma. In a nowhere-zero k-flow, every vertex is incident to an even number of edges of odd weight.

Proof: Since at each vertex the total weight on entering edges equals the total weight on exiting edges, the sum of the weights is even. ∎

7.3.24. Theorem. Let G be a cubic graph. If G has a nowhere-zero 4-flow, then G is 3-edge-colorable.

Proof: By Proposition 7.3.15, we may assume that G has a positive 4-flow (D, f), and thus $f(e) \in \{1, 2, 3\}$ for each edge e. By Lemma 7.3.23, each vertex is incident to exactly one edge of weight 2. Thus the edges of weight 2 form a 1-factor in G, and deleting them leaves a union of disjoint cycles. To complete a 1-factorization, it suffices to show that each of these cycles has even length.

Let C be such a cycle. The edges of weight 2 that are incident to vertices of C are chords or join $V(C)$ with $\overline{V(C)}$. The chords occupy an even size subset of $V(C)$. Thus it suffices to show that the number of edges between $V(C)$ and $\overline{V(C)}$ is even. These edges all have weight 2. Since the net flow out of $V(C)$ must be 0 and all edges between $V(C)$ and $\overline{V(C)}$ have flow 2, the number of edges leaving $V(C)$ must equal the number of edges entering it. ∎



Since the Petersen graph is not 3-edge-colorable, Theorem 7.3.24 implies that it is not 4-flowable. Existence of nowhere-zero k-flows is preserved by subdivision: when an edge e of weight j in a nowhere-zero k-flow is subdivided, replacing it with a path of length 2 oriented in the same direction with weight j on both edges yields a nowhere-zero k-flow in the new graph. Thus subdivisions of the Petersen graph also have no nowhere-zero 4-flows.

The converse of Theorem 7.3.24 is true but not trivial, since it may not be possible to treat the color classes as edge sets of fixed weight and orient the graph to make this a 4-flow. In the graph $C_3 \square K_2$ of Example 7.3.21, there is essentially only one proper 3-edge-coloring, and when the color classes are labeled $1, 2, 3$ it is not possible to obtain a 4-flow. In the positive 4-flow in Example 7.3.21, the edges of weight 1 do not form a matching.

Nevertheless, we can apply the next theorem to guarantee nowhere-zero 4-flows in cubic graphs. The characterization is more general, since it does not require regularity.

7.3.25. Theorem. A graph has a nowhere-zero 4-flow if and only if it is the union of two even graphs.

Proof: Let G_1, G_2 be even graphs with $G = G_1 \cup G_2$. Let D be an orientation of G, restricting to D_i on G_i. By Proposition 7.3.19 and Proposition 7.3.15, G_i has a nowhere-zero 2-flow (D_i, f_i). Extend f_i to $E(G)$ by letting $f_i(e) = 0$ for $e \in E(G) - E(G_i)$. Let $f = f_1 + 2f_2$. This weight function is odd on $E(G_1)$ and is ± 2 on $E(G) - E(G_1)$, so it is nowhere-zero. Its magnitude is always at most 3, and by Proposition 7.3.16 (D, f) is a flow; thus it is a nowhere-zero 4-flow.

Conversely, let (D, f) be a nowhere-zero 4-flow on G. Let $E_1 = \{e \in E(G)\colon f(e) \text{ is odd}\}$. By Lemma 7.3.23, E_1 forms an even subgraph of G. Thus there is a nowhere-zero 2-flow (D_1, f_1) on E_1, where D_1 agrees with D. Extend f_1 to $E(G)$ by letting $f_1(e) = 0$ for $e \in E(G) - E_1$; now (D, f_1) is a 2-flow on G.

Define f_2 on $E(G)$ by $f_2 = (f - f_1)/2$. By Proposition 7.3.16, (D, f_2) is a flow on G. It is an integer flow, since $f(e) - f_1(e)$ is always even. By Lemma 7.3.23, the set $E_2 = \{e \in E(G)\colon f_2(e) \text{ is odd}\}$ forms an even subgraph of G. For $e \in E(G) - E_1$, we have $f(e) = \pm 2$ and $f_1(e) = 0$, which yields $f_2(e) = \pm 1$, so $E(G) - E_1 \subseteq E_2$. Now G is the union of two even subgraphs. ∎

7.3.26. Corollary. If G is a cubic graph, then G is 3-edge-colorable if and only if G has a nowhere-zero 4-flow.

Proof: Every 3-edge-colorable cubic graph is the union of two even subgraphs: the edges of colors 1 and 2, and the edges of colors 1 and 3. ∎

In light of Theorem 7.3.22, Corollary 7.3.26 generalizes Tait's Theorem.

We have seen that subdivisions of the Petersen graph are not 4-flowable. Among bridgeless graphs, Tutte conjectured that excluding such subgraphs yields nowhere-zero 4-flows.

7.3.27. Conjecture. (Tutte's 4-flow Conjecture—Tutte [1966b]) Every bridgeless graph containing no subdivision of the Petersen graph is 4-flowable. ∎

Since every graph containing a subdivision of the Petersen graph is nonplanar, Tutte's 4-flow Conjecture implies the Four Color Theorem. Since nowhere-zero 4-flows are equivalent to 3-edge-colorings on cubic graphs, the 4-flow Conjecture also implies the 3-edge-coloring Conjecture (which has been proved). Researchers have hoped for an elegant proof of Tutte's 4-flow Conjecture as a way of obtaining a shorter proof of the Four Color Theorem.

We close this section by describing of several other famous conjectures related to these. Every nowhere-zero k-flow is a nowhere-zero $k+1$-flow, so conditions for nowhere-zero 3-flows or 5-flows should be more or less restrictive, respectively, than conditions for a nowhere-zero 4-flow. Statements of Tutte's 3-flow Conjecture appear in Steinberg [1976] and in Bondy–Murty [1976, Unsolved Problem 48].

7.3.28. Conjecture. (Tutte's 3-flow Conjecture) Every 4-edge-connected graph has a nowhere-zero 3-flow. ∎

7.3.29. Conjecture. (Tutte's 5-flow Conjecture—Tutte [1954b]) Every bridgeless graph has a nowhere-zero 5-flow. ∎

Kilpatrick [1975] and Jaeger [1979] proved that every bridgeless graph is 8-flowable. Seymour [1981] proved that these graphs are 6-flowable. We sketch the ideas of the 8-flow Theorem; details are requested in exercises.

Both proofs reduce to the 3-edge-connected case, by showing that a smallest bridgeless graph without a nowhere-zero k-flow is simple, 2-connected, and 3-edge-connected (Exercise 26). The main step is then to express a 3-edge-connected graph as a union of subgraphs with good flows. A generalization of Theorem 7.3.25 then applies: If G_1 has a nowhere-zero k_1-flow and G_2 has a nowhere-zero k_2-flow, then $G_1 \cup G_2$ has a nowhere-zero k_1k_2-flow (Exercise 24). (The converse also holds but is not needed.)

For the 8-flow Theorem, it then suffices to prove that a 3-edge-connected graph can be expressed as the union of three even subgraphs. First, adding an additional copy of each edge in G yields a 6-edge-connected graph G'. Then, the Tree–Packing Theorem of Nash-Williams (Corollary 8.2.59) yields three pairwise edge-disjoint spanning trees in G'. These correspond to three spanning trees in G. Since we obtained them as edge-disjoint trees in G', each edge of G appears in at most two of them.

Within a spanning tree of G, one can find a **parity subgraph** of G, meaning a spanning subgraph H such that $d_H(v) \equiv d_G(v) \bmod 2$ for all $v \in V(G)$ (Exercise 25). The complement within $E(G)$ of the edge set of a parity subgraph is an even subgraph of G. Since our three spanning trees have no common edge, the complements of their parity subgraphs express G as a union of three even subgraphs. By Proposition 7.3.19, each has a nowhere-zero 2-flow, and hence G has a nowhere-zero 8-flow.

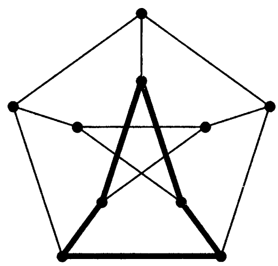
The approach in Seymour [1981] is similar; the task is to express a 3-edge-connected graph as a union of an even graph and a 3-flowable graph. This uses more subtle concepts, including a notion of "modular" flows originally introduced by Tutte [1949]. Seymour's proof was refined by Younger [1983] and Jaeger [1988]. We refer the reader to Zhang [1997] for an exposition.

Celmins [1984] proved that if the 5-flow Conjecture is false, then the smallest counterexample is a snark having girth at least 7 and no nontrivial edge cut with four edges.

We describe one additional conjecture and its relation to earlier topics. In a 2-edge-connected plane graph, all facial boundaries are cycles. Each edge lies in the boundary of two faces, so the facial cycles together cover every edge exactly twice. It is reasonable to ask whether such a covering can be obtained also for graphs that are not planar.

7.3.30. Definition. A **cover** of a graph G is a list of subgraphs whose union is G. A **double cover** is a cover with each edge appearing in exactly two subgraphs in the list. A **cycle double cover (CDC)** is a double cover consisting of cycles.

7.3.31. Example. Together with the outer 5-cycle, the 5 rotations of the 5-cycle illustrated below form a CDC of the Petersen graph. The Petersen graph also has CDCs using cycles of other lengths (Exercise 36). ∎



Since cut-edges appear in no cycles, only bridgeless graphs have CDCs.

7.3.32. Conjecture. (Cycle Double Cover Conjecture—Szekeres [1973], Seymour [1979b]) Every bridgeless graph has a cycle double cover. ∎

One might think that the CDC Conjecture follows immediately using embeddings on surfaces with handles, but such embeddings may have facial boundaries that traverse the same edge twice. The **Strong Embedding Conjecture** asserts that every 2-connected graph has an embedding (on some surface) in which the boundary of each face is a single cycle. Applying this to each block of a 2-edge-connected graph would yield the CDC Conjecture.

In discussing the CDC, we must alert the reader to an unfortunate conflict in terminology. Throughout this book, we use the definition of *cycle* that is common in discussing connectivity, girth, circumference, planarity, etc. In this language, a *circuit* is an equivalence class of closed trails (ignoring the starting vertex), and an *even graph* is a graph whose vertex degrees are all even. A circuit traverses a connected even graph.

The literature on cycle covers generally reverses this terminology, using "circuit" to mean what we call a cycle and "cycle" to mean what we call an even graph. Since the term "even graph" strongly evokes its definition, we hope that our usage will be clear.

The alternative usage arises from other contexts. In a matroid (Section 8.2), the circuits are the minimal dependent sets, and in the cycle matroid of a graph these are the edge sets of the cycles. The cycle space of a graph is a vector space (using scalars {0, 1}) where the coordinates are indexed by the edges and the vectors correspond to the even subgraphs.

The original CDC Conjecture states that every bridgeless graph has a double cover by even subgraphs. That phrasing is equivalent to ours, since every even graph is an edge-disjoint union of cycles.

Thus we might seek a double cover by using a small number of even subgraphs. The cycles in a cycle double cover are even subgraphs; when cycles are pairwise edge-disjoint, they can be combined to form a single even subgraph. This leads to the connection between integer flows and cycle double covers.

7.3.33. Proposition. A graph has a nowhere-zero 4-flow if and only if it has a cycle double cover forming three even subgraphs.

Proof: Theorem 7.3.25 states that a graph has a nowhere-zero 4-flow if and only if it is the union of two even subgraphs E_1, E_2. Let $E_3 = E_1 \triangle E_2$. At each vertex v the degree in E_3 is the sum of the degrees in E_1 and E_2 minus twice the number of common incident edges; hence it is even. Hence E_3 is an even subgraph, and it contains precisely the edges that appear in just one of $\{E_1, E_2\}$. Cycle decompositions of E_1, E_2, E_3 thus combine to yield a CDC.

Conversely, if a CDC forms three even subgraphs, then omitting one of them leaves the graph expressed as the union of two even subgraphs, and hence a nowhere-zero 4-flow exists. ■

Let **P** denote the family of graphs that do not contain a subdivision of the Petersen graph. By Proposition 7.3.33, Tutte's 4-flow Conjecture implies that every graph in **P** has a CDC. Alspach–Goddyn–Zhang [1994] proved a deep result that yields cycle double covers for graphs in **P**. (They proved that a stronger covering property holds for G if and only if $G \in \mathbf{P}$.) In light of Proposition 7.3.33, this is a partial result toward Tutte's 4-flow Conjecture.

The CDC Conjecture is also related to snarks. Goddyn [1985] proved that if the CDC Conjecture is false, then the smallest counterexample is a snark with girth at least 8.

EXERCISES

7.3.1. (−) Prove that every Hamiltonian 3-regular graph has a Tait coloring.

7.3.2. (−) Exhibit 3-regular simple graphs with the following properties.

a) Planar but not 3-edge-colorable.

b) 2-connected but not 3-edge-colorable.

c) Planar with connectivity 2, but not Hamiltonian.

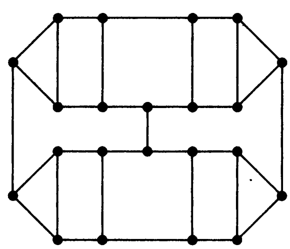
• • • • •

7.3.3. Prove that every maximal plane graph other than K_4 is 3-face-colorable.

7.3.4. Without using the Four Color Theorem, prove that every Hamiltonian plane graph is 4-face-colorable (nothing is assumed about the vertex degrees).

7.3.5. Prove that a 2-edge-connected plane graph is 2-face-colorable if and only if it is Eulerian.

7.3.6. Use Tait's Theorem (Theorem 7.3.2) to prove that $\chi'(G) = 3$ for the graph G below.

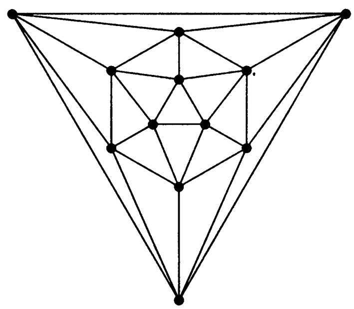


7.3.7. (!) Let G be a plane triangulation.

a) Prove that the dual G^* has a 2-factor.

b) Use part (a) to prove that the vertices of G can be 2-colored so that every face has vertices of both colors. (Hint: Use the idea in the proof of Theorem 7.3.2.) (Burštein [1974], Penaud [1975])

7.3.8. (+) It has been conjectured that every planar triangulation has edge-chromatic number $\Delta(G)$, and this has been proved when $\Delta(G)$ is high enough. Show that $\chi'(G) = \Delta(G)$ for the graph of the icosahedron, illustrated below.



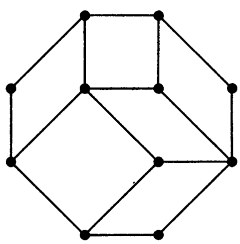
7.3.9. Prove that a proper 4-coloring of the icosahedron uses each color exactly 5 times.

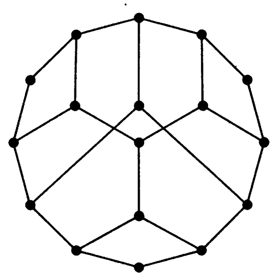
7.3.10. Whitney [1931] proved that every 4-connected planar triangulation is Hamiltonian. Use this to reduce the Four Color Problem to the problem of proving that every Hamiltonian planar graph is 4-colorable.

7.3.11. Find a 5-connected planar graph. Does there exist a 6-connected planar graph?

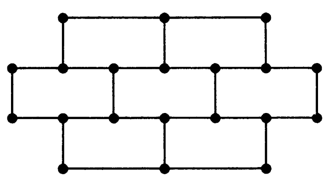
7.3.12. Let G be a planar graph with at least three faces. Prove that G has a vertex partition into two sets whose induced subgraphs are trees if and only if G^* is Hamiltonian.

7.3.13. (!) For each of the planar graphs below, present a Hamiltonian cycle or use planarity (Grinberg's condition) to prove that it is non-Hamiltonian.



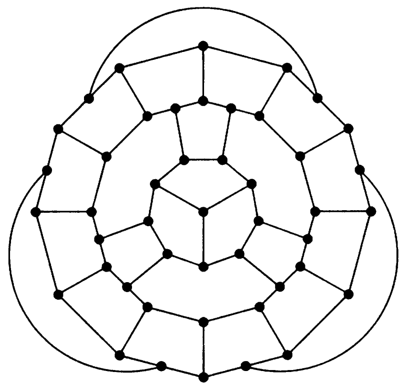


7.3.14. Let G be the graph below. Prove that G has no Hamiltonian cycle. Explain why Grinberg's Theorem cannot be used directly to prove this.

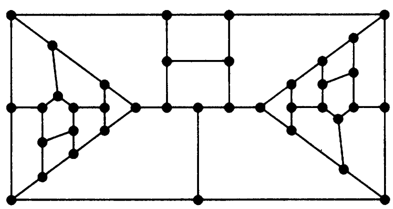


7.3.15. (!) Prove Grinberg's Theorem using Euler's Formula.

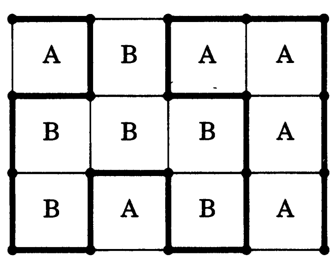
7.3.16. (!) Use Grinberg's condition to prove that the Grinberg graph (below) is not Hamiltonian.



7.3.17. (!) The smallest known 3-regular 3-connected planar graph that is not Hamiltonian has 38 vertices and appears below. Prove that this graph is not Hamiltonian. (Lederberg [1966], Bosák [1966], Barnette)



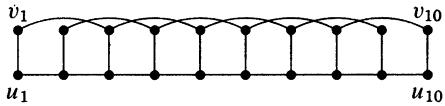
7.3.18. Let G be the grid graph $P_m \square P_n$. Let Q be a Hamiltonian path from the upper left corner vertex to the lower right corner vertex, such as that shown in bold below. Note that Q partitions the grid into regions, of which some open to the left or downward and others open to the right or upward. Prove that the total area of the up-right regions (B) equals the total area of the down-left regions (A). (Fisher–Collins–Krompart [1994])



7.3.19. (!) The **generalized Petersen graph** $P(n, k)$ is the graph with vertices $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ and edges $\{u_i u_{i+1}\}$, $\{u_i v_i\}$, and $\{v_i v_{i+k}\}$, where addition is modulo n. The Petersen graph itself is $P(5, 2)$.

a) Prove that the subgraph of $P(n, 2)$ induced by k consecutive pairs $\{u_i, v_i\}$ has a spanning cycle if $k \equiv 1 \bmod 3$ and $k \geq 4$.

b) Use part (a) to prove that $\chi'(P(n, 2)) = 3$ if $n \geq 6$.



7.3.20. (−) Let G be a 3-regular graph. Prove that if G is the union of three cycles, then G is 3-edge-colorable.

7.3.21. (+) *"Flower snarks"*. Let G_k and H_k be as constructed in (Example 7.3.13).

a) Prove that G_k is 3-edge-colorable.

b) Prove that H_k is not 3-edge-colorable when k is odd. (Isaacs [1975])

7.3.22. Prove that every edge cut of $K_k \square C_l$ that does not isolate a vertex has at least $2k$ edges.

7.3.23. (∗) Prove that applying the dot product operation (Definition 7.3.12) to two snarks yields a third snark. (Isaacs [1975])

7.3.24. (∗!) Let G_1 and G_2 be graphs. Prove that if G_1 has a nowhere-zero k_1-flow and G_2 has a nowhere-zero k_2-flow, then $G_1 \cup G_2$ has a nowhere-zero k_1k_2-flow.

7.3.25. (!) A **parity subgraph** of G is a subgraph H such that $d_H(v) \equiv d_G(v) \bmod 2$ for all $v \in V(G)$. Prove that every spanning tree of a connected graph G contains a partity subgraph of G. (Itai–Rodeh [1978])

7.3.26. (∗) For $k \geq 3$, prove that a smallest nontrivial 2-edge-connected graph G having no nowhere-zero k-flow must be simple, 2-connected, and 3-edge-connected. (Hint: First exclude loops and vertices of degree 2 and reduce to consideration of blocks. Then exclude multiple edges and finally edge cuts of size 2. In each case, compare G to a graph obtained from it by deleting or contracting edges.)

7.3.27. (∗) Prove that every Hamiltonian graph has a nowhere-zero 4-flow.

7.3.28. (∗) Prove that every bridgeless graph with a Hamiltonian path has a nowhere-zero 5-flow. (Jaeger [1978])

7.3.29. (∗) Embed K_6 on the torus, and let G be the dual graph. Find a nowhere-zero 5-flow on G.

7.3.30. (∗) Prove that a graph G is the union of r even subgraphs if and only if G has a nowhere-zero 2^r-flow. (Matthews [1978])

7.3.31. (∗) Let G be a graph having a cycle double cover forming 2^r even subgraphs. Prove that G has a nowhere-zero 2^r-flow. (Jaeger [1988])

7.3.32. (∗!) A **modular 3-orientation** of a graph G is an orientation D such that $d_D^+(v) \equiv d_D^-(v) \bmod 3$ for all $v \in V(G)$. Prove that a bridgeless graph has a nowhere-zero 3-flow if and only if it has a modular 3-orientation. (Steinberg–Younger [1989])

7.3.33. (∗) *Characterization of nowhere-zero k-flows.* Let G be a bridgeless graph, let D be an orientation of G, and let a and b be positive integers. Prove that the following statements are equivalent. (Hoffman [1958])

a) $\frac{a}{b} \leq \frac{|[S,\overline{S}]|}{|[\overline{S},S]|} \leq \frac{b}{a}$ for every nonempty proper vertex subset S.

b) G has an integer flow using weights in the interval $[a, b]$.
c) G has a real-valued flow using weights in the interval $[a, b]$.

7.3.34. $(*)$ Find cycle double covers for the graphs $C_m \vee K_1$, $C_m \vee 2K_1$, and $C_m \vee K_2$.

7.3.35. $(*)$ Find the cycle double covers with fewest cycles for every 3-regular simple graph with 6 vertices.

7.3.36. $(*-)$ Let G be the Petersen graph. Find a cycle double cover of G whose elements are not all 5-cycles. Find a double cover of G consisting of 1-factors. (Hint: Consider the drawing of G having a 9-cycle on the "outside". Comment: Fulkerson [1971] conjectured that every bridgeless cubic graph has a double cover consisting of 6 perfect matchings.)

7.3.37. $(*)$ Prove that any two 6-cycles in the Petersen graph must have at least two common edges. Conclude that the Petersen graph has no CDC consisting of five 6-cycles. Use this and Exercise 7.3.20 to conclude that the Petersen graph has no CDC consisting of even cycles. (C.Q. Zhang)

7.3.38. $(*!)$ A cycle double cover is **orientable** if its cycles can be oriented as directed cycles so that for each edge, the two cycles containing it traverse it in opposite directions. A digraph is **even** if $d^-(v) = d^+(v)$ for each vertex v.

a) Suppose that G has a nonnegative k-flow (D, f). Prove that f can be expressed as $\sum_{i=1}^{k-1} f_i$, where each (D, f_i) is a nonnegative 2-flow on G. (Hint: Use induction on k.) (Little–Tutte–Younger [1988])

b) Prove that a graph G has a positive k-flow (D, f) if and only if D is the union of $k-1$ even digraphs such that each edge e in D appears in exactly $f(e)$ of them. (Little–Tutte–Younger [1988])

c) Prove that a graph G has a nowhere-zero 3-flow if and only if it has an orientable cycle double cover forming three even subgraphs. (Tutte [1949])

7.3.39. $(*)$ Let G be a graph having a CDC formed from four even subgraphs. Prove that G also has a CDC formed from three even subgraphs. (Hint: Use symmetric differences.)

7.3.40. $(*)$ In the Petersen graph, prove that the solution to the Chinese Postman Problem has total length 20, but the minimum total length of cycles covering the Petersen graph is 21.

7.3.41. $(*)$ Let M be a perfect matching in the Petersen graph. Prove that there is no list of cycles in the Petersen graph that together cover every edge of M exactly twice and all other edges exactly once. (Itai–Rodeh [1978], Seymour [1979b])

7.3.42. $(*)$ Let G be a graph in which a shortest covering walk (that is, an optimal solution to the Chinese Postman Problem) decomposes into cycles. Prove that G has a cycle cover of total length at most $e(G) + n(G) - 1$. Determine the minimum length of a cycle cover of $K_{3,t}$ in terms of the number of edges and vertices.

Chapter 8

Additional Topics

In this chapter we explore more advanced or specialized material. Each section gives a glimpse of a topic that deserves its own chapter (or book). Several sections treat more difficult material near the end.

8.1. Perfect Graphs

We have discussed the lower bound $\chi(G) \geq \omega(G)$ for chromatic number; the vertices of a clique need different colors. In Section 5.3, we discussed graphs whose induced subgraphs all achieve equality in this bound.

8.1.1. Definition. A graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

When discussing perfect graphs, it is common to use **stable set** to mean an independent set of vertices. As before, a **clique** is a set of pairwise adjacent vertices. As usual, **maximum** means maximum-sized.

Since we focus on vertex coloring, again in this section we restrict our attention to simple graphs. Complementation converts cliques to stable sets and vice versa, so $\omega(\overline{H}) = \alpha(H)$. Properly coloring $\overline{H}$ means expressing $V(H)$ as a union of cliques in H; such a set of cliques in H is a **clique covering** of H. Thus for every graph G we have four optimization parameters of interest.

independence number	$\alpha(G)$	max size of a stable set
clique number	$\omega(G)$	max size of a clique
chromatic number	$\chi(G)$	min size of a coloring
clique covering number	$\theta(G)$	min size of a clique covering

Berge actually defined two types of perfection:

G is **γ-perfect** if $\chi(G[A]) = \omega(G[A])$ for all $A \subseteq V(G)$.
G is **α-perfect** if $\theta(G[A]) = \alpha(G[A])$ for all $A \subseteq V(G)$.

Our definition of perfect is the same as this definition of γ-perfect (Berge used $\gamma(G)$ for chromatic number). Since $\overline{G}[A]$ is the complement of $G[A]$, the definition of α-perfect can be stated in terms of $\overline{G}$ as "$\chi(\overline{G}[A]) = \omega(\overline{G}[A])$ for all $A \subseteq V(G)$". Thus "G is α-perfect" has the same meaning as "$\overline{G}$ is γ-perfect".

We now use only one definition of perfection, because Lovász [1972a] proved "G is γ-perfect if and only if G is α-perfect". In terms of our original definition of perfection, this becomes "G is perfect if and only if $\overline{G}$ is perfect". This statement is the **Perfect Graph Theorem (PGT)**.

Always $\chi(G) \geq \omega(G)$ and $\theta(G) \geq \alpha(G)$, since a clique and a stable set share at most one vertex. A statement of perfection for a class of graphs is thus an integral min-max relation. We observed in Example 5.3.21 that several familiar min-max relations are statements that bipartite graphs, their line graphs, and the complements of such graphs are perfect.

If $k \geq 2$, then $\chi(C_{2k+1}) > \omega(C_{2k+1})$ and $\chi(\overline{C}_{2k+1}) > \omega(\overline{C}_{2k+1})$ (Exercise 1). Thus odd cycles and their complements (except C_3 and $\overline{C}_3$) are imperfect.

8.1.2. Conjecture. **(Strong Perfect Graph Conjecture (SPGC)**—Berge [1960]) A graph G is perfect if and only if both G and $\overline{G}$ have no induced subgraph that is an odd cycle of length at least 5. ∎

The SPGC remains open. Since the condition in the conjecture is self-complementary, the SPGC implies the PGT.

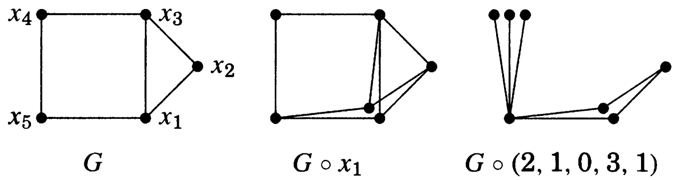
Having presented several classical families of perfect graphs in Section 5.3, our goal now is to prove the Perfect Graph Theorem. Later we also study properties of minimal imperfect graphs and classes of perfect graphs. For further reading, Golumbic [1980] provides a thorough introduction to the subject. Berge–Chvátal [1984] collects and updates many of the classical papers.

THE PERFECT GRAPH THEOREM

In 1960, Berge conjectured that γ-perfection and α-perfection are equivalent (see Berge [1961]). Lovász [1972a] stunned the world of combinatorics by proving this important and well-known conjecture at the age of 22. Fulkerson also studied it, reducing it to a statement he thought was too strong to be true. When Berge told him that Lovász had proved it, within hours he proved the missing lemma (Lemma 8.1.4), thus illustrating that a theorem becomes easier to prove when known to be true (Fulkerson [1971]).

We will prove the Perfect Graph Theorem using an operation that enlarges a graph without affecting the property of perfection.

8.1.3. Definition. **Duplicating a vertex** x of G produces a new graph $G \circ x$ by adding a vertex x' with $N(x') = N(x)$. The **vertex multiplication** of G by the nonnegative integer vector $h = (h_1, \ldots, h_n)$ is the graph $H = G \circ h$ whose vertex set consists of h_i copies of each $x_i \in V(G)$, with copies of x_i and x_j adjacent in H if and only if $x_i \leftrightarrow x_j$ in G.



8.1.4. Lemma. Vertex multiplication preserves γ-perfection and α-perfection.

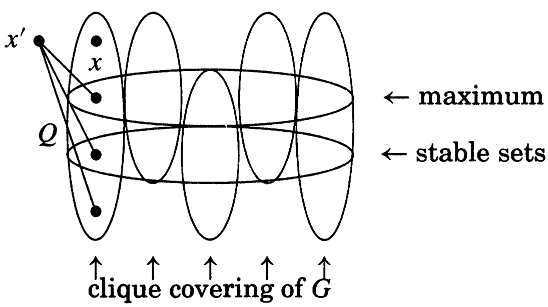
Proof: We first observe that $G \circ h$ can be obtained from an induced subgraph of G by successive vertex duplications. If every h_i is 0 or 1, then $G \circ h = G[A]$, where $A = \{i\colon h_i > 0\}$. Otherwise, start with $G[A]$ and perform duplications until there are h_i copies of x_i (for each i). Each vertex duplication preserves the property that copies of x_i and x_j are adjacent if and only if $x_i x_j \in E(G)$, so the resulting graph is $G \circ h$.

If G is α-perfect but $G \circ h$ is not, then some operation in the creation of $G \circ h$ from $G[A]$ produces a graph that is not α-perfect from an α-perfect graph. It thus suffices to prove that vertex duplication preserves α-perfection. The same reduction holds for γ-perfection. Since every proper induced subgraph of $G \circ x$ is an induced subgraph of G or a vertex duplication of an induced subgraph of G, we further reduce our claim to showing that $\chi(G \circ x) = \omega(G \circ x)$ when G is γ-perfect and that $\alpha(G \circ x) = \theta(G \circ x)$ when G is α-perfect.

When G is γ-perfect, we extend a proper coloring of G to a proper coloring of $G \circ x$ by giving x' the same color as x. No clique contains both x and x', so $\omega(G \circ x) = \omega(G)$. Hence $\chi(G \circ x) = \chi(G) = \omega(G) = \omega(G \circ x)$.

When G is α-perfect, we consider two cases. If x belongs to a maximum stable set in G, then adding x' to it yields $\alpha(G \circ x) = \alpha(G) + 1$. Since $\theta(G) = \alpha(G)$, we can obtain a clique covering of this size by adding x' as a 1-vertex clique to some set of $\theta(G)$ cliques covering G.

If x belongs to no maximum stable set in G, then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G. Since $\theta(G) = \alpha(G)$, Q intersects every maximum stable set in G. Since x belongs to no maximum stable set, $Q' = Q - x$ also intersects every maximum stable set. This yields $\alpha(G - Q') = \alpha(G) - 1$. Applying the α-perfection of G to the induced subgraph $G - Q'$ (which contains x) yields $\theta(G - Q') = \alpha(G - Q')$. Adding $Q' \cup \{x'\}$ to a set of $\alpha(G) - 1$ cliques covering $G - Q'$ yields a set of $\alpha(G)$ cliques covering $G \circ x$. ■



8.1.5. Lemma. In a minimal imperfect graph, no stable set intersects every maximum clique.

Proof: If a stable set S in G intersects every $\omega(G)$-clique, then perfection of $G - S$ yields $\chi(G - S) = \omega(G - S) = \omega(G) - 1$, and S completes a proper $\omega(G)$-coloring of G. This makes G perfect. ∎

8.1.6. Theorem. (The Perfect Graph Theorem (PGT) - Lovász [1972a, 1972b]) A graph is perfect if and only if its complement is perfect.

Proof: It suffices to show that α-perfection of G implies γ-perfection of G; applying this to $\overline{G}$ yields the converse. If the claim fails, then we consider a minimal graph G that is α-perfect but not γ-perfect. By Lemma 8.1.5, we may assume that every maximal stable set S in G misses some maximum clique $Q(S)$.

We design a special vertex multiplication of G. Let $\mathbf{S} = \{S_i\}$ be the list of maximal stable sets of G. We weight each vertex by its frequency in $\{Q(S_i)\}$, letting h_j be the number of stable sets $S_i \in \mathbf{S}$ such that $x_j \in Q(S_i)$. By Lemma 8.1.4, $H = G \circ h$ is α-perfect, yielding $\alpha(H) = \theta(H)$. We use counting arguments for $\alpha(H)$ and $\theta(H)$ to obtain a contradiction.

Let A be the 0,1-matrix of the incidence relation between $\{Q(S_i)\}$ and $V(G)$; we have $a_{i,j} = 1$ if and only if $x_j \in Q(S_i)$. By construction, h_j is the number of 1s in column j of A, and $n(H)$ is the total number of 1s in A. Since each row has $\omega(G)$ 1s, also $n(H) = \omega(G)\,|\mathbf{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) = \omega(G)$. Therefore, $\theta(H) \geq n(H)/\omega(H) = |\mathbf{S}|$.

We obtain a contradiction by proving that $\alpha(H) < |\mathbf{S}|$. Every stable set in H consists of copies of elements in some stable set of G, so a maximum stable set in H consists of all copies of all vertices in some maximal stable set of G. Hence $\alpha(H) = \max_{T \in \mathbf{S}} \sum_{i:\ x_i \in T} h_i$. The sum counts the 1s in A that appear in the columns indexed by T. If we count these 1s instead by rows, we obtain $\alpha(H) = \max_{T \in \mathbf{S}} \sum_{S \in \mathbf{S}} |T \cap Q(S)|$. Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$. Furthermore, T is disjoint from $Q(T)$. With $|T \cap Q(S)| \leq 1$ for all $S \in \mathbf{S}$, and $|T \cap Q(T)| = 0$, we have $\alpha(H) \leq |\mathbf{S}| - 1$. ∎

$$\begin{array}{c|ccc|} & & V(G) & \\ \hline Q(S_1) & \vdots & \vdots & \vdots \\ Q(T) & 0 & 0 & 0 \\ Q(S_n) & \vdots & \vdots & \vdots \\ \hline & \uparrow & \uparrow & \uparrow \\ & T & T & T \end{array}$$

8.1.7.* Remark. *Linear optimization and duality.* Clique-vertex incidence matrices also arise in expressing α and θ as integer optimization problems. A linear (maximization) program can be written as "maximize $c \cdot x$ over nonnegative vectors x such that $Ax \leq b$", where A is a matrix, b, c are vectors, and each row of $Ax \leq b$ is a linear constraint $a_i \cdot x \leq b_i$ on the vector x of variables. A vector x satisfying all the constraints is a **feasible solution**.

An **integer linear program** requires that each x_j also be an integer. Let A be the incidence matrix between maximal cliques and vertices in a graph G; we have $a_{i,j} = 1$ when $v_j \in Q_i$. By definition, $\alpha(G)$ is the solution to "max $\mathbf{1}_n \cdot x$ such that $Ax \leq \mathbf{1}_m$" when the variables are required to be nonnegative integers. In the solution, x_j is 1 or 0 depending on whether v_j is in the maximum stable set; the constraints prevent choosing adjacent vertices. Similarly, when B is the incidence matrix between maximal stable sets and vertices, $\omega(G)$ is the solution to "max $\mathbf{1}_n \cdot x$ such that $Bx \leq \mathbf{1}_p$" with integer variables.

Every maximization program has a dual minimization program. When the max program is "max $c \cdot x$ such that $Ax \leq b$", the dual is "min $y \cdot b$ such that $y^T A \geq c$". This program has a variable y_i for each original constraint and a constraint for each original variable x_j, and it interchanges c, max, $\leq$ with b, min, $\geq$. When stated in this form, the variables in both programs must be nonnegative. The integer programs dual to ω and α seek the minimum number of stable sets that cover the vertices and the minimum number of cliques that cover the vertices, respectively; this describes $\chi(G)$ and $\theta(G)$.

Using the nonnegativity of the variables, the constraints yield

$$c \cdot x \leq y^T Ax \leq y \cdot b.$$

The statement "$c \cdot x \leq y \cdot b$" for feasible solutions x, y is **weak duality**. The (strong) **Duality Theorem of Linear Programming** states that dual programs having feasible solutions have optimal solutions with the same value when integer solutions are not required.

The statements $\chi \geq \omega$ and $\theta \geq \alpha$ are statements of weak duality for dual pairs of linear programs. A guarantee of strong duality using solutions that have only integer values is a combinatorial min-max relation. We have presented many such relations and observed that they guarantee quick proofs of optimality. They also often lead to fast algorithms for finding optimal solutions, which is one motivation for studying families of perfect graphs. ■

8.1.8.* Example. *Fractional solutions for an imperfect graph.* For the 5-cycle, the linear programs for $\omega, \chi, \alpha, \theta$ all have optimal value 5/2. There are five maximal cliques and five maximal stable sets, each of size 2. Setting each $x_j = 1/2$ gives weight 1 to each clique and stable set, thereby satisfying the constraints for either maximization problem. Setting each $y_i = 1/2$ in the dual programs covers each vertex with a total weight of 1, so again the constraints are satisfied. These programs have no optimal solution in integers, and the integer programs have a "duality gap": $\chi = 3 > 2 = \omega$ and $\theta = 3 > 2 = \omega$. ■

CHORDAL GRAPHS REVISITED

Like trees, the more general class of chordal graphs has many characterizations. The definition by forbidding chordless cycles is a **forbidden substructure characterization**. A finite list of forbidden substructures such as induced subgraphs yields a fast algorithm for testing membership in the class, but for chordal graphs the list is infinite and other methods are needed.

A chordal graph can be built from a single vertex by iteratively adding a vertex joined to a clique; this is the reverse of a simplicial elimination ordering, and we have seen that greedy coloring with respect to such a construction ordering yields an optimal coloring. Many classes of perfect graphs have such a **construction procedure** that produces the graphs in the class and no others. A construction procedure or the reverse **decomposition procedure** may lead to fast algorithms for computations on graphs in the class.

Next we consider another type of characterization.

8.1.9. Definition. An **intersection representation** of a graph G is a family of sets $\{S_v\colon v \in V(G)\}$ such that $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$. If $\{S_v\}$ is an intersection representation of G, then G is the **intersection graph** of $\{S_v\}$.

Interval graphs are the graphs having intersection representations where each set in the family is an interval on the real line. Line graphs also form an intersection class; the allowed sets are pairs of natural numbers, corresponding to edges of the graph H such that $G = L(H)$. An intersection characterization for chordal graphs was found independently by Walter [1972, 1978], Gavril [1974], and Buneman [1974].

8.1.10. Lemma. If $T_1, \ldots, T_k$ are pairwise intersecting subtrees of a tree T, then there is a vertex belonging to all of $T_1, \ldots, T_k$.

Proof: (Lehel) We prove the contrapositive. If each vertex v misses some $T(v)$ among $T_1, \ldots, T_k$, we mark the edge that leaves v on the unique path to $T(v)$. If T has n vertices, then we make n marks, so some edge uw has been marked twice. Now $T(u)$ and $T(w)$ have no common vertex. ■

8.1.11. Theorem. A graph is chordal if and only if it has an intersection representation using subtrees of a tree (a **subtree representation**).

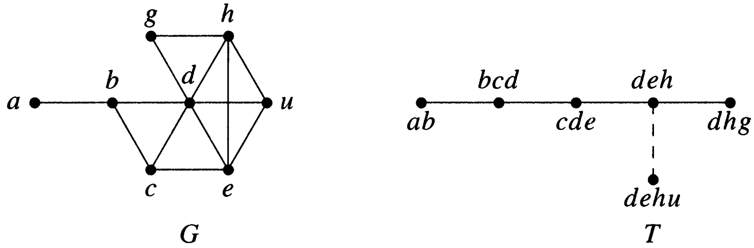
Proof: We prove that the condition is equivalent to the existence of a simplicial elimination ordering. We use induction, with trivial basis K_1.

Let $v_1, \ldots, v_n$ be a simplicial elimination ordering for G. Since $v_2, \ldots, v_n$ is a simplicial elimination ordering for $G - v_1$, the induction hypothesis yields a subtree representation of $G - v_1$ in a host tree T. Since v_1 is simplicial in G, the set $S = N_G(v_1)$ induces a clique in $G - v_1$. Therefore, the subtrees of T assigned to vertices of S are pairwise intersecting.

By Lemma 8.1.10, these subtrees have a common vertex x. We enlarge T to a tree T' by adding a leaf y adjacent to x, and we add the edge xy to the subtrees representing vertices of S. We represent v_1 by the subtree consisting only of y. This completes a subtree representation of G in T'.

Conversely, let T be a smallest host tree for a subtree representation of G, with each $v \in V(G)$ represented by $T(v) \subseteq T$. If $xy \in E(T)$, then G must have a vertex u such that $T(u)$ contains x but not y; otherwise, contracting xy into y would yield a representation in a smaller tree.

Let x be a leaf of T, and let u be a vertex of G such that $T(u)$ contains x but not its neighbor. The subtrees for neighbors of u in G must contain x and hence are pairwise intersecting. Thus u is simplicial in G. Deleting $T(u)$ yields a subtree representation of $G - u$. We complete a simplicial elimination order of G using such an ordering of $G - u$ given by the induction hypothesis. ■



Because the class of chordal graphs is hereditary, a simplicial elimination ordering can start with any simplicial vertex. Thus a brute-force approach to finding such an ordering would be to examine neighborhoods until we find a simplicial vertex, delete it, and iterate.

Rose–Tarjan–Lueker [1976] found a faster way, which was simplified further by Tarjan [1976]. The idea here, because there is always a simplicial vertex among the vertices farthest from a given vertex (proof of Theorem 5.3.17), is that a simplicial elimination ordering can *end* at any vertex. Thus we start with an arbitrary vertex and list the vertices clumped around it. The result is a simplicial construction ordering (the reverse of a simplicial elimination ordering) if and only if the graph is chordal. The algorithm was published with several applications in Tarjan–Yannakakis [1984]; we follow Golumbic [1984].

8.1.12. Algorithm. *Maximum Cardinality Search (MCS)*
Input: A graph G.
Output: A vertex numbering - a bijection $f: V(G) \to \{1, \ldots, n(G)\}$.
Idea: For each unnumbered vertex v, maintain a label $l(v)$ that is its degree among the vertices already numbered. The vertices at the end of a simplicial elimination ordering are those clumped around the last vertex, so in a simplicial construction ordering the vertices with high labels should be added first.
Initialization: Assign label 0 to every vertex. Set $i = 1$.
Iteration: Select any unnumbered vertex with maximum label. Number it i and add 1 to the label of its neighbors. Augment i and iterate. ■

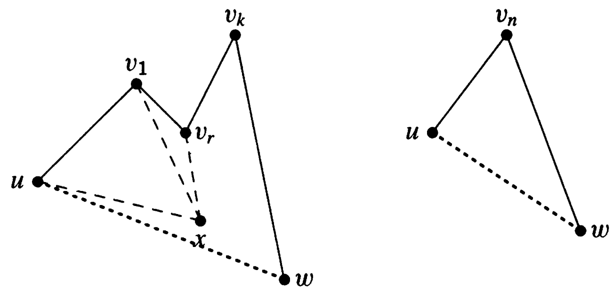
8.1.13. Example. The first vertex chosen in the MCS order is arbitrary. An application of MCS to the graph G above could start by setting $f(c) = 1$ and hence $l(b) = l(d) = l(e) = 1$. Next we could select $f(e) = 2$ and update $l(d) = 2$, $l(h) = l(u) = 1$. Now d is the only vertex with label as large as 2, and hence $f(d) = 3$. We update $l(b) = l(h) = l(u) = 2$, $l(g) = 1$, $l(a) = 0$. Continuing the procedure can produce the order c, e, d, b, h, g, a, u in increasing order of f. This is a simplicial construction ordering, and u, a, g, h, b, d, e, c is a simplicial elimination ordering. ■

8.1.14. Theorem. (Tarjan [1976]). A simple graph G is chordal if and only if the numbering $v_1, \dots, v_n$ produced by the Maximum Cardinality Search algorithm is a simplicial construction ordering of G.

Proof: If MCS produces a simplicial construction ordering, then G is chordal. Conversely, suppose that G is chordal, and let $f\colon V(G) \to [n]$ be the numbering produced by MCS. A *bridge* of f is a chordless path of length at least 2 whose lowest numbers occur at the endpoints. We prove first that f has no bridge. Otherwise, let $P = u, v_1, \dots, v_k, w$ be a bridge that minimizes $\max\{f(u), f(w)\}$. By symmetry, we may assume that $f(u) > f(w)$ (f is used as the vertical coordinate to position vertices in the illustration).

Since u is numbered in preference to v_k at time $f(u)$, and w is already numbered at that time, there exists a vertex $x \in N(u) - N(v_k)$ with $f(x) < f(u)$. Letting $v_0 = u$, set $r = \max\{j\colon x \leftrightarrow v_j\}$. The path $P' = x, v_r, \dots, v_k, w$ is chordless, since $x \leftrightarrow w$ would complete a chordless cycle. Since both of $f(x)$, $f(w)$ are less than $f(u)$, P' is a bridge that contradicts the choice of P. Hence f has no bridge.

With this claim, the proof follows by induction on $n(G)$. It suffices to show that v_n is simplicial, since the application of MCS to $G - v_n$ produces the same numbering $v_1, \dots, v_{n-1}$ that leaves v_n at the end. If v_n is not simplicial, then v_n has nonadjacent neighbors u, w, in which case u, v_n, w is a bridge of f. ■



The MCS algorithm runs in time $O(n(G) + e(G))$ with careful implementation. For each j, we maintain a doubly linked list of the vertices with label j. For each vertex we store its label and pointers to its neighbors and to its position in the lists. When v is numbered, in time $O(1 + d(v))$ we remove v from its list, augment its neighbors labels, and move its neighbors into the next higher lists. To complete the chordality test, we must also check whether the MCS order is a simplicial construction ordering (Exercise 10). Simplicial elimination or construction orderings quickly yield optimal colorings, cliques, stable sets, and clique coverings (Exercise 9).

The alternative algorithm found by Rose, Tarjan, and Leuker is known as Lexicographic Breadth First Search (LBFS). Closely related to the proof of Theorem 5.3.17, LBFS has been used for many applications in testing graph properties and computing graph parameters. Corneil–Olariu–Stewart [2000] provides a good introduction to this topic.

Given a simplicial elimination ordering, Theorem 8.1.14 computes a subtree representation. When the list of maximal cliques is known, Kruskal's algorithm (Theorem 2.3.3) can be used to compute a subtree representation without knowing a simplicial elimination ordering.

8.1.15. Definition. A tree T is a **clique tree** of G if there is a bijection between $V(T)$ and the maximal cliques of G such that for each $v \in V(G)$ the cliques containing v induce a subtree of T.

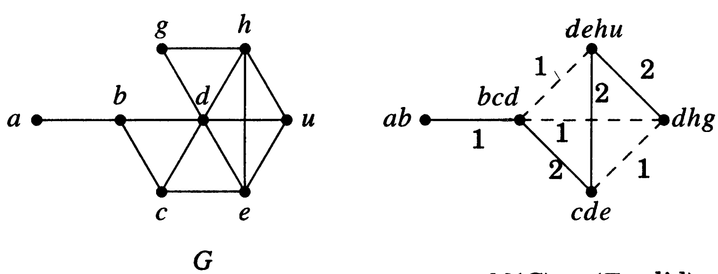
8.1.16. Lemma. Every tree of minimum order in which G has a subtree representation is a clique tree of G.

Proof: Let T be a host tree of minimum order for a subtree representation of G. By Lemma 8.1.10, the vertices of a maximal clique Q in G occur at a common vertex q of T. If the vertices of G assigned to some $q' \in V(T)$ form a proper subclique Q' of Q, then the subtrees for these vertices contain the entire q',q-path in T. The first edge of T on that path can be contracted without changing the intersection graph, which yields a smaller host tree. ■

The **weighted intersection graph** of a collection **A** of finite sets is a weighted clique in which the elements of **A** are the vertices and the weight of each edge AA' is $|A \cap A'|$.

8.1.17. Theorem. (Acharya–Las Vergnas [1982]) Let $M(G)$ be the weighted intersection graph of the set of maximal cliques $\{Q_i\}$ of a simple graph G. If T is a spanning tree of $M(G)$, then $w(T) \le \sum n(Q_i) - n(G)$, with equality if and only T is a clique tree.

Proof: (McKee [1993]) Let T be a spanning tree of $M(G)$. Let T_v be the subgraph of T induced by $\{Q_i\colon v \in Q_i\}$. Each vertex $v \in V(G)$ contributes once to the weight of T for each edge of T_v; hence $w(T) = \sum_{v \in V(G)} e(T_v)$. Each T_v is a forest, so $e(T_v) \le n(T_v) - 1$, with equality if and only if T_v is a tree. The term $n(T_v)$ contributes 1 to the size of each clique containing v. Summing the inequality for each vertex yields $w(T) \le \sum n(Q_i) - n(G)$. Equality holds if and only if each T_v is a tree, which is true if and only if T is a clique tree. ■



$M(G)$ (T solid)

$$w(T) = 7 = 15 - 8 = \sum n(Q_i) - n(G)$$

As a consequence of Theorem 8.1.17, we can test whether G is a chordal graph by finding the maximum weight of a spanning tree in $M(G)$. Furthermore, when G is chordal the clique trees are precisely the maximum-weight spanning trees of $M(G)$ (Bernstein–Goodman [1981], Shibata [1988]; see McKee [1993] for related material).

OTHER CLASSES OF PERFECT GRAPHS

Interval graphs are the intersection graphs of collections of intervals on a line. We proved directly in Proposition 5.1.16 that they are perfect; this also follows from being a subclass of the chordal graphs (Exercise 26). Interval graphs arise in linear scheduling problems having constraints on concurrent events (recall Example 5.1.15).

8.1.18. Example. *Classical applications of interval graphs.*

Analysis of DNA chains. Interval graphs were invented for the study of DNA. Benzer [1959] studied the linearity of the chain for higher organisms. Each gene is encoded as an interval, except that the relevant interval may contain a dozen or more irrelevant junk pieces called "introns" among the relevant pieces called "exons". Under the hypothesis that mutations arise from alterations of connected segments, changes in traits of microorganisms can be studied to determine whether their determining amino-acid sets could intersect. This establishes a graph with traits as vertices and "common alterations" as edges. Under the hypotheses of linearity and contiguity, the graph is an interval graph, and this aids in locating genes along the DNA sequence.

Timing of traffic lights. Given traffic streams at an intersection, a traffic engineer (or a person with common sense) can determine which pairs of streams may flow simultaneously. Given an "all-stop" moment in the cycle, the intersection graph of the green-light intervals must be an interval graph whose edges are a subset of the allowable pairs. These can be studied to optimize some criterion such as average waiting time (see Roberts [1978]).

Archeological seriation. Given pottery samples at an archeological dig, we seek a time-line of what styles were used when. Assume that each style was used during one time interval and that two styles appearing in the same grave were used concurrently. Let two styles be an edge if they appear together in a grave. If this graph is an interval graph, then its interval representations are the possible time-lines. Otherwise, the information is incomplete, and the desired interval graph requires additional edges. ■

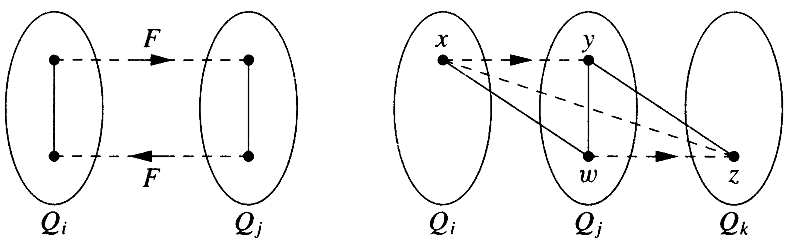
We present two characterizations of interval graphs. Property B in Theorem 8.1.20 is due to Gilmore and Hoffman [1964], and property C is due to Fulkerson and Gross [1965].

8.1.19. Definition. A 0,1-matrix has the **consecutive 1s property** (for columns) if its rows can be permuted so that the 1s in each column appear consecutively. The **clique-vertex incidence matrix** of G is the incidence matrix with rows indexed by the maximal cliques and columns indexed by $V(G)$.

8.1.20. Theorem. The following equivalent conditions on a graph G characterize the interval graphs.
A) G has an interval representation.
B) G is a chordal graph, and $\overline{G}$ is a comparability graph.
C) The clique-vertex incidence matrix has the consecutive 1s property.

Proof: We leave A $\Rightarrow$ B and A $\Leftrightarrow$ C to Exercises 26–27, proving B $\Rightarrow$ C here. Let G be a chordal graph such that $\overline{G}$ has a transitive orientation F. We use F and the absence of chordless cycles in G to establish an ordering on the maximal cliques of G that exhibits the consecutive 1s property for the clique-vertex incidence matrix M.

Let Q_i and Q_j be maximal cliques in G. By maximality, each vertex of one clique has a nonneighbor in the other. Suppose that under F, some edge of $\overline{G}$ points from Q_i to Q_j and some edge of $\overline{G}$ points from Q_j to Q_i. If these edges have a common vertex, then the transitivity of F forces an edge of a clique in G to belong to $\overline{G}$. Hence the situation is as on the left below, with the (dashed) edges of F having four distinct vertices. If the two remaining pairs among these four vertices form edges in G, then G has an induced C_4. Hence at least one diagonal is in $\overline{G}$, but each possible orientation of it in F contradicts transitivity. We conclude that all the edges of $\overline{G}$ between vertex sets Q_i and Q_j point in the same direction in F.



We can now define a tournament T with vertices corresponding to the maximal cliques of G. We put $Q_i \to Q_j$ in T when all edges of F between Q_i and Q_j point from Q_i to Q_j. By the preceding paragraph, T is an orientation of a complete graph. We claim that T is transitive. To prove this we need to show that $Q_i \to Q_j$ and $Q_j \to Q_k$ imply $Q_i \to Q_k$. Suppose that $x \to y$ and $w \to z$ in F with $x \in Q_i$, $y, w \in Q_j$, and $z \in Q_k$. If $y = w$, transitivity of F immediately implies $x \to z$. Otherwise, consider a pair xz as shown on the right above. Joining x and z in G would form an induced C_4 in G, so $x \not\leftrightarrow z$. Hence this pair appears in F, and it must be directed from $x \to z$ to avoid violating transitivity. We conclude that $Q_i \to Q_k$ in T.

A transitive tournament specifies a unique linear ordering of the vertices consistent with the edges; use the transitive tournament T to order the rows of M as $Q_1 \to \cdots \to Q_m$. Suppose that under this ordering there is some column x where the 1s do not appear consecutively. Then we have Q_i, Q_j, Q_k such that

$i < j < k$, $x \in Q_i, Q_k$, $x \notin Q_j$. Since $x \notin Q_j$, the clique Q_j must have some vertex y not adjacent to x, else Q_j could absorb x and would not be maximal. Now $x \in Q_i$ implies $x \to y$ in F, and $x \in Q_k$ implies $y \to x$ in F, which cannot both happen. ■

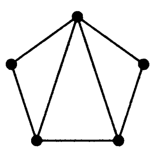
The interval graphs form a relatively small family of perfect graphs. We next discuss larger classes that maintain some of the nice properties of chordal graphs and comparability graphs.

8.1.21. Definition. *Classes of perfect graphs* (conditions on odd cycles apply only for length at least 5).
o-triangulated: every odd cycle has a noncrossing pair of chords.
parity: every odd cycle has a crossing pair of chords.
Meyniel: every odd cycle has at least two chords.
weakly chordal: no induced cycle of length at least 5 in G or $\overline{G}$.
strongly perfect: every induced subgraph has a stable set meeting all its maximal cliques.

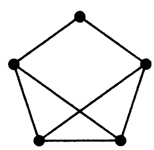
Gallai [1962] proved that o-triangulated graphs are perfect. Every chordal graph is o-triangulated (Exercise 34) and weakly chordal (Exercise 40). All o-triangulated and parity graphs are Meyniel graphs. Meyniel graphs are perfect (Meyniel [1976], Lovász [1983]) and also strongly perfect (Ravindra [1982]).

Parity graphs, shown to be perfect in Olaru [1969] and Sachs [1970], carry that name due to a later characterization by Burlet and Uhry [1984]: G is a parity graph if and only if, for every pair $x, y \in V(G)$, the chordless x, y-paths are all even or all odd (Exercise 36).

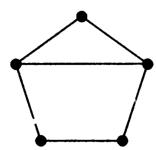
8.1.22. Example. The graphs below exhibit differences among these classes. Here **T**, **C**, **O**, **P**, **M**, **W** respectively denote the classes of chordal (**T**riangulated), comparability, o-triangulated, parity, Meyniel, and weakly chordal graphs. ■



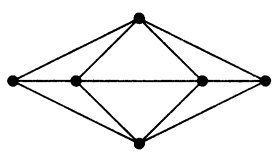
T − P



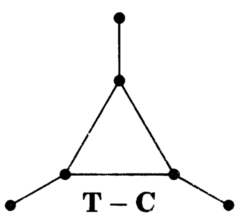
P − O



(W ∩ C) − M



M − O − P



T − C

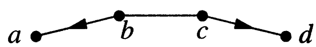
Strongly perfect graphs were introduced by Berge and Duchet [1984]. Changing maximal to maximum in the definition yields a weaker requirement equivalent to γ-perfection; a stable set meeting all maxim*um* cliques can be used as the first color class in an $\omega(G)$-coloring constructed inductively. Thus strongly perfect graphs are perfect.

The class of strongly perfect graphs does not contain all Meyniel graphs or all weakly chordal graphs (Exercises 39–40), but it does contain all chordal graphs and all comparability graphs. (As observed in Proposition 5.3.25, when G has a transitive orientation, each induced subgraph inherits a transitive orientation, and the vertices with indegree 0 in this orientation form a stable set that meets all the maximal cliques.)

Our next class is a subclass of the strongly perfect graphs (Exercises 37–38) that still contains all chordal graphs and comparability graphs. Introduced by Chvátal [1984], it has played an important role in the theory of perfect graphs.

8.1.23. Definition. A **perfect order** on a graph is a vertex ordering such that greedy coloring with respect to the ordering inherited by each induced subgraph produces an optimal coloring of that subgraph. A **perfectly orderable graph** is a graph having a perfect order.

In an orientation of G, an **obstruction** is an induced 4-vertex path a, b, c, d whose first and last edges are oriented toward the leaves. The orientation of G *associated with* a vertex ordering L orients each edge toward the vertex earlier in L: $u \leftarrow v$ if $u < v$. A vertex ordering is **obstruction-free** if its associated orientation has no obstruction.



The orientation associated with a perfect order is obstruction-free, because on an obstruction the greedy coloring would use three colors instead of two. Chvátal proved that a graph is perfectly orderable if and only if it has an obstruction-free ordering. The characterization implies that perfectly orderable graphs are perfect and that chordal graphs and comparability graphs are perfectly orderable.

8.1.24. Example. *Chordal graphs and comparability graphs are perfectly orderable.* The orientation of a chordal graph associated with a simplicial construction ordering has no induced $u \leftarrow v \rightarrow w$. A transitive orientation of a comparability graph has no induced $u \rightarrow v \rightarrow w$.

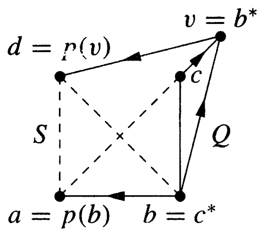
Every orientation with an obstruction has both an induced $u \rightarrow v \rightarrow w$ and an induced $u \leftarrow v \rightarrow w$. Hence if G is a comparability graph or a chordal graph, then G has an obstruction-free ordering. By Chvátal's characterization, such graphs are perfectly orderable. ■

8.1.25. Lemma. (Chvátal [1984]) Let G have a clique Q and a stable set S that are disjoint, and suppose that each $w \in Q$ is adjacent to some $p(w) \in S$. If

L is an obstruction-free ordering of G such that $p(w) < w$ for all $w \in Q$, then some $p(w) \in S$ is adjacent to all of Q.

Proof: We use induction on $n(G)$. For the basis step $n(G) = 1$, there is nothing to prove. Consider $n(G) > 1$. For each $w \in Q$, the graph $G - w$ satisfies the hypotheses using the clique $Q - w$ and the stable set $\{p(u) : u \in Q - w\}$. By the induction hypothesis, there is a vertex $w^* \in Q - w$ such that $p(w^*) \leftrightarrow Q - w$. We obtain $w \in Q$ such that $p(w^*) \leftrightarrow Q$ unless $p(w^*) \not\leftrightarrow w$ for all $w \in Q$. This assigns a unique w^* to every w, since $p(w^*)$ is nonadjacent only to w among Q. Mapping w to w^* thus defines a permutation on Q. Since $p(w) \leftrightarrow w$, the permutation has no fixed point.

We seek an obstruction in the orientation associated with L. Let v be the least vertex of Q in L. Let $b, c \in Q$ be the vertices such that $b^* = v$ and $c^* = b$ (possibly $c = v$). Let $a = p(b)$ and $d = p(v)$. Because $p(w^*) \not\leftrightarrow w$, we have $a \not\leftrightarrow c$ and $d \not\leftrightarrow b$, which implies $a \neq d$ in the stable set S and yields the picture below for the orientation associated with L.



Because $d = p(b^*)$, the only vertex of Q nonadjacent to d is b; thus $c \leftrightarrow d$. Since $d = p(v) < v \leq c$ in L, we have $d \leftarrow c$. Now a, b, c, d induce an obstruction, which contradicts the hypothesis for L. Hence $p(w^*) \leftrightarrow w$ for some w, and $p(w^*)$ is the desired vertex of S. ■

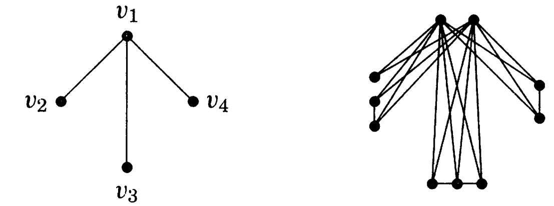
8.1.26. Theorem. (Chvátal [1984]) A vertex ordering of a simple graph G is a perfect order if and only if it is obstruction-free, and every graph with such an ordering is perfect.

Proof: We have observed that the condition is necessary. Since the class of graphs with obstruction-free orderings is hereditary (the inherited ordering for an induced subgraph is also obstruction-free), it suffices to show that the greedy coloring of G relative to an obstruction-free ordering L is optimal. Let k be the number of colors used by the greedy coloring relative to L. To prove optimality, we show that G has a k-clique; this also inductively proves perfection.

Let $f: V(G) \to [k]$ be the resulting coloring. Let i be the least integer such that G has a clique $w_{i+1}, \ldots, w_k$ such that $f(w_j) = j$. Since f uses color k on some vertex, i is well-defined. If $i = 0$, then G has a k-clique.

If $i > 0$, then for each w_j there is a vertex $p(w_j)$ such that $p(w_j) < w_j$ in L and $f(p(w_j)) = i$; otherwise the greedy coloring would use a lower color on w_j. Let $S = \{p(w_{i+1}), \ldots, p(w_k)\}$. Since all of S has the same color, S is a stable set. Hence the conditions of Lemma 8.1.25 are satisfied, and some vertex of S can be added to the clique to become w_i. This contradicts the minimality of i. ■

Next we consider a different way of generating perfect graphs. An operation that preserves perfection can enlarge a class of perfect graphs. Vertex multiplication, which expands each vertex into an independent set, is such a property. We generalize this. If $V(G) = \{v_1, \dots, v_n\}$, and $H_1, \dots, H_n$ are pairwise disjoint graphs, then the **composition** $G[H_1, \dots, H_n]$ is the graph $H_1 + \cdots + H_n$ together with $\{xy\colon x \in V(H_i),\, y \in V(H_j),\, v_iv_j \in E(G)\}$. The special case $G[\overline{K}_{h1}, \dots, \overline{K}_{hn}]$ is $G \circ h$. The example below uses $H_1 = 2K_1$, $H_2 = K_2 + K_1$, $H_3 = P_3$, $H_4 = K_2$, and $G = K_{1,3}$ with central vertex v_1.



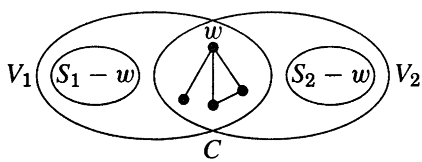
Lovász proved that composition preserves perfection. This is one corollary of Chvátal's Star-Cutset Lemma.

8.1.27. Definition. A **star-cutset** of G is a vertex cut S containing a vertex x adjacent to all of $S - \{x\}$. A **minimal imperfect graph** is an imperfect graph whose proper induced subgraphs are all perfect.

8.1.28. Lemma. (The Star-Cutset Lemma Lemma) If G has no stable set intersecting every maximum clique, and every proper induced subgraph of G is $\omega(G)$-colorable, then G has no star-cutset.

Proof: Suppose that G has a star-cutset C, with w adjacent to all of $C - \{w\}$. Since $G - C$ is disconnected, we can partition $V(G - C)$ into sets V_1, V_2 with no edge between them. Let $G_i = G[V_i \cup C]$, and let f_i be a proper $\omega(G)$-coloring of G_i. Let S_i be the set of vertices in G_i with the same color in f_i as w; this includes w but no other vertex of C. Since there are no edges between V_1 and V_2, the union $S = S_1 \cup S_2$ is a stable set.

If Q is a clique in $G - S$, then Q is contained in $G_1 - S_1$ or in $G_2 - S_2$. Since f_i provides an $\omega(G) - 1$-coloring of $G_i - S_i$, we have $|Q| \le \omega(G) - 1$. Since this applies to every clique Q in $G - S$, the stable set S meets every $\omega(G)$-clique of G, which contradicts the hypotheses. ■



8.1.29. Theorem. (The Star-Cutset Lemma, Chvátal [1985b]) No minimal imperfect graph has a star-cutset.

Proof: If G is a minimal imperfect graph, then $\chi(G) > \omega(G)$ and deletion of any stable set S leaves a perfect graph. Hence we have

$$1+\omega(G) \le \chi(G) \le 1+\chi(G-S) = 1+\omega(G-S) \le 1+\omega(G).$$

This yields $\omega(G-S) = \omega(G)$, which means that no stable set meets every maximum clique. Furthermore, since G is minimally imperfect, every proper induced subgraph G' satisfies $\chi(G') = \omega(G') \le \omega(G)$, making it $\omega(G)$-colorable. Lemma 8.1.28 now implies that G has no star-cutset. ■

The Replacement Lemma generalizes Lemma 8.1.4.

8.1.30. Corollary. (Replacement Lemma—Lovász [1972b]) Every composition of perfect graphs is perfect.

Proof: A composition can be constructed by a sequence of substitutions in which a single vertex v of G_1 is replaced by a graph G_2 and all edges added between $V(G_2)$ and $U = N_{G_1}(v)$ to form a graph G. Hence it suffices to prove that this operation preserves perfection. If the resulting graph G is not perfect, then it contains a minimal imperfect induced subgraph F. Such a subgraph cannot be contained in G_1 or G_2, which forces it to have at least two vertices of G_2 and at least one vertex of G_1.

If F has no vertex of G_1 outside U, then $F = F[U] \vee (F \cap G_2)$. The join operation preserves perfection, since $\chi(H \vee H') = \chi(H) + \chi(H')$ and $\omega(H \vee H') = \omega(H) + \omega(H')$ for all H, H'. Hence we may assume that F has a vertex of G_1 outside U. In this case, $V(F) \cap U$ together with one vertex of G_2 in F is a star-cutset of F. Hence the replacement of v with G_2 introduces no minimal imperfect subgraph F. ■

The Star-Cutset Lemma also yields perfection of weakly chordal graphs. Hayward [1985] proved that G or $\overline{G}$ has a star-cutset when G is a weakly chordal graph that is not a clique or stable set. With the Star-Cutset Lemma and the Perfect Graph Theorem, this implies that no weakly chordal graph is a minimal imperfect graph. Since the class is hereditary, it follows that every weakly chordal graph is perfect.

IMPERFECT GRAPHS

The **p-critical** graphs are the minimal imperfect graphs. The Strong Perfect Graph Conjecture (SPGC) states that the only p-critical graphs are the odd cycles (of length at least 5) and their complements. With enough properties of p-critical graphs, perhaps we could prove that only odd cycles and their complements have all these properties; this would prove the SPGC. We begin with simple observations about p-critical graphs, some used earlier in discussing star-cutsets. (This presentation was originally modeled after Shmoys [1981].)

8.1.31. Lemma. If G is p-critical, then G is connected, $\overline{G}$ is p-critical, $\omega(G) \geq 2$, and $\alpha(G) \geq 2$. Furthermore, for every $x \in V(G)$, $\chi(G-x) = \omega(G)$ and $\theta(G-x) = \alpha(G)$.

Proof: G is perfect if and only if every component of G is perfect, and G is perfect if and only if $\overline{G}$ is perfect. Cliques and their complements are perfect. Finally, we observed in proving Theorem 8.1.29 that deleting a stable set from a p-critical graph cannot decrease the clique number. Since $G-x$ is perfect, we thus have $\chi(G-x) = \omega(G-x) = \omega(G)$. The condition $\theta(G-x) = \alpha(G)$ is this statement for $\overline{G}$. ■

More subtle properties of p-critical graphs follow from Lovász's extension of the PGT.

8.1.32. Theorem. (Lovász [1972b]) A graph G is perfect if and only if $\omega(G[A])\alpha(G[A]) \geq |A|$ for all $A \subseteq V(G)$. ■

The property "$\omega(G[A])\alpha(G[A]) \geq |A|$ for all $A \subseteq V(G)$" was suggested by Fulkerson; we call it **β-perfection**. It is implied by α-perfection or γ-perfection; if we can color G with $\omega(G)$ stable sets, then some stable set has at least $n(G)/\omega(G)$ vertices. The converse involves counting arguments like those we gave for the PGT, but more delicate. Since β-perfection is unchanged under complementation, Theorem 8.1.32 immediately implies the PGT.

8.1.33. Theorem. If G is p-critical, then $n(G) = \alpha(G)\omega(G) + 1$. Furthermore, for every $x \in V(G)$, $G-x$ has a partition into $\omega(G)$ stable sets of size $\alpha(G)$ and a partition into $\alpha(G)$ cliques of size $\omega(G)$.

Proof: When G is p-critical, the condition for β-perfection fails only for the full vertex set $A = V(G)$. Hence for each $x \in V(G)$ we have

$$n(G) - 1 \leq \alpha(G-x)\omega(G-x) = \alpha(G)\omega(G) \leq n(G) - 1.$$

Therefore, $n(G) = \alpha(G)\omega(G) + 1$. Since $\chi(G-x) = \omega(G-x) = \omega(G)$, we can cover $G-x$ by $\omega(G)$ stable sets. Having size at most $\alpha(G)$, these sets partition the $\alpha(G)\omega(G)$ vertices of $G-x$ into $\omega(G)$ stable sets of size $\alpha(G)$. Similarly, $\theta(G-x) = \alpha(G-x) = \alpha(G)$ yields a partition of $V(G-x)$ into $\alpha(G)$ cliques of size $\omega(G)$. ■

Study of p-critical graphs has benefitted by enlarging the class to include other graphs satisfying the properties in Theorem 8.1.33. Structural properties of the larger class are useful when proving the SPGC for special classes of graphs. Padberg [1974] began the study of these graphs. Several definitions were suggested to extend the class of p-critical graphs but turned out to be alternative characterizations of the same class. The definition we use originates in Bland–Huang–Trotter [1979].

8.1.34. Definition. For integers $a, w \geq 2$, a graph G is **a, w-partitionable** if it has $aw + 1$ vertices and for each $x \in V(G)$ the subgraph $G - x$ has a partition into a cliques of size w and a partition into w stable sets of size a.

8.1.35. Theorem. (Buckingham–Golumbic [1983]) A graph G of order $aw + 1$ is a, w-partitionable if and only if $\chi(G - x) = w$ and $\theta(G - x) = a$ for every $x \in V(G)$. Furthermore, $\omega(G) = w$ and $\alpha(G) = a$ for such graphs, and the inequalities $\chi(G - x) \le w$ and $\theta(G - x) \le a$ suffice.

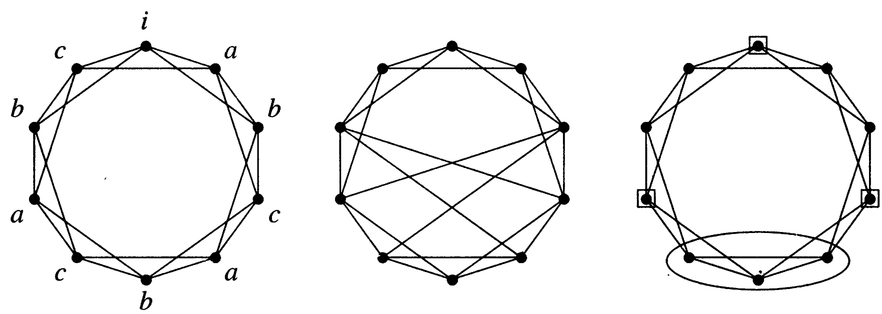
Proof: Let G be partitionable. Since $G - x$ is w-colorable and has a w-clique, $\chi(G - x) = w = \omega(G - x)$. Since $a \ge 2$, G is not a complete graph. Deleting a vertex x outside a maximum clique Q in G yields $\omega(G) = \omega(G - x) = w$. The same arguments for $\overline{G}$ yield the results for a.

Conversely, suppose that $\chi(G - x) \le w$ and $\theta(G - x) \le a$ for every x in $V(G)$. The latter inequality yields $\alpha(H) \le a$. Hence an optimal coloring of $G - x$ uses at most w stable sets of size at most a. Since $n(G - x) = aw$, such a coloring partitions $V(G - x)$ into w stable sets of size a. Similarly, a covering of $G - x$ by a cliques yields the desired clique partition. ■

By Theorem 8.1.33 and Theorem 8.1.35, every p-critical graph is partitionable and every partitionable graph is imperfect. Furthermore, G is a, w-partitionable if and only if $\overline{G}$ is w, a-partitionable.

8.1.36. Example. *Cycle-powers.* The graph C_n^d is constructed by placing n vertices on a circle and making each vertex adjacent to the d nearest vertices in each direction on the circle. When $d = 1$, $C_n^d = C_n$. We view the vertices as the integers modulo n, in order. The graph C_{10}^2, shown on the left below, is neither perfect nor p-critical (the vertices $0, 2, 4, 6, 8$ induce C_5), but C_{10}^2 is 3,3-partitionable. When i is removed, the unique partition of the remaining nine vertices into three triangles is $\{(i+1, i+2, i+3), (i+4, i+5, i+6), (i+7, i+8, i+9)\}$, and the unique partition into three stable sets is $\{(i+1, i+4, i+7), (i+2, i+5, i+8), (i+3, i+6, i+9)\}$.

Always C_{aw+1}^{w-1} is a, w-partitionable. Every w consecutive vertices in $G - x$ form a clique, and every a vertices spaced by jumps of length w form a stable set. Showing that C_{aw+1}^{w-1} is p-critical if and only if $w = 2$ or $a = 2$ reduces the SPGC to the statement that G is p-critical if and only if $G = C_{\alpha(G)\omega(G)+1}^{\omega(G)-1}$. ■



8.1.37. Example. *Other partitionable graphs.* Other partitionable graphs arise by adding unimportant edges to C_{aw+1}^{w-1}. In C_{10}^2, we can add any diagonal without changing the set of maximum cliques or the set of maximum stable sets, so the resulting graph is still partitionable. We will see that the SPGC would follow if all partitionable graphs came from cycle-powers by adding unimportant edges of this type.

Nevertheless, there are other partitionable graphs, such as the graph in the middle above (Chvátal–Graham–Perold–Whitesides [1979], Huang [1976]). Every edge in this graph belongs to a maximum clique, but it has two more edges than C_{10}^2. The partitions demonstrating that it is partitionable differ from those used for C_{10}^2 (Exercise 42). ■

8.1.38. Example. *Further properties of* C_{aw+1}^{w-1}. The graph C_{aw+1}^{w-1} has exactly n maximum cliques, each using w consecutive vertices on the cycle. Each vertex lies in w consecutive w-cliques. There are also exactly n maximum stable sets, each having $a-1$ gaps of length w and one gap of length $w+1$ between successive vertices. A maximum stable set containing x has a places for the larger gap, so each vertex x lies in a maximum stable sets.

Finally, a w-clique can avoid a maximum stable set only by fitting inside the gap of length $w+1$ (shown above Example 8.1.37 on the right). Thus there is a pairing $\{(Q_i, S_i)\}$ between the maximum stable sets and maximum cliques such that $Q_i \cap S_j = \emptyset$ if and only if $i = j$. ■

These "further properties" comprise the next characterization. The arguments are due to Padberg [1974], who used them in a polyhedral characterization of perfect graphs. Here combinatorial conclusions follow from properties of matrices in linear algebra. Other characterizations of partitionable graphs appeared in Bland–Huang–Trotter [1979], Golumbic [1980, p58-62], Tucker [1977], Chvátal–Graham–Perold–Whitesides [1979], and Buckingham [1980].

8.1.39. Theorem. A graph G of order $n = aw+1$ is a, w-partitionable if and only if both conditions below hold:

1) $\alpha(G) = a$ and $\omega(G) = w$, and each vertex of G belongs to exactly w cliques of size w and a stable sets of size a.

2) G has exactly n maximum cliques $\{Q_i\}$ and exactly n maximum stable sets $\{S_j\}$, with $Q_i \cap S_j = \emptyset$ if and only if $i = j$ (Q_i and S_i are **mates**).

Proof: *Necessity.* We have proved $\chi(G-x) = w = \omega(G)$ and $\theta(G-x) = a = \alpha(G)$ for each $x \in V(G)$. Choose a clique Q of size w. For each $x \in Q$, $G-x$ has a partition into a cliques of size w. Together, Q and these w partitions form a list of $n = aw+1$ maximum cliques $Q_1, \ldots, Q_n$. Each vertex outside Q appears in one clique in each partition. Each vertex in Q appears in Q and once in $w-1$ partitions. Hence every vertex appears in exactly w cliques in the list.

For each Q_i, we obtain a maximum stable set S_i disjoint from Q_i. Choose $x \in Q_i$. The w maximum stable sets that partition $V(G-x)$ can meet Q_i only at the $w-1$ vertices other than x. Therefore, one of these stable sets misses Q_i; call it S_i. We will show that these two lists contain all the cliques and stable sets and have the desired intersection properties.

Let A be the incidence matrix with $a_{i,j} = 1$ if $x_j \in Q_i$ and $a_{i,j} = 0$ otherwise. Let B be the matrix with $b_{i,j} = 1$ if $x_j \in S_i$ and $b_{i,j} = 0$ otherwise. The ijth entry of AB^T is the dot product of row i of A with row j of B, which equals $|Q_i \cap S_j|$. By proving that $AB^T = J - I$, where J is the matrix of all 1s, we obtain $Q_i \cap S_j \neq \emptyset$ if and only if $i \neq j$. Since $J - I$ is nonsingular, this will also imply that A and B are nonsingular. Nonsingular matrices have distinct rows, and hence $Q_1, \ldots, Q_n$ and $S_1, \ldots, S_n$ will be distinct.

By construction, $|Q_i \cap S_i| = 0$. Since cliques and stable sets intersect at most once, to prove that $AB^T = J - I$ we need only show that each column of AB^T sums to $n - 1$. Multiplying by the row vector $\mathbf{1}_n^T$ on the left computes these sums. We constructed A so that each column has w 1s (because each vertex appears in w cliques in the list) and B so that each row has a 1s (because each stable set has size a). Therefore,

$$\mathbf{1}_n^T(AB^T) = (\mathbf{1}_n^T A)B^T = w\mathbf{1}_n^T B^T = wa\mathbf{1}_n = (n-1)\mathbf{1}_n^T.$$

To prove that G has no other maximum cliques, we let q be the incidence vector of a maximum clique Q and show that q must be a row of A. Since A is nonsingular, its rows span $\mathbb{R}^n$, and we can write q as a linear combination: $q = tA$. To solve for t, we need A^{-1}. Since every row of A sums to ω, we have $A(\omega^{-1}J - B^T) = \omega^{-1}\omega J - (J - I) = I$, and hence $A^{-1} = \omega^{-1}J - B^T$. Thus,

$$t = qA^{-1} = q(\omega^{-1}J - B^T) = \omega^{-1}qJ - qB^T = \omega^{-1}\omega\mathbf{1}_n^T - qB^T.$$

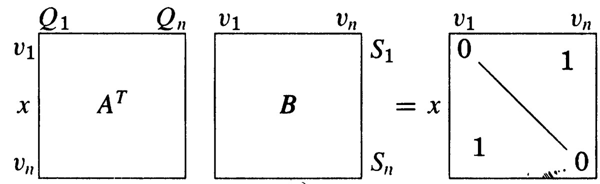
The ith column of B^T is the incidence vector of S_i; hence coordinate i of qB^T equals $|Q \cap S_i|$, which is 0 or 1. Hence t is a 0,1-vector and q is a sum of rows of A. Since q sums to ω, only one row can be used. Thus q is a row of A and $Q_1, \ldots, Q_n$ are the only maximum cliques.

The same argument applied to $\overline{G}$ shows that G has exactly n maximum stable sets, with each vertex appearing in a of them.

Sufficiency. By Theorem 8.1.35, we need only prove that $\chi(G - x) \leq w$ and $\theta(G - x) \leq a$ for all $x \in V(G)$. Given the cliques and stable sets as guaranteed by condition (2), define the incidence matrices A, B as above. By condition (1), each column of B has a 1s, and hence $JB = aJ = BJ$. The intersection requirements in condition (2) yield $AB^T = J - I$. This is nonsingular, so B is nonsingular and

$$A^TB = B^{-1}BA^TB = B^{-1}(J - I)B = B^{-1}BJ - I = J - I.$$

In the product $A^TB = J - I$, the row corresponding to $x \in V(G)$ states that $V(G - x)$ is covered by the mates of the w maximum cliques containing x (illustrated below), and hence $\chi(G - x) \leq w$. Similarly, the column corresponding to x states that $V(G - x)$ is covered by the mates of the a maximum stable sets containing x, and hence $\theta(G - x) \leq a$. ∎



8.1.40. Corollary. If G is α, ω-partitionable and $\omega = 2$, then $G = C_{2\alpha+1}$; if $\alpha = 2$, then $G = \overline{C}_{2\omega+1}$. Hence the SPGC reduces to showing that p-critical graphs have $\omega = 2$ or $\alpha = 2$.

Proof: If $\omega = 2$, then every vertex belongs to exactly two cliques of size 2, so G is 2-regular. Furthermore, G is connected and has odd order $(2\alpha + 1)$, so G is an odd cycle. For $\alpha = 2$, consider $\overline{G}$. ∎

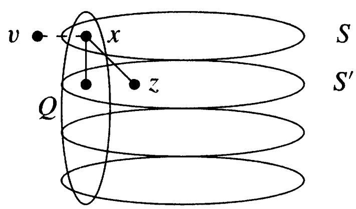
Henceforth we use $w, \omega(G), \omega$ interchangeably and $a, \alpha(G), \alpha$ interchangeably for partitionable graphs.

8.1.41. Theorem. (Tucker [1977]) Let x be a vertex in a partitionable graph G. The subgraph $G - x$ has a unique minimum coloring; denoted $\mathrm{X}(G - x)$, it consists of the mates of the maximum cliques containing x. Similarly, $G - x$ has a unique minimum clique covering $\mathrm{X}(G - x)$ consisting of the mates of the maximum stable sets containing x.

Proof: Since G is a, w-partitionable, $G - x$ is w-colorable using w stable sets of size a. Every w-clique containing x misses some color class, since the clique has only $w - 1$ vertices in $G - x$. Thus all w-cliques containing x have mates as color classes in the coloring. There are exactly w of these, so the coloring is unique. The other statement follows by complementation. ∎

8.1.42. Theorem. (Buckingham–Golumbic [1983]) If x is a vertex of an α, ω-partitionable graph G, then $2\omega - 2 \leq d(x) \leq n - 2\alpha + 1$.

Proof: Select a vertex $v \not\leftrightarrow x$ (see illustration above). Let S be the stable set in $\mathrm{X}(G - v)$ that contains x, and let S' be another stable set in $\mathrm{X}(G - v)$. Choose $z \in N(x) \cap S_2$. In $\Theta(G - z)$, some clique Q contains x. Since $v \not\leftrightarrow x$, Q has one vertex in each stable set of $\mathrm{X}(G - v)$, including S'. Since $Q \in \Theta(G - z)$ implies $z \notin Q$, this yields a second neighbor of x in S'. Thus x has at least two neighbors in each of the $\omega - 1$ stable sets in $\mathrm{X}(G - v)$, yielding $d(x) \geq 2\omega - 2$. The same argument in $\overline{G}$ yields $n - 1 - d(x) = |N_{\overline{G}}(x)| \geq 2\alpha - 2$. ∎



These bounds on vertex degrees in α, ω-partitionable graphs are sharp, as they hold with equality for powers of cycles.

8.1.43. Definition. An edge of a graph is **critical** if deleting it increases the independence number. A pair of nonadjacent vertices is **co-critical** if adding it increases the clique number.

The characterization of critical edges in partitionable graphs is implicit in the work of several authors.

8.1.44. Theorem. For an edge xy in a partitionable graph G, the following statements are equivalent.
A) xy is a critical edge.
B) $S \cup \{x\} \in \mathrm{X}(G-y)$.
C) xy belongs to $\omega - 1$ maximum cliques.

Proof: B $\Rightarrow$ A. $S \cup \{x, y\}$ is a stable set of size $\alpha + 1$ in $G - xy$.

A $\Rightarrow$ C. If xy is critical, then there is a set S such that $S \cup \{x\}$ and $S \cup \{y\}$ are maximum stable sets in G. Hence every maximum clique containing x but not y is disjoint from $S \cup \{y\}$. Since there are ω maximum cliques containing x and only one maximum clique disjoint from $S \cup \{y\}$, the remaining $\omega - 1$ maximum cliques containing x must also contain y.

C $\Rightarrow$ B. The stable sets in the unique coloring of $G - x$ are the mates of the cliques containing x. Since xy belongs to $\omega - 1$ maximum cliques, the mates of these $\omega - 1$ cliques belong to both $\mathrm{X}(G-x)$ and $\mathrm{X}(G-y)$. This leaves only $a+1$ vertices in the graph, consisting of the vertices x, y and a stable set S such that $S \cup \{y\} \in \mathrm{X}(G-x)$ and $S \cup \{x\} \in \mathrm{X}(G-y)$. ∎

8.1.45. Corollary. Let G be a partitionable graph. If xy is an edge appearing in no maximum clique, then $G - xy$ is partitionable. If x, y is a nonadjacent pair appearing in no maximum stable set, then $G + xy$ is partitionable.

Proof: By complementation, we need only prove the first statement. If we delete an edge appearing in no maximum clique, then by Theorem 8.1.44 it is not a critical edge, and we have $\omega(G - xy) = \omega(G)$ and $\alpha(G - xy) = \alpha(G)$. Since we have not destroyed any maximum clique and have not created a bigger stable set, we can use the optimal coloring and clique partition of $G - u$ to conclude that $\chi(G - xy - u) \leq \omega$ and $\theta(G - xy - u) \leq \alpha$. Hence $G - xy$ is partitionable, by Theorem 8.1.35. ∎

The discussion in Example 8.1.37 suggests that edges appearing in no maximum clique are uninteresting "junk". Corollary 8.1.45 assures us that "junk is junk". The partitionable cycle-powers have no junk.

THE STRONG PERFECT GRAPH CONJECTURE

We have been proving properties of partitionable graphs in a "top down" approach to the SPGC, trying to find enough properties to eliminate all but odd cycles and their complements as p-critical graphs. The "bottom up" approach is to verify that the SPGC holds on larger and larger classes of graphs, until all are included.

8.1.46. Definition. An **odd hole** or **odd antihole** in G is an induced subgraph of G that is C_{2k+1} or $\overline{C}_{2k+1}$ (for some $k \geq 2$), respectively. A graph having no odd hole or antihole is a **Berge graph**.

One way to prove that a class **G** satisfies the SPGC is to prove that every Berge graph in **G** is perfect. A hereditary class **G** satisfies the SPGC if the odd cycles and their complements are the only p-critical graphs in **G**.

The SPGC holds for planar graphs (Tucker [1973]), toroidal graphs (Grinstead [1981]), graphs with $\Delta(G) \le 6$ (Grinstead [1978]) or $\omega(G) \le 3$ (Tucker [1977]), and for various classes defined by forbidding fixed small induced subgraphs (Meyniel [1976], Tucker [1977], Parthasarathy–Ravindra [1976, 1979], Chvátal–Sbihi [1988], Olariu [1989], Sun [1991]). We consider three families.

8.1.47. Definition. A **circular-arc graph** is the intersection graph of a family of arcs of a circle. A **circle graph** is the intersection graph of a family of chords of a circle. A $K_{1,3}$**-free** graph is a graph not having $K_{1,3}$ as an induced subgraph.

Every cycle is both a circle graph and a circular-arc graph, but neither of these classes contains the other (Exercise 47).

One way to prove the SPGC for a class **G** is to show that every partitionable graph in **G** belongs to another class **H** where the SPGC is known to hold. In this role we use the class $\{C_n^d\}$.

8.1.48. Theorem. (Chvátal [1976]) Cycle-powers satisfy the SPGC. In particular, the graph C_{aw+1}^{w-1} is p-critical if and only if $w = 2$ or $a = 2$, in which case the graph is an odd hole or antihole.

Proof: It suffices to consider the partitionable graph $G = C_{aw+1}^{w-1}$. This is p-critical when $a = 2$ or $w = 2$, so we may assume $a, w > 2$. Let the vertices be $\{v_0, \dots, v_{aw}\}$, and let $S = \{v_{iw+1}, v_{(i+1)w}: 0 \le i \le a-1\}$. The subgraph $G[S]$ is a cycle, since the indices of consecutive vertices in S are separated by 1 or $w-1$ (except that v_{aw} and v_1 are separated by 2), and indices of nonconsecutive vertices differ by at least w. To obtain C_{2a-1} as a proper induced subgraph, we replace $\{v_{(a-1)w}, v_{aw}, v_1, v_w\}$ with $\{v_{(a-1)w+1}, v_0, v_{w-1}\}$ in S. We conclude that G is not p-critical. ■

8.1.49. Theorem. (Tucker [1975]) The SPGC holds for circular-arc graphs.

Proof: Recall that $N[v]$ denotes $N(v) \cup \{v\}$, the closed neighborhood of v (Definition 3.1.29). When G is partitionable with distinct vertices x, y, we claim that $N[x] \not\subseteq N[y]$. Consider the clique Q containing x in $\Theta(G-y)$; we have $Q \subseteq N[x]$. If $N[y]$ contains $N[x]$, then $Q \cup \{y\}$ is a clique of size $\omega(G)+1$.

Now, if G is a partitionable circular-arc graph, it suffices to show that $G = C_n^{\omega(G)-1}$, because the SPGC holds for cycle-powers (Theorem 8.1.48). Consider a circular-arc representation that assigns arc A_x to $x \in V$. Since $N[y]$ cannot contain $N[x]$, the arc A_x cannot lie within another arc A_y of the representation. If no arc contains another, then every arc that intersects A_x contains exactly one of its endpoints. Since the vertices corresponding to the arcs containing one point induce a clique, there are at most $\omega - 1$ other arcs containing each endpoint of A_x. Equality holds, since Theorem 8.1.42 requires $\delta(G) \ge 2\omega - 2$.

Starting from a given point p on the circle, let v_i be the vertex represented by the ith arc encountered moving clockwise from p. Since each arc meets exactly $\omega - 1$ others at each endpoint, v_i is adjacent to $v_{i+1}, \ldots, v_{i+\omega-1}$ (addition modulo n) for each i. Hence $G = C_n^{\omega-1}$. ■

The original proof of the SPGC for claw-free graphs (Parthasarathy–Ravindra [1976]) was quite intricate. Further study of p-critical graphs has shortened both it and the proof of the next theorem, which we will apply.

8.1.50. Theorem. (Giles–Trotter–Tucker [1984]) If a partitionable graph G has a cycle consisting of critical edges, then the subgraph G' obtained by deleting the edges belonging to no maximum clique is $C_n^{\omega-1}$

Proof: (Hartman [1995]) Suppose that G is a, w-partitionable. Deleting edges destroys no stable set. Deleting edges in no maximum clique destroys no maximum clique. Hence the coloring and clique covering of $G - x$ also yield $\chi(G' - x) \le w$ and $\theta(G' - x) \le a$ (regardless of whether $\alpha(G') > \alpha(G)$). By Theorem 8.1.35, G' is thus a, w-partitionable. Also, the clique coverings of $G' - x$ for various x force G' to be connected.

We next prove that if G has a u, v-path consisting of k critical edges, then u and v belong to at least $\omega - k$ common maximum cliques. We use induction on k, with Theorem 8.1.44 providing the basis step, $k = 1$. For $k > 1$, if y is the vertex before v on such a path, then the induction hypothesis puts u and y in $\omega - k + 1$ common maximum cliques. Since y belongs to exactly ω maximum cliques (by Theorem 8.1.39), and $\omega - 1$ of these contain v (by Theorem 8.1.44), at most one of the $\omega - k + 1$ cliques containing u and y can omit v.

Let C be a cycle of critical edges in G. Critical edges belong to maximum cliques, so C remains in G'. As shown above, ω vertices forming a path in G' induce a maximum clique in G'. If the length of C exceeds ω, then this establishes ω successive maximum cliques containing a given vertex x of C. By Theorem 8.1.39, these are all the maximum cliques of G containing x, and hence they include all the edges of G' incident to x. Hence C is a component of G', but G' is connected, so C contains all vertices of G'. This expresses G' as $C_n^{\omega-1}$.

If the length of C is at most ω, then $V(C)$ itself is a clique. If $x \in V(C)$, then the vertices of $C - x$ belong to distinct stable sets in the coloring $\mathrm{X}(G-x)$ defined by Theorem 8.1.41. Let $x_0, \ldots, x_k$ be the vertices of C in order. Let $S_1, \ldots, S_k$ be the stable sets in $G - V(C)$ such that $S_i \cup \{x_i\} \in \mathrm{X}(G - x_0)$. Because $x_i x_{i+1}$ is a critical edge, x_i and x_{i+1} belong to $\omega - 1$ common maximum cliques (Theorem 8.1.44), and hence by Theorem 8.1.41 the colorings $\mathrm{X}(G - x_i)$ and $\mathrm{X}(G - x_{i+1})$ have $\omega - 1$ common stable sets. The remaining set differs only in having x_i or x_{i+1}. Hence $\mathrm{X}(G - x_1)$ contains $S_i \cup \{x_i\}$ for $i \ge 2$, and it also contains $S_1 \cup \{x_0\}$.

Continuing these substitutions while following the edges of C, we find that $\mathrm{X}(G - x_k)$ contains $S_i \cup \{x_{i-1}\}$ for $1 \le i \le k$. Taking one more step to return to x_0, we find that $\mathrm{X}(G - x_0)$ contains $S_i \cup \{x_{i-1}\}$ for $2 \le i \le k$ and contains $S_1 \cup \{x_k\}$. Since $k \ge 2$ and $\alpha \ge 2$, these sets are different from our initial sets in $\mathrm{X}(G - x_0)$. Since the coloring $\mathrm{X}(G - x_0)$ is unique, we have obtained a contradiction, and the case $n(C) \le \omega$ does not arise. ■

8.1.51. Theorem. (Chvátal [1976]) If G is a p-critical graph such that the spanning subgraph G' obtained by deleting the edges of G belonging to no maximum clique is a cycle-power C_n^d, then G is an odd hole or odd antihole (and equals G').

Proof: A p-critical graph is partitionable. The stable sets and maximum cliques in G are stable sets and cliques in G', and by Theorem 8.1.35 we again conclude that G' is partitionable with $\alpha(G') = \alpha(G) = a$ and $\omega(G') = \omega(G) = w$. Hence $G' = C_{aw+1}^{w-1}$. We index the vertices so that the maximum cliques of G' (and G) consist of w cyclically consecutive vertices, and the maximum stable sets have the form $v_i, v_{i+w}, \dots, v_{i+aw}$. In particular, vertices separated by a multiple of w on the cycle $v_0, \dots, v_{aw}$ are nonadjacent in G' and in the full graph G.

If $G' = G$, then Theorem 8.1.48 implies that G is an odd hole or odd antihole. If $G' \neq G$, then $a, w > 2$, since otherwise deleting an edge increases the number of maximum stable sets or decreases the number of maximum cliques.

For $a, w \geq 3$, we exhibit an imperfect proper induced subgraph H of G (the induced odd cycle in G' obtained in Theorem 8.1.48 may have a chord in G). Let $S = \{v_{aw}, v_1, v_w, v_{w+2}\} \cup \{v_{iw+1} \colon 2 \leq i \leq a-1\}$, and let $T = \{v_{(a-1)w+1}, v_{aw}, v_1, v_w\} \cup \{v_{w+i} \colon 2 \leq i \leq w-1\}$. The sets S and T have sizes $a+2$ and $w+2$, and for $a, w \geq 3$ they share exactly the five vertices $\{v_{(a-1)w+}, v_{aw}, v_1, v_w, v_{w+2}\}$. Furthermore, S intersects every maximum clique of G' (and hence of G), and T intersects every maximum stable set of G' (and hence of G) (Exercise 49). Letting $H = G - (S \cup T)$, this yields $\alpha(H) = a - 1$ and $\omega(H) = w - 1$. Now imperfection follows from

$$n(H) \geq n(G) - (a + w + 4 - 5) > (a - 1)(w - 1). \qquad \blacksquare$$

8.1.52. Corollary. (Giles–Trotter–Tucker [1984]) If G is a p-critical graph and for each $v \in V(G)$ the minimum coloring $\mathrm{X}(G - v)$ has (at least) two sets that each contain exactly one neighbor of v, then G is an odd hole or an odd antihole.

Proof: When some set in $\mathrm{X}(G - v)$ has exactly one neighbor u of v, the edge uv is critical. Hence the hypothesis implies that the subgraph of critical edges has minimum degree at least 2 and therefore contains a cycle. By Theorem 8.1.50, the subgraph G' obtained by deleting the edges belonging to no maximum clique is $C_n^{\omega-1}$. By Theorem 8.1.51, G is an odd hole or an odd antihole. ■

8.1.53. Corollary. (Parthasarathy–Ravindra [1976]) The SPGC holds for $K_{1,3}$-free graphs.

Proof: (Giles–Trotter–Tucker [1984]) Let G be a p-critical $K_{1,3}$-free graph. For each $v \in V(G)$, $N(v)$ induces a perfect subgraph having no stable set of size 3. This means that $N(v)$ can be covered by two cliques, which implies $d(v) \leq 2\omega(G) - 2$. Each of the $\omega(G)$ stable sets in $\mathrm{X}(G - v)$ contains a neighbor of v, else adding v creates a larger stable set. With $d(v) \leq 2\omega(G) - 2$, at least two of these sets have exactly one neighbor of v. Hence G satisfies the hypothesis of Corollary 8.1.52, and G is an odd hole or antihole. ■

Corollary 8.1.53 also yields the SPGC for circle graphs (Exercise 50). The general SPGC remains open, but a result intermediate between it and the PGT is known (it is immediately implied by the SPGC and immediately implies the PGT). Chvátal conjectured that if G and H have the same vertex set and have the same 4-tuples of vertices that induce P_4, then G is perfect if and only if H is perfect. Reed [1987] proved this "Semi-Strong Perfect Graph Theorem".

EXERCISES

8.1.1. $(-)$ Compute $\chi(G)$ and $\omega(G)$ for the complement of the odd cycle C_{2k+1}.

8.1.2. $(-)$ Determine the smallest imperfect graph G such that $\chi(G) = \omega(G)$.

8.1.3. (!) P_4-free graphs are also called **cographs**, which stands for "complement reducible". A graph is **complement reducible** if it can be reduced to an empty graph by successively taking complements within components.

a) Prove that a graph G is P_4-free if and only if it is complement reducible.

b) Use part (a) and the Perfect Graph Theorem to prove that every P_4-free graph is perfect. (Seinsche [1974])

8.1.4. *Clique identification.* Suppose that $G = G_1 \cup G_2$, that $G_1 \cap G_2$ is a clique, and that G_1 and G_2 are perfect. Without using the Star-cutset Lemma, prove that G is perfect.

8.1.5. Find an imperfect graph G having a star-cutset C such that the C-lobes of G are perfect graphs. (Comment: Thus identification at star-cutsets does not preserve perfection, although no p-critical graph has a star-cutset.)

8.1.6. Let G be a cartesian product of complete graphs. Prove that $\alpha(G) = \theta(G)$. Prove that $K_2 \,\square\, K_2 \,\square\, K_3$ is not perfect.

8.1.7. Prove that $C_5 \vee K_1$ is the only color-critical 4-chromatic graph with six vertices.

8.1.8. $(+)$ Prove that G is an odd cycle if and only if $\alpha(G) = (n(G) - 1)/2$ and $\alpha(G - u - v) = \alpha(G)$ for all $u, v \in V(G)$. (Melnikov–Vizing [1971], Greenwell [1978])

8.1.9. Let $v_1, \ldots, v_n$ be a simplicial elimination ordering of G, and let $Q(v_i) = \{v_j \in N(v_i) \colon j > i\}$. Note that $Q(v_i)$ is the clique of neighbors of v_i at the time when v_i is deleted in the elimination ordering. Let $S = \{y_1, \ldots, y_k\}$ be the stable set obtained "greedily" from the ordering $v_1, \ldots, v_n$; that is, set $y_1 = v_1$, discard $N(y_1)$ from the remainder of the ordering, and proceed iteratively, at each step adding the least remaining element x to the stable set and discarding what remains of $Q(x)$.

a) Prove that applying the greedy coloring algorithm to the construction ordering $v_n, \ldots, v_1$ yields an optimal coloring and that $\omega(G) = 1 + \max \sum_{x \in V(G)} |Q(x)|$. (Fulkerson–Gross [1965])

b) Prove that S is a maximum stable set and that the sets $\{y_i\} \cup Q(y_i)$ form a minimum clique covering. (Gavril [1972])

8.1.10. Add a test to the MCS algorithm to check whether the resulting ordering is a simplicial elimination ordering. (Tarjan–Yannakakis [1984])

8.1.11. Prove directly (without using a simplicial elimination ordering) that the intersection graph of a family of subtrees of a tree has no chordless cycle.

8.1.12. (−) Prove that every graph is the intersection graph of a family of subtrees of some graph.

8.1.13. Prove that every chordal graph has an intersection representation by subtrees of a host tree with maximum degree 3.

8.1.14. Let Q be a maximal clique in a connected chordal graph G. For all $x \in V(G)$, prove that Q has two vertices whose distances from x are different. (Voloshin [1982])

8.1.15. *Intersection graphs of subtrees of a graph.* A **fraternal orientation** of a graph is an orientation such that any pair of vertices with a common successor are adjacent.

a) (−) Prove that a graph is chordal if and only if it has an acyclic fraternal orientation.

b) (−) Obtain a graph with no fraternal orientation.

c) A family of trees in a graph is *rootable* if the trees can be assigned roots so that a pair of them intersects if and only if at least one of the two roots belongs to both subtrees. Prove that G has a fraternal orientation if and only if G is the intersection graph of a rootable family of subtrees of some graph. (Gavril–Urrutia [1994])

8.1.16. (!) Prove that a simple graph G is a forest if and only if every pairwise intersecting family of paths in G has a common vertex. (Hint: For sufficiency, use induction on the number of paths in the family.)

8.1.17. (!) *Forbidden subgraph characterization of split graphs.* A graph is a **split graph** if its vertices can be partitioned into a clique and a stable set.

a) Prove that if G is a split graph, then G and $\overline{G}$ are chordal graphs. Observe that if G and $\overline{G}$ are chordal graphs, then G has no induced subgraph in $\{C_4, 2K_2, C_5\}$.

b) Prove that if G is a simple graph with no induced subgraph in $\{C_4, 2K_2, C_5\}$, then G is a split graph. (Hint: Among the maximum-sized cliques, let Q be one such that $G - Q$ has the minimum number of edges. Prove that $G - Q$ is a stable set, using the choice of Q and the forbidden subgraph conditions.) (Hammer–Simeone [1981])

8.1.18. Let $d_1 \ge \cdots \ge d_n$ be the degree sequence of a simple graph G, and let m be the largest value of k such that $d_k \ge k - 1$. Prove that G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$. (Comment: Compare with Exercise 3.3.28.) (Hammer–Simeone [1981])

8.1.19. (−) Determine the trees that are split graphs, and construct a pair of nonisomorphic split graphs with the same degree sequence.

8.1.20. The k**-trees** are the graphs that arise from a k-clique by 0 or more iterations of adding a new vertex joined to a k-clique in the old graph. Prove that G is a k-tree if and only if G satisfies the following three properties:

1) G is connected.
2) G has a k-clique but no $k+2$-clique.
3) Every minimal vertex separator of G is a k-clique.

8.1.21. Let G be an n-vertex chordal graph having no clique of order $k+2$. Prove that $e(G) \le kn - \binom{k+1}{2}$, with equality if and only if G is a k-tree.

8.1.22. (+) Generalize Theorem 2.2.3 (Cayley's Formula) by proving that the number of k-trees with vertex set $[n]$ is $\binom{n}{k}[k(n-k)+1]^{n-k-2}$. (Hint: Generalize the Prüfer code for *rooted* trees, which generates a list with $n-1$ entries and never deletes the root. In a k-tree, the vertices belonging to exactly one $k+1$-clique are the *leaves*. A k-tree can be grown using any k-clique as a root. The lists generated from k-trees with a fixed root have as symbols 0 and pairs ij, where i comes from some k-set and j from some $n-k$-set.) (Greene–Iba [1975]; other proofs in Beineke–Pippert [1969], Moon [1969])

8.1.23. Suppose that G is a chordal graph with $\omega(G) = r$. Prove that G has at most $\binom{r}{j} + \binom{r-1}{j-1}(n-r)$ cliques of size j, with equality (for all j simultaneously) if and only if G is an $r-1$-tree.

8.1.24. *The Helly property of the real line.* Suppose that $I_1, \dots, I_k$ are pairwise intersecting real intervals. Prove that $I_1, \dots, I_k$ have a common point.

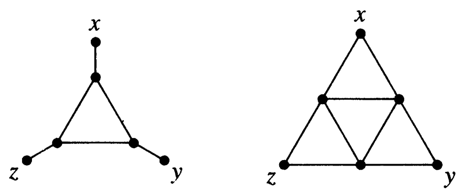
8.1.25. Prove directly that a tree is an interval graph if and only if it is a caterpillar (a tree having a path that contains at least one vertex of each edge).

8.1.26. (!) Let G be an interval graph. Prove that $\overline{G}$ is a comparability graph and that G is a chordal graph. (Hint: Establish a simplicial elimination ordering.)

8.1.27. Prove that a graph G has an interval representation if and only if the clique-vertex incidence matrix of G has the consecutive 1s property.

8.1.28. Prove that G is an interval graph if and only if the vertices of G can be ordered $v_1, \dots, v_n$ such that $v_i \leftrightarrow v_k$ implies $v_j \leftrightarrow v_k$ whenever $i < j < k$. (Jacobson–McMorris–Mulder [1991], for example)

8.1.29. An **asteroidal triple** in a graph is a triple of vertices x, y, z such that between any two there exists a path avoiding the neighborhood of the third. Prove that no asteroidal triple occurs in an interval graph. (Comment: Interval graphs are precisely the chordal graphs that have no asteroidal triples) (Lekkerkerker–Boland [1962]))



8.1.30. Six professors visited the library on the day the rare book was stolen. Each entered once, stayed for some time, and then left. For any two of them that were in the library at the same time, at least one of them saw the other. Detectives questioned the professors and gathered the following testimony:

PROFESSOR	CLAIMED TO HAVE SEEN
Abe	Burt, Eddie
Burt	Abe, Ida
Charlotte	Desmond, Ida
Desmond	Abe, Ida
Eddie	Burt, Charlotte
Ida	Charlotte, Eddie

In this situation, "lying" means providing false information, not omitting information. Assume that the culprit tried to frame another suspect by lying. If one professor lied, who was it? (Golumbic [1980, p20])

8.1.31. (+) Prove that G is a unit interval graph (representable by intervals of the same length) if and only if $A(G) + I$ has the consecutive 1s property. (Roberts [1968])

8.1.32. (+) Prove that G is a proper interval graph (representable by intervals such that none properly contains another) if and only if the clique-vertex incidence matrix of G has the consecutive 1s property for both rows and columns. (Fishburn [1985]

8.1.33. (−) Prove that every P_4-free graph is a Meyniel graph.

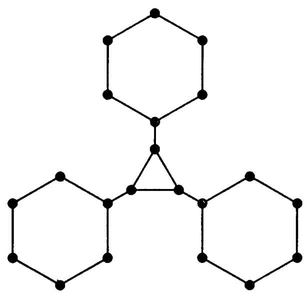
8.1.34. (!) Prove that every chordal graph is o-triangulated.

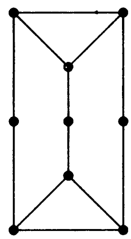
8.1.35. Let C be an odd cycle in a graph with no induced odd cycle. Prove that $V(C)$ has three pairwise-adjacent vertices such that paths joining them in C all have odd length.

8.1.36. (+) Prove that the conditions below are equivalent.
A) Every odd cycle of length at least 5 has a crossing pair of chords.
B) For every pair $x, y \in V(G)$, chordless x, y-paths are all even or all odd.
(Hint: For A $\Rightarrow$ B, consider a pair P_1, P_2 of x, y-paths with opposite parity such that the sum of their lengths is minimal.) (Burlet–Uhry [1984])

8.1.37. Prove that every perfectly orderable graph is strongly perfect. (Hint: Use Lemma 8.1.25) (Chvátal [1984])

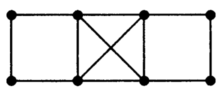
8.1.38. (!) Prove that the graphs below are strongly perfect but not perfectly orderable.





8.1.39. (−) Prove that the graph on the left above is a Meyniel graph but is not perfectly orderable. Prove that the graph $\overline{P_5}$ is perfectly orderable but is not a Meyniel graph.

8.1.40. (!) *Weakly chordal graphs.*
a) Prove that every chordal graph is weakly chordal.
b) Prove that the graph below is weakly chordal but not strongly perfect.



8.1.41. (−) A **skew partition** of G is a partition of $V(G)$ into two nonempty sets X, Y such that $G[X]$ is disconnected and $\overline{G}[Y]$ is disconnected. Chvátal [1985b] conjectured that no minimal imperfect graph has a skew partition. Prove that this implies the Star-Cutset Lemma and is implied by the SPGC.

8.1.42. Prove that the 10-vertex graph in Example 8.1.37 is 3, 3-partitionable. (Chvátal–Graham–Perold–Whitesides [1979])

8.1.43. (−) Let x and v be vertices of a partitionable graph G. Prove that if $x \not\leftrightarrow v$, then every maximum clique containing x consists of one vertex from each stable set that is the mate of a clique containing v. State the complementary assertion when $x \nleftrightarrow v$. (Buckingham–Golumbic [1983])

8.1.44. (+) Prove that no p-critical graph has **antitwins**, which are a pair of vertices such that every other vertex is adjacent to exactly one of them. (Hint: Given a p-critical graph with antitwins $\{x, y\}$, let S be the stable set containing y in the unique optimal coloring of $G - x$. Find among the vertices of the $\omega - 1$-colorable subgraph $G - x - S$ an $\omega - 1$ clique in $N(x)$ that doesn't extend into $N(y)$. Similarly, find a stable set in $N(y)$ that doesn't extend into $N(x)$. Now build an induced 5-cycle.) (Note: The partitionable graph of Example 8.1.37 has antitwins.) (Olariu [1988])

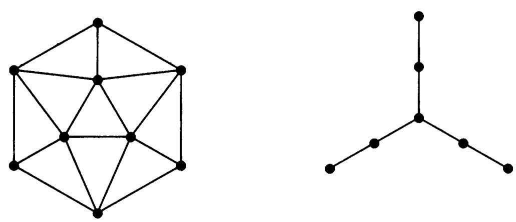
8.1.45. Vertices x, y form an **even pair** if every chordless x, y-path has even length (number of edges). **Twins** (nonadjacent vertices with the same neighborhood) are a special case.

a) Suppose that S_1, S_2 are maximum stable sets in a partitionable graph G. Prove that the subgraph of G induced by the symmetric difference of S_1 and S_2 is connected. (Bland–Huang–Trotter [1979])

b) Use part (a) to prove that no p-critical graph has an even pair. (Comment: Hence no p-critical graph has twins, which proves yet again that vertex duplication preserves perfection.) (Meyniel [1987], Bertschi–Reed [1988])

8.1.46. Let G be a partitionable graph, and let S_1, S_2 be stable sets in the optimal coloring of $G - x$. Use part (a) of the preceding problem to prove that the subgraph of G induced by $S_1 \cup S_2 \cup \{x\}$ is 2-connected. (Buckingham–Golumbic [1983])

8.1.47. Prove that one graph below is a circle graph but not a circular-arc graph, and prove that the other is a circular-arc graph but not a circle graph.



8.1.48. (!) The graph $K_{1,3} + e$ is the 4-vertex graph obtained by adding one edge to $K_{1,3}$. Using the perfection of Meyniel graphs, prove that $K_{1,3} + e$-free graphs satisfy the SPGC. (Meyniel [1976])

8.1.49. Let $G = C_{aw+1}^{w-1}$. Let $S = \{v_{aw}, v_1, v_w, v_{w+2}\} \cup \{v_{iw+1} \colon 2 \le i \le a - 1\}$, and let $T = \{v_{(a-1)w+1}, v_{aw}, v_1, v_w\} \cup \{v_{w+i} \colon 2 \le i \le w - 1\}$. Prove that S intersects every maximum clique of G and that T intersects every maximum stable set of G. (Chvátal [1976])

8.1.50. (!) *SPGC for circle graphs.* (Buckingham–Golumbic [1983])

a) Use Lemma 8.1.28 to prove that if x is a vertex in a partitionable graph G, then $G - N[x]$ is connected, where $N[x] = N(x) \cup \{x\}$.

b) Use part (a) to prove that partitionable circle graphs are $K_{1,3}$-free.

c) Conclude from part (b) and Corollary 8.1.53 that the SPGC holds for circle graphs.

8.2. Matroids

Many results of graph theory extend or simplify in the theory of matroids. These include the greedy algorithm for minimum spanning trees, the strong duality between maximum matching and minimum vertex cover in bipartite graphs, and the geometric duality relating planar graphs and their duals.

Matroids arise in many contexts but are special enough to have rich combinatorial structure. When a result from graph theory generalizes to matroids, it can then be interpreted in other special cases. Several difficult theorems about graphs have found easier proofs using matroids.

Matroids were introduced by Whitney [1935] to study planarity and algebraic aspects of graphs, by MacLane [1936] to study geometric lattices, and by van der Waerden [1937] to study independence in vector spaces. Most of the language comes from these contexts. Here we emphasize applications to graphs.

HEREDITARY SYSTEMS AND EXAMPLES

In many mathematical contexts, we study sets that avoid conflicts; often this is called "independence". Inherent in this notion is that subsets of independent sets are independent, and the empty set is independent.

8.2.1. Example. *Acyclic sets of edges.* Let E be the edge set of a graph G, and let $X \subseteq E$ be "independent" if it contains no cycle. Every subset of an independent set is independent, and the empty set is independent. The cycles are the minimal dependent sets.

Consider the kite $K_4 - e$, which has five edges. Since spanning trees of this graph have three edges, every set having more than three edges is dependent. Also the two triangles are dependent; this yields eight dependent sets and 24 independent sets among the subsets of E. There are three minimal dependent sets (the cycles) and eight maximal independent sets (the spanning trees). ■

8.2.2. Definition. A **hereditary family** or **ideal** is a collection of sets, $\mathbf{F}$, such that every subset of a set in $\mathbf{F}$ is also in $\mathbf{F}$. A **hereditary system** M on E consists of a nonempty ideal $\mathbf{I}_M$ of subsets of E and the various ways of specifying that ideal, called *aspects* of M.

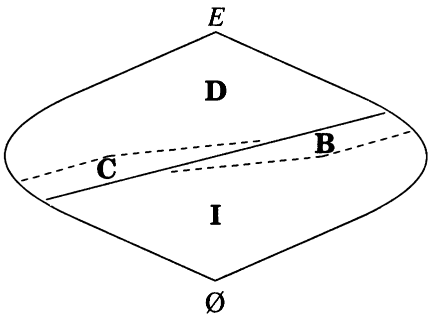
The elements of $\mathbf{I}_M$ are the **independent sets** of M. The other subsets of E (comprising the family $\mathbf{D}_M$) are **dependent**. The **bases** are the maximal independent sets, and the **circuits** are the minimal dependent sets; $\mathbf{B}_M$ and $\mathbf{C}_M$ denote these families of subsets of E.

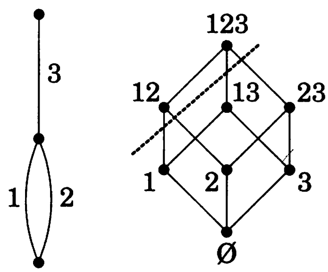
The **rank** of a subset of E is the maximum size of an independent set in it. The **rank function** r_M is defined by $r(X) = \max\{|Y| : Y \subseteq X,\ Y \in \mathbf{I}\}$.

8.2.3. Example. *Hereditary systems.* Label each vertex $a = (a_1, \ldots, a_n)$ of the hypercube Q_n by the corresponding set $X_a = \{i : a_i = 1\}$. Draw Q_n in the plane so that the vertical coordinates of vertices are in order by the size of the sets labeling them.

The diagram below illustrates the relationships among the independent sets, bases, circuits, and dependent sets of a hereditary system. The bases are the maximal elements of the family **I** and the circuits are the minimal elements not in **I**. In every hereditary system, Ø belongs to **I**. If every set is independent, then there is no circuit, but there is always at least one base.

In the example on the right, the independent sets are the acyclic edge sets in a graph with three edges. The only dependent sets are $\{1, 2\}$ and $\{1, 2, 3\}$, the only circuit is $\{1, 2\}$, and the bases are $\{1, 3\}$ and $\{2, 3\}$. The rank of an independent set is its size. For the dependent sets, we have $r(\{1, 2\}) = 1$ and $r(\{1, 2, 3\}) = 2$. ■





8.2.4. Remark. *Aspects of hereditary systems.* A hereditary system M is determined by any of $\mathbf{I}_M$, $\mathbf{B}_M$, $\mathbf{C}_M$, r_M, etc., because each aspect specifies the others. We have expressed $\mathbf{B}_M, \mathbf{C}_M, r_M$ in terms of $\mathbf{I}_M$. Conversely, if we know $\mathbf{B}_M$, then $\mathbf{I}_M$ consists of the sets contained in members of $\mathbf{B}_M$. If we know $\mathbf{C}_M$, then $\mathbf{I}_M$ consists of the sets containing no member of $\mathbf{C}_M$. If we know r_M, then $\mathbf{I}_M = \{X \subseteq E: r_M(X) = |X|\}$. ■

Hereditary systems are too general to behave nicely. We restrict our attention to hereditary systems having an additional property, and these we call matroids. We can translate any restriction on $\mathbf{I}_M$ into a corresponding restriction on some other aspect of the hereditary system. Because hereditary systems can be specified in many ways, we have many equivalent definitions of matroids. Using various motivating examples, we state several of these properties that characterize matroids. Later we prove that they are equivalent. We begin with the fundamental example from graphs.

8.2.5. Definition. The **cycle matroid** $M(G)$ of a graph G is the hereditary system on $E(G)$ whose circuits are the cycles of G. A hereditary system that is $M(G)$ for some graph G is a **graphic matroid**.

8.2.6. Example. *Bases in cycle matroids.* The bases of the cycle matroid $M(G)$ are the edge sets of the maximal forests in G. Each maximal forest contains a spanning tree from each component, so they have the same size. Consider $B_1, B_2 \in \mathbf{B}$ with $e \in B_1 - B_2$. Deleting e from B_1 disconnects some component of B_1; since B_2 contains a tree spanning that component of G, some edge $f \in B_2 - B_1$ can be added to $B_1 - e$ to reconnect it.

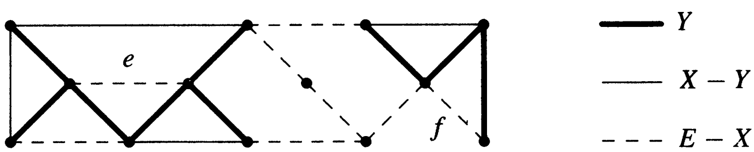
For a hereditary system M, the **base exchange property** is: if $B_1, B_2 \in \mathbf{B}_M$, then for all $e \in B_1 - B_2$ there exists $f \in B_2 - B_1$ such that $B_1 - e + f \in \mathbf{B}_M$. Matroids are the hereditary systems satisfying the base exchange property. ■

8.2.7. Remark. In this subject, we often discuss inclusion and omission of single elements from sets. For symmetry and simplicity, we use the symbols $+$ and $-$ instead of $\cup$ and $-$ for this, and we drop the set brackets on 1-element sets. ■

8.2.8. Example. *Rank function in cycle matroids.* Let G be a graph with n vertices. For $X \subseteq E(G)$, let G_X denote the spanning subgraph of G with edge set X. In $M(G)$, an independent subset of X is the edge set of a forest in G_X. When G_X has k components, the maximum size of such a forest is $n - k$. Hence $r(X) = n - k$. Below we show such a forest Y (bold) within X (bold and solid).

If $r(X + e) = r(X)$ for some $e \in E - X$, then the endpoints of e lie in a single component of G_X; adding e does not combine components. If we add two such edges, then again we do not combine components. Therefore, $r(X) = r(X + e) = r(X + f)$ implies $r(X) = r(X + e + f)$.

For a hereditary system M on E, the **(weak) absorption property** is: if $X \subseteq E$ and $e, f \in E$, then $r(X) = r(X+e) = r(X+f)$ implies $r(X+e+f) = r(X)$. Matroids are the hereditary systems satisfying the absorption property (name suggested by A. Kézdy). ■



Graphs may have loops and multiple edges. In cycle matroids, they lead to circuits of sizes 1 and 2. We use these terms for hereditary systems in general.

8.2.9. Definition. In a hereditary system, a **loop** is an element forming a circuit of size 1. **Parallel elements** are distinct non-loops forming a circuit of size 2. A hereditary system is **simple** if it has no loops or parallel elements.

8.2.10. Definition. The **vectorial matroid** on a set E of vectors in a vector space is the hereditary system whose independent sets are the linearly independent subsets of vectors in E. A matroid expressible in this way is a **linear matroid** (or **representable matroid**). The **column matroid** $M(A)$ of a matrix A is the vectorial matroid defined on its columns.

8.2.11. Example. *Circuits in vectorial matroids.* The set E may have repeated vectors; these would be parallel elements. The circuits are the minimal sets $\{x_1, \ldots, x_k\} \subseteq E$ such that $\sum c_i x_i = 0$ using coefficients not all zero. Minimality forces all $c_i \neq 0$.

Let C_1, C_2 be distinct circuits containing x. Using the equations of dependence for C_1 and C_2, we can write x as a linear combination in terms of $C_1 - x$ and in terms of $C_2 - x$. Equating these expressions yields an equation of dependence for $C_1 \cup C_2 - x$; thus $C_1 \cup C_2 - x$ contains a circuit.

For a hereditary system M on E, the **(weak) elimination property** is: whenever C_1, C_2 are distinct circuits and $x \in C_1 \cap C_2$, another member of $\mathbf{C}_M$ is contained in $C_1 \cup C_2 - x$. Matroids are the hereditary systems satisfying the weak elimination property.

The column matroid of the matrix below is also the cycle matroid $M(K_4 - e)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$ ■

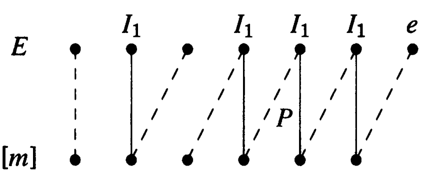
8.2.12. Definition. The **transversal matroid** induced by sets $A_1, \ldots, A_m$ with union E is the hereditary system on E whose independent sets are the systems of distinct representatives of subsets of $\{A_1, \ldots, A_m\}$. Equivalently, letting G be the $E, [m]$-bigraph defined by $e \leftrightarrow i$ if and only if $e \in A_i$, the independent sets are the subsets of E that are saturated by matchings in G.

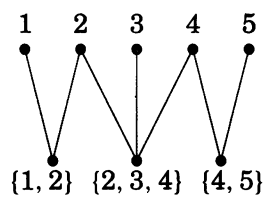
8.2.13. Example. *Independent sets in transversal matroids.* When M, M' are matchings in G and $|M'| > |M|$, the symmetric difference $M \triangle M'$ contains an M-augmenting path P (Theorem 3.1.10). Replacing $M \cap P$ with $M' \cap P$ yields a matching of size $|M| + 1$ that saturates all vertices of M plus the endpoints of P.

Consider independent sets I_1, I_2 in the transversal matroid generated by $A_1, \ldots, A_m$. In the associated bipartite graph, let M_1, M_2 be matchings saturating I_1, I_2, respectively (on the left below, M_1 is solid and M_2 is dashed). If $|I_2| > |I_1|$, then the matching obtained from M_1 by using an M_1-augmenting path in $M_2 \triangle M_1$ saturates I_1 plus an element $e \in I_2 - I_1$; this "augments" I_2.

For a hereditary system on E, the **augmentation property** is: for distinct $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, there exists $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathbf{I}$. Matroids are the hereditary systems satisfying the augmentation property.

The transversal matroid of the family $\mathbf{A} = \{\{1,2\}, \{2,3,4\}, \{4,5\}\}$, illustrated by the bipartite graph on the right, is again $M(K_4 - e)$. ■



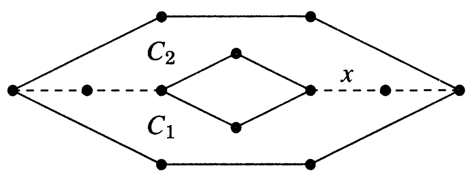


The name "transversal matroid" arises from the use of "transversal" in systems of distinct representatives. An SDR for a subset of $\{A_1, \ldots, A_m\}$ is a **partial transversal** for the full system. The independent sets of the transversal matroid on $\bigcup A_i$ are the partial transversals of $\{A_1, \ldots, A_m\}$. That these are matroids was discovered by Edmonds and Fulkerson [1965] and independently by Mirsky and Perfect [1967], who extended the result to infinite sets.

Every matroid must satisfy all properties of matroids. Once we show that the properties defined above are equivalent for hereditary systems, we need only verify one to use all. First we check that they all hold for cycle matroids.

8.2.14. Example. *Augmentation in cycle matroids.* Consider $I_1, I_2 \in \mathbf{I}_{M(G)}$. As in Example 8.2.8, the spanning subgraph G_{I_1} has $k = n - |I_1|$ components, and its largest forest has $n - k = |I_1|$ edges. Therefore, the forest I_2 has some edge with endpoints in two components of G_{I_1}. This edge can be added to I_1 to obtain a larger independent set. Hence the augmentation property holds. ■

8.2.15. Example. *Weak elimination in cycle matroids.* The circuits of $M(G)$ are the edge sets of cycles of G. Cycles have even degree at each vertex. If $C_1, C_2 \in \mathbf{C}$, then the symmetric difference $C_1 \triangle C_2$ also has even degree at each vertex. If $C_1 \neq C_2$, this implies that $C_1 \triangle C_2$ contains a cycle (see Proposition 1.2.27). This is stronger than the weak elimination property, since $C_1 \triangle C_2 \subseteq C_1 \cup C_2 - x$. In the figure below, C_1 and C_2 are face boundaries of length 9 sharing the dashed edges, and $C_1 \triangle C_2$ is the union of two disjoint cycles. ■



For transversal matroids, the base exchange property is similar to the augmentation property; Exercise 9 considers the weak elimination property. For linear matroids, directly verifying the augmentation or base exchange property requires the algebraic result that k linearly independent vectors cannot all be expressed as linear combinations of a smaller set. Instead, we can use Theorem 8.2.20. Since the weak elimination property holds for independent sets of vectors, many theorems of linear algebra follow from Theorem 8.2.20!

8.2.16. Remark. *Notational conventions*: Boldface $\mathbf{I}, \mathbf{B}, \mathbf{C}$ for families of subsets of E allows $I \in \mathbf{I}$, $B \in \mathbf{B}$, $C \in \mathbf{C}$ to denote members of the families. Roman letters I, B, C, R denote properties that yield matroids. We use e, f, x, y as elements of E, and we use X, Y, F as subsets of E. ■

Every hereditary family is the collection of independent sets of a hereditary system. A collection $\mathbf{B}$ is realizable as the set of bases of a hereditary system if and only if $\mathbf{B}$ is nonempty and no element of $\mathbf{B}$ contains another. A collection

C is realizable as the set of circuits of a hereditary system if and only if the elements of **C** are nonempty and no element of **C** contains another.

The characterization of rank functions is more subtle. It includes two properties (r1, r2 below) that we will need, plus an additional technical condition that forces r to be the rank function of the hereditary system M defined by $I_M = \{X \subseteq E\colon r(X) = |X|\}$.

8.2.17. Lemma. For the rank function r of a hereditary system on E,

(r1) $r(\emptyset) = 0$.

(r2) $r(X) \le r(X+e) \le r(X) + 1$ whenever $X \subseteq E$ and $e \in E$.

Proof: From the definition $r(X) = \max\{|Y|\colon Y \subseteq X, Y \in \mathbf{I}\}$, we have $r(\emptyset) = 0$. Because $X + e$ contains every independent subset of X, also $r(X+e) \ge r(X)$. Because the independent subsets of $X + e$ not contained in X consist of e plus an independent subset of X, we have $r(X+e) \le r(X) + 1$. ■

PROPERTIES OF MATROIDS

We have remarked that many equivalent conditions on hereditary systems yield matroids. We can show that a hereditary system is a matroid by verifying any of them, after which we can employ them all without additional proof. We obtained the same benefit from equivalent characterizations of trees.

Adding an edge to a forest creates at most one cycle. More generally, adding one element to an independent set in a matroid creates at most one circuit. Our proof of the greedy algorithm for spanning trees (Theorem 2.3.3) used *only* this property of graphs. This "induced circuit" property is one of the conditions that characterize matroids, as is the effectiveness of the greedy algorithm itself! Both properties appear in our list.

Given weights on the elements of a matroid, the **greedy algorithm** is the process of iteratively including an element of largest nonnegative weight whose addition to the independent set already selected yields a larger independent set. Rado [1957] proved that matroids are precisely the hereditary systems for which the greedy algorithm selects a maximum-weighted independent set regardless of the choice of weights.

8.2.18. Definition. A hereditary system M on E is a **matroid** if it satisfies any of the following additional properties, where $\mathbf{I}, \mathbf{B}, \mathbf{C}$, and r are the independent sets, bases, circuits, and rank function of M.

I: **augmentation**—if $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, then $I_1 + e \in \mathbf{I}$ for some $e \in I_2 - I_1$.

U: **uniformity**—for every $X \subseteq E$, the maximal subsets of X belonging to **I** have the same size.

B: **base exchange**—if $B_1, B_2 \in \mathbf{B}$, then for all $e \in B_1 - B_2$ there exists $f \in B_2 - B_1$ such that $B_1 - e + f \in \mathbf{B}$.

R: **submodularity**—$r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$ whenever $X, Y \subseteq E$.

A: **weak absorption**—$r(X) = r(X+e) = r(X+f)$ implies $r(X+e+f) = r(X)$ whenever $X \subseteq E$ and $e, f \in E$,

A′: **strong absorption**—if $X, Y \subseteq E$, and $r(X + e) = r(X)$ for all $e \in Y$, then $r(X \cup Y) = r(X)$.
C: **weak elimination**—for distinct circuits $C_1, C_2 \in \mathbf{C}$ and $x \in C_1 \cap C_2$, there is another member of $\mathbf{C}$ contained in $(C_1 \cup C_2) - x$.
J: **induced circuits**—if $I \in \mathbf{I}$, then $I + e$ contains at most one circuit.
G: **greedy algorithm**—for each nonnegative weight function on E, the greedy algorithm selects an independent set of maximum total weight.

The base exchange property implies that all bases have the same size: if $|B_1| < |B_2|$ for some $B_1, B_2 \in \mathbf{B}$, then we can iteratively replace elements of $B_1 - B_2$ by elements of $B_2 - B_1$ to obtain a base of size $|B_1|$ contained in B_2, but no base is contained in another.

8.2.19.* Remark. The rank of a set $X \subseteq E$ in a vectorial matroid is the dimension of the space spanned by X. Hence for vectorial matroids the submodularity inequality says that $\dim U \cap V + \dim U \oplus V \le \dim U + \dim V$, where $U, V, U \oplus V$ are the spaces spanned by subsets $X, Y, X \cup Y$ of E, respectively. The usual proof of this is the vector space statement of our proof of U⇒R below. Exercise 10 obtains submodularity directly for cycle matroids.

Various of these properties (together with requirements for a hereditary system) have been used as the defining condition for matroids. Examples include I (Welsh [1976], Schrijver [to appear]), U (Edmonds [1965b,c], Bixby [1981], Nemhauser–Wolsey [1988]), A (Whitney [1935]), C (Tutte [1970]), G (Papadimitriou–Steiglitz [1982]), and others (van der Waerden [1937], Rota [1964], Crapo–Rota [1970], Aigner [1979]). ■

Many authors include basic properties of hereditary systems in the set of axioms characterizing some aspect of a matroid. This can distract from the special additional properties of matroids and lead to extra work. Starting with hereditary systems yields more concise proofs. All properties of hereditary systems are always available.

8.2.20. Theorem. For a hereditary system M, the conditions defining matroids in Definition 8.2.18 are equivalent.

Proof: U ⇒ B. By uniformity for $X = E$, all bases have the same size. We then apply uniformity to the set $(B_1 - e) \cup B_2$. This yields an augmentation of the independent set $B_1 - e$ from B_2 to reach size $|B_2|$.

B ⇒ I. Given independent sets $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, choose $B_1, B_2 \in \mathbf{B}$ such that $I_1 \subseteq B_1$, $I_2 \subseteq B_2$. We use base exchange to replace elements of $B_1 - I_1$ outside B_2 with elements of B_2. Hence we may assume that $B_1 - I_1 \subseteq B_2$. If $B_1 - I_1 \subseteq B_2 - I_2$, then $|B_1| < |B_2|$, which is forbidden by the base exchange property as remarked above. Hence I_2 has an element in $B_1 - I_1$, and we use such an element to augment I_1.

I ⇒ A. Suppose that $r(X) = r(X+e) = r(X+f)$. If $r(X+e+f) > r(X)$, then let I_1, I_2 be maximum independent subsets of X and of $X+e+f$. Now $|I_2| > |I_1|$, and we can augment I_1 from I_2. Since I_1 is a maximum independent subset of

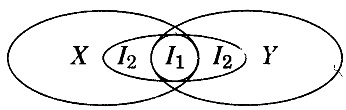
X, the augmentation can only add e or f, which contradicts the hypothesis that $r(X) = r(X + e) = r(X + f)$.

A $\Rightarrow$ A$'$. We use induction on $|Y - X|$. The statement is trivial when $|Y - X| = 1$. When $|Y - X| > 1$, choose $e, f \in Y - X$, and let $Y' = Y - e - f$. Applying the induction hypothesis to proper subsets of Y yields $r(X) = r(X \cup Y') = r(X \cup Y' + e) = r(X \cup Y' + f)$. Now weak absorption yields $r(X) = r(X \cup Y)$.

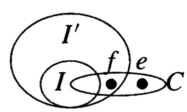
A$'$ $\Rightarrow$ U. If Y is a maximal independent subset of X, then $r(Y + e) = r(Y)$ for all $e \in X - Y$. By strong absorption, $r(X) = r(Y) = |Y|$. Hence all such Y have the same size.

U $\Rightarrow$ R. Given $X, Y \subseteq E$, choose a maximum independent set I_1 from $X \cap Y$. By uniformity, I_1 can be enlarged to a maximum independent subset of $X \cup Y$; call this I_2. Consider $I_2 \cap X$ and $I_2 \cap Y$, these are independent subsets of X and Y, and each includes I_1. Hence

$$r(X \cap Y) + r(X \cup Y) = |I_1| + |I_2| = |I_2 \cap X| + |I_2 \cap Y| \le r(X) + r(Y).$$



U$\Rightarrow$R



J$\Rightarrow$G

R $\Rightarrow$ C. Consider distinct circuits $C_1, C_2 \in \mathbf{C}$ with $x \in C_1 \cap C_2$. We have $r(C_1) = |C_1| - 1$ and $r(C_2) = |C_2| - 1$. Also $r(C_1 \cap C_2) = |C_1 \cap C_2|$, since every proper subset of a circuit is independent. If $(C_1 \cup C_2) - x$ does not contain a circuit, then $r((C_1 \cup C_2) - x) = |C_1 \cup C_2| - 1$, and hence $r(C_1 \cup C_2) \ge |C_1 \cup C_2| - 1$. Applying submodularity to C_1 and C_2 yields the contradiction

$$|C_1 \cap C_2| + |C_1 \cup C_2| - 1 \le |C_1| + |C_2| - 2.$$

C $\Rightarrow$ J. If $I + e$ contains $C_1, C_2 \in \mathbf{C}$ for some $I \in \mathbf{I}$, then C_1, C_2 both contain e. Now weak elimination guarantees a circuit in $(C_1 \cup C_2) - e$. On the other hand, $(C_1 \cup C_2) - e$ is independent, being contained in I.

J $\Rightarrow$ G. For weight function w, let I be the output of the greedy algorithm. Among the maximum-weight independent sets, let I^* be one having largest intersection with I. The algorithm cannot end with $I \subset I^*$. If $I \ne I^*$, then let e be the first element of $I - I^*$ chosen by the algorithm. By the choice of I^*, $I^* + e$ is dependent; hence it has a unique circuit C. Since $C \not\subseteq I$, we may choose $f \in C - I$. Since $I^* + e$ has no other circuit, $I^* + e - f \in \mathbf{I}$. The optimality of I^* yields $w(f) \ge w(e)$. Since f and the elements of I chosen earlier than e all lie in I^*, f does not complete a circuit with them. Thus f was available when the algorithm selected e, which yields $w(f) \le w(e)$. Now $w(f) = w(e)$ and $w(I^* + e - f) = w(I^*)$. With $|I^* + e - f \cap I| > |I^* \cap I|$, this contradicts the choice of I^*. Thus $I^* = I$.

G $\Rightarrow$ I. Given $I_1, I_2 \in \mathbf{I}$ with $k = |I_1| < |I_2|$, we design a weight function for which the success of the greedy algorithm yields the desired augmentation. Let $w(e) = k + 2$ for $e \in I_1$, and let $w(e) = k + 1$ for $e \in I_2 - I_1$. Let $w(e) =$

0 for $e \notin I_1 \cup I_2$. Now $w(I_2) \geq (k+1)^2 > k(k+2) = w(I_1)$, so I_1 is not a maximum-weighted independent set. However, the greedy algorithm chooses every element of I_1 before any element of $I_2 - I_1$. Because it finds a maximum-weighted independent set, it continues after absorbing I_1 and adds an element $e \in I_2 - I_1$ such that $I_1 + e \in \mathbf{I}$. ■

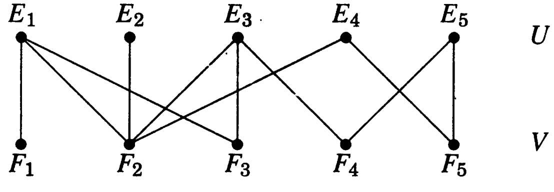
The property most often used to show that a hereditary system is a matroid is the augmentation property.

8.2.21. Example. The **uniform matroid** of rank k, denoted $U_{k,n}$ when $|E| = n$, is defined by $\mathbf{I} = \{X \subseteq E\colon |X| \leq k\}$. This immediately satisfies the base exchange and augmentation properties. The **free matroid** is the uniform matroid of rank $|E|$. Uniform matroids are used in building more interesting matroids and in characterizing classes of matroids. Few uniform matroids are graphic, and few graphic matroids are uniform (Exercise 6). Neither $M(K_4 - e)$ nor $M(K_4)$ is a uniform matroid.

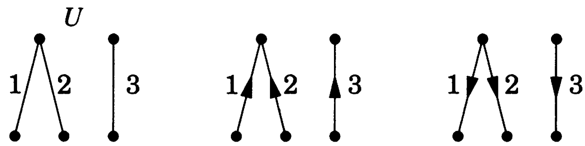
A linear matroid representable over the field $\mathbb{Z}_2$ or $\mathbb{Z}_3$ is **binary** or **ternary**, respectively. Every graphic matroid is binary (Exercise 43); $U_{2,4}$ is ternary (Exercise 44) but not binary (and hence not graphic). ■

8.2.22. Example. The **partition matroid** on E induced by a partition of E into blocks $E_1, \ldots, E_k$ is defined by $\mathbf{I} = \{X \subseteq E\colon |X \cap E_i| \leq 1 \text{ for all } i\}$. Since $\emptyset \in \mathbf{I}$, and since $X \in \mathbf{I}$ when its elements lie in distinct blocks, $\mathbf{I}$ is a hereditary family. Given $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, the set I_2 must intersect more blocks than I_1; an element of I_2 in a block that I_1 misses yields the desired augmentation of I_1. Alternatively, $r(X)$ is the number of blocks having elements in X; this satisfies the absorption property. (Note: $M(K_4 - e)$ is *not* a partition matroid.)

Given a U, V-bigraph G, the incidences with $U = u_1, \ldots, u_k$ define a partition matroid on $E(G)$ (this differs from the transversal matroid on U induced by G). The blocks are the sets $E_i = \{e \in E(G)\colon u_i \in e\}$. A set $X \subseteq E(G)$ is a matching in G if and only if X is independent in the partition matroid induced by U *and* in the partition matroid induced by V. This is the motivation for our later discussion of matroid intersection.



When G has an odd cycle, G has no set of vertices whose incident sets partition $E(G)$. In a digraph, however, each edge has a head and a tail, and we can define the **head partition matroid** and the **tail partition matroid** using the edge partitions induced by incidences with heads and by incidences with tails. (Example: The matroid of Example 8.2.3 arises as the partition matroid on E induced by U in the bipartite graph below, as the head partition matroid in the first digraph, and as the tail partition matroid in the second digraph.) ■



THE SPAN FUNCTION

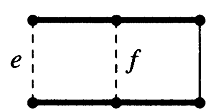
We next introduce several additional aspects of hereditary systems and matroid properties involving them. We use these aspects to illuminate matroid duality, which will lead to a characterization of planar graphs using matroids.

The algebraic concept of the space "spanned" by a set of vectors extends to hereditary systems. The definition is suggested by cycle matroids; a set spans itself and the elements that complete circuits with its subsets.

8.2.23. Definition. The **span function** of a hereditary system M is the function σ_M on the subsets of E defined by $\sigma_M(X) = X \cup \{e \in E\colon Y + e \in \mathbf{C}_M$ for some $Y \subseteq X\}$. If $e \in \sigma(X)$, then X **spans** e.

In a hereditary system, X is a dependent set if and only if it contains a circuit, which by Definition 8.2.23 holds if and only if $e \in \sigma(X - e)$ for some $e \in X$. We can therefore find the independent sets from the span function via $\mathbf{I} = \{X \subseteq E\colon (e \in X) \Rightarrow (e \notin \sigma(X - e))\}$. The properties of span functions that we use in studying matroids are (s1, s2, s3) below (an additional technical condition is needed to characterize the span functions of hereditary systems). First we illustrate property (s3) using graphs.

8.2.24. Example. In the cycle matroid $M(G)$, the meaning of $e \notin \sigma(X)$ is that X has no path between the endpoints of e. If also $e \in \sigma(X + f)$, then adding f completes such a path. The path completes a cycle with e, and hence also $f \in \sigma(X + e)$. In the figure below, X consists of the four bold edges. ■



8.2.25. Proposition. If σ is the span function of a hereditary system on E, and $X, Y \subseteq E$, then the following properties hold.

s1) $X \subseteq \sigma(X)$ (σ is **expansive**).

s2) $Y \subseteq X$ implies $\sigma(Y) \subseteq \sigma(X)$ (σ is **order-preserving**).

s3) $e \notin \sigma(X)$ and $e \in \sigma(X + f)$ imply $f \in \sigma(X + e)$ (**Steinitz exchange**).

Proof: Definition 8.2.23 implies immediately that σ is expansive and order-preserving. If $e \in \sigma(X+f)$, then e belongs to a circuit C in $X+f+e$. If also $e \notin \sigma(X)$, then $f \in C$. This circuit yields $f \in \sigma(X+e)$, and hence σ satisfies the Steinitz exchange property. ■

Properties of the span function lead to a short proof of a stronger form of the elimination property. The weak elimination property states that when $e \in C_1 \cap C_2$, there is a circuit in $(C_1 \cup C_2) - e$. Cycle matroids have the much stronger property that $C_1 \triangle C_2$ is an edge-disjoint union of cycles, since every vertex in $C_1 \triangle C_2$ has even degree. General matroids have the intermediate property that all elements of the symmetric difference belong to cycles in $(C_1 \cup C_2) - e$ when $e \in C_1 \cap C_2$ (Property C′ below).

We need a property relating rank and span in hereditary systems. The truth of the converse is our next characterization of matroids.

8.2.26.* Lemma. In a hereditary system, $[r(X+e) = r(X)] \Rightarrow e \in \sigma(X)$.

Proof: Let Y be a maximum independent subset of X. Since $|Y| = r(X) = r(X+e)$, also Y is a maximum independent subset of $X+e$. Hence e completes a circuit with some subset of X contained in Y, and $e \in \sigma(X)$. ■

8.2.27.* Theorem. If M is a hereditary system, then each condition below is necessary and sufficient for M to be a matroid.

P: **incorporation**—$r(\sigma(X)) = r(X)$ for all $X \subseteq E$.
S: **idempotence**—$\sigma^2(X) = \sigma(X)$ for all $X \subseteq E$.
T: **transitivity of dependence**—if $e \in \sigma(X)$ and $X \subseteq \sigma(Y)$, then $e \in \sigma(Y)$.
C′: **strong elimination**—whenever $C_1, C_2 \in \mathbf{C}$, $e \in C_1 \cap C_2$, and $f \in C_1 \triangle C_2$, there exists $C \in \mathbf{C}$ such that $f \in C \subseteq C_1 \cup C_2 - e$.

Proof: U $\Rightarrow$ P. Every element in $\sigma(X) - X$ completes a circuit with a subset of of X and thus lies in the span of every set between X and $\sigma(X)$. Thus it suffices to prove that $r(Y+e) = r(Y)$ when $e \in \sigma(Y)$. Let Z be a subset of Y such that $Z + e \in \mathbf{C}$. Augment Z to a maximal independent subset I of $Y+e$. By the uniformity property, $|I| = r(Y+e)$. Since $Z+e \in \mathbf{C}$, we have $e \notin I$. Thus $I \subseteq Y$, and we have $r(Y) \geq |I| = r(Y+e)$. (Absorption can be used instead.)

P $\Rightarrow$ S. Since σ is expansive, $\sigma^2(X) \supseteq \sigma(X)$, and we need only show that $e \in \sigma^2(X)$ implies $e \in \sigma(X)$. By the incorporation property, $r(\sigma(X)+e) = r(\sigma(X))$ and $r(\sigma(X)) = r(X)$. Since $X \subseteq \sigma(X)$, monotonicity of r yields $r(X) \leq r(X+e) \leq r(\sigma(X)+e) = r(X)$. Since equality holds throughout, Lemma 8.2.26 yields $e \in \sigma(X)$.

S $\Rightarrow$ T. If $X \subseteq \sigma(Y)$, then the order-preserving and idempotence properties of σ imply $\sigma(X) \subseteq \sigma^2(Y) = \sigma(Y)$.

T $\Rightarrow$ C′. Given distinct $C_1, C_2 \in \mathbf{C}$ with $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$, we want $f \in \sigma(Y)$, where $Y =)C_1 \cup C_2) - e - f$. We have $f \in \sigma(X)$, where $X = C_1 - f$. By T, it suffices to show $X \subseteq \sigma(Y)$. Since $X - e \subseteq Y \subseteq \sigma(Y)$, we need only show $e \in \sigma(Y)$. Since σ is order-preserving, we have $e \in \sigma(C_2 - e) \subseteq \sigma(Y)$.

C′ $\Rightarrow$ C. C is a less restrictive statement than C′. ■

Like uniqueness of induced circuits (J), the incorporation property (P) relates two aspects of hereditary systems. These are well-known properties of matroids, and in the approach via hereditary systems they become characterizations. The equivalence of C and C′ was first proved by Lehman [1964].

Idempotence occurs naturally for graphic and linear matroids. The span of a set of vectors contains nothing additional in its span; similarly, every edge that can be added to the span of a set of edges joins two components. This suggests related aspects of hereditary systems.

8.2.28. Definition. The **spanning sets** of a hereditary system on E are the sets $X \subseteq E$ such that $\sigma(X) = E$. The **closed sets** are the sets $X \subseteq E$ such that $\sigma(X) = X$ (also called **flats** or **subspaces**). The **hyperplanes** are the maximal proper closed subsets of E.

8.2.29.* Remark. The span function of a matroid is also called its **closure function**. A **closure operator** is an expansive, order-preserving, idempotent function from the family of subsets of a set to itself. A closure operator is the span function of a matroid if and only if it has the Steinitz exchange property.

In every hereditary system, the span function satisfies Steinitz exchange. Thus treating matroids as hereditary systems with additional properties is not well suited for studying closure operators. The span function of a hereditary system M is a closure operator if and only if M is a matroid. Matroids are developed from lattice theory in MacLane [1936], Rota [1964], and Aigner [1979].

We have not considered all relationships among aspects of matroids. Brylawski [1986] presents a matrix describing the transformations among about a dozen aspects of matroids, calling these maps **cryptomorphisms**. ■

THE DUAL OF A MATROID

Duality in matroids generalizes the notion of duality for planar graphs. Every connected plane graph G has a natural dual graph G^* such that $(G^*)^* = G$. The dual is formed by associating a vertex of G^* with each face of G and including a dual edge e^* in G^* for each edge of G, such that the endpoints of the edge e^* are the vertices for the faces on the two sides of e.

A set of edges in a plane graph G forms a spanning tree in G if and only if the duals to the remaining edges form a spanning tree in G^* (Exercise 6.1.21). Hence the bases in the cycle matroid $M(G^*)$ are the complements of the bases in $M(G)$. We define duality for matroids and hereditary systems so that the properties of duality in planar graphs generalize.

8.2.30. Definition. The **dual** of a hereditary system M on E is the hereditary system M^* whose bases are the complements of the bases of M. The aspects $\mathbf{B}^*(\mathbf{B}_{M^*})$, $\mathbf{C}^*$, $\mathbf{I}^*$, r^*, σ^*, of M^* are the **cobases**, **cocircuits**, etc., of M.

The **supbases** $\mathbf{S}$ of M are the sets containing a base. The **hypobases** $\mathbf{H}$ are the maximal subsets containing no base. We write $\overline{X}$ for $E - X$.

8.2.31. Lemma. If M is a hereditary system, then

a) $\mathbf{B}^* = \{\overline{B}: B \in \mathbf{B}\}$ and $(M^*)^* = M$.

b) $\mathbf{I}^* = \{\overline{S}: S \in \mathbf{S}\}$ and $\mathbf{S}^* = \{\overline{I}: I \in \mathbf{I}\}$.

c) $\mathbf{C}^* = \{\overline{H}: H \in \mathbf{H}\}$ and $\mathbf{H}^* = \{\overline{C}: C \in \mathbf{C}\}$.

Proof: The statement about $\mathbf{B}^*$ is the definition of M^*. It immediately yields $(M^*)^* = M$ and both parts of (b). Also, X is a maximal (proper) subset of E containing no base (a hypobase of M) if and only if $\overline{X}$ is a minimal nonempty set contained in no cobase, which is a circuit of M^*. Similarly, the hypobases of M^* are the complements of the circuits of M. ■

We have chosen "supbase" and "hypobase" to share initials with "spanning" and "hyperplane", because for matroids the spanning sets and supbases are the same, and the hyperplanes and hypobases are the same.

8.2.32. Lemma. If M is a matroid, then the supbases are the spanning sets, and the hypobases are the hyperplanes.

Proof: A set X is spanning if and only if $\sigma(X) = E$. By the incorporation property, this is equivalent to $r(X) = r(E)$. By the uniformity property, this is equivalent to X containing a base. For hyperplanes, see Exercise 32.) ■

Consider $B_1, B_2 \subseteq E$. If neither of B_1, B_2 contains the other, then also neither of $\overline{B}_1, \overline{B}_2$ contains the other. Therefore, the dual of a hereditary system is a hereditary system. The notion of duality becomes useful when we prove that the dual of a matroid is a matroid. This follows easily from a dual version of the base exchange property.

8.2.33. Lemma. If M is a matroid and $B_1, B_2 \in \mathbf{B}$, then for each $e \in B_1 - B_2$ there exists $f \in B_2$ such that $B_2 + e - f$ is a base.

Proof: Since B_2 is a base, $B_2 + e$ contains exactly one circuit C. Since B_1 is independent, C also contains an element $f \in B_2 - B_1$. Now $B_2 + e - f$ contains no circuit and has size $r(E)$. ■

8.2.34. Theorem. (Whitney [1935]) The dual of a matroid M on E is a matroid with rank function $r^*(X) = |X| - (r(E) - r(\overline{X}))$.

Proof: We have observed that M^* is a hereditary system; now we prove the base exchange property for M^*. If $\overline{B}_1, \overline{B}_2 \in \mathbf{B}^*$ and $e \in \overline{B}_1 - \overline{B}_2$, then $B_1, B_2 \in \mathbf{B}$, with $e \in B_2 - B_1$. By Lemma 8.2.33, there exists $f \in B_1 - B_2$ such that $B_1 + e - f \in \mathbf{B}$. Now $\overline{B}_1 - e + f \in \mathbf{B}^*$ is the desired exchange.

To compute $r^*(X)$, let Y be a maximal coindependent subset of X, so $r^*(X) = r^*(Y) = |Y|$. By Lemma 8.2.31, $\overline{Y}$ is a minimal superset of $\overline{X}$ that contains a base of M. Since $\overline{Y}$ arises from $\overline{X}$ by augmenting a maximal independent subset of $\overline{X}$ to become a base, we have $|\overline{Y}| - |\overline{X}| = r(E) - r(\overline{X})$. With $|\overline{Y}| - |\overline{X}| = |X| - |Y|$, this yields the desired formula

$$r^*(X) = |Y| = |X| - (|\overline{Y}| - |\overline{X}|) = |X| - (r(E) - r(\overline{X})).$$ ■

We can restate any matroid property using dual aspects. Exercises 33–34 request characterizations of hyperplanes and closed sets by this method. More subtle results involve relationships between a matroid and its dual.

8.2.35. Proposition. (Dual augmentation property) Let M be a matroid. If $X \in \mathbf{I}$ and $X' \in \mathbf{I}^*$ are disjoint, then there are disjoint $B \in \mathbf{B}$ and $B' \in \mathbf{B}^*$ such that $X \subseteq B$ and $X' \subseteq B'$.

Proof: Since X' is coindependent in M, $\overline{X'}$ is spanning in M. Hence every maximal independent subset of $\overline{X'}$ is a base; we augment $X \subseteq \overline{X'}$ to a base B contained in $\overline{X'}$. The cobase $B' = \overline{B}$ contains X'. ■

We will use cycle matroids to characterize planar graphs. The next result enables us to describe the cocircuits of a cycle matroid.

8.2.36. Proposition. Cocircuits of a matroid are the minimal sets intersecting every base. Bases are the minimal sets intersecting every cocircuit.

Proof: The cocircuits are the minimal sets contained in no cobase. Because the cobases are the complements of the bases, a set is contained in no cobase if and only if it intersects every base. Similarly, the cobases are the maximal sets containing no cocircuit, so the complements of the cobases are the minimal sets intersecting every cocircuit. ■

8.2.37. Corollary. The cocircuits of the cycle matroid $M(G)$ are the bonds of G.

Proof: By Proposition 8.2.36, the cocircuits are the minimal sets intersecting every maximal forest. Hence they are the minimal sets whose deletion increases the number of components; these are the bonds. ■

8.2.38. Definition. The **bond matroid** or **cocycle matroid** of a graph G is the hereditary system whose circuits are the bonds of G.

By Corollary 8.2.37, the bond matroid of G is the the dual of the cycle matroid $M(G)$. Weak elimination now applies to bonds. Since a cycle must return to its starting point, it cannot intersect a bond in exactly one edge. This generalizes to matroids as another characterization of cocircuits.

8.2.39. Theorem. The cocircuits of a matroid M on E are the minimal nonempty sets $C^* \subseteq E$ such that $|C^* \cap C| \neq 1$ for every $C \in \mathbf{C}$.

Proof: To show that every cocircuit has this property, suppose that $C \in \mathbf{C}$, $C^* \in \mathbf{C}^*$, $C^* \cap C = e$. Then $C - e \in \mathbf{I}$ and $C^* - e \in \mathbf{I}^*$, and the dual augmentation property yields $B \in \mathbf{B}$ and $\overline{B} \in \mathbf{B}^*$ such that $C - e \subseteq B$ and $C^* - e \subseteq \overline{B}$. Since e must appear in B or B^*, we obtain $C \in \mathbf{I}$ or $C^* \in \mathbf{I}^*$.

For the converse, we show that every nonempty set in $\mathbf{I}*$ meets some $C \in \mathbf{C}$ in one element; since cocircuits do not, every *minimal* set that does not is a cocircuit. Choose $X^* \in \mathbf{I}^*$. Let B^* be a cobase containing X^*, and let $B = \overline{B^*}$. For each $e \in X^*$, $B + e$ contains a circuit C, and $X^* \cap C = \{e\}$. ■

MATROID MINORS AND PLANAR DUALS

From a graph G we can obtain smaller graphs by repeatedly deleting and/or contracting edges. The resulting graphs are the **minors** of G. Wagner [1937] proved that G is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor (Exercise 6.2.12). Hadwiger [1943] conjectured that G is k-colorable if G has no minor isomorphic to K_{k+1}. A simple graph is a forest if and only if it does not have C_3 as a minor.

To generalize these operations to matroids, we need to know how deletion and contraction affect cycle matroids. The acyclic subsets of $E(G-e)$ are precisely the acyclic subsets of $E(G)$ that omit e. The acylic subsets of $E(G \cdot e)$ are the subsets of $E(G) - e$ whose union with e is acyclic in G. A dual description of contraction is more convenient: X contains a spanning tree of each component of $G \cdot e$ if and only if $X + e$ contains a spanning tree of each component of G.

We also want the notation to extend in a natural way. This causes difficulty, because discussion of graph minors often emphasizes the edges removed, while discussion of matroid minors emphasizes the elements that remain. We compromise by using matroid notation for the matroid on the set that remains while extending graph notation to describe matroids obtained by deleting or contracting one element.

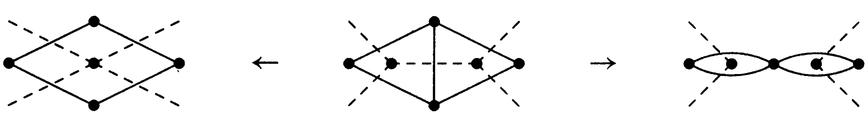
8.2.40. Definition. For a hereditary system M on E, the **restriction** *of M to $F \subseteq E$*, denoted $M|F$ and obtained by **deleting** $\overline{F}$, is the hereditary system defined by $\mathbf{I}_{M|F} = \{X \subseteq F \colon X \in \mathbf{I}_M\}$. The **contraction** *of M to $F \subseteq E$*, denoted $M.F$ and obtained by **contracting** $\overline{F}$, is the hereditary system defined by $\mathbf{S}_{M.F} = \{X \subseteq F \colon X \cup \overline{F} \in \mathbf{S}_M\}$. When $F = E - e$, we write $M - e = M|F$ and $M \cdot e = M.F$. The **minors** of M are the hereditary systems arising from M using deletions and contractions.

The definitions imply that $M|F$ and $M.F$ are hereditary systems. The operations of restriction and contraction commute (Exercise 41). The definition of contraction via supbases yields a natural duality between these operations.

8.2.41. Proposition. For hereditary systems, restriction and contraction are dual operations: $(M.F)^* = (M^*|F)$ and $(M|F)^* = (M^*.F)$.

Proof: $\mathbf{I}_{(M.F)^*} = \{X \subseteq F \colon F - X \in \mathbf{S}_{M.F}\} = \{X \subseteq F \colon (F - X) \cup \overline{F} \in \mathbf{S}_M\}$
$= \{X \subseteq F \colon \overline{X} \in \mathbf{S}_M\} = \{X \subseteq F \colon X \in \mathbf{I}_{M^*}\} = \mathbf{I}_{M^*|F}$.
For the second statement, apply the first to M^* and take duals. ∎

The duality between deletion and contraction is most intuitive for plane graphs. Deleting an edge e in a plane graph G contracts the corresponding dual edge in G^*; contracting e deletes the edge in the dual.



8.2.42. Corollary. Under deletion or contraction of an edge e in a graph G, the cycle matroid and bond matroid behave as listed below.

$$\begin{array}{ll} M(G-e) = M(G) - e & M^*(G-e) = M^*(G)\cdot e \\ M(G\cdot e) = M(G)\cdot e & M^*(G\cdot e) = M^*(G) - e \end{array}$$

Proof: Matroid deletion and contraction are defined so that the statements in the first column describe the behavior of cycle matroids. Using these and Proposition 8.2.41, we compute

$$M^*(G-e) = [M(G-e)]^* = [M(G)-e]^* = M^*(G)\cdot e, \text{ and}$$
$$M^*(G\cdot e) = [M(G\cdot e)]^* = [M(G)\cdot e]^* = M^*(G) - e. \qquad \blacksquare$$

As desired, restrictions and contractions of matroids are matroids.

8.2.43. Theorem. Given $F \subseteq E$ and a matroid M on E, both $M|F$ and $M.F$ are matroids on F. In terms of r_M, their rank functions are $r_{M|F}(X) = r_M(X)$ and $r_{M.F}(X) = r_M(X \cup \overline{F}) - r_M(\overline{F})$.

Proof: The augmentation property from M applies to any pair of sets in $\mathbf{I}_{M|F}$; thus $M|F$ satisfies the augmentation property and is a matroid. Using duality, $M.F = (M^*|F)^*$ is also a matroid. The rank function for $M|F$ follows from the definition of $\mathbf{I}_{M|F}$. This and repeated application of Theorem 8.2.34 to $(M^*|F)^*$ yields the rank function for $M.F$ (Exercise 42). $\blacksquare$

The formula for $r_{M.F}$ yields a description of the independent sets: $X \in \mathbf{I}_{M.F}$ if and only if adding X to $\overline{F}$ increases the rank by $|X|$.

A set of edges in a plane graph G forms a cycle if and only if the corresponding dual edges form a bond in G^* (Theorem 6.1.14). Using the natural bijection between edges and dual edges, this tells us that the cycle matroid of a plane graph G is (isomorphic to) the bond matroid of G^*. By Corollary 8.2.37, the bond matroid of a graph H is $[M(H)]^*$. Applying this to G and to G^* tells us that the bond matroid of G is (isomorphic to) the cycle matroid of G^*. Thus the bond matroid of a planar graph G is graphic. Using Kuratowski's Theorem, we will prove that this condition characterizes planarity.

Whitney [1933a] approached this by defining a non-geometric notion of dual. Changing his definition slightly, we say that H is an **abstract dual** of G if there is a bijection ϕ: $E(G) \to E(H)$ such that $X \subseteq E(G)$ is a bond in G if and only if $\phi(X)$ is the edge set of a cycle in H. With this definition, saying that G has an abstract dual is the same as saying that the bond matroid of G is graphic; the bijection ϕ establishes an isomorphism between $M^*(G)$ and $M(H)$.

8.2.44. Theorem. (Whitney [1933a]) A graph G is planar if and only if its bond matroid $M^*(G)$ is graphic.

Proof: We first prove that existence of an abstract dual is preserved under deletion and contraction of edges. Suppose that G has an abstract dual H, so that $M(H) \cong M^*(G)$. Let e' be the edge of H corresponding to e under the bijection. To prove that $H \cdot e'$ is an abstract dual of $G - e$ and that $H - e'$ is an abstract dual of $G \cdot e$, we use Corollary 8.2.42 to compute

$$M^*(G - e) = M^*(G) \cdot e \cong M(H) \cdot e' = M(H \cdot e'), \text{ and}$$
$$M^*(G \cdot e) = M^*(G) - e \cong M(H) - e' = M(H - e').$$

We have demonstrated that planar graphs have abstract duals. By Kuratowski's Theorem, a nonplanar graph contains a subdivision K_5 or $K_{3,3}$. Hence K_5 or $K_{3,3}$ is a minor of it. Since existence of abstract duals is preserved under deletion and contraction, showing that K_5 and $K_{3,3}$ have no abstract dual implies that every nonplanar graph has no abstract dual.

If H is an abstract dual of G, then also G is an abstract dual of H, since $M^*(G) \cong M(H)$ if and only if $M(G) \cong M^*(H)$. If G has girth g, then bonds of H have size at least g, so $\delta(H) \geq g$. Also $e(H) = e(G)$, and the degree-sum formula yields $n(H) \leq \lfloor 2e(H)/\delta(H) \rfloor \leq \lfloor 2e(G)/g \rfloor$.

Let H be an abstract dual of K_5. Since K_5 has girth 3, $n(H) \leq \lfloor 20/3 \rfloor = 6$. Since all bonds of K_5 have four or six edges, all cycles of H have four or six edges, and thus H is a simple bipartite graph. However, no simple bipartite graph with at most six vertices has ten edges.

Let H be an abstract dual of $K_{3,3}$. Since $K_{3,3}$ has girth 4, $n(H) \leq \lfloor 18/4 \rfloor = 4$. Since all bonds of $K_{3,3}$ have at least three edges, all cycles of H have at least three edges, and thus H is a simple graph. However, no simple graph with at most four vertices has nine edges. ■

The argument that bond matroids of plane graphs are graphic shows that every "geometric" dual of a planar graph is an abstract dual. We have seen that the geometric dual need not be unique. Nevertheless, the cycle matroid of every graph dual to G must be $M^*(G)$; hence all geometric duals of G have the same cycle matroid. Whitney [1933b] determined when graphs have the same cycle matroid (see Exercise 45, also Kelmans [1980, 1987, 1988]).

Minors have many applications. They will soon help us prove the Matroid Intersection Theorem. They are used in characterizing classes of matroids by forbidden substructures; for example, a matroid is binary if and only if it does not have $U_{2,4}$ as a minor. Minors also are used to produce a winning strategy for a matroid generalization of Bridg-it (Theorem 2.1.17).

8.2.45.* Definition. Given $e \in E$ and a matroid M on E, the **Shannon Switching Game** (M, e) is played by the Spanner and the Cutter. The Cutter deletes elements of $E - e$ and the Spanner seizes them, one per move. The Spanner aims to seize a set that spans e, and the Cutter aims to prevent this. The Cutter moves first.

Having the Spanner move first can be simulated by adding an element e' such that $\{e, e'\}$ is a circuit; the Cutter must begin by deleting e' to avoid losing immediately. Bridg-it occurs by letting M be the cycle matroid of the graph in Theorem 2.1.17 with e the "auxiliary edge" and e' an extra auxiliary edge. The spanning tree strategy for the Spanner results from the following sufficient condition for a winning strategy. The condition is also necessary, but proving that requires the Matroid Union Theorem (Theorem 8.2.55).

8.2.46.* Theorem. (Lehman [1964]) In the Shannon Switching Game (M, e), the Spanner has a winning strategy if there are disjoint subsets X_1, X_2 of $E - e$ such that $e \in \sigma(X_1) = \sigma(X_2)$.

Proof: We use X_1, X_2 to produce a winning strategy. Let $X = \sigma(X_1) = \sigma(X_2)$. Since the Spanner can ignore deletions outside X and play in $M|(X+e)$, we may assume that X_1, X_2 are disjoint bases. If the Cutter plays g and the Spanner plays f, then g is no longer available and f cannot be deleted; the effect is deletion and contraction. Letting $M' = (M - g) \cdot f$, we have $e \in \sigma_{M'}(X)$ if and only if $g \notin X$ and $e \in \sigma_M(X + f)$. The Spanner wins if e is a loop in M', which is equivalent to $e \in \sigma_M(F)$, where F is the set seized by the Spanner.

If $|E| = 1$, then e is a loop and the Spanner wins; we proceed by induction on $|E|$. It suffices to provide an immediate answer f to g so that $M' = (M-g)\cdot f$ has two disjoint bases. If the Cutter deletes g not in X_1 or X_2, then the Spanner seizes an arbitrary f, and the two sets $X_1 - g - f$ and $X_2 - g - f$ are disjoint and spanning in M'. Hence we may assume that $g \in X_1$. The base exchange property yields $f \in X_2$ such that $X' = X_1 - g + f \in \mathbf{B}$. Now $X' - f$ and $X_2 - f$ are disjoint bases avoiding e in the game (M', e). ■

MATROID INTERSECTION

Matroid theory took a great leap forward with the proof of the Matroid Intersection and Union Theorems by Edmonds. This provided a unified context for many well-known min-max relations, which became corollaries. We have proved some of these in earlier chapters. Yielding a simple unified proof for many important theorems, the Matroid Intersection Theorem can be considered among the most beautiful theorems of combinatorics.

The Matroid Intersection Theorem is a min-max relation for common independent sets in two matroids on the same ground set. We can view the intersection of two matroids as a hereditary system, but *not* as a matroid. For multiple matroids on a set E, we typically use subscripts to distinguish corresponding aspects, as in $\mathbf{B}_i$ for the bases of M_i, etc. We still use $\overline{X}$ to denote the complement of X within the ground set E.

8.2.47. Definition. Given hereditary systems M_1, M_2 on E, the **intersection** of M_1 and M_2 is the hereditary system whose independent sets are $\{X \subseteq E \colon X \in \mathbf{I}_1 \cap \mathbf{I}_2\}$.

For example, the intersection of the two natural partition matroids on the edges of a bipartite graph G has as its independent sets the matchings of G. These are generally not the independent sets of a matroid (see Exercises 1–2), and thus the greedy algorithm does not solve maximum-weighted matching.

Recall that a *loop* is an element forming a nonempty set of rank 0.

8.2.48. Theorem. (Matroid Intersection Theorem, Edmonds [1970]) For matroids M_1, M_2 on E, the size of a largest common independent set satisfies

$$\max\{|I| : I \in \mathbf{I}_1 \cap \mathbf{I}_2\} = \min_{X \subseteq E}\{r_1(X) + r_2(\overline{X})\}.$$

Proof: (Seymour [1976]) For weak duality, consider arbitrary $I \in \mathbf{I}_1 \cap \mathbf{I}_2$ and $X \subseteq E$. The sets $I \cap X$ and $I \cap \overline{X}$ are also common independent sets, and $|I| = |I \cap X| + \left|I \cap \overline{X}\right| \le r_1(X) + r_2(\overline{X})$.

To achieve equality, we use induction on $|E|$; when $|E| = 0$ both sides are 0. If every element of E is a loop in M_1 or in M_2, then $\max |I| = 0 = r_1(X) + r_2(\overline{X})$, where X consists of all loops in M_1. Hence we may assume that $|E| > 0$ and that some $e \in E$ is a non-loop in both matroids. Let $F = E - e$, and consider the matroids $M_1|F$, $M_2|F$, $M_1.F$, and $M_2.F$.

Let $k = \min_{X \subseteq E}\{r_1(X) + r_2(\overline{X})\}$; we seek a common independent k-set in M_1 and M_2. If there is none, then $M_1|F$ and $M_2|F$ have no common independent k-set, and $M_1.F$ and $M_2.F$ have no common independent $k-1$-set. The induction hypothesis and rank formulas (Theorem 8.2.43) yield

$$\begin{aligned} r_1(X) + r_2(F - X) &\le k - 1 && \text{for some } X \subseteq F, \text{ and} \\ r_1(Y + e) - 1 + r_2(F - Y + e) - 1 &\le k - 2 && \text{for some } Y \subseteq F. \end{aligned}$$

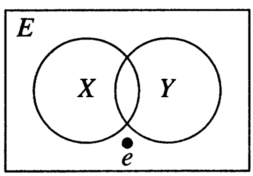
We use $(F - Y) + e = \overline{Y}$ and $F - X = \overline{X + e}$ and sum the two inequalities:

$$r_1(X) + r_2(\bar{X} + e) + r_1(Y + e) + r_2(\overline{Y}) \le 2k - 1.$$

Now we apply submodularity of r_1 to X and $Y + e$ and submodularity of r_2 to $\overline{Y}$ and $\overline{X + e}$. For clarity, write $U = X + e$ and $V = Y + e$. Applying this to the preceding inequality yields

$$r_1(X \cup V) + r_1(X \cap V) + r_2(\overline{Y} \cup \overline{U}) + r_2(\overline{Y} \cap \overline{U}) \le 2k - 1.$$

Since $\overline{Y} \cap \overline{U} = \overline{X \cup V}$ and $\overline{Y} \cup \overline{U} = \overline{X \cap V}$, the left side sums two instances of $r_1(Z) + r_2(\overline{Z})$, and the hypothesis $k \le r_1(Z) + r_2(\overline{Z})$ for all $Z \subseteq E$ yields $2k \le 2k - 1$. Hence M_1 and M_2 do have a common independent k-set. ■



It can be helpful to restrict the range of the minimization.

8.2.49. Corollary. The maximum size of a common independent set in matroids M_1, M_2 on E is the minimum of $r_1(X_1) + r_2(X_2)$ over sets X_1, X_2 such that $X_1 \cup X_2 = E$ and each X_i is closed in M_i.

Proof: The incorporation property implies that $r_i(\sigma_i(X)) = r_i(X)$. ■

We have proved special cases of the Matroid Intersection Theorem by other means. We proved the König–Egerváry Theorem in various ways, and we proved the Ford–Fulkerson characterization of CSDRs from Menger's Theorem in Theorem 4.2.25. Whenever we have two matroids on the same set, the Matroid Intersection Theorem tells us that there must be a min-max relation for the maximum size of a common independent set, tells us what the result should be, and provides a proof.

8.2.50. Corollary. (König [1931], Egerváry [1931]) In a bipartite graph, the largest matching and smallest vertex cover have equal size.

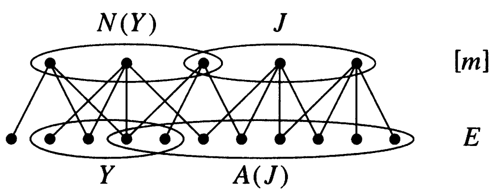
Proof: When M_1 and M_2 are the partition matroids on $E(G)$ induced by the partite sets U_1, U_2 of G, the matchings are the common independent sets. For $X_1, X_2 \subseteq E$, the rank $r_i(X_i)$ counts the vertices of U_i incident to edges in X_i. Hence if $X_1 \cup X_2 = E$, then G has a vertex cover of size $r_1(X_1) + r_2(X_2)$, using vertices of U_i to cover X_i. Conversely, if $T_1 \cup T_2$ is a vertex cover with $T_i \subseteq U_i$, let X_i be the set of edges incident to T_i; we have $X_1 \cup X_2 = E$ with X_i closed in M_i and $r_1(X_1) + r_2(X_2) = |T_1| + |T_2|$. We conclude that

$$\alpha'(G) = \max\{|I|: I \in \mathbf{I}_1 \cap \mathbf{I}_2\} = \min\{r_1(X_1) + r_2(X_2)\} = \beta(G). \qquad \blacksquare$$

The next corollary uses the rank function for transversal matroids.

8.2.51. Example. *Transversal matroids* (see Example 8.2.13). Suppose that $A_1 \cup \cdots \cup A_m = E$, and let G be the corresponding incidence graph with partite sets E and $[m]$. Consider $X \subseteq E$. If $|N(Y)| < |Y|$ for some $Y \subseteq X$, then Y forces at least $|Y| - |N(Y)|$ unsaturated elements in X. Hall's Condition applied to X yields $r(X) = \min\{|X| - (|Y| - |N(Y)|): Y \subseteq X\}$ (Exercise 51).

We obtain another expression for $r(X)$ (see Ore [1955]). Let $A(J) = \bigcup_{i \in J} A_i$; in terms of the graph, $A(J) = N(J)$. By applying Hall's Condition to $[m]$ instead of E, we can write the maximum size of a matching as $r(M) = \min\{m - (|J| - |A(J)|): J \subseteq [m]\}$. To determine the maximum number of elements in $X \subseteq E$ that can be matched, we discard the elements of $E - X$, obtaining $r(X) = \min_{J \subseteq [m]}\{|A(J) \cap X| - |J| + m\}$.



The first formula for $r(X)$ uses neighborhoods of subsets of E; the second uses neighborhoods of subsets of $[m]$. Exercise 53 shows directly that the second rank formula is the rank function of a matroid, without relying on results from bipartite matching. Further material on transversals appears in Mirsky [1971] and in Lovász–Plummer [1986]. $\blacksquare$

8.2.52. Corollary. (Ford–Fulkerson [1958]) Families $\mathbf{A} = \{A_1, \ldots, A_m\}$ and $\mathbf{B} = \{B_1, \ldots, B_m\}$ have a common system of distinct representatives (CSDR) if and only if, for each $I, J \subseteq [m]$,

$$\left|(\bigcup_{i\in I} A_i) \cap (\bigcup_{j\in J} B_j)\right| \geq |I| + |J| - m.$$

Proof: A common partial SDR is a common independent set in the two transversal matroids M_1, M_2 induced on E by $\mathbf{A}$ and $\mathbf{B}$. To determine when there is a complete CSDR, we need only restate the condition $r_1(X) + r_2(\overline{X}) \geq m$ to find the appropriate condition on the set systems.

The rank formulas from Example 8.2.51 yield

$$r_1(X) + r_2(\overline{X}) = \min_{I\subseteq[m]}\{|A(I) \cap X| - |I| + m\} + \min_{J\subseteq[m]}\{\left|B(J) \cap \overline{X}\right| - |J| + m\}.$$

Hence $r_1(X) + r_2(\overline{X}) \geq m$ for all X if and only if

$$|A(I) \cap X| + \left|B(J) \cap \overline{X}\right| \geq |I| + |J| - m \text{ for all } X \subseteq E \text{ and } I, J \subseteq [m].$$

Given I, J, consider the contribution of an element of E to the left side. Each element of $A(I) \cap B(J)$ counts once whether it belongs to X or $\overline{X}$. Elements of $A(I) - B(J)$ count if and only if they belong to X, and elements of $B(J) - A(I)$ count if and only if they belong to $\overline{X}$. Hence the left side is minimized for I, J when $A(I) - B(J) \subseteq \overline{X}$ and $B(J) - A(I) \subseteq X$. In this case the left side equals $|A(I) \cap B(J)|$, which yields the Ford–Fulkerson condition. ■

The augmenting path approach to maximum bipartite matching generalizes to matroid intersection. The algorithm yields a common independent set I of maximum size and a set X such that $r_1(X) + r_2(\overline{X}) = |I|$ (see Lawler [1976], Edmonds [1979], Faigle [1987]). Finding a maximum common independent set in three matroids is NP-complete (??s –).

MATROID UNION

The intersection of two matroids is seldom a matroid, but a natural concept of matroid union does always yield a matroid. Together with a useful min-max relation for the rank function, this is the content of the Matroid Union Theorem. The Matroid Intersection and Union Theorems are equivalent; they can be derived from each other. Welsh [1976] proves the Matroid Union Theorem first; here we obtain it from the Matroid Intersection Theorem.

8.2.53. Definition. The **union** $M_1 \cup \cdots \cup M_k$ of hereditary systems $M_1, \ldots, M_k$ on E is the hereditary system M on E defined by $\mathbf{I}_M = \{I_1 \cup \cdots \cup I_k \colon I_i \in \mathbf{I}_i\}$. The **direct sum** $M_1 \oplus \cdots \oplus M_k$ of hereditary systems $M_1, \ldots, M_k$ on disjoint sets $E_1, \ldots, E_k$ is the hereditary system M on $E_1 \cup \cdots \cup E_k$ defined by $\mathbf{I}_M = \{I_1 \cup \cdots \cup I_k : I_i \in \mathbf{I}_i\}$.

The direct sum $M_1 \oplus \cdots \oplus M_k$ on $E_1, \ldots, E_k$ can be expressed as the union of $M'_1, \ldots, M'_k$ on $E' = E_1 \cup \cdots \cup E_k$ by letting M'_i be a copy of M_i with the additional elements of $E' - E_i$ added as loops. When each M_i is a uniform matroid, the direct sum is a **generalized partition matroid**. Here $E_1, \ldots, E_k$ partition E, there are positive integers $r_1, \ldots, r_k$, and $X \in \mathbf{I}$ if $|X \cap E_i| \le r_i$. The partition matroids defined earlier arise when all $r_i = 1$.

8.2.54. Proposition. Given matroids $M_1, \ldots, M_k$ on disjoint sets $E_1, \ldots, E_k$, the direct sum $M = M_1 \oplus \cdots \oplus M_k$ is a matroid.

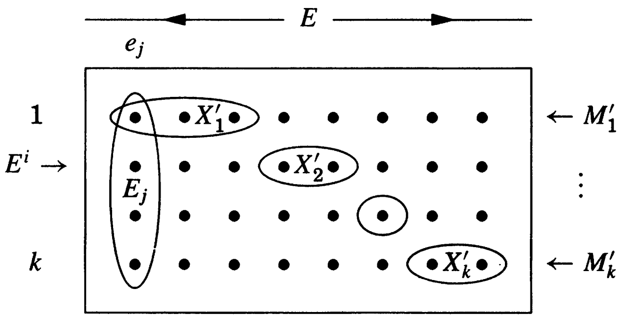
Proof: Since the $E_1, \ldots, E_k$ are pairwise disjoint, the intersection of any $I \in \mathbf{I}$ with each E_i is independent in M_i. If $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, then $|I_2 \cap E_i| > |I_1 \cap E_i|$ for some i. Since both sets are independent in M_i, we can augment $I_1 \cap E_i$ from $I_2 \cap E_i$ and therefore I_1 from I_2. Hence $M_1 \oplus \cdots \oplus M_k$ satisfies the augmentation property. ■

Using a direct sum, we prove that the union of matroids is always a matroid, and we compute the rank function.

8.2.55. Theorem. (Matroid Union Theorem—Edmonds–Fulkerson [1965], Nash-Williams [1966]) If $M_1, \ldots, M_k$ are matroids on E with rank functions $r_1, \ldots, r_k$, then the union $M = M_1 \cup \cdots \cup M_k$ is a matroid with rank function $r(X) = \min_{Y \subseteq X}(|X - Y| + \sum r_i(Y))$.

Proof: (following Schrijver [to appear]). After proving the formula for the rank function, we will verify the submodularity property to prove that M is a matroid. First we reduce the computation of the rank function to the computation of $r(E)$. In the restriction of the hereditary system M to the set X, we have $\mathbf{I}_{M|X} = \{Y \subseteq X \colon Y \in \mathbf{I}_M\}$ and $r_{M|X}(Y) = r_M(Y)$ for $Y \subseteq X$. Thus $M|X = \cup_i(M_i|X)$, and applying the formula for the rank of the full union to $M|X$ yields $r_M(X)$.

Consider a k by $|E|$ grid of elements E' in which the jth column E_j consists of k copies of the element $e_j \in E$. We define two matroids N_1, N_2 on E' such that the maximum size of a set independent in both N_1 and N_2 equals the maximum size of a set independent in M. We then compute $r_M(E)$ by applying the Matroid Intersection Theorem to N_1 and N_2. Let M'_i be a copy of M_i defined on the elements E^i of row i in E'. Let N_1 be the direct sum matroid $M'_1 \oplus \cdots \oplus M'_k$, and let N_2 be the partition matroid induced on E' by the column partition $\{E_j\}$.



Each set $X \in \mathbf{I}_M$ has a decomposition as a disjoint union of subsets $X_i \in \mathbf{I}_i$, because $\mathbf{I}_i$ is a hereditary family. Given a decomposition $\{X_i\}$ of $X \in \mathbf{I}_M$, let X_i' be the copy of X_i in E^i. Since $\{X_i\}$ are disjoint, $\cup X_i'$ is independent in N_2, and $X_i \in \mathbf{I}_i$ implies that $\cup X_i'$ is also independent in N_1. From $X \in \mathbf{I}_M$, we have constructed $\cup X_i'$ of size $|X|$ in $\mathbf{I}_{N_1} \cap \mathbf{I}_{N_2}$. Conversely, any $X' \in \mathbf{I}_{N_1} \cap \mathbf{I}_{N_2}$ corresponds to a decomposition of a set in $\mathbf{I}_M$ of size $|X'|$ when the sets $X' \cap E^i$ are transferred back to E, because N_2 forbids multiple copies of elements.

Hence $r(E) = \max\{|I| : I \in \mathbf{I}_{N_1} \cap \mathbf{I}_{N_2}\}$. To compute this, let the rank functions of N_1, N_2 be q_1, q_2, and let r_i' be the rank function of the copy M_i' of M_i on E^i. We have $q_1(X') = \sum r_i'(X' \cap E^i)$, and $q_2(X')$ is the number of elements of E that have copies in X'. The Matroid Intersection Theorem yields $r(E) = \min_{X' \subseteq E'}\{q_1(X') + q_2(E' - X')\}$.

By Corollary 8.2.49, the minimum is achieved by a set X' such that $E' - X'$ is closed in N_2. The closed sets in the partition matroid N_2 are the sets that contain all or none of the copies of each element—the unions of full columns of E'. Given X' with $E' - X'$ closed in N_2, let $Y \subseteq E$ be the set of elements whose copies comprise X'. Then $q_2(E' - X') = |E - Y|$, and X' contains all copies of the elements of Y, so $q_1(X') = \sum r_i'(X' \cap E^i) = \sum r_i(Y)$. We conclude that $r(E) = \min_{Y \subseteq E}\{|E - Y| + \sum r_i(Y)\}$.

To show that M is a matroid, we verify submodularity for r. Given $X, Y \subseteq E$, the formula for r yields $U \subseteq X$ and $V \subseteq Y$ such that

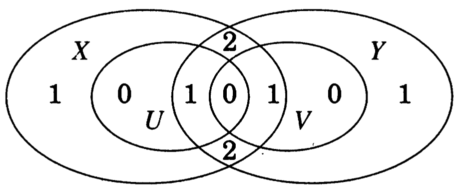
$$r(X) = |X - U| + \sum r_i(U); \qquad r(Y) = |Y - V| + \sum r_i(V).$$

Since $U \cap V \subseteq X \cap Y$ and $U \cup V \subseteq X \cup Y$, we also have

$$r(X \cap Y) \le |(X \cap Y) - (U \cap V)| + \sum r_i(U \cap V);$$

$$r(X \cup Y) \le |(X \cup Y) - (U \cup V)| + \sum r_i(U \cup V).$$

After applying the submodularity of each r_i and the diagram below, these inequalities yield $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$. ■



$$|(X \cap Y) - (U \cap V)| + |(X \cup Y) - (U \cup V)| = |X - U| + |Y - V|$$

In applying the Matroid Intersection Theorem, we needed N_1 to be a matroid, which required $\{M_i\}$ to be matroids. Hence this rank formula does not apply for unions of arbitrary hereditary systems.

The Matroid Union Theorem yields short proofs of min-max relations for packing and covering problems. In each formula below, the optimal subset is closed, since switching from X to $\sigma(X)$ improves the numerator without changing the denominator. The graph corollaries originally had difficult ad hoc proofs.

8.2.56. Corollary. (Matroid Covering Theorem—Edmonds [1965b]) In a loopless matroid M on E, the minimum number of independent sets whose union is E is $\max_{X \subseteq E} \left\lceil \frac{|X|}{r(X)} \right\rceil$.

Proof: Let $M_1, \ldots, M_k$ be copies of M on E. The set E is the union of k independent sets in M if and only if E is independent in $M' = M_1 \cup \cdots \cup M_k$. By the Matroid Union Theorem, $r'(E) \geq |E|$ is equivalent to $|E| - |Y| + \sum r_i(Y) \geq |E|$ for all $Y \subseteq E$. Since $r_i(Y) = r(Y)$ for all i, we conclude that E is the union of k independent sets if and only if $kr(Y) \geq |Y|$ for all $Y \subseteq E$. ■

8.2.57. Corollary. (Nash-Williams [1964]) The minimum number of forests needed to cover the edges of a graph G (its **arboricity**) is $\max_{H \subseteq G} \left\lceil \frac{e(H)}{n(H)-1} \right\rceil$.

Proof: (Edmonds [1965b]) This follows immediately by applying Corollary 8.2.56 to $M(G)$. The best lower bound arises from a connected induced subgraph H (corresponding to a closed set in $M(G)$). ■

8.2.58. Corollary. (Matroid Packing Theorem—Edmonds [1965c]) Given a matroid M on E, the maximum number of pairwise disjoint bases equals $\min_{X:\, r(X)<r(E)} \left\lfloor \frac{|E|-|X|}{r(E)-r(X)} \right\rfloor$.

Proof: The set E contains k disjoint bases if and only if $r'(E) \geq kr(E)$ in the union M' of k matroids $M_1, \ldots, M_k$ that are copies of M on E. By the Matroid Union Theorem, this requires $|E| - |Y| + \sum r_i(Y) \geq kr(E)$ for all $Y \subseteq E$. Since $r_i(Y) = r(Y)$ for all i, we conclude that k disjoint bases exist if and only if $|E| - |Y| \geq k(r(e) - r(Y))$ for all $Y \subseteq E$. ■

8.2.59. Corollary. (Nash-Williams [1961], Tutte [1961a]) A graph G has k pairwise edge-disjoint spanning trees if and only if, for every vertex partition P, there are at least $k(|P| - 1)$ edges with endpoints in different sets of P.

Proof: (Edmonds [1965c]) We may assume that G is connected. By applying Corollary 8.2.58 to $M(G)$, we must determine when $|E| - |X| \geq k(r(E) - r(X))$ for each closed set X. The closed sets correspond to partitions of $V(G)$ into vertex sets inducing connected subgraphs. For each such partition $V_1, \ldots, V_p$, the corresponding closed set X is $\bigcup E(G[V_i])$ with rank $n - p$. Since $|E| - |X|$ counts the edges between sets of the partition and $r(E) - r(X) \geq p - 1$, the graph has k disjoint spanning trees if and only if the condition holds. ■

EXERCISES

8.2.1. (−) Show that the stable sets of a graph need not be the independent sets of a matroid by finding vertex-weighted graphs where the ratio between the maximum weight of a stable set and the weight of a stable set found greedily is arbitrarily large.

8.2.2. (−) Characterize the graphs whose stable sets form the family of independent sets of a matroid on the set of vertices.

8.2.3. (−) Show that every partition matroid is a transversal matroid.

8.2.4. Modify the greedy algorithm to obtain (with proof) an algorithm for finding the maximum-weighted independent set in a matroid with arbitrary real weights (not necessarily nonnegative) on the elements.

8.2.5. Characterize the graphs whose matchings form the family of independent sets of a matroid on the set of edges.

8.2.6. (!) Determine which uniform matroids are graphic. Characterize the graphs whose cycle matroids are uniform matroids.

8.2.7. (!) Determine which partition matroids are graphic. Characterize the graphs whose cycle matroids are partition matroids.

8.2.8. Using only linear dependence, prove that vectorial matroids satisfy the induced circuit property: adding an element to a linearly independent set of vectors creates at most one minimal dependent set.

8.2.9. Describe the circuits of a transversal matroid M in terms of the corresponding bipartite graph G. Using only properties of bipartite graphs, prove that M satisfies the weak elimination property.

8.2.10. Let $M(G)$ be the cycle matroid of G. Let $k(X)$ be the number of components of the spanning subgraph G_X with edge set X; so $r(X) = n - k(X)$. Let U and V be the sets of components in G_X and G_Y, respectively. Let H be the U, V-bigraph with $u \leftrightarrow v$ when the components corresponding to u and v intersect.

a) Count the vertices and components of H in terms of the numbers $k(X)$, $k(Y)$, and $k(X \cap Y)$. Prove that $k(X \cup Y) \geq e(H)$.

b) Use part (a) to prove the submodularity property for $M(G)$ without using other properties of matroids. (Aigner [1979])

8.2.11. Use the König–Egerváry Theorem to prove directly that the rank function of a transversal matroid is submodular.

8.2.12. Let D be a digraph with distinguished source s and sink t. Let $E = V(D) - \{s, t\}$. For $X \subseteq E$, let $r(X)$ be the number of edges from $s \cup X$ to $\overline{X} \cup t$. Prove that r is submodular.

8.2.13. (−) For an element x in a hereditary system, prove that the following properties are equivalent and characterize loops.

a) $r(x) = 0$.
b) $x \in \sigma(\emptyset)$.
c) x is a circuit.
d) x belongs to no base.
e) Every set containing x is dependent.
f) x belongs to the span of every $X \subseteq E$.

8.2.14. (−) Prove equivalence of the following characterizations of parallel elements, assuming that $x \neq y$ and neither is a loop.

a) $r(x, y) = 1$.
b) $\{x, y\} \in \mathbf{C}$.
c) $x \in \sigma(y)$, $y \in \sigma(x)$, $r(x) = r(y) = 1$.

Furthermore, show that if x, y are parallel and $x \in \sigma(X)$, then $y \in \sigma(X)$.

8.2.15. (−) Suppose that $r(X) = r(X \cap Y)$ for some $X, Y \subseteq E$ in a matroid on E. Prove that $r(X \cup Y) = r(Y)$. Does the converse hold?

8.2.16. Let M be a hereditary system with nonnegative weights on E. Prove directly that if M satisfies the base exchange property (B), then the greedy algorithm always generates a maximum-weighted base.

8.2.17. *Alternative matroid axiomatics.* Let M be a hereditary system. Prove the following implications directly for M.

a) (−) Submodularity (R) implies weak absorption (A).

b) Strong absorption (A′) implies submodularity (R) (without using uniformity). (Hint: Use induction on $|X \triangle Y|$.)

c) Base exchange (B) implies uniqueness of induced circuits (J).

d) (−) Uniqueness of induced circuits (J) implies weak elimination (C).

e) Uniqueness of induced circuits (J) implies augmentation (I). (Hint: Use J and induction on $|I_1 - I_2|$ to obtain the augmentation.)

8.2.18. Prove that a hereditary system is a matroid if and only if it satisfies the "ultra-weak" augmentation property: If $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$ and $|I_1 - I_2| = 1$, then $I_1 + e \in \mathbf{I}$ for some $e \in I_2 - I_1$. (Chappell [1994a])

8.2.19. (−) Let M be a matroid on E, and fix $A \subseteq E$. Obtain $\mathbf{I}'$ from $\mathbf{I}$ by deleting the sets that intersect A. Prove that $\mathbf{I}'$ is the family of independent sets of a matroid on E.

8.2.20. For a matroid on E with $e \notin B \in \mathbf{B}$, let $C(e, B)$ be the unique circuit in $B + e$.

a) For $e \notin B$, prove that $B - f + e$ is a base if and only if f belongs to $C(e, B)$.

b) For $e \in C \in \mathbf{C}$, prove that $C = C(e, B)$ for some base B.

8.2.21. (−) Let B_1, B_2 be bases of a matroid such that $|B_1 \triangle B_2| = 2$. Prove that there is a unique circuit C such that $B_1 \triangle B_2 \subseteq C \subseteq B_1 \cup B_2$.

8.2.22. (−) Let B_1, B_2 be bases in a matroid M. Given $X_1 \subseteq B_1$, prove that there exists $X_2 \subseteq B_2$ such that $(B_1 - X_1) \cup X_2$ and $(B_2 - X_2) \cup X_1$ are both bases of M. (Greene [1973])

8.2.23. (!) Let B_1, B_2 be distinct bases of a matroid M.

a) Let G be a B_1, B_2-bigraph with $e \in B_1$ adjacent to $f \in B_2$ when $B_2 + e - f \in \mathbf{B}$. Prove that G has a perfect matching.

b) Conclude from part (a) that there exists a bijection $\pi\colon B_1 \to B_2$ such that for each $e \in B_1$, the set $B_2 - \pi(e) + e$ is a base of M.

8.2.24. (!) Let B_1, B_2 be distinct bases of a matroid M.

a) Prove that for each $e \in B_1$, there is $f \in B_2$ such that $B_1 - e + f$ and $B_2 - f + e$ are bases. (Hint: Use the incorporation property. Note: This generalizes Exercise 2.1.34.)

b) Use the cycle matroid $M(K_4)$ to show that there may be no bijection $\pi\colon B_1 \to B_2$ such that e and $f = \pi(e)$ satisfy part (a) for all $e \in B_1$.

8.2.25. (−) A collection of $|E| - r(E)$ circuits of a matroid on E form a **fundamental set of circuits** if it is possible to order the elements $e_1, \ldots, e_n$ in such a way that C_i contains $e_{r(E)+i}$ but no higher-indexed element. Prove that every matroid has a fundamental set of circuits. (Whitney [1935])

8.2.26. (−) Given k distinct circuits $\{C_i\}$ with none contained in the union of the others, and given a set X with $|X| < k$, prove that $\bigcup_{i=1}^{k} C_i - X$ contains a circuit. (Welsh [1976])

8.2.27. (+) For a hereditary system, prove directly that the weak elimination property implies the strong elimination property, using induction on $|C_1 \cup C_2|$. (Lehman [1964])

8.2.28. (!) *Min-max relation for weighted independent set.* Let M be a matroid on E, with each $e \in E$ having nonnegative integer weight $w(e)$. Let $\mathbf{A}$ be the set of chains $X_1 \subseteq X_2 \subseteq \cdots$ such that each $e \in E$ appears in at least $w(e)$ sets in the chain (sets may repeat in the chain). Use the greedy algorithm to prove that

$$\max_{I \in \mathbf{I}} \sum_{e \in I} w(e) = \min_{\{X_i\} \in \mathbf{A}} \sum_i r(X_i).$$

8.2.29. (−) Let r and σ be the rank function and span function of a matroid. Prove that $r(X) = \min\{|Y|: Y \subseteq X, \sigma(Y) = \sigma(X)\}$.

8.2.30. Prove that a matroid of rank r has at least 2^r closed sets. (Lazarson [1957])

8.2.31. Prove that a matroid is simple if and only if 1) no element appears in every hyperplane, and 2) from every distinct pair of elements some hyperplane contains exactly one. Prove that these conditions also suffice for a family of sets to be the collection of hyperplanes of a simple matroid.

8.2.32. Prove that in a matroid, a set is a hypobase if and only if it is a hyperplane.

8.2.33. Use the weak elimination property to characterize when a family of sets is the family of hyperplanes of some matroid.

8.2.34. Prove that the closed sets of a matroid are the complements of the unions of cocircuits.

8.2.35. Let X be a closed set in a matroid M.

a) Let Y be a closed set contained in X such that $r(Y) = r(X) - 1$. Prove that M has a hyperplane H such that $Y = X \cap H$. (Hint: Given a maximal independent subset Z of Y, augment it by $e \in X$ and then to a base B, and let $H = \sigma(B - e)$.)

b) Prove that X is the intersection of $r(M) - r(X)$ distinct hyperplanes.

8.2.36. Prove the following properties of closed sets in a matroid.

a) The intersection of two closed sets is a closed set.

b) The span of a set is the intersection of all closed sets containing it. (Comment: Hence $\sigma(X)$ is the unique minimal closed set containing X.)

c) The union of two closed sets need not be a closed set.

8.2.37. Prove that $M.X$ has no loops if and only if $\overline{X}$ is closed.

8.2.38. (!) *Bases and cocircuits in matroids.*

a) Prove that when e belongs to a base B in a matroid M, there is exactly one cocircuit of M disjoint from $B - e$, and it contains e.

b) Use part (a) to prove that if C is a circuit of a matroid M and x, y are distinct elements of C, then there is a cocircuit $C^* \in \mathbf{C}^*$ with $C^* \cap C = \{x, y\}$. (Minty [1966])

c) Explain why part (b) is trivial for cycle matroids.

8.2.39. (−) Show that the dual of a simple matroid (no loops or parallel elements) need not be simple. Determine whether a set can be both a circuit and a cocircuit in a matroid.

8.2.40. (!) Use matroid duality to prove Euler's Formula for connected plane graphs.

8.2.41. Prove that any minor of a matroid obtained by restricting and then contracting can also be obtained by contracting and then restricting. In particular, if M is a matroid on E and $Y \subseteq X \subseteq E$, prove that $(M|X).Y = (M.\overline{X - Y})|Y$ and $(M.X)|Y = (M|\overline{X - Y}).Y$.

8.2.42. (!) Use duality and matroid restriction to prove that $r_{M.F}(X) = r_M(X \cup F) - r_M(F)$. Also derive the formula directly by proving that X is independent in $M.F$ if and only if adding X to $\overline{F}$ increases the rank by $|X|$.

8.2.43. Prove that the cycle matroid $M(G)$ is the column matroid over $\mathbb{Z}_2$ of the vertex-edge incidence matrix of G. (Hence every graphic matroid is binary.)

8.2.44. Tutte [1958] proved that a matroid if and only if it has no $U_{2,4}$-minor.

a) Prove that the matrix $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ represents $U_{2,4}$ over $\mathbb{Z}_3$.

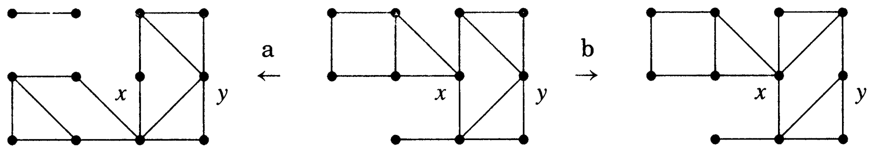
b) Prove that $U_{2,4}$ has no representation over $\mathbb{Z}_2$.

8.2.45. Prove that the three operations below preserve the cycle matroid of G.

a) Decompose G into its blocks $B_1, \ldots, B_k$, and reassemble them to form another graph G' with blocks $B_1, \ldots, B_k$.

b) In a block B of G that has a two-vertex cut $\{x, y\}$, interchange the neighbors of x and y in one of the components of $B - \{x, y\}$.

c) Add or delete isolated vertices.



(Comment: Whitney's 2-Isomorphism Theorem [1933b] states that G and H have the same cycle matroid if and only if some sequence of these operations turns G into H. Thus every 3-connected planar graph has only one dual graph, meaning essentially only one planar embedding. See also Kelmans [1980])

8.2.46. Construct a graph without isolated vertices that is an abstract dual of the graph below but is not a geometric dual of this graph. (Hint: Consider the operations of Exercise 8.2.45.) (Woodall, in Welsh [1976], p91–92)

8.2.47. The **matroid basis graph** is the graph having a vertex for each base of a matroid, with bases adjacent when their symmetric difference has size 2. Prove that every matroid basis graph has a spanning cycle, and interpret the result for graphic matroids and for uniform matroids. (Hint: Use contraction and restriction inductively to establish a spanning cycle through any edge.) (Holzmann–Harary [1972], Kung [1986, p72])

8.2.48. Use weak duality of linear programming to prove the weak duality property for matroid intersection: $|I| \le r_1(X) + r_2(\overline{X})$ for any $I \in \mathbf{I}_1 \cap \mathbf{I}_2$ and $X \subseteq E$. (Hint: Consider the discussion of dual pairs of linear programs in Remark 8.1.7.)

8.2.49. Let M_1, M_2 be two matroids on E.

a) Prove that the minimum size of a set in E that is spanning in both M_1 and M_2 is $\max_{X\subseteq E}(r_1(E) - r_1(X) + r_2(E) - r_2(\overline{X}))$.

b) Apply part (a) to prove that in a bipartite graph with no isolated vertices the minimum number of edges needed to cover all the vertices equals the maximum number of vertices with no edges among them. (König's "other" theorem)

c) From part (a), prove that the maximum size of a common independent set plus the minimum size of a common spanning set equals $r_1(E)+r_2(E)$. In particular, conclude Gallai's Theorem for bipartite graphs: in a bipartite graph with no isolated vertices, the maximum size of a matching plus the minimum number of edges needed to cover the vertices equals the number of vertices.

8.2.50. Use the Matroid Intersection Theorem to prove that in every acyclic orientation of G the vertices can be covered with at most $\alpha(G)$ pairwise-disjoint paths. (Chappell [1994b]) (Comment: This is the special case of Theorem 8.4.33 for acyclic digraphs.)

8.2.51. (−) Let M be the transversal matroid on $E = \cup A_i$ induced by sets $A_1, \ldots, A_m$. Use Hall's Theorem for matchings in bipartite graphs to derive the rank function as $r(X) = \min_{Y\subseteq X}\{|X| - (|Y| - |N(Y)|)\}$.

8.2.52. Let G be an $E, [m]$-bigraph without isolated vertices. For $X \subseteq E$, let $r(X) = \min\{|N(J) \cap X| - |J| + m\colon J \subseteq [m]\}$. Prove that the following are equivalent for X.

A) Hall's Condition holds ($|N(S)| \ge |S|$ for all $S \subseteq X$).

B) $r(X) \ge |X|$.

C) X is saturated by some matching in G.

(Hint: The proof of B $\Rightarrow$ C uses paths from unsaturated vertices that alternate between edges outside and within a specified matching.)

8.2.53. (!) Let G be an $E, [m]$-bigraph without isolated vertices. For $X \subseteq E$ and $J \subseteq [m]$, let $g(X, J) = |N(J) \cap X| - |J|$, and let $r(X) = \min\{g(X, J) + m \colon J \subseteq [m]\}$. Say that J is *X-optimal* if $r(X) = g(X, J) + m$.

a) Prove that $r(\emptyset) = 0$ and that $r(X) \le r(X + e) \le r(X) + 1$.

b) Prove that r satisfies the weak absorption property.

8.2.54. Prove that restrictions and unions of transversal matroids are transversal matroids, but that contractions and duals of transversal matroids need not be.

8.2.55. *Gammoids.* Let D be a digraph, and let F, E be subsets of $V(D)$. The **gammoid** on E induced by D, F is the hereditary system given by $\mathbf{I} = \{X \subseteq E \colon$ there exist $|X|$ pairwise disjoint paths from F to $X\}$; equivalently, $r(X)$ is the maximum number of pairwise disjoint F, X-paths.

a) Verify that every transversal matroid is a gammoid.

b) (+) Prove that every gammoid is a matroid. (Hint: Use Menger's Theorem to verify the submodularity property. Verifying the augmentation property is also possible but somewhat longer.) (Mason [1972])

8.2.56. *Strict gammoids.* Let D be a directed graph, let F, E be subsets of the vertices of D, and let M be the gammoid on E induced by D, F (Exercise 8.2.55). When E consists of all vertices of D, the gammoid is a **strict gammoid**. Prove that a matroid is a strict gammoid if and only if it is the dual of a transversal matroid. (Hint: Use a natural correspondence between directed graphs on n vertices and bipartite graphs on $2n$ vertices.) (Ingleton–Piff [1973])

8.2.57. (−) Since the union of two matroids is a matroid, there should be a dual operation yielding its dual. Given matroids M_1, M_2 with spanning sets $\mathbf{S}_1, \mathbf{S}_1$, let $M_1 \wedge M_2$ be the hereditary system whose spanning sets are $\{X_1 \cap X_2 \colon X_1 \in \mathbf{S}_1, X_2 \in \mathbf{S}_2\}$. Prove that $M_1 \wedge M_2$ is the matroid $(M_1^* \cup M_2^*)^*$.

8.2.58. *Generalized transversal matroids.*

a) Let M be a matroid on E, and let $\mathbf{A} = \{A_1, \dots, A_m\}$ be a set system on E. Let M' be the hereditary system on $[m]$ whose independent sets are the subsets of $\mathbf{A}$ having transversals that belong to $\mathbf{I}_M$. Prove that M' is a matroid with rank function $r'(X) = \min_{Y \subseteq X}\{|X - Y| + r(A(Y))\}$.

b) Let E, F be finite sets, and let f be a function from E to F. For $X \subseteq E$, let $f(X)$ be the set of images of elements of X. Let M be a matroid on E. Let M' be the hereditary system on F defined by $\mathbf{I}_{M'} = \{f(X) \colon X \in \mathbf{I}_M\}$. Prove that M' is a matroid. Prove also that $r'(X) = \min_{Y \subseteq X}\{|X - Y| + r(f^{-1}(Y))\}$ when f is surjective.

8.2.59. Apply matroid sum and Exercise 8.2.58 to prove the Matroid Union Theorem.

8.2.60. (!) Prove that the maximum size of a common independent set in matroids M_1 and M_2 on E is $r_{M_1 \cup M_2^*}(E) - r_{M_2^*}(E)$. Use this to prove the Matroid Intersection Theorem by applying the Matroid Union Theorem to $M_1 \cup M_2^*$. (Comment: Thus these two theorems are equivalent.)

8.2.61. Let G be an n-vertex weighted graph, and let $E_1, \dots, E_{n-1}$ be a partition of $E(G)$ into $n - 1$ sets. Is there a polynomial-time algorithm to compute a spanning tree of minimum weight among those that have exactly one edge in each subset E_i?

8.2.62. (!) Use the characterization of graphs having k pairwise edge-disjoint spanning trees (Corollary 8.2.59) to prove that every $2k$-edge-connected graph has k pairwise edge-disjoint spanning trees. Exhibit for each k a $2k$-edge-connected graph that does not have $k+1$ pairwise edge-disjoint spanning trees. (Nash-Williams [1961])

8.2.63. Given matroids $M_1, \dots, M_k$ on E, the **Matroid Partition Problem** is the problem of deciding whether an input set $X \subseteq E$ partitions into sets $I_1, \dots, I_k$ with $I_i \in \mathbf{I}_i$.

a) Use the Matroid Union Theorem to show that X is partitionable if and only if $|X - Y| + \sum r_i(Y) \geq |X|$ for all $Y \subseteq X$, and that all maximal partitionable sets are maximum partitionable sets.

b) Let M' be the union of k copies of a matroid M on E, and let X be a maximum partitionable set. Prove that there are disjoint sets $F_1, \dots, F_k \subseteq X$ such that $\{F_i\} \subseteq \mathbf{I}$ and $\overline{X} \subseteq \sigma(F_1) = \cdots = \sigma(F_k)$.

8.3. Ramsey Theory

"Ramsey theory" refers to the study of partitions of large structures. Typical results state that a special substructure must occur in some class of the partition. Motzkin described this by saying that "Complete disorder is impossible". The objects we consider are merely sets and numbers, and the techniques are little more than induction.

Ramsey's Theorem generalizes the pigeonhole principle, which itself concerns partitions of sets. We study applications of the pigeonhole principle, prove Ramsey's Theorem, and then focus on Ramsey-type questions for graphs. Finally, we discuss Sperner's Lemma about labelings of triangulations; like Ramsey's Theorem, it guarantees a special substructure.

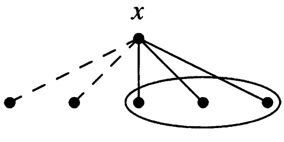
THE PIGEONHOLE PRINCIPLE REVISITED

The pigeonhole principle (Lemma A.57) states that if m objects are partitioned into n classes, then some class has at least $\lceil m/n \rceil$ objects (and some class has at most $\lfloor m/n \rfloor$ objects). This is a discrete version of the statement that every set of numbers contains a number at least as large as the average (and one at least as small). The concept is simple, but the applications can be quite subtle. The difficulty is how to define a partitioning problem relevant to the desired application. We illustrate this with four examples.

8.3.1. Proposition. Among six persons it is possible to find three mutual acquaintances or three mutual non-acquaintances.

Proof: (Exercise 1.1.29). In the language of graph theory, we are asked to show that for every simple graph G with six vertices, there is a triangle in G or in $\overline{G}$. The degrees of vertex x in G and $\overline{G}$ sum to 5, so the pigeonhole principle implies that one of them is at least 3.

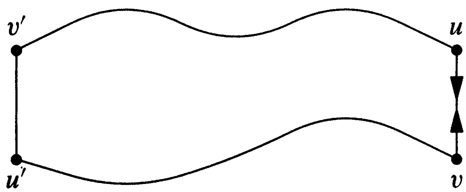
By symmetry, we may that assume $d_G(x) \geq 3$. If two neighbors of x are adjacent, then they form a triangle in G with x; otherwise, three neighbors of x form a triangle in $\overline{G}$. ■



8.3.2. Theorem. (Graham–Entringer–Székely [1994]) If T is a spanning tree of the k-dimensional cube Q_k, then there is an edge of Q_k outside T whose addition to T creates a cycle of length at least $2k$.

Proof: For each vertex v of Q_k, expressed as a binary k-tuple, there is a complementary vertex v' that differs from v in each position. There is a unique v, v'-path in T; orient its first edge toward v'. Since $n(Q_k) = e(T) + 1$, doing this for each vertex orients some edge twice, by the pigeonhole principle.

Since this edge uv receives an orientation from u and from v, we have v on the u, u'-path and u on the v, v'-path in T. Hence the u, v'-path and the v, u'-path in T are disjoint. Each has length at least $k - 1$, since the distance in Q_k between a vertex and its complement is k. Finally, $u \leftrightarrow v$ in Q_k implies also $u' \leftrightarrow v'$, which completes a cycle of length at least $2k$. ■



Theorem 8.3.2 implies that every spanning tree of Q_k has diameter at least $2k - 1$ (Graham–Harary [1992]).

8.3.3. Theorem. (Erdős–Szekeres [1935]) Every list of more than n^2 distinct numbers has a monotone sublist of length more than n.

Proof: Let $a = a_1, \ldots, a_{n^2+1}$ be the list. Assign position k the label (x_k, y_k), where x_k is the length of a longest increasing sublist ending at a_k, and y_k is the length of a longest decreasing sublist ending at a_k. If a has no monotone sublist of length $n + 1$, then x_k and y_k never exceed n, and there are only n^2 possible labels.

Since the list has length $n^2 + 1$, the pigeonhole principle now implies that two labels must be the same. This is impossible when the elements of a are distinct. When $i < j$ and $a_i < a_j$, we can append a_j to the longest increasing sequence ending at a_i. When $i < j$ and $a_i > a_j$, we can append a_j to the longest decreasing sequence ending at a_i. (See Exercise 5.1.43 for a generalization.) ■

a:	7	4	1	8	5	2	9	6	3	0
x, y:	1, 1	1, 2	1, 3	2, 1	2, 2	2, 3	3, 1	3, 2	3, 3	4, 1

8.3.4. Theorem. (Graham–Kleitman [1973]) In every labeling of $E(K_n)$ using distinct integers, there is a trail of length at least $n-1$ along which the labels strictly increase.

Proof: We assign each vertex a weight equal to the length of the longest increasing trail ending there. If we can show that these n weights sum to at least $n(n-1)$, then the pigeonhole principle guarantees a vertex with a large enough weight. The problem is how to compute the weights and their sum.

We grow the graph from the trivial graph by adding the edges in order, updating the weights and their sum at each step. The vertex weights begin at 0. If the next edge joins two vertices whose weights were both i, then their weights both become $i+1$. If it joins two vertices of weights i and j with $i<j$, then their weights become $j+1$ and j.

In either case, each time an edge is added, the sum of the weights of the vertices increases by at least 2. Therefore, when the construction is finished, the sum of the vertex weights is at least $n(n-1)$. ∎

Finally, we note that the thresholds in the classes may differ.

8.3.5. Theorem. If $\sum p_i - k + 1$ objects are partitioned into k classes with quotas $\{p_i\}$, then some class must meet its quota.

Proof: If not, then at most $\sum(p_i - 1)$ objects can be accommodated. ∎

RAMSEY'S THEOREM

The pigeonhole principle guarantees a class with many objects when we partition objects into classes. The famous theorem of Ramsey [1930] makes a similar statement about partitioning the r-element subsets of objects into classes. Roughly put, Ramsey's Theorem says that whenever we partition the r-sets in a sufficiently large set S into k classes, there is a p-subset of S whose r-sets all lie in the same class.

A partition is a separation of a set into subsets, and the set we want to partition consists of subsets of another set, so for clarity we use the language of coloring instead of the language of partitioning. Recall that a **k-coloring** of a set is a partition of it into k classes. A class or its label is a **color**. Typically we use $[k]$ as the set of colors, in which case a k-coloring of X can be viewed as a function $f\colon X \to [k]$.

8.3.6. Definition. Let $\binom{S}{r}$ denote the set of r-element subsets (**r-sets**) of a set S. A set $T \subseteq S$ is **homogeneous** under a coloring of $\binom{S}{r}$ if all r-sets in T receive the same color; it is **i-homogeneous** if that color is i.

Let r and $p_1, \dots, p_k$ be positive integers. If there is an integer N such that every k-coloring of $\binom{[N]}{r}$ yields an i-homogeneous set of size p_i for some i, then the smallest such integer is the **Ramsey number** $R(p_1, \dots, p_k; r)$.

Ramsey's Theorem states that such an integer exists for every choice of r and $p_1, \ldots, p_k$ (the latter are called **thresholds** or **quotas**). When the quotas all equal p, the theorem states that every k-coloring of the r-sets of a sufficiently large set has a p-set whose r-sets receive the same color. A thorough study of Ramsey's Theorem and other partitioning theorems appears in Graham–Rothschild–Spencer [1980, 1990].

Before proving the theorem, we consider the case $r = k = 2$, which is easy to describe in terms of edge-coloring of graphs. The proof for this case has the same structure as for the general case.

When $r = 2$, a k-partition of $\binom{S}{r}$ is merely a k-edge-coloring of the complete graph with vertex set S (not a proper edge-coloring). When $k = 2$, the time-honored tradition in Ramsey theory is that color 1 is "red" and color 2 is "blue".

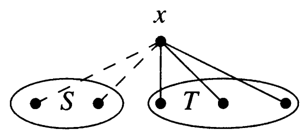
By Proposition 8.3.1, $R(3, 3; 2) \leq 6$; we extend the argument to prove that

$$R(p_1, p_2; 2) \leq R(p_1 - 1, p_2; 2) + R(p_1, p_2 - 1; 2).$$

Assuming that $R(p_1 - 1, p_2; 2)$ and $R(p_1, p_2 - 1; 2)$ exist, let N be their sum. Proving the bound for $R(p_1, p_2; 2)$ means showing that every red/blue-coloring of the edges of a complete graph with N vertices yields a p_1-set of vertices within which all edges are red or a p_2-set of vertices within which all edges are blue.

Consider a red/blue-coloring of K_N, and choose a vertex x. Let $s = R(p_1 - 1, p_2; 2)$ and $t = R(p_1, p_2 - 1; 2)$; there are $s + t - 1$ vertices other than x. Theorem 8.3.5 implies that x has at least s incident red edges or at least t incident blue edges.

By symmetry, we may assume that x has at least N incident red edges. By the definition of s, the complete subgraph induced by the neighbors of x along these edges has a blue p_2-clique or a red $p_1 - 1$-clique. The latter would combine with x to form a red p_1-clique. In either case, we obtain an i-homogeneous set of size p_i for some i. We postpone discussion of the resulting bound on $R(p_1, p_2; 2)$.



$|S| \geq R(p_1, p_2 - 1; 2)$ or $|T| \geq R(p_1 - 1, p_2; 2)$

8.3.7. Theorem. (Ramsey [1930]) Given positive integers r and $p_1, \ldots, p_k$, there exists an integer N such that every k-coloring of $\binom{[N]}{r}$ yields an i-homogeneous set of size p_i for some i.

Proof: The proof is a "double" induction. We use induction on r, but the proof of the induction step itself uses induction on $\sum p_i$.

Basis step: $r = 1$. By Theorem 8.3.5, $R(p_1, \ldots, p_k; 1)$ exists.

Induction step: $r > 1$. We assume that the claim in the theorem statement holds for k-colorings of the $r - 1$-subsets of a set, no matter what the thresholds are. We prove the same statement for k-colorings of the r-subsets of a set by induction on the sum of the quotas, $\sum p_i$.

Basis step: some quota p_i is less than r. In this case, a set of p_i objects contains no r-sets, so vacuously its r-sets all have color i. Hence $R(p_1, \ldots, p_k; r) = \min\{p_1, \ldots, p_k\}$ when $\min\{p_1, \ldots, p_k\} < r$.

For clarity, we state the induction step only for $k = 2$; the argument for general k is similar (Exercise 17). Write (p, q) for (p_1, p_2). Let

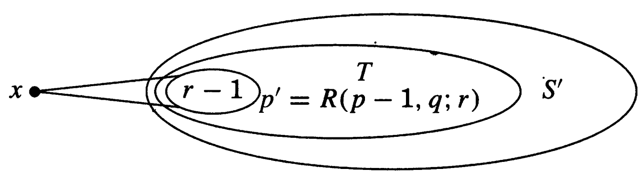
$$p' = R(p-1, q; r), \quad q' = R(p, q-1; r), \quad N = 1 + R(p', q'; r-1).$$

By the induction hypothesis of the inner induction, p' and q' exist. By the induction hypothesis of the outer induction, N also exists. Note that p' and q' may be very large; this is why we need the double induction.

Let S be a set of N elements, and choose $x \in S$. Consider a 2-coloring f of $\binom{S}{r}$. With colors (red,blue), we need to show that f has a red-homogeneous p-set or a blue-homogeneous q-set.

We use f to induce a 2-coloring f' of the $r-1$-sets of $S' = S - x$. This is the reason for our choice of $|S'|$ as a Ramsey number for $r-1$-sets. Define f' by assigning color i to an $(r-1)$-set in S' if its union with x has color i under f. Since $|S'| = R(p', q'; r-1)$, the induction hypothesis implies that some color meets its quota (p' or q') under f' (when $r = 2$, this step was the invocation of the pigeonhole principle). By symmetry, we may assume that the red quota is met. Let T be a p'-element subset of S' whose $r-1$-sets are red under f'.

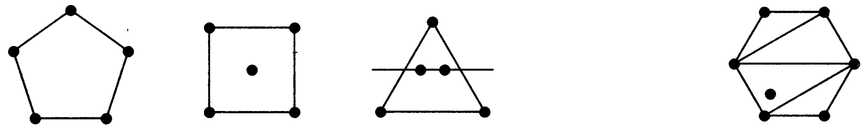
We return to the original coloring f on the r-sets in T. Since $|T| = p' = R(p-1, q; r)$, under f there is a red-homogeneous $p-1$-set or a blue-homogeneous q-set in T. If there is a blue-homogeneous q-set, then we are done. If there is a red-homogeneous $p-1$-set P, then consider $P \cup \{x\}$. From the definition of T, the $(r-1)$-sets of P are all red under f', which means their unions with x are red under f. Hence $P \cup \{x\}$ is a red-homogeneous p-set under f. ∎



Like the pigeonhole principle, Ramsey's Theorem has subtle and fascinating applications. Ramsey's Theorem typically gives an elegant existence proof but a horribly large bound.

8.3.8. Theorem. (Erdős–Szekeres [1935]) Given an integer m, there exists a (least) integer $N(m)$ such that every set of at least $N(m)$ points in the plane with no three collinear contains an m-subset forming a convex m-gon.

Proof: We need two facts. *(1) Among five points in the plane, four determine a convex quadrilateral* (if no three are collinear). Construct the convex hull of the five points. If it is a pentagon or a quadrilateral, then the result follows immediately. If it is a triangle, then the other two points lie inside. By the pigeonhole principle(!), two of the vertices of the triangle are on one side of the line through the two inside points. These two vertices together with the two points inside form a convex quadrilateral, as illustrated below.



In a convex m-gon, any four corners determine a convex quadrilateral. We need the converse: *(2) If every 4-subset of m points in the plane forms a convex quadrilateral, then the m points form a convex m-gon.* If the claim fails, then the convex hull of the m points consists of t points, for some $t < m$. The remaining points lie inside the t-gon. When we triangulate the t-gon, as illustrated on the right above, a point inside lies in one of the triangles. With the vertices of that triangle, it forms a 4-set that does not determine a convex quadrilateral.

To prove the theorem, let $N = R(m, 5; 4)$. Given N points in a plane with no three on a line, color each 4-set by convexity: red if it determines a convex quadrilateral, blue if it does not. By fact (1), there cannot be five points whose 4-subsets are all blue. By Ramsey's Theorem, this means there are m points whose 4-subsets are all red. By fact (2), they form a convex m-gon. Hence $N(m)$ exists and is at most $R(m, 5; 4)$. ∎

The bound $R(m, 5; 4)$ is very loose. It is exact for $m = 4$, where fact (1) implies that $N(4) = 5 = R(4, 5; 4)$. In contrast, $N(5) = 9$ (Exercise 10), but $R(5, 5; 4)$ is enormous. Erdős and Szekeres conjectured that $N(m) = 2^{m-2} + 1$ and proved that $2^{m-2} \le N(m) \le \binom{2m-4}{m-2} + 1$.

Another application concerns search strategies for numbers stored in tables. From a set U, a subset of size n is stored in a table of size n according to some rule for storing n-sets. Yao [1981] used Ramsey's Theorem to prove that when U is large, the strategy minimizing the worst-case number of probes required to test whether some element of U is in the table is to store the chosen set in sorted order and test membership by binary search. (For small U, this strategy is not best!) The value that Ramsey's Theorem yields for "large" is probably much larger than needed.

RAMSEY NUMBERS

Ramsey's Theorem defines the Ramsey numbers $R(p_1, \dots, p_k; r)$. No exact formula is known, and few Ramsey numbers have been computed. To prove that $R(p_1, \dots, p_k; r) = N$, we must exhibit a k-coloring of the r-sets among $N-1$ points that meets no quota (or show that one exists without constructing it), *and* we must show that *every* coloring on N points meets some quota.

In principle, we could use a computer to examine all k-colorings of $\binom{[n]}{r}$ for successive n until we find the first N such that every such coloring meets a quota p_i for some i. Even for 2-color Ramsey numbers, $2^{\binom{n}{2}}$ rapidly becomes too large to contemplate. Erdős joked that if an alien being threatened to destroy us unless we told it the exact value of $R(5,5)$, then we should set all the computers in the world to work on an exhaustive solution. If we were asked for $R(6,6)$, then his advice was to try to destroy the alien.

When $r = 2$, we abbreviate the notation $R(p_1, \ldots, p_k; r)$ to $R(p_1, \ldots, p_k)$. When $p = p_1 = \cdots = p_k$, we abbreviate it to $R_k(p; r)$. For $r > 2$, little is known other than $R(4,4;3) = 13$ (McKay–Radziszowski [1991]). Even for $r = 2$, only one Ramsey number is known exactly when $k > 2$, which is $R(3,3,3) = 17$. The table below contains the known values of $R(p,q)$ and the best known upper and lower bounds for several other values as of July 1999. Several of these bounds have improved slightly since the first edition of this book. The current bounds are maintained in Radziszowski [1995], which is periodically updated.

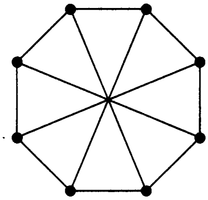
	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	35/41	49/61	55/84	69/115
5			43/49	58/87	80/143	95/216	116/316
6				102/165	109/298	122/495	153/780

The computations of $R(3,9)$ (Grinstead–Roberts [1982]), $R(3,8)$ (McKay–Zhang [1992]), and $R(4,5)$ (McKay–Radziszowski [1995]) are recent; the others are much older (due primarily to Greenwood–Gleason [1955], Kalbfleisch [1967], and Graver–Yackel [1968]).

We prove only the first two of these results (see Exercise 16 for $R(3,5)$). When $r = k = 2$, we simplify terminology by using two colors called "in" and "out". Ramsey's Theorem for this case then becomes: "There exists a minimum integer $R(p,q)$ such that every graph on $R(p,q)$ vertices has a clique of size p or an independent set of size q".

8.3.9. Example. $R(3,3) = 6$. We showed earlier that $R(3,3) \le 6$. Since the 5-cycle has no triangle and no independent 3-set, $R(3,3) \ge 6$. ■

8.3.10. Example. $R(3,4) = 9$. The graph below has no K_3 and no $\overline{K}_4$, since four independent vertices on an 8-cycle include pairs of opposite vertices on the cycle. Hence $R(3,4) \ge 9$.



Given a vertex x in a graph G, we can add x to two adjacent neighbors to form a triangle or add x to an independent 3-set of nonneighbors to form an independent 4-set. Since $R(2,4) = 4$ and $R(3,3) = 6$, we conclude that if x has four neighbors or has six nonneighbors, then G has a triangle or an independent 4-set. Avoiding both possibilities limits x to at most three neighbors and at most five nonneighbors, which yields $n(G) \leq 9$. If this occurs for a 9-vertex graph, then *every* vertex has exactly three neighbors. Since the degree-sum formula forbids 3-regular graphs of order 9, we obtain $R(3, 4) = 9$. ■

The proof of Ramsey's Theorem yields a (very large) recursive upper bound on $R(p, q; r)$. Graham–Rothschild–Spencer [1980, 1990] explains how large.

8.3.11. Theorem. $R(p, q) \leq R(p-1, q) + R(p, q-1)$. If both summands on the right are even, then the inequality is strict.

Proof: If a vertex in an arbitrary graph has $R(p-1, q)$ neighbors or $R(p, q-1)$ nonneighbors, then the graph has a p-clique or an independent q-set. With $R(p-1, q) + R(p, q-1)$ points altogether in the graph, the pigeonhole principle guarantees that one of these possibilities occurs. Equality in the bound requires a regular graph with $R(p-1, q) + R(p, q-1) - 1$ vertices. If both summands are even, this requires a regular graph of odd degree on an odd number of vertices, which is impossible. ■

Since $R(p, 2) = R(2, p) = p$, Theorem 8.3.11 yields $R(p, q) \leq \binom{p+q-2}{p-1}$ (Exercise 15). The lack of exact answers has led to study of asymptotics. For fixed q and large p, $R(p, q) \leq cp^{q-1} \log\log p / \log p$ (Graver–Yackel [1968], Chung–Grinstead [1983]). For $q = 3$, the answer is known within a constant factor:

$$c' p^2 / \log p \leq R(p, 3) \leq cp^2 / \log p.$$

The upper bound is due to Ajtai–Komlós–Szemerédi [1980]; the lower to Kim [1995]. All these bounds use probabilistic methods (Section 8.5).

Ramsey numbers for equal quotas are called **diagonal Ramsey numbers**. Asymptotically, the upper bound of $\binom{2p-2}{p-1}$ for $R(p, p)$ is $c4^p/\sqrt{p}$. Exercise 14 presents a constructive lower bound that is polynomial in p. The best known constructive lower bound grows faster than every polynomial in p but slower than every exponential in p (Frankl–Wilson [1981], Exercise 29).

An exponential lower bound can be proved by counting methods. It yields

$$\sqrt{2} \leq \liminf R(p, p)^{1/p} \leq \limsup R(p, p)^{1/p} \leq 4.$$

Determination of this limit (and whether it exists) is the foremost open problem about Ramsey numbers.

8.3.12. Theorem. (Erdős [1947]). $R(p, p) > (e\sqrt{2})^{-1} p 2^{p/2}(1 + o(1))$.

Proof: Consider the graphs with vertex set $[n]$. Each possible p-clique occurs in $2^{\binom{n}{2}-\binom{p}{2}}$ of these $2^{\binom{n}{2}}$ graphs. Similarly, each p-set occurs as an independent set in $2^{\binom{n}{2}-\binom{p}{2}}$ of these graphs. Discarding this amount for each possible p-clique and each possible independent p-set leaves a lower bound on the number of graphs having no p-clique or independent p-set.

Since there are $\binom{n}{p}$ ways to choose p vertices, the inequality $2\binom{n}{p}2^{-\binom{p}{2}} < 1$ thus implies $R(p, p) > n$. Rough approximations yield $\binom{n}{p}2^{1-\binom{p}{2}} < 1$ whenever $n < 2^{p/2}$. More careful approximations (using Stirling's formula to approximate the factorials) lead to the result claimed. ■

GRAPH RAMSEY THEORY

Ramsey's Theorem for $r = 2$ says that k-coloring the edges of a large enough complete graph forces a monochromatic complete subgraph. A monochromatic p-clique contains a monochromatic copy of every p-vertex graph. Perhaps monochromatic copies of graphs with fewer edges can be forced by coloring a smaller graph than needed to force K_p. For example, 2-coloring the edges of K_3 always yields a monochromatic P_3, although six points are needed to force a monochromatic triangle. This suggests many Ramsey number questions, some easier to answer than the questions for cliques.

8.3.13. Definition. Given simple graphs $G_1, \dots, G_k$, the **(graph) Ramsey number** $R(G_1, \dots, G_k)$ is the smallest integer n such that every k-coloring of $E(K_n)$ contains a copy of G_i in color i for some i. When $G_i = G$ for all i, we write $R_k(G)$ for $R(G_1, \dots, G_k)$.

Burr [1983] determined $R(G, G)$, called the "Ramsey number of G", for all 113 graphs with at most six edges and no isolated vertices. Nice formulas are known for $R(G_1, G_2)$ in some cases. Again our two colors are red and blue.

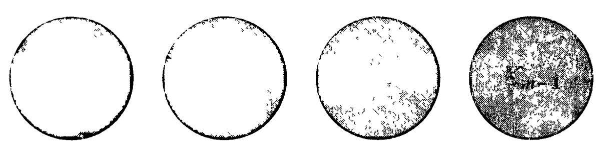
8.3.14. Theorem. (Chvátal [1977]) If T is an m-vertex tree, then $R(T, K_n) = (m-1)(n-1)+1$.

Proof: For the lower bound, color $K_{(m-1)(n-1)}$ by letting the red graph be $(n-1)K_{m-1}$. With red components of order $m-1$, there is no red m-vertex tree. The blue edges form an $n-1$-partite graph and hence cannot contain K_n.

The proof of the upper bound uses induction on each parameter, focusing on the neighbors of one vertex. Our presentation uses induction on n, invoking a property of trees that we proved in Chapter 2 by induction on m. The basis step is $n = 1$; no edges are needed to obtain K_1.

Given a 2-coloring of $E(K_{(m-1)(n-1)+1})$, consider a vertex x. If x has more than $(m-1)(n-2)$ neighbors along blue edges, then the induction hypothesis yields a red T or a blue K_{n-1} among them. This yields a red T or a blue K_n (with x) in the full coloring.

Otherwise, every vertex has at most $(m-1)(n-2)$ incident blue edges and thus at least $m-1$ incident red edges. This yields a red T, because every graph with minimum degree at least $m-1$ contains T (Proposition 2.1.8). ■

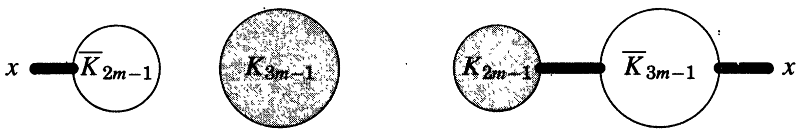


Whenever the largest component of G has m vertices and $\chi(H) = n$, the construction in Theorem 8.3.14 yields $R(G, H) \geq (m-1)(n-1)+1$ (Chvátal–Harary [1972]). Burr and Erdős [1983] conjectured that $R(G, K_n) = (m-1)(n-1)+1$ when m is sufficiently large relative to $n(H)$ and $\max_{F\subseteq G} \frac{e(F)}{n(F)}$. Although this holds (Burr [1981]) when G has many vertices of degree 2 and in some other cases, Brandt [2000] showed that for every nonbipartite graph H (such as K_n) and every $h \in \mathbb{R}$, there is a threshold d_0 such that $R(G, H) > hn(G)$ for almost every d-regular graph G with $d > d_0$.

In the upper bound for Theorem 8.3.14, it is crucial that the color classes in H are single vertices. When this fails, the lower bound can be very weak. When $G = H = mK_3$, for example, the Chvátal–Harary result yields $R(G, H) \geq (3-1)(3-1)+1 = 5$, but the correct value is $5m$. Here the coloring for the lower bound is surprisingly asymmetric, considering the symmetry of the inputs.

8.3.15. Theorem. (Burr–Erdős–Spencer [1975]) $R(mK_3, mK_3) = 5m$ for $m \geq 3$.

Proof: Let the red graph be $K_{3m-1} + K_{1,2m-1}$, as shown below. Every triangle in this graph uses three vertices from the $3m-1$-clique, but the clique does not have enough vertices to make m disjoint triangles. The complementary blue graph is $(K_{2m-1} + K_1) \vee \overline{K}_{3m-1}$. Every blue triangle has at least 2 vertices in the copy of K_{2m-1}, so there cannot be m disjoint blue triangles.



For the upper bound, we use induction on m. Basis step: $m = 2$. This requires a case analysis that is fairly short if phrased carefully (Exercise 26).

Induction step: $m \geq 3$. Since $5m > R(3,3) = 6$, we know that every 2-coloring contains a monochromatic triangle. Discarding vertices of triangles as we find them, we can continue to find monochromatic triangles while at least six vertices remain. Since $5m - 3m \geq 6$ for $m \geq 3$, we find m disjoint monochromatic triangles. If these all have the same color, then we are done.

Otherwise, we have at least one triangle in each color. Let abc be a red triangle, and let def be a blue triangle disjoint from it. Of the nine edges between them, we may assume by symmetry that at least five are red. Some pair of these must have a common endpoint in def.

Now we have a red triangle and a blue triangle with a common vertex; together they have five vertices. Since $m > 2$, the induction hypothesis for the coloring on the remaining $5m-5$ vertices yields $(m-1)K_3$ in one color. We add the appropriately colored triangle from the five special vertices. ■

Readers worried about the omission of the basis step in Theorem 8.3.15 may consider coloring K_{11}. Avoiding $2K_3$ forces a bowtie (monochromatic triangles with a common vertex) as above, but then we find another monochromatic triangle among the remaining six points. This completes a proof that $R(mK_3, mK_3) \leq 5m + 1$. Related results appear in Exercises 27–28.

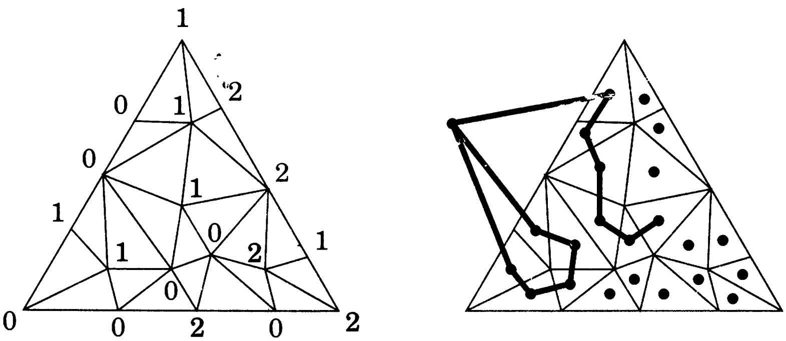
We mention one remarkable result. The Ramsey number of an arbitrary graph may be exponential in the number of vertices, such as for K_n. Chvátal, Rödl, Szemerédi, and Trotter [1983] proved that for the class of graphs with maximum degree d, the Ramsey number grows at most linearly in the number of vertices! In other words, $R(G, G) \leq cn(G)$, where c is a constant depending only on d. Of course, the constant is a fast-growing function of d, but it does not depend on $n(G)$. The proof uses the Szemerédi Regularity Lemma [1978], itself a difficult result with many applications.

SPERNER'S LEMMA AND BANDWIDTH

Although Sperner's Lemma is not generally considered part of Ramsey theory, we include this material in this section because Sperner's Lemma has the flavor of Ramsey theory: every labeling of a triangulation that satisfies certain boundary conditions contains a piece with a special labeling (one element from *each* class). Like Ramsey's Theorem, Sperner's Lemma uses very simple ideas but has subtle applications; Ramsey's Theorem relies on the pigeonhole principle and induction, while Sperner's Lemma uses only a parity argument (and induction for a generalization to higher dimensions).

8.3.16. Definition. A **simplicial subdivision** of a large triangle T is a partition of T into triangular **cells** such that every intersection of two cells is a common edge or corner. We call the corners of cells **nodes**. A **proper labeling** of a simplicial subdivison of T assigns labels from $\{0, 1, 2\}$ to the nodes, avoiding label i on the ith edge of T, for $i \in \{0, 1, 2\}$. A **completely labeled cell** is a cell having all three labels on its corners.

In a proper labeling, each label appears at one corner of T, and label i avoids the edge of T joining the corners not labeled i. The figure below illustrates a simplicial subdivision and the graph we will obtain from it to prove that it has a completely labeled cell.



8.3.17. Theorem. (Sperner's Lemma [1928]) Every properly labeled simplicial subdivision has a completely labeled cell.

Proof: We prove the stronger result that there are an odd number of completely labeled cells. We seek such a cell by beginning outside T and entering a cell by crossing an edge with labels 0 and 1. If we reach a cell whose third label is 2, we are finished. If not, then the third label is 0 or 1, and the cell has another 0,1-edge. By crossing it, we enter a new cell and can continue looking for a cell with the third label.

This suggests defining a graph G encoding the possible steps. We include a vertex for each cell plus one vertex for the outside region. Two vertices of G are adjacent if those regions share a boundary edge with endpoints labeled 0 and 1. The graph on the right above results from the proper labeling on the left.

A completely labeled cell becomes a vertex of degree 1 in G. A cell with no 0 or no 1 becomes a vertex of degree 0. The remaining cells have corners labeled $0, 0, 1$ or $0, 1, 1$ and become vertices of degree 2. Hence the desired cells become vertices of degree 1 in G, and these are the only cells that become vertices of odd degree. We have transformed the original problem into the problem of showing that G has such a vertex of degree 1.

The vertex v for the outside region also has odd degree. As we travel from the 0-corner to the 1-corner along the edge of T that avoids label 2, we cross an edge of G involving v every time we switch from a 0 to a 1 or back again. Since we start with 0 and end with 1, we switch an odd number of times. Hence v has odd degree. Since the number of vertices of odd degree in every graph is even, the number of vertices other than v having odd degree is odd, so there are an odd number of completely labeled cells. ∎

8.3.18. Application. *Brouwer Fixed-Point Theorem.* Brouwer's Theorem (for two dimensions) can be interpreted as saying that a continuous mapping from a triangular region T to itself must have a fixed point. Suppose that the corners of T are the points (vectors) v_0, v_1, v_2. Just as we can express a point on a segment uniquely as a weighted average of its endpoints, so we can express each $v \in T$ uniquely as a weighted average of the corners: $v = a_0v_0 + a_1v_1 + a_2v_2$, where $\sum a_i = 1$ and each $a_i \geq 0$ (Exercise 37). We can specify v by its vector of coefficients $a = (a_0, a_1, a_2)$.

Define sets S_0, S_1, S_2 from the mapping f by placing $a \in S_i$ if $a_i' \leq a_i$, where $f(a) = a'$. Since the coefficients of each point sum to 1, every point in T belongs to some S_i, and a point belongs to all three sets if and only if it is a fixed point for f. We want to show that the three sets have a common point.

Given a simplicial subdivision of T, for each node a choose a label i such that $a \in S_i$. Points on the edge of T opposite v_i have ith coordinate 0. Their ith coordinate cannot decrease under f, so we can choose a label other than i for each point on that edge. The resulting labeling is proper, and Sperner's Lemma guarantees a completely labeled cell. Repeating the process using triangulations with successively smaller cells yields a sequence of successively smaller completely labeled triangles. Let the jth triangle have corners x_j, y_j, z_j with labels 0,1,2, respectively. In each S_i, we obtain an infinite sequence of points.

The remaining details are topological; we merely suggest the steps. Since f is continuous, each S_i is closed and bounded. Every infinite sequence of points in a closed and bounded set has a convergent subsequence. Hence $\{x_1, x_2, \ldots\}$ has a convergent subsequence; let x_{i_k} be its kth entry. Because the distance from x_{i_k} to y_{i_k} and z_{i_k} approaches 0, these subsequences also converge to the same point. Since S_0, S_1, S_2 are closed and bounded, this limit point belongs to all three of them and is a fixed point of f. ■

We also apply Sperner's Lemma to solve a problem on the "triangular grid".

8.3.19. Definition. When the vertices of G are numbered with distinct integers, the **dilation** is the maximum difference between integers assigned to adjacent vertices. The **bandwidth** $B(G)$ of a graph G is the minimum dilation of a numbering of G.

Dilation is always minimized when there are no gaps in the numbering, but it can be convenient to allow gaps (Exercise 42). The name "bandwidth" comes from matrix theory; the optimal numbering describes a permutation of the rows and columns of the adjacency matrix so that the 1's appear only in diagonal bands close to the main diagonal; arranging the matrix in this order can speed up computation of the inverse. Another motivation is to minimize the delay between adjacent vertices when the vertices must be processed in a linear order. Computation of bandwidth is NP-hard even for trees with maximum degree 3 (Garey–Graham–Johnson–Knuth [1978]).

We present two fundamental lower bounds on bandwidth.

8.3.20. Lemma. $B(G) \geq \max_{H\subseteq G} \frac{n(H)-1}{\operatorname{diam} H}$.

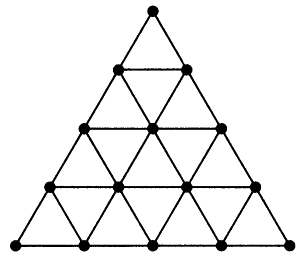
Proof: Every numbering of G contains a numbering of each subgraph of G. On every subgraph H, two numbers differing by at least $n(H) - 1$ are used. By the pigeonhole principle, some edge on a path between the two corresponding vertices has dilation at least $n(H) - 1$ divided by the distance between them. ■

8.3.21. Lemma. (Harper [1966]) $B(G) \geq \max_k \min\{|\partial S| \colon |S| = k\}$, where ∂S denotes the subset of vertices in a set $S \subseteq V(G)$ that have at least one neighbor outside S.

Proof: For every value of k, some set S of k vertices must be the first k vertices in the optimal numbering of G. The bandwidth of G must be at least $|\partial S|$, because the vertex among ∂S that has the least label has an edge of dilation at least $|\partial S|$ to its neighborhood above S. ■

Chung [1988] named the first bound the **local density** bound. The computation of Harper's bound is usually difficult. For the cube Q_k, the value is $\sum_{i=0}^{n-1} \binom{i}{\lfloor i/2 \rfloor}$. For the grid $P_m \square P_n$, the value of Harper's lower bound is $\min\{m, n\}$, which can be achieved (Exercise 43).

8.3.22. Example. *The triangular grid.* The triangular grid T_l consists of vertices (i, j, k) such that i, j, k are nonnegative integers summing to l, with two vertices adjacent if the total of the absolute differences in corresponding coordinates is 2. Below we show T_4. Numbering the vertices by rows produces an upper bound of $l+1$ for $B(T_l)$. This is optimal, but the local density bound is only about $l/2$, and Harper's bound is about $l/\sqrt{2}$. Sperner's Lemma can be used to prove that $l+1$ is optimal. ■



Let G be the graph formed by a simplicial subdivision of a triangle. The outer boundary of G is a cycle, the bounded regions are triangles, and the cycle is partitioned into three paths by the corners of the large triangle. We say that a **connector** is a vertex set inducing a connected subgraph that contains a vertex of each boundary path.

8.3.23. Lemma. (Hochberg–McDiarmid–Saks [1995]) Let T be a simplicial subdivision in which each vertex is assigned red or blue. Let R, B be the subgraphs induced by the red and by the blue vertices, respectively. For each such coloring, exactly one of R, B contains a connector.

Proof: For each vertex v, consider the vertices reachable from v using vertices with the same color as v. If the three sides are not all reachable, label v with the smallest index of a side not reachable from v. For the vertices on the ith side, the label i does not appear. If there is no connector, then each node has a label, and this is a proper labeling of T.

By Sperner's Lemma, there is a completely labeled cell. Since the cell has three corners and we only used two colors R, B, two of the corners of this cell have the same color. Since they are adjacent, they can reach the same set of vertices in their color. Hence the least side unreachable from them cannot be different. This contradiction means that we could not have constructed the specified labeling. Hence there is a vertex from which every side is reachable.

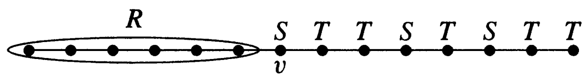
If one color has a connector, it partitions the remaining vertices into sets such that from each set at least one side is unreachable. Hence there cannot be connectors in both colors. ■

8.3.24. Theorem. (Hochberg–McDiarmid–Saks [1995]) Let G be a graph that triangulates a region bounded by a cycle C partitioned into three paths. If k is the minimum over $v \in V(G)$ of the sum of the distances from v to each of the three paths, then $B(G) \geq k+1$.

Proof: Let f be a numbering of G. Let t be the maximum index such that the subgraph induced by the vertices numbered $1, \ldots, t$ does not have a component meeting all three paths. Let R be this vertex set, let S be the set of vertices outside R having neighbors in R, and let T be the remaining vertices.

By construction, the vertex v with $f(v) = t + 1$ belongs to S. Since $R \cup \{v\}$ contains a connector, $R \cup S$ contains a connector, and T does not. Since there is no edge between R and T and R contains no connector, $R \cup T$ contains no connector. Now Lemma 8.3.23 implies that S contains a connector. The set S equals $\partial(S \cup T)$ for the terminal segment $S \cup T$ in the numbering. Therefore, the numbering has difference at least $|S|$ on some edge from S to R.

A connector contains walks from each of its vertices to each of the three boundary paths. By hypothesis, the sum of the lengths of these walks from any fixed vertex is at least k. There exists a vertex in S for which these walks in S are disjoint paths. Hence $|S| \geq k + 1$. ■



8.3.25. Corollary. The triangular grid T_l has bandwidth $l + 1$.

Proof: For each vertex (i, j, k) in T_l, the distances to the three sides are i, j, k, respectively, so the sum of the distances is l. By Theorem 8.3.24, the bandwidth is at least $l + 1$, which we have observed is achievable. ■

EXERCISES

8.3.1. (−) Each of two concentric discs has 20 radial sections of equal size. For each disc, 10 sections are painted red and 10 blue, in some arrangement. Prove that the two discs can be aligned so that at least 10 sections on the inner disc match colors with the corresponding sections on the outer disc.

8.3.2. For $n \in \mathbb{N}$, let S be a set of $n + 1$ elements in $\{1, \ldots, 2n\}$. Prove that S has two elements with greatest common factor 1 and has two elements such that one divides the other. For each conclusion, exhibit a subset of size n where it does not hold; hence these conclusions are best possible.

8.3.3. Use partial sums and the pigeonhole principle to prove the following statements.

a) Every set of n integers contains a nonempty subset whose sum is divisible by n. (Also exhibit a collection of $n - 1$ integers with no such subset.)

b) Given $x \in \mathbb{R}$, prove that at least one of $\{x, 2x, \ldots, (n-1)x\}$ differs by at most $1/n$ from an integer.

8.3.4. (!) A private club has 90 rooms and 100 members. Keys are given to the members so that any 90 members have access to the rooms in the sense that each of these 90 members will have a key to a different room. (They do not share their keys.) Prove that at least 990 keys are needed and that 990 suffice.

8.3.5. Let T be a tree. Use the technique of Theorem 8.3.2 to prove that the center of T consists of one vertex or two adjacent vertices (this proves Theorem 2.1.13 again). (Jordan [1869], Graham–Entringer–Székely [1994])

8.3.6. Prove that every set of $2^m + 1$ integer lattice points in $\mathbb{R}^m$ contains a pair of points whose centroid (mean vector) is also an integer lattice point.

8.3.7. Prove that every 2-coloring of the integer lattice points in $\mathbb{R}^m$ has a collection of n points with the same color whose centroid (mean vector) is an integer lattice point also having that color. (Hint: Ramsey's Theorem is not needed; there is a short proof using only the pigeonhole principle.) (Bòna [1990])

8.3.8. Let S be a collection of $n+1$ positive integers summing to k. For $k \le 2n+1$, prove that S has a subset with sum i for each $i \in [k]$. For each n, exhibit a collection for which this fails when $k = 2n + 2$.

8.3.9. For even n, construct an ordering of $E(K_n)$ so that the maximum length of an increasing trail is $n - 1$. (Comment: This proves that Theorem 8.3.4 is best possible when n is even. It also is best possible when n is odd and at least 9, but the construction is much more difficult.) (Graham–Kleitman [1973])

8.3.10. Let S be a set of nine points in the plane (no three collinear). Prove that S contains the vertex set of a convex 5-gon. Exhibit a set of eight points without this property.

8.3.11. (!) Let S be a set of $R(m, m; 3)$ points in the plane no three of which are collinear. Prove that S contains m points that form a convex m-gon. (Tarsi)

8.3.12. Recall that a digraph is *simple* if no two edges have the same ordered pair of endpoints. A **monotone tournament** is a tournament in which the orientation of the edges always agrees with the order of the indices on the vertices or always disagrees with that order. A **complete loopless digraph** has one copy of each ordered pair of distinct vertices as an edge. Given m, prove that if N is sufficiently large, then every simple loopless digraph with vertex set $[N]$ has an independent set of order m or a monotone tournament of order m or a complete loopless digraph of order m.

8.3.13. (!) *Schur's Theorem.* (Schur [1916])

a) Given $k > 0$, prove that there exists a least integer s_k such that every k-coloring of the integers $1, \ldots, s_k$ yields a monochromatic x, y, z (not necessarily distinct) such that $x + y = z$. (Hint: Apply Ramsey's Theorem for $r = 2$.)

b) Prove constructively that $s_k \ge 3s_{k-1} - 1$ and hence that $s_k \ge (3^k - 1)/2$.

8.3.14. (!) The **composition** or **lexicographic product** of two simple graphs G and H is the simple graph $G[H]$ whose vertex set is $V(G) \times V(H)$, with edges given by $(u, v) \leftrightarrow (u', v')$ if and only if (1) uu' is an edge of G, or (2) $u = u'$ and vv' is an edge of H.

a) Prove that $\alpha(G[H]) = \alpha(G)\alpha(H)$.

b) Prove that the complement of $G[H]$ is $\overline{G}[\overline{H}]$.

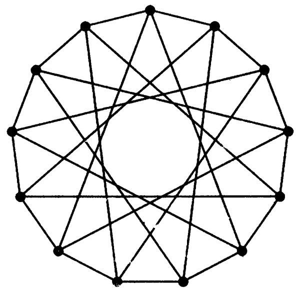
c) Use parts (a) and (b) to prove by construction that

$$R(pq + 1, pq + 1) - 1 \ge [R(p + 1, p + 1) - 1] \times [R(q + 1, q + 1) - 1].$$

d) Deduce that $R(2^n + 1, 2^n + 1) \ge 5^n + 1$ for $n \ge 0$ and compare this lower bound to the nonconstructive lower bound for $R(k, k)$. (Abbott [1972])

8.3.15. (−) Verify that $R(p, 2) = R(2, p) = p$. Use this and Theorem 8.3.11 to prove that $R(p, q) \le \binom{p+q-2}{p-1}$.

8.3.16. (−) Use the graph below to prove that $R(3,5) = 14$.



8.3.17. *Ramsey numbers for $r = 2$ and multiple colors.*

a) Let $p = (p_1, \dots, p_k)$, and let q_i be obtained from p by subtracting 1 from p_i but leaving the other coordinates unchanged. Prove that $R(p) \le \sum_{i=1}^{k} R(q_i) - k + 2$.

b) Prove that $R(p_1 + 1, \dots, p_k + 1) \le \frac{(p_1 + \cdots + p_k)!}{p_1! \cdots p_k!}$.

8.3.18. Let $r_k = R_k(3; 2)$ (this is the value of n such that k-coloring $E(K_n)$ forces a monochromatic triangle).

a) Show that $r_k \le k(r_{k-1} - 1) + 2$.

b) Use part (a) to show that $r_k \le \lfloor k!e \rfloor + 1$, so that $r_3 \le 17$. (Comment: $r_3 = 17$, but the lower bound requires a clever 3-coloring of K_{16} that arises from the finite field $GF(2^4)$).

8.3.19. Prove that $R_k(p; r+1) \le r + k^M$, where $M = \binom{R_k(p;r)}{r}$.

8.3.20. (+) *Off-diagonal Ramsey numbers.*

a) Prove that $R(k,l) > n$ if $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1$ for some $p \in (0,1)$. Prove that $R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l}(1-p)^{\binom{l}{2}}$ for all $n \in \mathbb{N}$ and $p \in (0,1)$.

b) Use part (a) to prove $R(3,k) > k^{3/2+o(1)}$. What lower bound on $R(3,k)$ can be obtained from the first part of (a)? (Spencer [1977])

c) Use part (a) to obtain a lower bound for $R_k(q)$.

8.3.21. (!) Determine the Ramsey number $R(K_{1,m}, K_{1,n})$. (Hint: The answer depends on whether m and n are even or odd.)

8.3.22. (!) Let T be a tree with m vertices. Given that $m-1$ divides $n-1$, determine the Ramsey number $R(T, K_{1,n})$. (Burr [1974])

8.3.23. If $p > (m-1)(n-1)$, prove that every 2-coloring of $E(K_p)$ in which the red graph is transitively orientable contains a red m-clique or a blue n-clique, and prove that this is best possible. (Brozinsky–Nishiura) (Hint. Use perfect graphs.)

8.3.24. Show that $R(T, K_{n_1}, \dots, K_{n_k}) = (m-1)(R(n_1, \dots, n_k) - 1) + 1$ when T is a tree with m vertices. (Burr)

8.3.25. Prove that $R(C_4, C_4) = 6$. (Comment: There are many proofs.)

8.3.26. Prove that $R(2K_3, 2K_3) = 10$. (Hint: Reduce to the case of a bowtie with triangles of both colors plus monochromatic 5-cycles; then use symmetry.)

8.3.27. (!) Prove that $R(mK_2, mK_2) = 3m - 1$.

8.3.28. (!) For $1 \le i \le k$, let G_i be a graph on p_i vertices, and fix a multiplicity m_i. Prove that $R(m_1G_1, \dots, m_kG_k) \le \sum(m_i - 1)p_i + R(G_1, \dots, G_k)$.

8.3.29. Frankl and Wilson [1981] explicitly constructed graphs with n vertices that have no clique or independent set with size exceeding $2^{c\sqrt{\log n \log\log n}}$, where c is a particular constant. Prove that this gives a lower bound for $R(p, p)$ that grows faster than every polynomial in p but slower than every exponential in p.

8.3.30. (!) For every simple graph G, determine $R(P_3, G)$ as a function only of the number of vertices of G and the maximum size of a matching in $\overline{G}$.

8.3.31. (!) Let r and s be natural numbers with $r + s \not\equiv 0 \mod 4$. Prove that every 2-coloring of $E(K_{r,s})$ has a monochromatic connected graph with at least $\lceil r/2 \rceil + \lceil s/2 \rceil$ vertices. Conclude that every 3-coloring of $E(K_{r+s})$ contains a monochromatic connected subgraph with more than $(r+s)/2$ vertices. Show that this fails when 4 divides $r + s$.

8.3.32. *Forcing 4-cycles.*
a) Prove that if $\sum_{v \in V(G)} \binom{d(v)}{2} > \binom{n(G)}{2}$, then G contains a 4-cycle.
b) Prove that if $e(G) > \frac{n(G)}{4}(1 + \sqrt{4n(G) - 3})$, then G contains a 4-cycle.
c) Prove that $R_k(C_4) \le k^2 + k + 2$. (Chung–Graham [1975])

8.3.33. (!) Bondy [1971a] proved that $x \not\leftrightarrow y$ implies $d(x) + d(y) \ge n(G)$, then $G = K_{t,t}$ or G has a cycle of each length from 3 to n. Use this to prove that $R(C_m, K_{1,n}) = \max\{m, 2n+1\}$, except possibly if m is even and at most $2n$. (Lawrence [1973])

8.3.34. (!) Prove that every 2-coloring of $E(K_n)$ has a Hamiltonian cycle that is monochromatic or consists of two monochromatic paths. (Hint: Use induction on n.) (Lovász [1979, p85, p482 - attributed to H. Raynaud])

8.3.35. (+) Let f be a 2-coloring of $E(K_n)$, and suppose that $k \ge 3$. Prove the following:
a) If f has a monochromatic C_{2k+1}, then f also has a monochromatic C_{2k}.
b) If f has a monochromatic C_{2k}, then f also has a monochromatic C_{2k-1} or $2K_k$.
c) If $m \ge 5$, then $R(C_m, C_m) \le 2m - 1$ (see Exercise 8.3.25 for $m = 4$). (Hint: Use parts (a) and (b) and the result of Erdős–Gallai [1959] (Theorem 8.4.35) that $e(G) > (m-1)(n(G)-1)/2$ forces a cycle of length at least m in G. There remains one difficult case).

8.3.36. The **Ramsey multiplicity** of G is the minimum number of monochromatic copies of G in a 2-coloring of the edges of a clique on $R(G, G)$ vertices. Show that the Ramsey multiplicity of K_3 is 2.

8.3.37. Prove that each point in a triangular region has a unique expression as a convex combination of the vertices of the triangle (convex combinations are linear combinations where the coefficients are nonnegative and sum to 1).

8.3.38. *Sperner's Lemma in higher dimensions.* A k-**dimensional simplex** consists of the convex combinations of $k+1$ points in $\mathbb{R}^k$ not lying in a hyperplane. A **simplicial subdivision** expresses a k-dimensional simplex as a union of k-dimensional simplices (cells) such that any two cells intersect in the simplex determined by their common corners. A **completely labeled** cell has $\{0, \dots, k\}$ at its corners.

State a general definition of "proper labeling" so that every proper labeling of a simplicial subdivision of a k-simplex contains a completely labeled cell. Prove this theorem. (Hint: The proof of Sperner's Lemma in two dimensions (Theorem 8.3.17) is an instance of the induction step for a proof by induction on k.)

8.3.39. ($-$) Compute the bandwidths of P_n, K_n, and C_n.

8.3.40. Compute the bandwidth of $K_{n_1,\dots,n_k}$. (Eitner [1979])

8.3.41. (!) Prove that every tree with k leaves is the union of $\lceil k/2 \rceil$ pairwise intersecting paths (Exercise 2.1.37). Use this to prove that the bandwidth of a tree with k leaves is at most $\lceil k/2 \rceil$. (Ando–Kaneko–Gervacio [1996])

8.3.42. (+) Let G be a caterpillar (Definition 2.2.17), and let m be an integer such that $\lceil \frac{n(H)-1}{\operatorname{diam} H} \rceil \le m$ for all $H \subseteq G$. Prove that $B(G) \le m$. (Hint: Prove that G has a numbering f in which $f(v)$ is a multiple of m whenever v is on the spine and $|f(u) - f(v)| \le m$ for all $u \leftrightarrow v$.) (Sysło–Zak [1982], Miller [1981])

8.3.43. *Bandwidth of grids.*

a) Compute the local density lower bound for the bandwidth of $P_m \square P_n$.

b) Let S be a k-set of vertices in $P_n \square P_n$ with a_i vertices in the ith row and b_j vertices in the jth column. Prove that $|\partial T| \le |\partial S|$ if T is the set consisting of the first a_i vertices in the ith row for each i.

c) Prove that $|\partial S|$ is minimized over k-sets in $V(P_n \square P_n)$ by some S such that $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$. Conclude that Harper's lower bound for $B(P_n \square P_n)$ is n.

d) Conclude that $B(P_m \square P_n) = \min\{m, n\}$. (Chvátalová [1975])

8.3.44. (+) Let G be a simple graph with order n and bandwidth b.

a) For $e \in \overline{G}$, prove that $B(G + e) \le 2b$.

b) Prove that if $n \ge 6b$, then $B(G + e)$ can be as large as $2b$.

(Comment: The maximum of $B(G+e)$ is $b+1$ if $n \le 3b+4$ and is $\lceil (n-1)/3 \rceil$ if $3b+5 \le n \le 6b-2$.) (Wang–West–Yao [1995])

8.4. More Extremal Problems

Extremal graph theory is a huge area. In Section 1.3 we described the distinction between optimization problems (find an extremal structure in the input graph) and extremal problems (find an extremal instance over a class of graphs), and we have studied both types of problems throughout this book. In this section we study the latter type. The archetypal example is the Turán problem: find the maximum number of edges in a graph not containing H as a subgraph. We list one additional example from each chapter.

Objective	*Class of graphs*	*Answer*	*Reference*
max $e(G)$	n vertices and k components	$\binom{n-k+1}{2}$	Exercise 1.3.40
max girth	diameter k and not a tree	$2k+1$	Exercise 2.1.61
max $\beta(G)$	$\alpha'(G) \le k$	$2k$	Exercise 3.3.10
min $\alpha(G)$	$\kappa(G) = k$ and diameter d	$\lceil (d+1)/2 \rceil$	Exercise 4.2.22
max $\chi(G)$	$2K_2$-free and $\omega(G) = k$	$\binom{k+1}{2}$	Exercise 5.2.11
max $\chi(G)$	outerplanar	3	Exercise 6.3.12
max $e(G)$	$n(G) = n$ and non-Hamiltonian	$\binom{n-1}{2} + 1$	Exercise 7.2.26
max $n(G)$	$\omega(G) < p$ and $\alpha(G) < q$	$R(p,q) - 1$	Section 8.3

With such enormous variety of extremal problems, we can only hope in this section to exhibit a small sample of interesting results.

ENCODINGS OF GRAPHS

We first consider parameters related to three types of graph encoding. Each model of encoding involves assigning vectors to vertices, and the parameter is the minimum length of vectors that suffice. We study the maximum of this parameter over n-vertex graphs. The parameters are intersection number, product dimension, and squashed-cube dimension.

8.4.1. Definition. An **intersection representation** of **length** t assigns each vertex a 0,1-vector of length t such that $u \leftrightarrow v$ if and only if their vectors have a 1 in a common position. Equivalently, it assigns each $x \in V(G)$ a set $S_x \subseteq [t]$ such that $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$. The **intersection number** $\theta'(G)$ is the minimum length of an intersection representation of G.

The elements of $[t]$ in a representation correspond to complete subgraphs that cover $E(G)$. This motivates our use of θ' for intersection number: $\theta(G)$ is the minimum number of cliques needed to cover $V(G)$.

8.4.2. Proposition. (Erdős–Goodman–Pósa [1966]) The intersection number equals the minimum number of complete subgraphs needed to cover $E(G)$.

Proof: We define a natural correspondence between representations of length t and coverings of $E(G)$ by t complete subgraphs. Each $i \in [t]$ generates a clique $\{v \in V(G): i \in S_v\}$. The resulting complete subgraphs cover $E(G)$, since $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$.

Conversely, if complete subgraphs $Q_1, \dots, Q_t$ cover $E(G)$, then assigning $\{i: v \in V(Q_i)\}$ to each vertex v yields an intersection representation. ∎

Hence $\theta'(G) = e(G)$ if G is triangle-free, and $\theta'(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = \lfloor n^2/4 \rfloor$. In fact, this is the unique n-vertex graph maximizing $\theta'(G)$. Exercise 1 suggests a direct proof of the bound; here we present a stronger result.

Let **F** be a family of graphs. For an input graph G, the **F-decomposition** problem is to decompose G into the minimum number of graphs in **F**. When **F** is not closed under taking subgraphs, **F**-decomposition may require more subgraphs than **F-covering**. For example, we can cover the kite with two complete subgraphs, but three complete subgraphs are needed to decompose it.

Proving $\theta'(G) \le \lfloor n^2/4 \rfloor$ for n-vertex graphs means showing that every n-vertex graph can be covered with $\lfloor n^2/4 \rfloor$ complete subgraphs; we prove the stronger result that there is always a decomposition using at most this many complete subgraphs. In fact, we can find such a decomposition greedily.

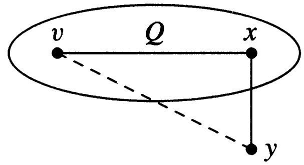
8.4.3. Theorem. (McGuinness [1994]) Every greedy clique decomposition of an n-vertex graph uses at most $\lfloor n^2/4 \rfloor$ cliques.

Proof: We use induction on n. The claim is obvious for $n \le 2$; consider $n > 2$. Let $\mathbf{Q} = Q_1, \dots, Q_m$ be a greedy decomposition of G, meaning that each Q_i is a maximal complete subgraph in $G - \bigcup_{j<i} E(Q_j)$. Note that deleting Q_j from the list $\mathbf{Q}$ leaves a greedy decomposition of $G - E(Q_j)$.

If each Q_i has at least three edges, then $m < n^2/6$, so we may assume that some Q_j is an edge xy. Let R consist of the elements of $\mathbf{Q}-\{Q_j\}$ that are incident to x, and let S consist of those incident to y. The set $\mathbf{Q}' = \mathbf{Q} - (R \cup S \cup \{Q_j\})$ is a greedy decomposition of a subgraph of $G - x - y$. By the induction hypothesis, $|\mathbf{Q}'| \le (n-2)^2/4$. Hence it suffices to prove that $|R| + |S| \le n - 2$.

We prove this by choosing distinct vertices in $V(G) - \{x, y\}$ from the vertex sets of the elements of $R \cup S$. Since each edge is deleted exactly once, each $v \notin \{x, y\}$ appears once in R if $v \in N(x)$ and once in S if $v \in N(y)$. Consider $Q \in R$. If Q uses a vertex $v \notin N(y)$, then we choose such a v for Q. If $V(Q) \subseteq N(y)$, then we choose for Q a vertex $v \in Q$ such that vy belongs to the earliest element of $\mathbf{Q}$ that contains both y and some vertex of Q. Call this element Q'; note that Q' is the only element of S containing v. Since Q and xy are maximal when chosen, Q' precedes both of these in $\mathbf{Q}$. For elements of S, choose vertices by reversing the roles of x and y.

We have shown that if v belongs to some $Q \in R$ and to some $Q' \in S$, and v is chosen for one of them, then the one for which it is chosen occurs after the other one in the list $\mathbf{Q}$. Hence no vertex is chosen twice. We conclude that $|R| + |S| \le n - 2$ and $m \le n^2/4$. ■



Both Chung [1981] and Győri–Kostochka [1979] strengthened the decomposition bound, proving that every n-vertex graph has a decomposition into complete subgraphs whose orders sum to at most $\lfloor n(G)^2/2 \rfloor$.

Now we consider the second encoding model.

8.4.4. Definition. A **product representation** of **length** t assigns the vertices distinct vectors of length t so that $u \leftrightarrow v$ if and only if their vectors differ in every position. The **product dimension** pdim G is the minimum length of such a representation of G.

By devoting one coordinate to each $e \in E(\overline{G})$, in which the vertices of e have value 0 and other vertices have distinct positive values, we obtain pdim $G \le e(\overline{G})$ (if G is not a complete graph).

8.4.5. Example. Every complete graph has product dimension 1. For $\overline{K}_n$, each pair of vertices must agree in some coordinate, but we cannot assign two vertices the same vector. Hence two coordinates are needed, and assigning $(0, j)$ to v_j for each j suffices.

For $K_1 + K_{n-1}$, the vectors for the clique must differ in each coordinate. The vector for the isolated vertex must agree with each of the others somewhere, but it cannot agree with more than one in a single coordinate. Hence at least $n-1$ coordinates are needed. This suffices, by using $(1, 2, \ldots, n-1)$ for the isolated vertex and $(i, i, \ldots, i)$ for the ith vertex of the clique. ■

Again we can describe the parameter using complete graphs.

8.4.6. Definition. An **equivalence** on G is a spanning subgraph of G whose components are complete graphs.

8.4.7. Proposition. The product dimension of G is the minimum number of equivalences $E_1, \ldots, E_t$ such that $\bigcup E_i = \overline{G}$ and $\bigcap E_i = \emptyset$.

Proof: Again there is a natural bijection. Given a product representation, the ith coordinate generates E_i, with a component for each value used in the ith coordinate. Every nonadjacent pair agrees in some coordinate, so every edge of $\overline{G}$ is covered.

Conversely, given $E_1, \ldots, E_t$, each component of E_i becomes a fixed value in the ith coordinate of a representation. The requirement $\cap E_i = \emptyset$ is the requirement of using distinct vectors in the product representation. ■

8.4.8. Lemma. If $\chi'(\overline{G}) > 1$, then $\text{pdim}\, G \le \chi'(\overline{G})$, with equality if $\overline{G}$ is triangle-free.

Proof: Every matching is a disjoint union of complete graphs and becomes an equivalence by the addition of isolated vertices; hence $\chi'(\overline{G})$ equivalences cover $\overline{G}$. If $\chi'(\overline{G}) > 1$, then these equivalences have no common edge.

If $\overline{G}$ is triangle-free, then every equivalence used in a cover of $\overline{G}$ is a matching plus isolated edges, and thus $\chi'(\overline{G}) \le \text{pdim}\, G$. ■

8.4.9. Corollary. For $n \ge 3$, the maximum product dimension of an n-vertex graph is $n-1$.

Proof: Let G be an n-vertex graph. By Lemma 8.4.8 and Vizing's Theorem (Theorem 7.1.10), $\text{pdim}\, G \le \chi'(\overline{G}) \le \Delta(\overline{G}) + 1 \le n$. Furthermore, the bound improves to $n-1$ unless $\Delta(\overline{G}) = n-1$. Let S be the set of vertices of degree $n-1$ in $\overline{G}$; we may assume that $|S| = k \ge 1$.

By Lemma 8.4.8 and Vizing's Theorem again, $\text{pdim}\,(G - S) \le n - k$. By duplicating coordinates if needed, we obtain a product representation of $G - S$ of length $n-k$. Let x^i be the vector assigned to v_i in this representation.

Each vertex of S is isolated in G. We now assign to each $v \in S$ the vector whose ith coordinate, for $1 \le i \le n-k$, is the ith coordinate of x^i. If $k = 1$, then this completes a representation of G with length $n-1$. If $k > 1$, then we have assigned the same vector to all of S; we add one coordinate using distinct values to complete a representation of length $n-k+1$, which is less than $n-1$.

Since $\text{pdim}\,(K_1 + K_{n-1}) = n-1$ (Example 8.4.5), the bound is sharp. ■

Lovász–Nešetřil–Pultr [1980] characterized the n-vertex graphs with product dimension $n-1$ (Exercise 4). They also proved a general lower bound using a dimension argument in linear algebra.

8.4.10. Theorem. (Lovász–Nešetřil–Pultr [1980]) Let $u_1, \dots, u_r$ and $v_1, \dots, v_r$ be two lists of vertices (not necessarily distinct) in a graph G. If $u_i \leftrightarrow v_j$ for $i = j$ and $u_i \not\leftrightarrow v_j$ for $i < j$, then pdim $G \ge \lceil \lg r \rceil$.

Proof: Let G have a representation of length d. Let $x^1, \dots, x^r$ and $y^1, \dots, y^r$ be the vectors for $u_1, \dots, u_r$ and $v_1, \dots, v_r$, respectively. The vectors x^i and y^i differ in every coordinate, but x^i and y^j agree in some coordinate if $i \neq j$. Hence $\prod_{k=1}^{d}(x_k^i - y_k^j)$ is nonzero if and only if $i = j$.

We use this product property to construct r linearly independent vectors in $\mathbb{R}^{2^d}$; this proves that $r \le 2^d$ and hence that pdim $G \ge \lceil \lg r \rceil$. Expansion of $\prod_{k=1}^{d}(w_k - z_k)$ for $w, z \in \mathbb{R}^d$ yields the sum $\sum_{S \subseteq [d]} \prod_{i \in S} w_i \prod_{j \in \overline{S}}(-z_j)$. To relate r to 2^d, we view this as a dot product in $\mathbb{R}^{2^d}$, with coordinates indexed by the subsets of $[d]$. For each $w \in \mathbb{R}^d$, define two vectors in $\mathbb{R}^{2^d}$ by setting $\overline{w}_S = \prod_{i \in S} w_i$ and $\hat{w}_S = \prod_{i \notin S}(-w_i)$ for the coordinate $S \subseteq [d]$. With this definition, the dot product $\overline{w} \cdot \hat{z}$ equals $\prod_{k=1}^{d}(w_k - z_k)$. The conditions on the x's and y's thus imply that $\overline{x}^i \cdot \hat{y}^j$ is nonzero if and only if $i = j$.

We claim that $\overline{x}^1, \dots, \overline{x}^r$ are independent. Consider a linear dependence $\sum_{i=1}^{r} c_i \overline{x}^i = \mathbf{0}$. Taking the dot product of $\hat{y}^r$ with both sides kills all terms below $i = r$, yielding $c_r \overline{x}^r \cdot \hat{y}^r = 0$. Since $\overline{x}^r \cdot \hat{y}^r \neq 0$, we have $c_r = 0$. We can now apply the same argument using $\hat{y}^{r-1}$. Knowing that $c_r = 0$ yields $c_{r-1}\overline{x}^{r-1} \cdot \hat{y}^{r-1} = 0$. Successively decreasing the index yields $c_j = 0$ for all j. We conclude that $\overline{x}^1, \dots, \overline{x}^r$ are independent, which requires $2^d \ge r$. ■

8.4.11. Example. *Matchings:* pdim $(n/2)K_2 = \lceil \lg n \rceil$. Given k coordinates, the graph encoded by using all 2^k binary k-tuples as codes is $2^{k-1}K_2$, since only with its complement does a vector disagree in each position. If n is not a power of 2, then we can discard complementary pairs to obtain a construction. The lower bound follows from Theorem 8.4.10, using each vertex in each list (for example, set $u_i = v_{n+1-i}$). ■

In our third encoding model, we want to recover more detailed information: distance between vertices. This arises from an addressing problem in communication networks. Each message should travel a shortest path to its destination. Without central control, a vertex receiving a message must determine where to send it using only the name of the destination. If the vectors for two vertices yield the distance between them in G, then a vertex can compare the destination vector with the vectors for its neighbors and send the message to a neighbor closest to the destination.

For a connected graph G, we want to assign vectors to vertices such that the distance between vertices is the number of positions where their vectors differ. This is an **isometric** or **"distance-preserving"** embedding of G into $H = K_{n_1} \square \cdots \square K_{n_t}$, meaning a mapping $f\colon V(G) \to V(H)$ such that $d_G(u, v) = d_H(f(u), f(v))$. However, many connected graphs have no isometric embedding in a cartesian product of cliques; C_{2k+1} for $k \ge 2$ is an example (Exercise 11).

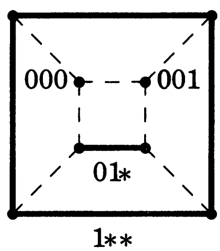
Hence we introduce a "don't care" symbol $*$. Let $S = \{0, 1, *\}$, and define a symmetric function d by $d(0, 1) = 1$ and $d(0, *) = 0 = d(1, *)$. Let S^N denote the set of N-tuples (vectors) with entries in S, and for $a, b \in S^N$ let $d_S(a, b) = \sum d(a_i, b_i)$. For each graph G, we obtain for some N an encoding $f\colon V(G) \to S^N$ so that $d_G(u, v) = d_S(f(u), f(v))$ for all $u, v \in V(G)$.

Each $a \in S^N$ corresponds to a subcube of Q_N, the N-dimensional cube; the dimension of the subcube is the number of $*$s in a. For $a, b \in S^N$, the minimum distance between vertices of the corresponding subcubes is $d_S(a, b)$. The vectors assigned to distinct vertices correspond to disjoint subcubes, else their distance would be 0. If we contract the edges of each assigned subcube, we obtain a "squashed cube" H. The distance-preserving map $f\colon V(G) \to S^N$ is an isometric embedding of G in H.

8.4.12. Definition. A **squashed-cube embedding** of **length** N is a map $f\colon V(G) \to S^N$ such that $d_G(u, v) = d_S(f(u), f(v))$. The **squashed-cube dimension** qdim G is the minimum length of such an embedding of G.

8.4.13. Example. The vectors 000, 001, 01$*$, and 1$**$ form a squashed-cube embedding of K_4 with length 3. Two adjacent vertices of the 3-cube remain unchanged, an edge adjacent to both collapses, and the entire opposite face collapses. The resulting graph is K_4. The image subcubes appear below in bold. The construction generalizes to embed K_n in a squashed $n-1$-dimensional cube.

The path P_n embeds isometrically in Q_{n-1} without squashings, using $00\cdots00$, $10\cdots00$, $11\cdots00, \ldots, 11\cdots10$, $11\cdots11$. No shorter embedding exists, because the distance between the endpoints of P_n is $n-1$, and each coordinate contributes at most 1 to the distance between vectors. ∎



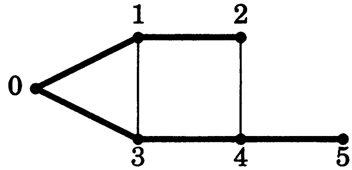
8.4.14. Proposition. For a graph G, qdim $(G) \le \sum_{i<j} d_G(v_i, v_j)$.

Proof: For each pair i, j with $i < j$, we dedicate a block of $d_G(v_i, v_j)$ coordinates. Set these coordinates to 0 for v_i, to 1 for v_j, and to $*$ for other vertices. Given two vertices, the only coordinates where neither contains $*$ are the coordinates dedicated to the pair, so $d_G(v_i, v_j) = d_S(f(v_i), f(v_j))$. ∎

Using an eigenvalue technique (Exercise 8.6.14), Graham and Pollak [1971, 1973] proved a general lower bound on qdim (G) that yields qdim $K_n = n-1$. Hence K_n and P_n both have squashed-cube dimension $n-1$; Graham and Pollak conjectured that qdim $G \le n-1$ for every n-vertex connected graph. Graham offered \$100 for a proof, and Winkler found an encoding scheme to prove this "Squashed-Cube Conjecture".

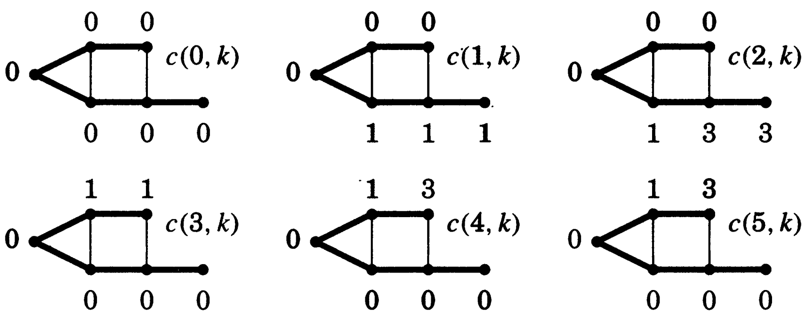
Winkler's proof generates an explicit $n-1$-dimensional squashed-cube encoding for each connected n-vertex graph G. We begin by indexing the vertices; choose v_0 arbitrarily. Next, find a spanning tree T such that $d_T(v, v_0) = d_G(v, v_0)$ for all $v \in V(G)$ (T can be generated by a breadth-first search from v_0). Now, number the vertices by a *depth*-first search in T. In other words, having chosen the indexing for $v_0, \ldots, v_i$, let v_{i+1} be an unvisited child of v_i in T, if one exists; otherwise backtrack toward the root until a vertex with such a child is found. The resulting indices increase along every path from v_0 in T.

8.4.15. Example. *Depth-first numbering of a breadth-first spanning tree.* Below, the bold edges belong to T and the solid edges to $G - T$. We will use this example to illustrate several steps in the proof. ■



We henceforth fix T and this ordering and refer to vertices by their index in this ordering. Let P_i be the vertex set of the $i, 0$-path in T, let i' be the parent of i in T (the next vertex on the path from i to 0), and let $i \wedge j = \max(P_i \cap P_j)$ be the vertex at which the $i, 0$-path and $j, 0$-path meet. Given a depth-first numbering of a breadth-first tree T in G, let $c(i, j) = d_T(i, j) - d_G(i, j)$ be the **discrepancy** of two vertices i, j.

8.4.16. Example. In the graph marked $c(i, k)$ below, we record at each vertex k the discrepancy $c(i, k)$ for the tree T in Example 8.4.15. ■



8.4.17. Lemma. (Winkler [1983]). Discrepancy has the following properties.

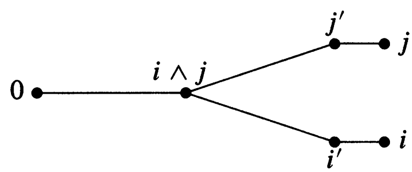
a) $c(i, j) = c(j, i) \geq 0$.

b) If $i \in P_j$, then $c(i, j) = 0$.

c) If neither $i \in P_j$ nor $j \in P_i$, then $c(i, j') \leq c(i, j) \leq c(i, j') + 2$.

Proof: (a) Distance in graphs is symmetric, and the shortest i, j-path in G is no longer than the path between them in T. (b) The preservation of distances to

v_0 implies that the i, j-path in T is a shortest i, j-path in G. (c) Since j' belongs to the i, j-path in T, we have $d_T(i, j) - d_T(i, j') = 1$. Since $jj' \in E(G)$, we have $|d_G(i, j) - d_G(i, j')| \le 1$. Thus $c(i, j) - c(i, j')$ is 0, 1, or 2. ∎



With this notion of discrepancy, we can give an overview of how Winkler's encoding works. We use a search tree because it gives us $n - 1$ natural coordinates. Distance in the tree is an "approximation" to distance in the graph; it needs to be adjusted (reduced) by the discrepancy. Winkler's encoding puts a 1 in coordinate k for exactly one of vertices i and j for exactly $d_T(i, j)$ values of k. We want the other code to have a 0 in exactly $d_G(i, j)$ of these coordinates, so we perform the adjustment by having $*$ in exactly $c(i, j)$ of the coordinates where one code has a 1. The problem is to design the encoding to achieve this simultaneously for all pairs of vertices.

8.4.18. Theorem. (Winkler [1983]) Every connected n-vertex graph G has squashed-cube dimension at most $n - 1$.

Proof: Choose a tree T and numbering $0, \dots, n - 1$ as described above. We define an encoding $f(i) = (f_1(i), \dots, f_{n-1}(i))$ and verify that $d_G(i, j) = d_S(a_i, a_j)$. The encoding is

$$f_k(i) = \begin{cases} 1 & \text{if } k \in P_i \\ * & \text{if } c(i,k) - c(i,k') = 2 \\ * & \text{if } c(i,k) - c(i,k') = 1 \text{ and } i < k \text{ and } c(i,k) \text{ is even} \\ * & \text{if } c(i,k) - c(i,k') = 1 \text{ and } i > k \text{ and } c(i,k) \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

(The vectors in the encoding for Example 8.4.16 are $f(0) = 00000$, $f(1) = 10000$, $f(2) = 110{*}0$, $f(3) = {*}0100$, $f(4) = {*}{*}110$, and $f(5) = {*}{*}111$.)

To prove that $d_S(f(i), f(j)) = d_G(i, j)$, we count the coordinates where one of $f(i), f(j)$ has a 1 and the other has a 0. Such coordinates k belong to $P_i \cup P_j$, where all the 1's are located. By symmetry, we may assume that $i < j$. Hence $j \notin P_i$, and we consider the two cases $i \in P_j$ and $i \notin P_j$.

If $i \in P_j$, then $d_G(i, j) = d_T(i, j) = |P_j - P_i|$, and $f_k(i) = f_k(j) = 1$ if and only if $k \in P_i$. The coordinates where exactly one of $f(i), f(j)$ has a 1 all lie in $P_j - P_i$. For $k \in P_j - P_i$, we have $f_k(i) = 0$, and thus $d_G(i, j) = d_S(f(i), f(j))$.

If $i \notin P_j$, then exactly one of $\{f_i(k), f_j(k)\}$ equals 1 precisely when $k \in (P_j - P_i) \cup (P_i - P_j)$. We need to prove that the other vector has $*$ in exactly $c(i, j)$ of these coordinates. This will yield

$$d_S(f(i), f(j)) = |P_j - P_i| + |P_i - P_j| - c(i, j) = d_T(i, j) - c(i, j) = d_G(i, j).$$

In Example 8.4.16, $(P_5 - P_2) \cup (P_2 - P_5)$ is all five coordinates; since $f(2)$ and $f(5)$ together have $*$ in three of these coordinates, we have $d_S(f(2), f(5)) = d_G(2, 5) = 2$, as desired.

To locate the $*$'s in these positions, consider the change in discrepancies as we bring either of i, j to the point where P_i, P_j meet. Consider two lists:

$$0 = c(i, i \wedge j) \leq \cdots \leq c(i, j') \leq c(i, j)$$
$$0 = c(i \wedge j, j) \leq \cdots \leq c(i', j) \leq c(i, j).$$

We will obtain one $*$ in $f(i)$ for each even m with $0 < m \leq c(i, j)$ and one $*$ in $f(j)$ for each odd m with $0 < m \leq c(i, j)$.

For even m with $0 < m \leq c(i, j)$, let j_m be the unique vertex such that $c(i, j_m) \geq m$ and $c(i, j_m') < m$. Even when the value m is not in the first list, j_m is well-defined. Because c changes by at most 2 with each step, the values of j_m are distinct. Furthermore, the depth-first ordering guarantees $i < k$ for all $k \in P_j - P_i$. Thus $f_k(i) = *$ for $k \in P_j - P_i$ if and only if $k = j_m$ for some even m. In Example 8.4.16, for $(i, j) = (2, 5)$ we have $j_2 = 4$ and $f_4(2) = *$.

Similarly, for odd m with $0 < m \leq c(i, j)$, let i_m be the unique vertex such that $c(i_m, j) \geq m$ and $c(i_m', j) < m$. As before, the values of i_m are distinct and well-defined. The depth-first ordering guarantees $j > k$ for all $k \in P_i - P_j$, so $a_j(k) = *$ for $k \in P_i - P_j$ if and only if $k = i_m$ for some odd m. In Example 8.4.16, for $(i, j) = (2, 5)$ we have $i_1 = 1$, $i_3 = 2$, and $f_1(j) = f_3(j) = *$.

Thus, we have counted the $*$'s in $P_i - P_j \cup P_j - P_i$. Their number is the number of even integers between 1 and $c(i, j)$ plus the number of odd integers between 1 and $c(i, j)$, which together equals $c(i, j)$. ■

BRANCHINGS AND GOSSIP

We have studied the problem of finding the maximum number of pairwise edge-disjoint spanning trees in a graph; this equals the maximum k such that for every vertex partition P, there are at least $k(|P| - 1)$ edges crossing between sets of P (Corollary 8.2.59). Here we consider an analogous problem for digraphs that is related to Menger's Theorem (Exercise 14). Menger's Theorem is a min-max theorem that focuses on vertex pairs. We examine "connectedness" from a single vertex to the rest of the digraph.

8.4.19. Definition. An r-**branching** in a digraph is a rooted tree "branching out" from r. Vertex r has indegree 0, all other vertices have indegree 1, and all other vertices are reachable from r. Let $\kappa'(r; G)$ denote the minimum number of edges whose deletion makes some vertex unreachable from r.

Deleting the edges entering a set $X \subseteq V(G) - \{r\}$ makes each vertex of X unreachable from r. On the other hand, a minimal set whose deletion makes some vertex unreachable includes all edges leaving the set of reached vertices. Hence $\kappa'(r; G)$ equals the minimum, over nonempty $X \subseteq V(G) - \{r\}$, of the number of edges entering X.

In a set of pairwise edge-disjoint r-branchings, each must use at least one edge entering X. Thus there are at most $\kappa'(r; G)$ pairwise edge-disjoint r-branchings in G. Edmonds proved that this bound is achievable. Our discussion allows multiple edges.

8.4.20. Theorem. (Edmonds' Branching Theorem [1973]) For a vertex r in a digraph G, the maximum number of pairwise edge-disjoint r-branchings in G is $\kappa'(r; G)$.

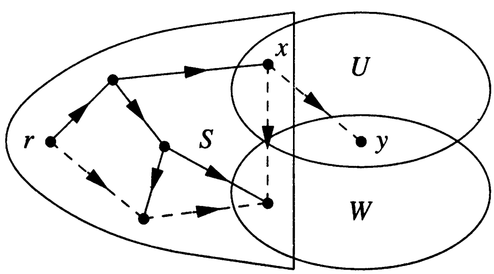
Proof: (Lovász [1976]) Let V be the vertex set of G. The upper bound holds since each subset of $V - r$ is entered by at least one edge in every r-branching. We prove the existence of $\kappa'(r; G)$ edge-disjoint r-branchings by induction on $k = \kappa'(r; G)$. For $k = 1$, a breadth-first search suffices to grow an r-branching, since every vertex is reachable. For $k > 1$, we seek an r-branching T such that $\kappa'(r; G - E(T)) = k - 1$; the induction hypothesis then supplies $k - 1$ additional r-branchings.

A *partial* r-branching is an r-branching of an induced subgraph of G. Let T be a partial r-branching of maximum order such that $\kappa'(r; G - E(T)) \geq k - 1$. The vertex r itself is such a branching, with $E(T) = \emptyset$. Let $S = V(T)$. If $S = V$, then we are done, so we may assume that $S \neq V$.

For $X \subseteq V - r$, let e_X denote the number of edges in $G - E(T)$ that enter X. If $e_X \geq k$ for every $X \subseteq V - r$ that intersects $V - S$, then we can extend T by adding any edge from S to $V - S$. Hence we can choose a smallest set $U \subseteq V - r$ that intersects $V - S$ and is entered by exactly $k - 1$ edges. (In the illustration, T consists of the solid edges.)

Because $\kappa'(r; G) = k$ and we have deleted no edge entering $U - S$, we still have $e_{U-S} \geq k$. However, $e_U = k - 1$, so there must be an edge xy from $S \cap U$ to $U - S$. We claim that xy can be added to enlarge T, contradicting the maximality of T. We need only verify that at least $k - 1$ edges still enter each $W \subseteq V - r$ when we delete xy from $G - E(T)$. This holds trivially unless $x \in V - W$ and $y \in W$. It suffices to prove that $e_W \geq k$ for such a W.

The quantity $e_W + e_U$ counts edges entering W and entering U. Except for the edges between $U - W$ and $W - U$, these enter $W \cup U$, and those entering $W \cap U$ are counted twice. Thus $e_W + e_U \geq e_{W\cup U} + e_{W\cap U}$. We have $e_{W\cup U} \geq k - 1$ by the defining property of T, $e_U = k - 1$ by construction, and $e_{W\cap U} \geq k$ by $x \in U - W$ and the minimality of U. Hence $e_W \geq k - 1 - (k - 1) + k = k$, as desired. ∎



Lovász's proof can be converted to an algorithm for finding the maximum number of pairwise disjoint r-branchings; Tarjan [1974/75] gave another algorithm. We might call $\kappa'(r; G)$ the **local-global edge-connectivity**. Theorem 8.4.20 has several equivalent forms:

8.4.21. Corollary. If G is a directed graph, r is a vertex of G, and $k \geq 0$, then the following statements are equivalent.
A) G has k pairwise edge-disjoint r-branchings.
B) $\kappa'(r; G) \geq k$; equivalently, $|[\overline{X}, X]| \geq k$ for all $X \subseteq V(G) - \{r\}$.
C) For each $s \neq r$ there exist k pairwise edge-disjoint r, s-paths.
D) There exist k pairwise edge-disjoint spanning trees of the underlying (undirected) graph that for each $s \neq r$ contain among them exactly k edges of the digraph G entering s.

Proof: A $\Leftrightarrow$ B is Edmonds' Theorem, B $\Leftrightarrow$ C is Menger's Theorem, and A $\Rightarrow$ D is immediate. For D $\Rightarrow$ B, assume that the trees exist and consider $U \subseteq V - r$. Each spanning tree has at most $|U| - 1$ edges within U, so the trees together have at most $k(|U| - 1)$ edges within U. By hypothesis, the edges of the digraph G corresponding to these trees contain exactly $k\,|U|$ edges with heads in U, so at least k edges enter U. ■

Schrijver observed that Edmonds' Branching Theorem can also be proved using matroid union and matroid intersection. Discard the edges entering the root r. Let M_1 be the union of k copies of the cycle matroid on the underlying undirected graph. Let M_2 be the matroid in which a set of edges is independent if and only if no $k + 1$ of them have the same head (this is the direct sum of uniform matroids of rank k). There exist k disjoint r-branchings if and only if these two matroids have a common independent set of size $k(n(G) - 1)$.

Pairwise edge-disjoint r-branchings provide a fault-tolerant static protocol for message transmissions from r; alternative trees are available. We next consider a static protocol for transmissions from each vertex to every other. Each transmission is two-way, but they are performed in a specified order.

The resulting question is the **gossip problem**. Consider n gossips, each having a tidbit of information. Being gossips, each wants to know all the information, and when two communicate they tell each other everything they know. How many telephone calls are needed to transmit all the information? Several solutions were published in the early 1970s.

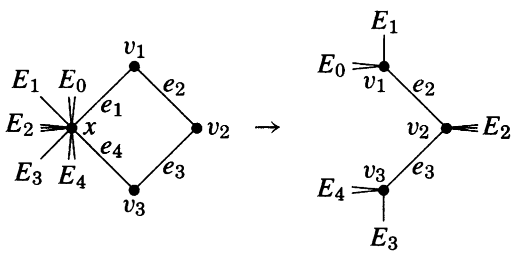
Succeeding with $2n - 3$ calls is easy: everyone calls x, and then x calls everyone back, saving one call by combining the last call in and first call out. When $n \geq 4$, $2n - 4$ calls suffice: first the others call in to a set S of four people, then S shares the information in two successive pairings, and finally the others receive calls back from S, using a total of $(n - 4) + 4 + (n - 4) = 2n - 4$ calls. Using a graph model, we show that this is optimal.

8.4.22. Definition. An **ordered graph** is a graph with an ordering of the edges (multiple edges allowed). An **increasing path** is a path via successively later edges. A **gossip scheme** is an ordered graph having an increasing path from each vertex to every other vertex. A gossip scheme **satisfies NOHO** ("No One Hears his or her Own information") if it has no increasing x, y-path plus a later edge between x and y.

8.4.23. Theorem. For $n \geq 4$, the minimum number of edges in a gossip scheme on n vertices is $2n - 4$.

Proof: (Baker–Shostak [1972]). We freely use "calls" in place of "edges" to emphasize the ordering and the possibility of repeated edges. The scheme described above uses $2n - 4$ calls, and case analysis shows that it is optimal for $n = 4$. This provides the basis for a proof by induction on n. For $n > 4$, we may assume that every gossip scheme with $n-1$ vertices uses at least $2n-6$ calls. If $2n - 4$ is not optimal for n vertices, then we can add calls to the optimal scheme (if necessary) to obtain an n-vertex gossip scheme G with exactly $2n - 5$ calls.

Claim 1. G satisfies NOHO. Otherwise, G has an increasing path from x to v_k along edges $e_1, \ldots, e_k$ followed by a call $e_{k+1} = v_k x$. Delete e_1 and e_{k+1}. Partition the other calls involving x into $k + 2$ sets: E_0 consists of those before e_1, E_i for $1 \leq i \leq k$ consists of those between e_i and e_{i+1}, and E_{k+1} consists of those after e_{k+1}. In each edge $e \in E_i$, replace x by v_1, v_i, or v_k in the cases $i = 0$, $1 \leq i \leq k$, or $i = k + 1$, respectively (see illustration). Now $E(G) - \{e_1, e_{k+1}\}$ is a gossip scheme on $V(G) - \{x\}$, because every increasing path through x is replaced by an increasing path that consists of the same edges and perhaps additional edges from $\{e_i\}$. The scheme has $2(n-1)-5$ edges, which contradicts the induction hypothesis.



Claim 2. $d(x)-3$ calls are useless to x, and hence $\delta(G) \geq 3$. Let $O(x)$ be the set of calls on which some vertex is reached for the first time by an increasing path "Out" from x; these calls form a tree. The tree $I(x)$ of edges useful "In" to x is $O(x)$ for the reverse order on $E(G)$. We show that $O(x) \cap I(x)$ is the set of edges incident to x. If an increasing x, y-path reaches $y \in N(x)$ before the edge xy, then x violates NOHO; hence $xy \in O(x)$. Similarly, $xy \in I(x)$. Conversely, if $O(x) \cap I(x)$ contains an edge e not incident to x, then an increasing path from x containing e and an increasing path to x containing e combine to violate NOHO for x. Hence $|O(x) \cap I(x)| = d(x)$. The edges "useless to x" are those not in $O(x) \cup I(x)$. We have

$$\left|\overline{O(x) \cup I(x)}\right| = 2n - 5 - (n-1) - (n-1) + d(x) = d(x) - 3.$$

Since this counts a set of edges, $\delta(G) \geq 3$.

Claim 3. The subgraph obtained by deleting the first call and the last call made by each vertex has at least five components and has no isolated vertex. Let xy be the first call involving x. If the first call involving y is yz with $z \neq x$, then by definition it occurs before xy, and these two calls do not communicate from x to z. After yz and xy, an increasing x, z-path violates NOHO at z. Hence the set F of first calls is a matching, and there are $n/2$ of them. Similarly, the set L of last calls is a matching of size $n/2$. The graph $G - F - L$ has $n - 5$ edges and hence at least five components, by Proposition 1.2.11. It has no isolated vertex, since $\delta(G) \geq 3$.

The contradiction. Since $e(G) = 2n - 5 < 2n$, some vertex x has degree at most 3. Let C_1, C_2, C_3 be the components of $G - F - L$ containing x, its first neighbor, and its last neighbor, respectively (its middle neighbor is in C_1). Edges of $G - F - L$ can belong to $O(x)$ only via paths that start with the first or middle edge involving x and avoid $F \cup L$, so they belong to C_1 or C_2. Similarly, edges of $G - F - L$ belonging to $I(x)$ appear only in C_1 or C_3. The edges of the remaining components, of which there are at least two, are useless to x ($G - F - L$ has no isolated vertex), but Claim 3 allows only $d(x) - 3 = 0$ edges useless to x. ■

In practical applications, we might wish to minimize the total length of the messages or the total time (assuming that each vertex participates in at most one call per time unit). We can also restrict the pairs that are allowed to call each other. Gossiping can be completed in $2n - 4$ if and only if the graph of allowable calls is connected and has a 4-cycle (Bumby [1981], Kleitman–Shearer [1980]). Other variations consider digraphs (Exercises 15–16), fault-tolerance, conference calls, etc.

LIST COLORING AND CHOOSABILITY

List coloring is a more general version of the vertex coloring problem. We still pick a single color for each vertex, but the set of colors available at each vertex may be restricted. This model was introduced independently in Vizing [1976] and Erdős–Rubin–Taylor [1979].

8.4.24. Definition. For each vertex v in a graph G, let $L(v)$ denote a list of colors available at v. A **list coloring** or **choice function** is a proper coloring f such that $f(v) \in L(v)$ for all v. A graph G is **k-choosable** or **list k-colorable** if every assignment of k-element lists to the vertices permits a proper list coloring. The **list chromatic number**, **choice number**, or **choosability** $\chi_l(G)$ is the minimum k such that G is k-choosable.

Since the lists could be identical, $\chi_l(G) \geq \chi(G)$. If the lists have size at least $1 + \Delta(G)$, then coloring the vertices in succession leaves an available color at each vertex. This argument is analogous to the greedy coloring algorithm and proves that $\chi_l(G) \leq 1 + \Delta(G)$ (see Exercise 22 for other analogues with

$\chi(G)$). It is not possible, however, to place an upper bound on $\chi_l(G)$ in terms of $\chi(G)$; there are bipartite graphs with arbitrarily large list chromatic number.

8.4.25. Proposition. (Erdős–Rubin–Taylor [1979]) If $m = \binom{2k-1}{k}$; then $K_{m,m}$ is not k-choosable.

Proof: Let X, Y be the bipartition of $G = K_{m,m}$. Assign the distinct k-subsets of $[2k-1]$ as the lists for the vertices of X, and do the same for Y. Consider a choice function f. If f uses fewer than k distinct choices in X, then there is a k-set $S \subseteq [2k-1]$ not used, which means that no color was chosen for the vertex of X having S as its list. If f uses at least k colors on vertices of X, then there is a k-set $S \subseteq [2k-1]$ of colors used in X, and no color can be properly chosen for the vertex of Y with list S. ■

List chromatic number is more difficult to compute than chromatic number, because the statements of the upper bound and lower bound both involve universal quantifiers. Determining the 3-choosable complete bipartite graphs was difficult. For $3 \le m \le n$, $K_{m,n}$ is 3-choosable if and only if

$m = 3$ and $n \le 26$ (Erdős–Rubin–Taylor [1979]), or
$m = 4$ and $n \le 20$ (Mahadev–Roberts–Santhanakrishnan [1991]), or
$m = 5$ and $n \le 12$ (Shende–Tesman [1994]), or
$m = 6$ and $n \le 10$ (O'Donnell [1995]).

Alon and Tarsi [1992] used a polynomial associated with a graph to obtain upper bounds on $\chi_l(G)$ (see also Alon [1993]). Fleischner and Stiebitz [1992] used the technique to prove that adding n disjoint triangles to a $3n$-cycle yields a 3-colorable graph; they proved the stronger result that it is 3-choosable.

There is also an edge-coloring variant, where we assign lists to the edges and must choose a proper edge-coloring.

8.4.26. Definition. Let $L(e)$ denote the list of colors available for e. A **list edge-coloring** is a proper edge-coloring f with $f(e)$ chosen from $L(e)$ for each e. The **edge-choosability** $\chi_l'(G)$ is the minimum k such that every assignment of lists of size k yields a proper list edge-coloring. Equivalently, $\chi_l'(G) = \chi_l(L(G))$, where $L(G)$ is the line graph of G.

The argument for $\chi'(G) \le 2\Delta(G) - 1$ also yields $\chi_l'(G) \le 2\Delta(G) - 1$ (Exercise 22) and thus $\chi_l'(G) < 2\chi'(G)$. As in ordinary coloring, the use of line graphs expresses the edge version as a special case of the vertex version, and it behaves much better. Even so, the conjectured bound for edge-choosability is surprising. It was suggested independently by many researchers, including Vizing, Gupta, Albertson, Collins, and Tucker, and it seems to have been published first in Bollobás–Harris [1985] (see also Bollobás [1986]).

8.4.27. Conjecture. (List Coloring Conjecture) $\chi_l'(G) = \chi'(G)$ for all G. ■

For simple graphs, this conjecture and Vizing's Theorem (Theorem 7.1.10) would yield $\chi_l'(G) \le \Delta(G) + 1$. Bollobás and Harris [1985] proved that $\chi_l'(G) <$

$c\Delta(G)$ when $c > 11/6$ for sufficiently large $\Delta(G)$. This and later improvements used probabilistic methods. Kahn [1996] proved the conjecture asymptotically: $\chi_l'(G) \le (1 + o(1))\Delta(G)$. Häggkvist and Janssen [1997] sharpened the error term: $\chi_l'(G) \le d + O(d^{2/3}\sqrt{\log d})$ when $d = \Delta(G)$. Molloy and Reed [1999] further sharpened (and generalized) the bound.

The special case of the List Coloring Conjecture for $G = K_{n,n}$ was posed by Dinitz in 1979. (Janssen [1993] proved it for $K_{n,n-1}$.) The Dinitz Conjecture became popular in its matrix formulation: If each position of an n by n grid contains a set of size n, then it is possible to choose one element from each set so that the elements chosen in each row are distinct and the elements chosen in each column are distinct.

Galvin [1995] proved the List Coloring Conjecture for bipartite graphs, which includes the Dinitz Conjecture (see also Slivnik [1996]). Here we prove only the Dinitz Conjecture, using the Stable Matching Problem (Section 3.2).

8.4.28. Definition. A **kernel** of a digraph is an independent set S having a successor of every vertex outside S. A digraph is **kernel-perfect** if every induced subdigraph has a kernel. Given a function $f\colon V(G) \to \mathbb{N}$, the graph G is **f-choosable** if a proper coloring can be chosen from the lists at the vertices whenever $|L(x)| = f(x)$ for each x.

We used the concept of "kernel" in Application 1.4.14 (digraphs without odd cycles, for example, have kernels). An f-choosable graph is k-choosable for $k = \max f(x)$, since adding colors to a list cannot make the choice more difficult.

8.4.29. Lemma. (Bondy–Boppana–Siegel) If D is a kernel-perfect orientation of G and $f(x) = 1 + d_D^+(x)$ for all $x \in V(G)$, then G is f-choosable.

Proof: We use induction on $n(G)$; the statement is trivial for $n(G) = 1$. For $n(G) > 1$, consider an assignment of lists, with the list $L(x)$ having size $f(x)$. Choose a color c appearing in some list. Let $U = \{v\colon c \in L(v)\}$. Let S be the kernel of the induced subdigraph $D[U]$. Assign color c to all of S, which is permissible since S is independent.

Delete c from $L(v)$ for each $v \in U - S$. Delete additional colors arbitrarily from other lists to reduce $L(x)$ for each $x \in V(D) - S$ to size $f'(x)$, where $f'(x) = 1 + d_{D-S}^+(x)$. Since each vertex not in S has a successor in S, we have $f'(x) < f(x)$ for $x \in V(D) - S$, which accommodates the deletion of c from the lists. By the induction hypothesis, D' is $f'(x)$-choosable, so we can complete a list coloring for G by adding a list coloring of D' to the use of c on S. ■

8.4.30. Theorem. (Galvin [1995]) $\chi_l'(K_{n,n}) = n$.

Proof: Since $\chi_l'(G) = \chi_l(L(G))$, it suffices by Lemma 8.4.29 to prove that $L(K_{n,n})$ has a kernel-perfect orientation with each vertex having indegree and outdegree $n - 1$. The graph $L(K_{n,n})$ is the cartesian product $K_n \square K_n$ (Exercise 7.1.8); placed in an n by n grid, vertices are adjacent if and only if they are in the same row or in the same column.

Assign labels $1, 2, \ldots, n$ so that vertex (r, s) has label $r+s-1 \bmod n$. Define an orientation D of $K_n \square K_n$ by directing edges from vertex (r, s) with label i to the vertices in column s with lower labels and the vertices in row r with higher labels. Since i is higher than $i-1$ other labels, (r, s) has $i-1$ successors in its column and $n-i$ successors in its row. Hence $d^+(r, s) = d^-(r, s) = n-1$.

We prove that D is kernel-perfect. Given $U \subseteq V(D)$, we obtain a kernel for the subdigraph $D[U]$ by solving a stable matching problem. When $(r, b) \in U$ and $(r, s) \to (r, b)$ in D, we want r to prefer b to s. Thus for row r, the preferences among columns begin with $\{s\colon (r, s) \in U\}$ in decreasing order of vertex labels, followed by any order among $\{s\colon (r, s) \notin U\}$. Similarly, for column s, the preferences among rows begin with $\{r\colon (r, s) \in U\}$ in increasing order of vertex labels, followed by any order among $\{r\colon (r, s) \notin U\}$.

The Gale–Shapley Proposal Algorithm (Algorithm 3.2.17) yields a stable matching M for these preferences. Viewing the matched pairs in M as positions in the grid, let $S = M \cap U$. Because M is a matching, S has no two positions in the same row or column; hence S is an independent set in D. We show that each $x \in U - S$ has a successor in S.

Let i be the label of position $x = (r, s) \in U - S$. Since $S = M \cap U$, we have $x \notin M$. Thus M has a position $y = (r, b)$ with some label j and a position $z = (a, s)$ with some label k. Because M is stable, we cannot have both r preferring s to b and s preferring r to a. From this statement we deduce by the steps below that x has y or z as a successor in S. ■

	b	s
a		$z : k$
r	$y : j$	$x : i$

$$
\begin{array}{ll}
\text{not} & \big[(r \text{ prefers } s \text{ to } b) \ \text{ and } \ (s \text{ prefers } r \text{ to } a)\big] \\
\text{not} & \big[(y \notin U \ \text{ or } \ i > j) \ \text{ and } \ (z \notin U \ \text{ or } \ i < k)\big] \\
 & (y \in U \ \text{ and } \ i < j) \ \text{ or } \ (z \in U \ \text{ and } \ i > k) \\
 & \qquad (x \to y \in S) \ \text{ or } \ (x \to z \in S)
\end{array}
$$

8.4.31. Remark. The List Coloring Conjecture relates to another conjecture. A **total coloring** of G assigns a color to each vertex and to each edge so that colored objects have different colors when they are adjacent or incident. The Total Coloring Conjecture (Behzad [1965]) states that every simple graph G has a total coloring with at most $\Delta(G)+2$ colors. Rosenfeld [1971] and Behzad [1971] provide results on special classes. The List Coloring Conjecture would yield an upper bound of $\Delta(G)+3$, since every graph G has a total coloring with at most $\chi'_l(G)+2$ colors (Exercise 25). ■

The List Coloring Conjecture has been studied for planar graphs. Ellingham and Goddyn [1996] proved that every k-regular k-edge-colorable planar graph is k-edge-choosable (using the Four Color Theorem).

The discussion of planar graphs brings us back to list coloring of vertices. Although planar graphs have chromatic number at most 4, Vizing [1976] and

Erdős–Rubin–Taylor [1979] conjectured that the maximum choice number on this class is 5. Voigt [1993] constructed a non-4-choosable planar graph with 238 vertices; Mirzakhani [1996] (Exercise 26) reduced this to 63 vertices (both examples generalize to infinite families). In fact, there are 3-colorable planar graphs that are not 4-choosable (Gutner [1996], Voigt–Wirth [1997]).

Thomassen [1994b] proved the upper bound (and also [1995] that planar graphs of girth 5 are 3-choosable). Often in inductive proofs for planar graphs, the vertices on the unbounded face ("external vertices") play a special role.

8.4.32. Theorem. (Thomassen [1994b]) Planar graphs are 5-choosable.

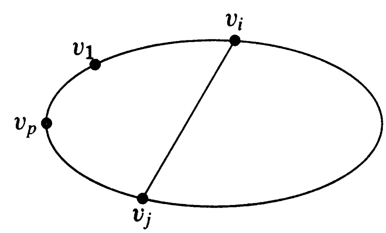
Proof: Adding edges cannot reduce the list chromatic number, so we may restrict our attention to plane graphs where the outer face is a cycle and every bounded face is a triangle. By induction on $n(G)$, we prove the stronger result that a coloring can be chosen even when two adjacent external vertices have distinct lists of size 1 and the other external vertices have lists of size 3. For the basis step ($n = 3$), a color remains available for the third vertex.

Now consider $n > 3$. Let v_p, v_1 be the vertices with fixed colors on the external cycle C. Let $v_1, \dots, v_p$ be $V(C)$ in clockwise order.

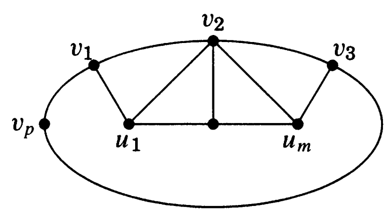
Case 1: C has a chord $v_i v_j$ with $1 \le i \le j-2 \le p-2$. We apply the induction hypothesis to the graph consisting of the cycle $v_1, \dots, v_i, v_j, \dots, v_p$ and its interior. This selects a proper coloring in which v_i, v_j receive some fixed colors. Next we apply the induction hypothesis to the graph consisting of the cycle $v_i, v_{i+1}, \dots, v_j$ and its interior to complete the list coloring of G.

Case 2: C has no chord. Let $v_1, u_1, \dots, u_m, v_3$ be the neighbors of v_2 in order ($3 = p$ is possible). Because bounded faces are triangles, G contains the path P with vertices $v_1, u_1, \dots, u_m, v_3$. Since C is chordless, $u_1, \dots, u_m$ are internal vertices, and the outer face of $G' = G - v_2$ is bounded by a cycle C' in which P replaces v_1, v_2, v_3.

Let c be the color assigned to v_1. Since $|L(v_2)| \ge 3$, we may choose distinct colors $x, y \in L(v_2) - \{c\}$. We reserve x, y for possible use on v_2 by forbidding x, y from $u_1, \dots, u_m$. Since $|L(u_i)| \ge 5$, we have $|L(u_i) - \{x, y\}| \ge 3$. Hence we can apply the induction hypothesis to G', with $u_1, \dots, u_m$ having lists of size at least 3 and other vertices having the same lists as in G. In the resulting coloring, v_1 and $u_1, \dots, u_m$ have colors outside $\{x, y\}$. We extend this coloring to G by choosing for v_2 a color in $\{x, y\}$ that does not appear on v_3 in the coloring of G'. ■



Case 1



Case 2

PARTITIONS USING PATHS AND CYCLES

We have considered the **F-decomposition** problem: partitioning $E(G)$ into the minimum number of subgraphs in a family **F**. This has been studied for many families **F**, such as cliques (Theorem 8.4.3), bipartite graphs (Exercise 3), complete bipartite graphs (Theorem 8.6.20), stars (vertex cover number—Section 3.1), and forests (arboricity—Corollary 8.2.57). Before considering extremal problems for decomposition of graphs into paths and cycles, we discuss an easier problem: covering the vertices of a digraph using the fewest paths.

Comparability graphs are those having transitive orientations; a digraph is **transitive** if $x \to y$ and $y \to z$ imply $x \to z$. The vertices of a path in a transitive digraph induce a tournament. Comparability graphs are perfect (Proposition 5.3.25), meaning that a transitive digraph D in which the largest tournament has ω vertices can be properly ω-colored. By the Perfect Graph Theorem (Theorem 8.1.6), we also know that $V(D)$ can be covered using $\alpha(D)$ tournaments in D, where $\alpha(D)$ is the maximum size of an independent set.

Letting paths be "chains" and independent sets be "antichains", this becomes Dilworth's Theorem for transitive loopless digraphs: The maximum size of an antichain equals the minimum number of chains needed to partition $V(D)$. In addition to following from the Perfect Graph Theorem, Dilworth's Theorem is equivalent to the König–Egerváry Theorem (Exercise 27), and a generalization of it follows from the Matroid Intersection Theorem (Exercise 8.2.50). Here we present a further generalization that has a short and self-contained proof.

8.4.33. Theorem. (Gallai–Milgram [1960]) The vertices of a digraph D can be covered using at most $\alpha(D)$ pairwise disjoint paths.

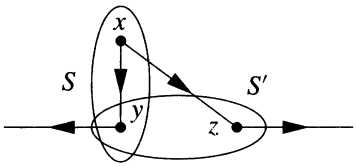
Proof: Since $V(D)$ can be covered using n disjoint paths of length 0, it suffices to prove a stronger claim: If **C** is a set of pairwise disjoint paths covering $V(D)$, and S is the set of sources (initial vertices) of these paths, then $V(D)$ can be covered using at most $\alpha(D)$ pairwise disjoint paths with sources in S. The proof is by induction on $n(D)$, with a trivial basis step for $n(D) = 1$. The added statement about the sources helps the induction step work.

Suppose that $n > 1$ and that **C** is a covering of $V(D)$ by k paths with source set S. The claim holds unless $|\mathbf{C}| = k > \alpha(D)$, in which case we construct a cover using fewer paths, all with sources in S. Since $k > \alpha$, there exists an edge xy with $x, y \in S$. Let A and B be the paths in **C** starting with x and y, respectively. We may assume that A has an edge xz, else we could add x to the beginning of B and save one path.

By deleting x from the start of A, we obtain a cover $\mathbf{C}'$ of $V(D - x)$ by k paths having sources in $S' = S - x + z$. Since $\alpha(D - x) \le \alpha(D)$, the induction hypothesis yields a cover $\mathbf{C}''$ of $V(D - x)$ using fewer than k paths, all with sources in S'. All elements of S' belong to S except z.

If z is the source of a path in $\mathbf{C}''$, then we add x at the beginning of that path. If z is not a source but y is, then we add x at the beginning of the path starting with y. If neither y nor z is a source, then at most $|S'| - 2 = k - 2$ paths have been used. and we can add x as a path by itself to obtain the desired cover of $V(D)$ using $k-1$ paths. In all cases, the resulting paths are pairwise disjoint and have sources in S.

By repeating this argument as long as $k > \alpha$, we can reduce the number of paths to α. ■



We return to the decomposition problem. Gallai conjectured that every n-vertex graph can be decomposed using $\lceil n/2 \rceil$ paths. Equality holds for cliques (Exercise 28). Other graphs have fewer edges, but the lack of connections could require more paths. Hajós conjectured analogously that an n-vertex even graph can be decomposed into $\lfloor n/2 \rfloor$ cycles. Both conjectures remain open, but Lovász proved the optimal bound when both paths and cycles are allowed. The **size** of a decomposition is the number of subgraphs used.

8.4.34. Theorem. (Lovász [1968b]) Every n-vertex graph can be decomposed into $\lfloor n/2 \rfloor$ paths and cycles.

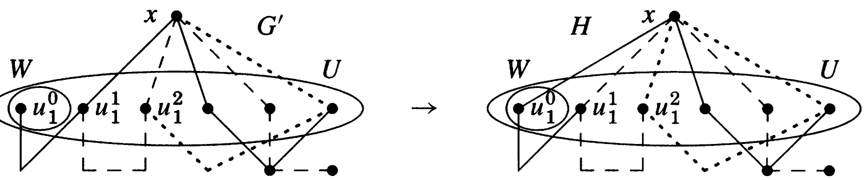
Proof: Let $\mathbf{F}$ be the family of all paths and cycles, and let $n'(G)$ be the number of non-isolated vertices in a graph G. By induction on $\lambda(G) = 2e(G) - n'(G)$, we prove that G has an $\mathbf{F}$-decomposition of size at most $\lfloor n'(G)/2 \rfloor$. Each component of G with more than one edge contributes positively to $\lambda(G)$. Hence $\lambda(G) \geq 0$, with equality only when each nontrivial component is an edge. The claim holds with equality when $\lambda(G) = 0$.

In the induction step, $\lambda(G) > 0$. We consider two cases. **Case 1**: If G has a vertex y of positive even degree, choose $x \in N(y)$, and let $W = \{z \in N(x)\colon d(z)$ is even$\}$. In this case, let $G' = G - \{xz : z \in W\}$. In obtaining G', we lose at least one edge (xy) and we isolate at most one vertex (x), so $\lambda(G') < \lambda(G)$. **Case 2**: If G has no vertex of positive even degree, then $\lambda(G) > 0$ forces $\Delta(G) > 1$. Let x be a vertex of degree at least 3, and form G^+ by introducing a new vertex y to subdivide an edge xx'. Let $W = \{y\}$, and let $G' = G^+ - xy$. Now $e(G') = e(G)$, but $n'(G') > n'(G)$, so $\lambda(G') < \lambda(G)$.

In each case, the induction hypothesis yields an $\mathbf{F}$-decomposition $\mathbf{D}$ of G' with $|\mathbf{D}| \leq \lfloor n'(G')/2 \rfloor$. We convert $\mathbf{D}$ into an $\mathbf{F}$-decomposition of size $|\mathbf{D}|$ for the graph H obtained from G' by adding edges from x to W. In Case 1, $H = G$ and $n'(G') \leq n'(G)$, so this is the desired decomposition. In Case 2, $H = G^+$ and $n'(G') = n'(G^+)$. Since $n'(G)$ is even, $\lfloor n'(G)/2 \rfloor = \lfloor n'(G^+)/2 \rfloor$. In an $\mathbf{F}$-decomposition of G^+, the $n'(G)$ vertices of odd degree must all be endpoints of paths; thus the added vertex y of degree 2 cannot be the end of a path. This means that xy and yx' belong to the same subgraph and can be replaced by xx' to obtain the desired decomposition of G.

The two cases now combine; we need only obtain the decomposition of H from $\mathbf{D}$. Let $U = N_H(x)$. Every vertex of U has odd degree in G', so for each

$u \in U$ there is a path $P(u)$ in $\mathbf{D}$ with endpoint u. For $u \in W$, we would like to extend $P(u)$ to absorb ux. This cannot be done if $P(u)$ reaches but does not end at x, since then the subgraph $P(u) \cup ux$ is not in $\mathbf{F}$. The idea is to cut the edge $u'x$ on which $P(u)$ reaches x, use the path $P(u) \cup ux - u'x$, and use $u'x$ to extend $P(u')$ instead. This generates a sequence of changes from each $u \in W$. We must show that the sequences terminate and do not conflict with each other.

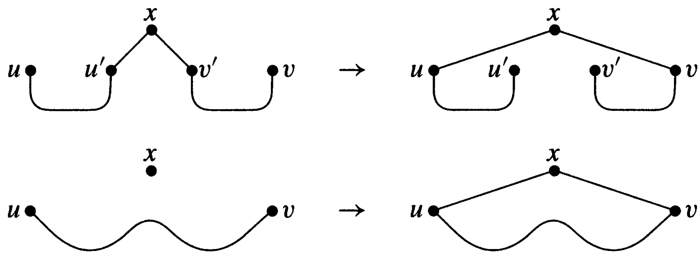


Let $W = w_1, \ldots, w_t$. For $w_i \in W$, we form a list $u_i^0, u_i^2, \cdots$ with $u_i^0 = w_i$ and each $u_i^j \in U$. If in the ith list we have chosen a vertex u_i^j, we check whether x is an internal vertex of $P(u_i^j)$. If not, then we stop and do not define u_i^{j+1}. If so, then we set u_i^{j+1} to be the vertex on $P(u_i^j)$ just before x; this is the "u'" suggested above. The path $P(u_i^j)$ for $j \geq 1$ cannot start along the edge $u_i^j x$, because that edge is internal to $P(u_i^{j-1})$. (Our picture of G' shows three successive paths: $P(u_1^0)$ solid, $P(u_1^1)$ dashed, $P(u_1^2)$ dotted.)

We prove next that no vertex of U appears twice in the lists. Since $xu_i^j \in E(G')$ if $j \geq 1$, the vertices of W appear only as initial vertices. Let u_i^j, u_k^l be a repeated vertex with $\min\{j, l\}$ minimal; we have shown that $j, l > 0$. By minimality, $u_i^{j-1} \neq u_k^{l-1}$, and hence the paths $P(u_i^{j-1})$ and $P(u_k^{l-1})$ start at distinct vertices. If $u_i^j = u_k^l$, then the two paths share the edge $u_i^j x$ and must be the same path. This happens from distinct vertices only if u_i^{j-1} and u_k^{l-1} are opposite ends of the path, but then they cannot both visit u_i^j before x. Hence no repetition occurs.

Let $W' = \{u_i^j\}$. If $u = u_i^j$ and u is not the end of its list, let $u' = u_i^{j+1}$. We define an $\mathbf{F}$-decomposition of G consisting of one path or cycle Q' corresponding to each $Q \in \mathbf{D}$. If $Q \neq P(u)$ for some $u \in W'$, let $Q' = Q$. If $Q = P(u)$, let $Q' = Q + ux$ or $Q' = Q + ux - u'x$ depending on whether u is or is not the last vertex in its list. Always Q' is a path, except that Q' is a cycle when Q ends at x (and then u' is not defined). The union of the new paths corresponding to $\{P(u_i^j)\}$ is the same as $\cup P(u_i^j)$, except that the edges $\{xw_i\}$ are absorbed. Since $u \in W'$ appears only once in the lists, the edge ux winds up in only one of the new paths, and $\{Q'\colon Q \in \mathbf{D}\}$ is a decomposition of H. ∎

Note that in this proof Q may be the selected path from each of its endpoints $u, v \in W'$. This is not a problem, because the adjustments to Q made from the two ends do not conflict. The path may visit x (thus defining u' and v') or not, as sketched below.



CIRCUMFERENCE

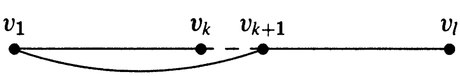
When a sufficient condition for Hamiltonian cycles fails slightly, we might expect that the graph still must have a fairly long cycle. The length of the longest cycle in G is the **circumference** $c(G)$. We first consider the number of edges needed to force a cycle of length at least c in an n-vertex graph. In this section, $P(v, w)$ denotes the v, w-portion of a path P containing v and w. Also P, Q denotes the concatenation of paths P and Q when the last vertex of P is the first vertex of Q.

8.4.35. Theorem. (Erdős–Gallai [1959]) For $m \geq 2$, every simple n-vertex graph with more than $m(n-1)/2$ edges has a cycle of length more than m.

Proof: (Woodall [1972]) We use induction on n for fixed m. When $n = m+1$, fewer than $(n-1)/2$ edges are missing, so $\delta(G) \geq n/2$ and G is Hamiltonian. Suppose that $n > m+1$ and $c(G) \leq m$. If $d(x) \leq m/2$, then $e(G-x) \geq m(n-2)/2$. Applying the induction hypothesis to $G-x$ yields $c(G-x) > m$. Hence we may assume that $\delta(G) > m/2$. Similarly, we may assume that G is connected.

Among all longest paths in G, choose $P = v_1, \dots, v_l$ to maximize the degree d of v_1; since G is connected, we have $v_1 \not\leftrightarrow v_l$ (otherwise an edge from $V(P)$ to $V(G) - V(P)$ would yield a longer cycle). Let $W = \{v_i \colon v_1 \leftrightarrow v_{i+1}\}$. All neighbors of v_1 lie on P, so $|W| = d$. For $v_k \in W$, the path $P(v_k, v_1), v_1v_{k+1}, P(v_{k+1}, v_l)$ also has length l; hence $N(v_k) \subseteq V(P)$, and the choice of P yields $d(v_k) \leq d$. Furthermore, no $v_k \in W$ has a neighbor v_j such that $j > m$, because then we could complete the long cycle by adding v_jv_k to $P(v_k, v_1), v_1v_{k+1}, P(v_{k+1}, v_j)$.

By limiting the edges incident to W, we force many edges into $G - W$. Let $Z = \{v_1, \dots, v_r\}$, where $r = \min\{l, m\}$. For each $v_k \in W$, we have shown that $N(v_k) \subseteq Z$. Hence there are $|[W, Z-W]| + e(G[W])$ edges incident to W. For fixed degree-sum in W, this is maximized when $[W, Z-W]$ is a complete bipartite graph. We further maximize by letting each vertex of W have degree d. The resulting count is $\frac{1}{2}\,|W|\,(d+|Z-W|) = dr/2 \leq dm/2$. Therefore, $G - W$ has $n-d$ vertices and more than $m(n-d-1)/2$ edges. By the induction hypothesis $c(G-W) > m$. (If the number of edges forced into $G - W$ is too large to exist, then this case cannot occur, and an earlier case applies.) ■



Most sufficient conditions for Hamiltonian cycles have "long cycle" versions. The long cycle version of Dirac's Theorem says that a 2-connected graph G has a cycle of length at least $\min\{n(G), 2\delta(G)\}$ (Dirac [1952b]). Requiring 2-connectedness eliminates the example $K_1 \vee 2K_\delta$ with circumference $\delta + 1$.

The long cycle version of Ore's Theorem [1960] came much later. It is implicit in Bondy [1971b] and was made explicit in Bermond [1976] and in Linial [1976]. The fundamental argument used in many long cycle results appears in Bondy [1971b]. It strengthens the Ore/Dirac switching argument (Theorem 7.2.8) by considering "gaps".

8.4.36. Lemma. (Bondy [1971b]) If $P = v_1, \ldots, v_l$ is a longest path in a 2-connected graph G, then $c(G) \geq \min\{n(G), d(v_1) + d(v_l)\}$.

Proof: (See also Linial [1976]). Let $m = d(v_1) + d(v_l)$, and suppose that $c(G) < \min\{n(G), m\}$. Since G is connected, an l-cycle would yield a longer path; thus $v_1 \not\leftrightarrow v_l$. If $v_1 \leftrightarrow v_j$ and $v_i \leftrightarrow v_l$ for some $i < j$, then i, j is a *crossover* with *gap* $j - i$. If we add v_1v_j and v_lv_i to $P(j, l)$ and $P(i, 1)$, we obtain a cycle with length $l - (j - i - 1)$. Hence $l - (j - i - 1) < m$ when i, j is a crossover.

Let $x = v_1$ and $y = v_l$. If P has a crossover, let i, j be one with smallest gap. Thus x and y have no neighbors between v_i and v_j on P. Also $N(y)$ contains no predecessor on P of a neighbor of x, since an l-cycle yields a longer path. Hence $N(y)$ lies in $V(P) - \{y\}$ but avoids $\{v_{i+1}, \ldots, v_{j-2}\}$ and $\{v_{r-1} : v_r \leftrightarrow x\}$. Thus $d(y) \leq (l - 1) - (j - 2 - i) - d(x)$. Since $l - (j - i - 1) < m$, we have $d(x) + d(y) < m$, which contradicts the hypothesis. Hence there is no crossover.

With $t_0 = \max\{i : x \leftrightarrow v_i\}$ and $u = \min\{i : y \leftrightarrow v_i\}$, we have proved that $t_0 \leq u$. We will construct a cycle containing x and y and all their neighbors. Since the absence of crossovers implies that $|N(x) \cap N(y)| \leq 1$, such a cycle has length at least $d(x) + d(y) + 1 > m$.

We iteratively define paths $P_1, P_2, \cdots$. Given t_{i-1}, we choose integers $s_i < t_{i-1} < t_i$ to maximize t_i such that G has a v_{s_i}, v_{t_i}-path P_i internally disjoint from P. Such a path exists because $G - v_{t-1}$ is connected. These paths are disjoint; if P_i shares a vertex with a later path P_j, then we can choose P_i as an s_i, t_j-path, which contradicts the maximality of t_i. Similarly, $s_{i+1} \geq t_{i-1}$, since otherwise P_{i+1} would be chosen instead of P_i.

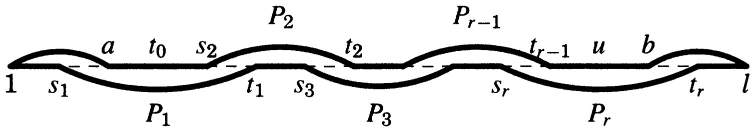
Let r be the smallest index such that $t_r > u$. Set

$$a = \min\{j : x \leftrightarrow v_j \text{ and } j > s_1\}, \qquad b = \max\{j : y \leftrightarrow v_j \text{ and } j < t_r\}.$$

Since $s_1 < t_0$ and $t_r > u$, the indices a, b are well-defined. We use the even-indexed paths P_i to build one x, y-path and the odd-indexed paths to build another x, y-path. When r is odd, the two paths are formed by the following concatenations.

$$xv_a, P(a, s_2), P_2, P(t_2, s_4), P_4, \ldots, P(t_{r-1}, b), v_by$$

$$P(1, s_1), P_1, P(t_1, s_3), P_3, P(t_3, s_5), \ldots, P_r, P(t_r, l)$$



When r is even, the path starting with xv_a reaches t_r and ends with $P(t_r, l)$, while the other path reaches v_b and ends with $v_b y$.

We have observed that $s_{i+1} \geq t_{i-1}$. Hence

$$s_1 < a \leq t_0 \leq s_2 < t_1 \leq s_3 < t_2 \cdots < t_{r-1} \leq u \leq b < t_r$$

This implies that the two concatenations described are paths and that their union is a cycle. By the definition of a, we have $N(x) \subseteq P(1, s_1) \cup P(a, t_0)$, and similarly $N(y) \subseteq P(u, b) \cup P(t_r, l)$. With x and y themselves, the cycle thus has length at least $2 + d(x) + d(y) - 1 > m$. ■

Ore proved that G is Hamiltonian if $d(u) + d(v) \geq n(G)$ when $u \nleftrightarrow v$. Bondy's Lemma implies the long cycle version of this, which strengthens the long cycle version of Dirac's Theorem.

8.4.37. Theorem. (Bondy [1971b], Bermond [1976], Linial [1976]) If G is 2-connected and $d(u) + d(v) \geq s$ for every nonadjacent pair $u, v \in V(G)$, then $c(G) \geq \min\{n(G), s\}$.

Proof: Ore's Theorem guarantees a Hamiltonian cycle if $s \geq n$, so we may assume that $s < n$. Suppose that P is a longest path in G, with endpoints x and y. Since G is connected, the maximality of P implies that $x \nleftrightarrow y$. Now the condition $d(x) + d(y) \geq s$ allows us to invoke Lemma 8.4.36. ■

Bermond extended this to a "long cycle" combination of Chvátal's condition and Las Vergnas' condition. The technique of edge-switches involving an endpoint of a longest path was used in Theorem 8.4.35. Our statement is slightly weaker than that of Bermond but has a simpler proof.

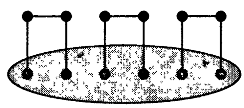
8.4.38. Theorem. (Bermond [1976]) Let G be a 2-connected graph with degree sequence $d_1 \leq \cdots \leq d_n$. If G has no nonadjacent pair x, y with degrees i, j such that $d_i \leq i < c/2$, $d_{j+1} \leq j$, and $i + j < c$, then $c(G) \geq c$.

Proof: Among the longest paths in G, let $P = v_1, \ldots, v_l$ with endpoints $x = v_1$ and $y = v_l$ be chosen to maximize $d(v_1) + d(v_l)$. If $d(x) + d(y) \geq c$, then we apply Bondy's Lemma. If $d(x) + d(y) < c$, then we claim that x, y contradicts the hypotheses. As usual, an l-cycle would yield a longer path (since G is connected), so $x \nleftrightarrow y$. We may assume that $d(x) \leq d(y)$ and set $i = d(x)$ and $j = d(y)$.

All neighbors of x and y lie in P. If $x \leftrightarrow v_k$, then $P(v_{k-1}, x)$, xv_k, $P(v_k, y)$ is another longest path ending at y; thus $d(v_{k-1}) \leq d(x) = i$, by the choice of P. Since this holds for each of the i neighbors of x, we have $d_i \leq i$. Similarly, the j neighbors of y each have degree at most j. Also $d(x) \leq j$, so $d_{j+1} \leq j$. By hypothesis, $i + j = d(x) + d(y) < c$, which completes the contradiction. ■

G.-H. Fan [1984] strengthened Theorem 8.4.37 by weakening the degree condition and by requiring it only for nonadjacent pairs with common neighbors. T. Feng [1988] used Bondy's Lemma to shorten the proof. The result includes a sufficient condition for Hamiltonian cycles that does not require the closure to be complete.

8.4.39. Example. *A Hamiltonian graph.* For even n, let $G_1 = K_{n/2}$ and $G_2 = (n/4)K_2$, and form G by adding a matching between disjoint copies of G_1 and G_2. The Hamiltonian closure of G is G itself, so our previous sufficient conditions do not apply. Even though G has $n/2$ vertices of degree 2, Fan's Theorem implies that G is Hamiltonian. ■

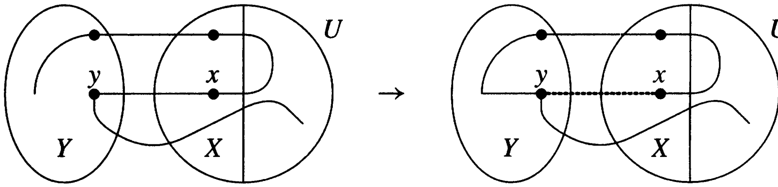


8.4.40. Theorem. (Fan [1984]) If G is 2-connected, and $d_G(u, v) = 2$ implies $\max\{d(u), d(v)\} \geq c/2$, then $c(G) \geq \min\{n(G), c\}$.

Proof: (Feng [1988]) Let $U = \{v \in V(G)\colon d(v) \geq c/2\}$. By Bondy's Lemma, it suffices to find a longest path having both endpoints in U. Among the paths of maximum length, let $P = v_1, \ldots, v_m$ be one that has the maximum number of endpoints in U. If P fails to have both endpoints in U, then we will find a longer path or a path of the same length with more of its endpoints in U. We may assume that $v_1 \notin U$.

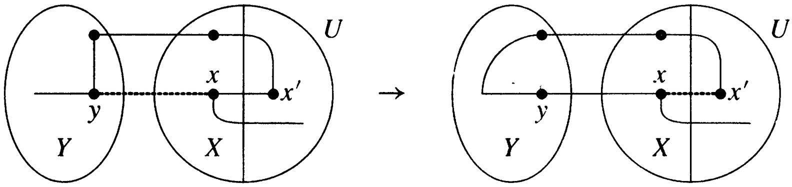
Since $d(v) < c/2$ for all $v \notin U$, the hypothesis on pairs with distance 2 implies that $G - U$ is a disjoint union of complete graphs. Let Y be the one containing v_1. Let X be the set of vertices in U having neighbors in Y. By the hypotheses, vertices of X have neighbors only in $Y \cup U$. Also $|X| \geq 2$, because G is 2-connected.

Let $r = |Y|$. We first show that P begins by visiting all of Y. If P omits some vertex of Y, then we can absorb it before the first exit from Y. If P leaves and returns to Y, then it returns via an edge xy. Because $G[Y]$ is complete, we can replace xy in P with v_1y, obtaining an x, v_m-path having the same length as P but more endpoints in U. Hence we may assume that $Y = \{v_1, \ldots, v_r\}$.



Consider $x \in X - v_{r+1}$. Suppose first that x has a neighbor $y \in Y$ other than the exit vertex v_r of P. If $x \notin V(P)$, then we can instead start with xy, absorb the rest of Y up to v_r, and thus complete an x, v_l-path longer than P. If

$x \in V(P)$, then we let x' be the vertex before x on P. Since $x \neq v_{r+1}$, we have $x' \in U$. We replace $x'x$ in P with yx, obtaining an x', v_l-path with the same length as P but more endpoints in U.



Hence we may assume for $x \in X - v_{r+1}$ that x has no neighbor in Y other than v_r. If $|Y| \geq 2$, this makes v_r a cut-vertex unless v_{r+1} has another neighbor $y \in Y - v_r$. Now we rearrange P to start with $v_r, \dots, y, v_{r+1}$ instead of $v_1, \dots, v_r, v_{r+1}$. This puts us in the case just discussed.

The remaining case is $|Y| = 1$ and $N(v_1) = X$. With $x \in X - v_{r+1}$ as before, we append x to the beginning of P or replace $x'x$ with xv_1. ■

Finally, we present one result about digraphs that strengthens Ghouilà-Houri's sufficient condition (Theorem 7.2.22) for Hamiltonian cycles. We consider only loopless digraphs having at most one copy of each ordered pair as an edge; call these **strict** digraphs. For digraphs, we use "u, v nonadjacent" to mean $uv, vu \notin E(G)$. Also, we define $d(v) = d^+(v) + d^-(v)$.

Ghouilà-Houri [1960] actually proved that a digraph G is Hamiltonian if $d(v) \geq n(G)$ for each v; this is stronger than Theorem 7.2.22 as stated. Woodall [1972] proved that it suffices to have $d^+(u) + d^-(v) \geq n(G)$ whenever u, v are nonadjacent. This generalizes Ore's Theorem for undirected graphs (Exercise 33). Meyniel [1973] proved that a strict strong digraph G is Hamiltonian if $d(u)+d(v) \geq 2n(G)-1$ for all nonadjacent pairs u, v. Meyniel's Theorem implies Ghouilà-Houri's Theorem and Woodall's Theorem (Exercise 33).

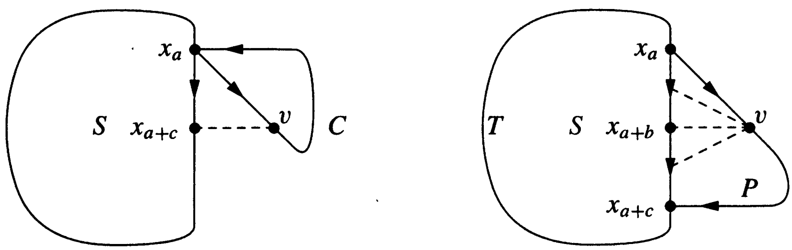
8.4.41. Example. *Meyniel's Theorem is best possible.* Let G consist of two doubly-directed cliques sharing a vertex. The digraph is strongly connected, and the only pairs of nonadjacent vertices consist of one vertex from each clique. If the cliques have order k and order $n+1-k$, then the total degrees for any nonadjacent pair are $2k-2$ and $2n-2k$, which sum to $2n-2$. ■

8.4.42. Theorem. (Meyniel [1973]) If G is a strict strongly connected digraph such that $d(u) + d(v) \geq 2n - 1$ whenever u, v are distinct nonadjacent vertices, then G is Hamiltonian.

Proof: (Bondy–Thomassen [1977]). We prove a technical lemma: if $T = v_1, \dots, v_k$ is a path that cannot absorb the vertex v internally (between two of its vertices), then the number of edges from v to T plus the number of edges from T to v is at most $k+1$. This follows by counting. For $1 \leq i \leq k-1$, only one of the edges $v_i v$ and vv_{i+1} is permitted. Also vv_1 and $v_k v$ are permitted; there is no restriction on absorption at the end.

We use this to prove the following statement: If G is a strict strong non-Hamiltonian digraph, and S is a maximal vertex subset having a spanning cycle $(x_1, \dots, x_m)$ in G, then there exist $v \in \overline{S}$ and integers a, b with $1 \le a \le m$ and $1 \le b < m$ such that (1) $x_a v \in E(G)$, (2) v is not adjacent to any x_{a+i} with $1 \le i \le b$, and (3) $d(v) + d(x_{a+b}) \le 2n - 1 - b$. Since $b \ge 1$, the conclusion of this statement is impossible under the hypothesis of the theorem, which will imply that the only maximal vertex set having a spanning cycle is $V(G)$.

Suppose first that no path leaves S and returns to it. Since G is strong and $S \ne V(G)$, some cycle C of length at least 2 shares exactly one vertex with S. Let this vertex be x_a, and let v be the successor of x_a on C. By the path condition, there is no path between v and $S - \{x_a\}$ in either direction. In particular, each vertex outside $S \cup \{v\}$ is incident to at most two edges also incident to v or v_{a+1}. Furthermore, v is incident to at most two edges also incident to S (the other endpoint must be v_a). Finally, each vertex of $S - v_{a+1}$ is incident to at most two edges also incident to v_{a+1}. Summing the allowed contributions yields $d(v) + d(x_{a+1}) \le 2n - 2$. Hence the desired condition holds with $b = 1$.



Now suppose that some path leaves S and returns to it. Choose such a path P so that the distance c along S from the start of P to the end of P is minimal. Let x_a be the start of P, and let v be its successor on P. The maximality of S implies that $c > 1$. Let T be the portion of S from x_{a+c} to x_a; this has $m - c + 1$ vertices. The maximality of S implies that v cannot be absorbed internally by T. Hence our technical lemma implies that v belongs to at most $m - c + 2$ edges incident to T. The minimality of c makes v nonadjacent to $x_{a+1}, \dots, x_{c-1}$.

Let b be the largest integer in $[c]$ such that G has a path from x_{a+c} to x_a with vertex set $S - \{x_{a+b}, \dots, x_{a+c-1}\}$. Let R be such a path (the path T with $b = 1$ implies that R exists.) Since $P \cup R$ is a cycle, the maximality of S yields $b < c$. By the maximality of b, x_{a+b} is not absorbed internally by R. Hence, by our technical lemma, x_{a+b} belongs to at most $m - c + b + 1$ edges incident to R.

Now we count $d(v) + d(x_{a+b})$. Each vertex outside $S \cup \{v\}$ is incident to at most two edges also incident to $\{v, x_{a+b}\}$, because the minimality of c prevents a path of length 2 between v and x_{a+b} (in either direction) using a vertex not in S. We have observed that v belongs to at most $m - c + 2$ edges incident to S. We have observed that v_{a+b} belongs to at most $m - c + b + 2$ edges incident to R. Finally, x_{a+b} belongs to at most $2(c - b - 1)$ edges incident to $S - R$. Hence $d(v) + d(x_{a+b}) \le 2(n - m - 1) + (m - c + 2) + (m - c + b + 1) + 2(c - b - 1) = 2n - 1 - b$. Again we have obtained the desired condition. ■

EXERCISES

8.4.1. Let $m = \lfloor n^2/4 \rfloor$. Prove that every n-vertex graph has an intersection representation using subsets of $[m]$ such that each element of $[m]$ appears in at most three sets. Equivalently, every n-vertex graph decomposes into at most m edges and triangles.

8.4.2. Prove that the following conditions on a graph G with no isolated vertices are equivalent. (Choudom–Parthasarathy–Ravindra [1975])

A) $\theta'(G) = \alpha(G)$.
B) $\theta'(G \vee G) = (\theta'(G))^2$.
C) $\theta'(G) = \theta(G)$.
D) Every clique in a minimum clique cover of $E(G)$ uses a simplicial vertex of G.

8.4.3. (+) Let $b(G)$ be the minimum number of bipartite graphs needed to partition $E(G)$ (called **biparticity**). Let $a(G)$ denote the minimum number of classes needed to partition $E(G)$ such that every cycle of G contains a non-zero even number of edges from some class. Prove that these parameters both equal $\lceil \lg \chi(G) \rceil$. (Hint: Prove $\lg \chi(G) \le b(G) \le a(G) \le \lceil \lg \chi(G) \rceil$.) (Harary–Hsu–Miller [1977], Alon–Egawa [1985])

8.4.4. Determine all the n-vertex graphs that have product dimension $n-1$. (Lovász–Nešetřil–Pultr [1980])

8.4.5. Prove that pdim $G \le 2$ if and only if G is the complement of the line graph of a bipartite graph (Lovász–Nešetřil–Pultr [1980])

8.4.6. Given r, compute pdim $(K_r + mK_1)$ for all $m \ge 1$. (Lovász–Nešetřil–Pultr [1980])

8.4.7. (−) Compute the product dimension of the three-dimensional cube.

8.4.8. Obtain upper and lower bounds on the product dimension of the Petersen graph that differ by 1 (the upper bound will most likely be the correct value, but showing that it cannot be improved is tedious).

8.4.9. Let $f(n)$ be the maximum value of pdim $G \cdot$ pdim $\overline{G}$ over all graphs on n vertices. Prove that $\lfloor n^2/4 \rfloor \le f(n) \le (n-1)^2$.

8.4.10. For $n \ge 4$, prove that pdim $P_n = \lceil \lg(n-1) \rceil$. For $n \ge 3$, prove that pdim $C_{2n} = 1 + \lceil \lg(n-1) \rceil$ and $1 + \lceil \lg n \rceil \le$ pdim $C_{2n+1} \le 2 + \lceil \lg n \rceil$. (Lovász–Nešetřil–Pultr [1980]) (Comment: Evans–Fricke–Maneri–McKee–Perkel [1994] showed that pdim $C_{2n+1} = 1 + \lceil \lg n \rceil$ except possibly when n is a power of 2.)

8.4.11. Prove that C_{2k+1} is not isometrically embeddable in any cartesian product of cliques if $k > 1$.

8.4.12. Determine the squashed-cube dimension of C_5.

8.4.13. (+) Determine the squashed-cube dimension of $K_{3,3}$. (Hint: Use symmetry to reduce case analysis.)

8.4.14. (!) Use Edmonds' Branching Theorem (Theorem 8.4.20) to prove the edge version of Menger's Theorem in digraphs: $\lambda'(x,y) = \kappa'(x,y)$. (Hint: Devise an appropriate graph transformation to obtain a short proof.)

8.4.15. (!) The gossip problem is also called the "telephone problem", and the corresponding problem for directed graphs is called the "telegraph problem". As a function of n, determine the minimum number of one-way transmissions among n people so that each person has a transmission path to every other. (Harary–Schwenk [1974])

8.4.16. Let D be a digraph solving the telegraph problem in which each vertex receives information from each other vertex exactly once. Prove that in D at least $n - 1$ vertices hear their own information. For each n, construct such a D in which only $n - 1$ vertices hear their own information, but for each $x \neq y$ there is exactly one increasing x, y-path. (Seress [1987])

8.4.17. *The NOHO property.*

a) Let G be a connected graph with $2n - 4$ edges having a linear ordering that solves the gossip problem and satisfies NOHO (no increasing cycle). Suppose also that $n(G) > 8$ and that at most two vertices have degree 2. Prove that the graph obtained by deleting the first calls and last calls of vertices in G has 4 components, of which two are isolated vertices and two are caterpillars having the same size. (West [1982a])

b) For every even $n \geq 4$, construct a connected ordered graph with $2n - 4$ edges that satisfies the NOHO property. (Hint: Make use of the structural properties proved in part (a) to guide the search.)

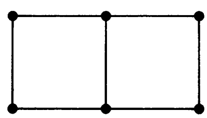
8.4.18. A **NODUP scheme** (NO DUPlicate transmission) is a connected ordered graph that has exactly one increasing path from each vertex to every other.

a) (−) Prove that every NODUP scheme has the NOHO property.

b) Prove that there is no NODUP scheme when $n \in \{6, 10, 14, 18\}$. (Comment: Seress [1986] proved that these are the only even values of n for which NODUP schemes do not exist, constructing them for all other values. For $n = 4k$, West [1982b] constructed NODUP schemes with $9n/4 - 6$ calls, and Seress [1986] proved that these are optimal.)

8.4.19. A vertex in a simple graph G wishes to broadcast information to all other vertices. In each time unit, each vertex that already knows the information can make one call to a neighbor that does not know the information. The time required to broadcast from v is the minimum number of time units in which all vertices can learn the information. Construct an n-vertex graph G with fewer than $2n$ edges such that every vertex of G can broadcast in time at most $1 + \lg n$. (Grigni–Peleg [1991])

8.4.20. (!) Prove that the graph below is not 2-choosable.



8.4.21. Prove that $K_{k,m}$ is k-choosable if and only if $m < k^k$ (Erdős–Rubin–Taylor [1979])

8.4.22. Prove that $\chi_l(G) \leq 1 + \max_{H \subseteq G} \delta(G)$ and that $\chi_l(G) + \chi_l(\overline{G}) \leq n + 1$. Prove also that $\chi_l'(G) \leq 2\Delta(G) - 1$.

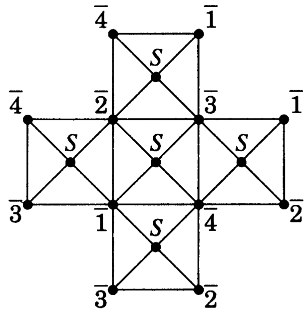
8.4.23. Prove that every chordal graph G is $\chi(G)$-choosable.

8.4.24. Prove that a connected graph G has a proper list coloring from lists such that $|L(v)| \geq d(v)$ for all v if there is strict inequality for at least one vertex.

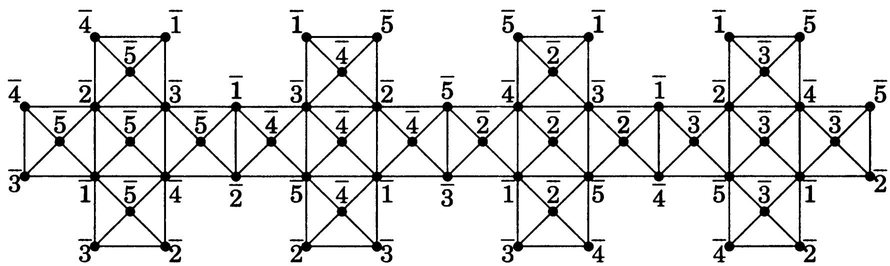
8.4.25. (!) Prove that G has a total coloring (Remark 8.4.31) with at most $\chi_l'(G) + 2$ colors.

8.4.26. (!) *Non-4-choosable planar graph of order 63.*

a) In the list assignments for the graph below, S denotes [4] and $\bar{i}$ denotes $S - \{i\}$. Prove that this graph has no proper coloring chosen from these lists.



b) In the list assignments for the graph G below, $\bar{i}$ denotes $[5] - \{i\}$; each list has size 4. Let G' be the graph obtained from G by adding one vertex with list $\overline{1}$ adjacent to all vertices on the outside face of this drawing of G. Prove that G' has no proper coloring chosen from these lists. (Mirzakhani [1996])



8.4.27. (!) *Equivalence of Dilworth's Theorem and König–Egerváry Theorem.*

a) Given a bipartite graph G, apply Dilworth's Theorem to a transitive orientation of it to obtain the König–Egerváry Theorem.

b) Given a transitive digraph D, let G be the split of D as defined in Definition 1.4.20. Apply the König–Egerváry Theorem to G to obtain Dilworth's Theorem for D.

8.4.28. (!) Prove that K_n decomposes into $\lceil n/2 \rceil$ paths. Prove that K_n decomposes into $\lfloor n/2 \rfloor$ cycles when n is odd.

8.4.29. (!) *Decomposition of K_n into spanning connected subgraphs.*

a) Prove that if K_n decomposes into k spanning connected subgraphs, then $n \geq 2k$.

b) Prove that K_{2k} decomposes into k spanning trees of diameter 3. (Hint: Let the central edges of these trees form a perfect matching.) (Palumbíny [1973])

8.4.30. Prove that every 2-edge-connected 3-regular simple planar graph decomposes into paths of length 3. Prove the same statement for planar triangulations. (Jünger–Reinelt–Pulleyblank [1985])

8.4.31. Prove that Theorem 8.4.35 is best possible when $m - 1$ divides $n - 1$.

8.4.32. Let G be a graph such that $\overline{G}$ is triangle-free and not a forest. Prove that G has a cycle of length at least $n(G)/2$. (Hint: Use Theorem 8.4.37.) (N. Graham)

8.4.33. Use Woodall's Theorem to prove Ore's Theorem, and use Meyniel's Theorem to prove Woodall's Theorem.

8.4.34. Use Meyniel's Theorem to prove that a strict n-vertex digraph has a spanning path if $d(u) + d(v) \geq 2n - 3$ for every pair u, v of distinct nonadjacent vertices.

8.5. Random Graphs

In its simplest form, the probabilistic method is used to prove the existence of desired combinatorial objects without constructing them. An appropriate probability model is defined on a large class of objects. The occurrence of the desired structure is an event. If this event has positive probability, then some object with the desired structure exists. Designing the model and applying probabilistic and asymptotic techniques may involve considerable art.

We discuss these methods in the context of random graphs. The study of random graphs is itself motivated by the modeling of physical properties and by the analysis of algorithms in computer science.

8.5.1. Example. *Melting points.* The behavior of random graphs suggests a mathematical explanation for melting points. Think of a solid as a three-dimensional grid of molecules, with neighboring molecules joined by bonds. For example, consider the graph $P_l \square P_m \square P_n$, with bonds corresponding to edges.

Adding energy excites molecules and breaks bonds. We assume that bonds break at random as we raise the temperature (energy level). Each temperature corresponds to some fraction of bonds broken. While the graph remains largely connected, the material seems solid. Breaking off small pieces doesn't change this, but when all the components are small the global nature of the material changes. Small components of molecules float freely, like a liquid or gas.

Mathematically, there is a threshold for the number of bonds to be broken (in terms of the size of the grid) such that almost every way of breaking somewhat fewer bonds leaves a giant component, and almost every way of breaking somewhat more bonds leaves all components being tiny. Just below the threshold temperature the material will almost certainly be a solid, and just above it the material will almost certainly not be a solid. ■

8.5.2. Example. *Analysis of algorithms.* Worst-case complexity is the maximum running time for an algorithm over all inputs of size n (see Appendix B). For difficult problems, we may seek an algorithm that takes many steps on a few bizarre graphs while running quickly on most graphs. We need a way to describe the usefulness of such algorithms.

The answer is **probabilistic analysis**. We assume a probability distribution on the inputs and study the expected running time with respect to this distribution. Choosing a realistic distribution can be difficult. In practice, we choose a probability distribution that makes the analysis feasible. We cannot define a probability distribution over infinitely many graphs, so we define a distribution on the graphs of each order. This is consistent with viewing the expected running time as a function of the input size. ■

Erdős and Rényi [1959] introduced random graphs. The subject developed rapidly in the 1980s, with books by Bollobás [1985], by Palmer [1985], and by Alon and Spencer [1992] (the last treats broader combinatorial applications of probabilistic methods). The book Janson–Łuczak–Ruciński [2000] emphasizes later developments.

More sophisticated probabilistic techniques than we can present here are now being applied to random graphs. We describe the basic techniques and suggest the flavor of the subject, with no attempt at exhaustive treatment.

EXISTENCE AND EXPECTATION

We begin by showing how probabilistic methods can prove existence statements. Suppose we want to prove that an object with some desired property exists. We define a probability space where occurrence of the desired property is an event A. If A has positive probability, then the desired object exists.

8.5.3. Definition. A discrete **probability space** or **probability model** is a finite or countable set S together with nonnegative weights on the elements that sum to 1. **An event** is a subset of S. The **probability** $P(A)$ of an event A is the sum of the weights of the elements of A. Events A and B are **independent** if $P(A \cap B) = P(A)P(B)$.

Erdős popularized the probabilistic method in 1947 by using it to prove lower bounds on Ramsey numbers (Definition 8.3.6). We phrased this combinatorially in Theorem 8.3.12; here we present the same proof in probabilistic language. It uses the observation that $P(\bigcup_i A_i) \le \sum_i P(A_i)$. Note that **in this section, all graphs are simple.**

8.5.4. Theorem. (Erdős [1947]) If $\binom{n}{p}2^{1-\binom{p}{2}} < 1$, then $R(p,p) > n$.

Proof: It suffices to show that when $\binom{n}{p}2^{1-\binom{p}{2}} < 1$ there is an n-vertex graph G with $\omega(G) < p$ and $\alpha(G) < p$. We define a probability model on graphs with vertex set $[n]$ by letting each edge appear independently with probability .5. If the probability of the event Q = "no p-clique or independent p-set" is positive, then the desired graph exists.

Each possible p-clique occurs with probability $2^{-\binom{p}{2}}$, since obtaining the complete graph requires obtaining all its edges, and they occur independently. Hence the probability of having at least one p-clique is bounded by $\binom{n}{p}2^{-\binom{p}{2}}$. The same bound holds for independent p-sets. Hence the probability of "not Q" is bounded by $\binom{n}{p}2^{1-\binom{p}{2}}$, and the given inequality guarantees that $P(Q) > 0$. ■

8.5.5. Remark. Existence arguments can be used as probabilistic construction algorithms. The probability that a random 1024-vertex graph has a 10-clique or independent 10-set is less than $2^{11}/20!$. If the first random one doesn't work, generate another; the probability of continued failure is the *product* of these small numbers and soon becomes incomprehensibly small. ■

The lower bound in Theorem 8.5.4 is roughly $\sqrt{2}^k$; the inductive upper bound in Theorem 8.3.11 is roughly 4^k. The gap is large. More sophisticated probabilistic methods have achieved only small improvements in the lower bound. Nevertheless, the constructive bounds are much weaker, so this is a triumph for the probabilistic method. The proof is essentially just a counting argument. Many probabilistic arguments with finite sample spaces can be rephrased as weighted counting arguments, but the proofs are simpler in the language of probability.

The introduction of random variables adds considerable power. We assign values to the elements of our probability space.† We have already used the comparison between the average and maximum values of a random variable to prove inequalities.

8.5.6. Definition. A **random variable** is a function assigning a real number to each element of a probability space. We use $X = k$ to denote the event consisting of all elements where the variable X has the value k.

The **expectation** $E(X)$ of a random variable X is the weighted average $\sum_k kP(X = k)$. The **pigeonhole property** of the expectation is the statement that there exists an element of the probability space for which the value of X is as large as (or as small as) $E(X)$.

Applying the pigeonhole property requires a value or bound for $E(X)$. Often the computation applies the **linearity of expectation** to an expression for X in terms of simpler random variables. For our purposes, we generally restrict our attention to probability models on finite sets and sum only finitely many random variables. Analogous results hold in continuous probability spaces.

8.5.7. Lemma. (Linearity property) If X and the finite set $\{X_i\}$ are random variables on the same space and $X = \sum X_i$, then $E(X) = \sum E(X_i)$. Also $E(cX) = cE(X)$ for $c \in \mathbb{R}$.

Proof: In a discrete probability space, each element contributes the same amount to each side of the desired equations. ■

We often apply Lemma 8.5.7 to random variables that count substructures. Such a random variable is a sum of variables indicating whether one of the possible things being counted actually occurs. These **indicator variables** take values in $\{0, 1\}$ (they are also called 0, 1-variables). The expectation of an indicator variable is the probability that it equals 1. These properties facilitate what was perhaps the first use of the probabilistic method.

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†We consider only discrete probability spaces, but analogous concepts hold for continuous probability spaces.

8.5.8. Theorem. (Szele [1943]) Some n-vertex tournament has at least $n!/2^{n-1}$ Hamiltonian paths.

Proof: Generate tournaments on $[n]$ randomly by choosing $i \to j$ or $j \to i$ with equal probability for each pair $\{i, j\}$. Let X be the number of Hamiltonian paths; X is the sum of $n!$ indicator variables for the possible Hamiltonian paths. Each Hamiltonian path occurs with probability $1/2^{n-1}$, so $E(X) = n!/2^{n-1}$. In some tournament, X is at least as large as the expectation. ■

This simple bound using expectation gives almost the right answer for the maximum number of Hamiltonian paths in an n-vertex tournament; Alon [1990] proved that it is at most $n!/(2-o(1))^n$. When almost all instances have a value near the extreme, probabilistic arguments are especially effective.

Many inequalities can be interpreted as statements about the expected value of a random variable. This often yields a shorter proof than combinatorial methods. Exercise 3.1.42 requests a combinatorial proof of the next result.

8.5.9. Theorem. (Caro [1979], Wei [1981]) $\alpha(G) \ge \sum_{v\in V(G)} \frac{1}{d(v)+1}$ for every graph G.

Proof: (Alon–Spencer [1992, p81]) Given an ordering of the vertices of G, the set of vertices that appear before all their neighbors form an independent set. When the ordering is chosen uniformaly at random, the probability that v appears before all its neighbors is $1/(d(v)+1)$. Thus the right side of the inequality is the expected size of the independent set formed by choosing the vertices appearing before their neighbors in a random vertex ordering. ■

When a randomly generated object is close to having a desired property, a slight alteration may produce it. This technique is called the **deletion method**, the **alteration method**, or the **two-step method**. Ramsey numbers furnish a classical application (Exercise 16). We provide two others.

Recall that $S \subseteq V(G)$ is a *dominating set* in G if every vertex outside S has a neighbor in S (Definition 3.1.26). When G is k-regular, every vertex dominates $k+1$ vertices (including itself), so every dominating set has at least $n(G)/(k+1)$ vertices. The alteration method yields a dominating set close to that bound in every graph with minimum degree k. The argument, like many involving these techniques, uses the fundamental inequality $1 - p < e^{-p}$ (Exercise 2).

8.5.10. Theorem. (Alon [1990]) Every n-vertex graph with minimum degree $k > 1$ has a dominating set of size at most $n\frac{1+\ln(k+1)}{k+1}$.

Proof: In such a graph G, select a random set $S \subseteq V(G)$ by including each vertex independently with probability $p = \ln(k+1)/(k+1)$. Given S, let T be the set of vertices outside S having no neighbor in S; adding T to S yields a dominating set. We seek the expected size of $S \cup T$.

Since each vertex appears in S with probability p, linearity yields $E(|S|) = np$. The random variable $|T|$ is the sum of n indicator variables for whether individual vertices belong to T. We have $v \in T$ if and only if v and its neighbors all fail to be in S. This has probability bounded by $(1-p)^{k+1}$, since v has degree at least k. Since $(1-p)^{k+1} < e^{-p(k+1)}$, we have $E(|S|+|T|) \le np + ne^{-p(k+1)} = n\frac{1+\ln(k+1)}{k+1}$. The pigeonhole property of the expectation completes the proof. ■

This easy bound yields almost the smallest s_k such that every graph G with minimum degree k has a dominating set of size at most $s_k n(G)$ (Alon [1990]). A greedy algorithm proves the same result constructively (Theorem 3.1.30).

A striking and famous application of the deletion method is the existence of graphs with large girth and chromatic number. Explicit constructions came much later (Lovász [1968a], Nešetřil–Rödl [1979], Kriz [1989]). We present a simplification of the original proof (Alon–Spencer [1992, p35]). It uses a property of the expectation that we will prove in Lemma 8.5.17.

8.5.11. Theorem. (Erdős [1959]) Given $m \ge 3$ and $g \ge 3$, there exists a graph with girth at least g and chromatic number at least m.

Proof: We generate graphs with vertex set $[n]$ by letting each pair be an edge with probability p, independently. A graph with no large independent set has large chromatic number, since $\chi(G) \ge n(G)/\alpha(G)$. We therefore choose p large enough to make large independent sets unlikely. We also choose p small enough to make the expected number of short cycles (length less than g) small. Given a graph satisfying both conditions, we can delete a vertex from each short cycle to obtain the desired graph.

To make it unlikely that we generate more than $n/2$ short cycles, we let $p = n^{t-1}$, where $t < 1/g$. Each of the possible cycles of length j occurs with probability p^j. Since there are $n_{(j)}/(2j)$ of these for each j, the total number X of cycles of length less than g has expectation

$$E(X) = \sum_{i=3}^{g-1} n_{(i)} p^i/(2i) \le \sum_{i=3}^{g-1} n^{ti}/(2i).$$

Since $tg < 1$, this implies that $E(X)/n \to 0$ as $n \to \infty$. In Markov's Inequality we will complete the details of concluding from this that $P(X \ge n/2) \to 0$ as $n \to \infty$. For n large enough, $P(X \ge n/2) < 1/2$.

Since $\alpha(G)$ cannot grow when we delete vertices, at least $(n - -X)/\alpha(G)$ independent sets are needed to color the vertices remaining when we delete a vertex of each cycle. If $X < n/2$ and $\alpha(G) \le n/(2k)$, then at least k colors are needed for the graph remaining. With $r = \lceil 3 \ln n/p \rceil$, we have

$$P(\alpha(G) \ge r) \le \binom{n}{r}(1-p)^{\binom{r}{2}} < [ne^{-p(r-1)/2}]^r.$$

This approaches 0 as n grows.

Since $r = \lceil 3n^{1-t} \ln n \rceil$ and k is fixed, we can choose n large enough to obtain $r < n/(2k)$. If we also choose n large enough so that $P(X \ge n/2) < 1/2$ and $P(\alpha(G) \ge r) < 1/2$, then there will exist an n-vertex graph G such that $\alpha(G) \le n/(2k)$ and such that G has fewer than $n/2$ cycles of length less than g. We delete a vertex from each short cycle and retain a graph with girth at least g and chromatic number at least k. ■

PROPERTIES OF ALMOST ALL GRAPHS

We have proposed studying properties that "almost always" hold. This phrase has meaning in the context of a probability model.

8.5.12. Definition. Given a sequence of probability spaces, let q_n be the probability that property Q holds in the nth space. Property Q **almost always** holds if $\lim_{n\to\infty} q_n = 1$.

For us, the nth space is a probability distribution over n-vertex graphs. When property Q almost always holds, we say "almost every graph has property Q". Making all graphs with vertex set $[n]$ equally likely is equivalent to letting each vertex pair appear as an edge with probability $1/2$. Models where edges arise independently with the same probability are the most common for random graphs because they lead to the simplest computations. We allow this probability to depend on n.

8.5.13. Definition. Model A: Given n and $p = p(n)$, generate graphs with vertex set $[n]$ by letting each pair be an edge with probability p, independently. Each graph with m edges has probability $p^m(1-p)^{\binom{n}{2}-m}$. The random variable G^p denotes a graph drawn from this probability space. "*The* random graph" means Model A with $p = 1/2$, which makes all graphs with vertex set $[n]$ equally likely.

Computations are much simpler for graphs with a fixed vertex set ("labeled" graphs) than for random isomorphism classes. Since inputs to algorithms are graphs with specified vertex sets, this model is consistent with applications.

We often measure running times of algorithms in terms of the number of vertices and number of edges; hence we may want to control the number of edges. This suggests a model in which the n-vertex labeled graphs with m edges are equally likely. (We use m to count edges in this section because the number $e = 2.71828...$ plays an important role in asymptotic arguments.)

8.5.14. Definition. Model B: Given n and $m = m(n)$, let each graph with vertex set $[n]$ and m edges occur with probability $\binom{N}{m}^{-1}$, where $N = \binom{n}{2}$. The random variable G^m denotes a graph generated in this way.

These two are the most common of many models studied. Model B seems more pertinent for applications. We ask questions like "as a function of n, how many edges are needed to make a graph almost surely connected?" In Model A we would say, "as a function of n, what edge probability is needed to make a graph almost surely connected?" Unfortunately, calculations needed to answer such questions are messier in Model B than in Model A.

Fortunately, Model B is accurately described by Model A when n is large and $p = m/\binom{n}{2}$, because the actual number of edges generated in Model A is almost always very close to the resulting expectation m. The correspondence is valid for most properties of interest. The proof of this requires detailed use of the binomial distribution for the number of edges. A graph property Q is **convex** if G satisfies Q whenever $F \subseteq G \subseteq H$ and F, H satisfy Q.

8.5.15. Theorem. (Bollobás [1985, p34-35]) If Q is convex and $p(1-p)\binom{n}{2} \to \infty$, then almost every G^p satisfies Q if and only if, for every fixed x, almost every G^m satisfies Q, where $m = \lfloor p\binom{n}{2} + x[p(1-p)\binom{n}{2}]^{1/2} \rfloor$. ■

Theorem 8.5.15 justifies restricting our attention to Model A. It also motivates letting p be a function of n; to study graphs with a linear number of edges, we must let p vanish at a rate like c/n, where c is constant. Constant p yields dense graphs.

Proving $P(Q) \to 1$ is usually much easier than computing $P(Q)$; this distinction is important. Exact computation of probabilities is difficult, unnecessary, and avoided wherever possible. Instead we use asymptotic analysis, which rests on limits. We write $a_n \to L$ for $\lim_{n\to\infty} a_n = L$. To compare growth rates of sequences, we use "big O" and "little o" notation (see Appendix B for definitions). We write $a_n = b_n(1+o(1))$ when $\langle a \rangle$ and $\langle b \rangle$ differ by a sequence that grows more slowly than $\langle b \rangle$; equivalently, $a_n/b_n \to 1$. When $a_n/b_n \to 1$, we say that a_n is **asymptotic** to b_n, written $a_n \sim b_n$.

We use asymptotic statements to discard lower-order terms that don't affect whether $\lim_{n\to\infty} P(Q) = 1$. Computing $P(Q)$ first and then proving that the formula tends to 1 is harder and is unnecessary. We need only show that $P(\neg Q)$ is *bounded* by something tending to 0. Many asymptotic arguments are "sloppy" in this sense; we don't care how loose the bound is as long as it tends to 0. Experience refines our intuition about what can be discarded safely.

8.5.16. Theorem. (Gilbert [1959]) When p is constant, almost every G^p is connected.

Proof: We can make G disconnected by picking a vertex partition into two sets and forbidding edges between the two sets. Occurrence of edges within the sets is irrelevant. We bound the probability q_n that G^p is disconnected by summing $P([S,\overline{S}] = \emptyset)$ over all bipartitions $S, \overline{S}$. Graphs with many components are counted many times. When $|S| = k$, there are $k(n-k)$ possible edges in $[S, \overline{S}]$. Each has probability $1-p$ of not appearing, independently, so $P([S,\overline{S}] = \emptyset) = (1-p)^{k(n-k)}$. Considering all S generates each partition from each side, so $q_n \le \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(1-p)^{k(n-k)}$.

This formula is symmetric in k and $n-k$; hence q_n is bounded by $\sum_{k=1}^{\lfloor n/2 \rfloor}\binom{n}{k}(1-p)^{k(n-k)}$. We loosen the bound to simplify it. Using $\binom{n}{k} < n^k$ and $(1-p)^{n-k} \le (1-p)^{n/2}$ (for $k \le n/2$) yields $q_n < \sum_{k=1}^{\lfloor n/2 \rfloor}(n(1-p)^{n/2})^k$. For large enough n, we have $n(1-p)^{n/2} < 1$. This makes our bound the initial portion of a convergent geometric series. We obtain $q_n < x/(1-x)$, where $x = n(1-p)^{n/2}$. Since $n(1-p)^{n/2} \to 0$ when p is constant, our bound on q_n approaches 0 as $n \to \infty$. ■

We avoid struggling with probability formulas by introducing integer-valued random variables and techniques involving expectation. If X is a nonnegative random variable such that $X = 0$ when G^p has property Q, then $E(X) \to 0$ implies that almost every G^p satisfies Q. This is a special case of the following lemma. We prove it only for integer variables, but it also holds for continuous variables.

8.5.17. Lemma. (Markov's Inequality) If X takes only nonnegative values, then $P(X \geq t) \leq E(X)/t$. In particular, if X is integer-valued, then $E(X) \to 0$ implies $P(X = 0) \to 1$.

Proof: $E(X) = \sum_{k\geq 0} kp_k \geq \sum_{k\geq t} kp_k \geq t \sum_{k\geq t} p_k = tP(X \geq t)$. ■

For connectedness, we can define $X(G^p)$ by $X = 1$ if G is disconnected and $X = 0$ otherwise. The expectation of an indicator variable is the probability that it equals 1. We proved $P(X = 1) \to 0$ (when p is constant) to prove that almost every G^p is connected. With a different random variable we can simplify the proof and strengthen the result. We still want G to satisfy Q if $X = 0$ (in order to apply Markov's Inequality), but we don't need $(X = 0) \Leftrightarrow (G$ satisfies $Q)$. We use a sum X of many indicator variables, such that G satisfies Q if $X = 0$. The linearity of expectation and convenience of $E(X_i) = P(X_i = 1)$ for the indicator variables simplify the task of proving $E(X) \to 0$.

8.5.18. Theorem. If p is constant, then almost every G_p has diameter 2 (and hence is connected).

Proof: Let $X(G^p)$ be the number of unordered vertex pairs with no common neighbor. If there are none, then G_p is connected and has diameter 2. By Markov's Inequality, we need only show $E(X) \to 0$. We express X as the sum of $\binom{n}{2}$ indicator variables $X_{i,j}$, one for each vertex pair $\{v_i, v_j\}$, where $X_{i,j} = 1$ if and only if v_i, v_j have no common neighbor.

When $X_{i,j} = 1$, the $n-2$ other vertices fail to have edges to both of these, so $P(X_{i,j} = 1) = (1-p^2)^{n-2}$ and $E(X) = \binom{n}{2}(1-p^2)^{n-2}$. When p is fixed, $E(X) \to 0$, and hence almost every G_p has diameter 2. ■

The intuition behind this argument, made precise by Markov's Inequality, is that if we expect almost no bad pairs, then almost every graph has none. The summation disappears, and for the limit we need only know that $(1 - p^2)^{n-2}$ tends to 0 faster than any polynomial function of n.

THRESHOLD FUNCTIONS

Roughly speaking, random graphs with constant edge probability are connected because they have many more edges than needed to be connected. To improve Theorem 8.5.18, we want to make $p(n)$ as small as possible to have almost every G^p connected. We need the notion of a threshold probability function. By the relationship between Model A and Model B, a threshold edge probability also yields a threshold number of edges.

8.5.19. Definition. A **monotone property** is a graph property preserved by addition of edges. A **threshold probability function** for a monotone property Q is a function $t(n)$ such that $p(n)/t(n) \to 0$ implies that almost no G^p satisfies Q, and $p(n)/t(n) \to \infty$ implies that almost every G^p satisfies Q. **Threshold edge function** is defined similarly for Model B.

This is a broad notion of threshold function; it allows a property to have many threshold functions. A threshold function $t(n)$ is "sharper" when the "almost surely" behavior occurs when $p(n)/t(n)$ approaches nonzero constants. Still sharper is a threshold $t(n)$ such that this behavior occurs when $p(n)$ differs from $t(n)$ by the subtraction or addition of a lower-order term.

Markov's Inequality does half the job of deriving a threshold function. If $X = 0$ implies property Q and we prove that $E(X) \to 0$, then $P(Q) \to 1$. We obtain candidates for threshold functions by determining which functions $p(n)$ yield $E(X) \to 0$. Often we obtain $p(n)$ such that $E(X) \to 0$ or $E(X) \to \infty$, depending on the value of a parameter c. The property $E(X) \to \infty$ suggests that $P(X = 0) \to 0$, but this does not always follow. For example, $E(X) \to \infty$ when $P(X = 0) = .5$ and $P(X = n) = .5$. To obtain $P(X = 0) \to 0$, we must prevent the probability from spreading out like this.

8.5.20. Definition. The rth **moment** of X is the expectation of X^r. The **variance** of X, written $Var(X)$, is the quantity $E[(X - E(X))^2]$. The **standard deviation** of X is the square root of $Var(X)$.

8.5.21. Lemma. (Second Moment Method) If X is a random variable, then $P(X = 0) \le \frac{E(X^2)-E(X)^2}{E(X)^2}$. In particular, $P(X = 0) \to 0$ when $\frac{E(X^2)}{E(X)^2} \to 1$.

Proof: Applied to the variable $(X - E(X))^2$ and the value t^2, Markov's Inequality yields $P[(X - E(X))^2 \ge t^2] \le E[(X - E(X))^2]/t^2$. We rewrite this as $P[|X - E(X)| \ge t] \le Var(X)/t^2$ (Chebyshev's Inequality). Since

$$E[(X - E(X))^2] = E[X^2 - 2XE(X) + (E(X))^2] = E(X^2) - (E(X))^2,$$

Chebyshev's Inequality becomes $P[|X - E(X)| \ge t] \le (E(X^2) - E(X)^2)/t^2$. Since $X = 0$ only when $|X - E(X)| \ge E(X)$, setting $t = E(X)$ completes the proof. ∎

Intuitively, if the mean grows and the standard deviation grows more slowly, then all the probability is pulled away from 0, and $P(X = 0) \to 0$ results. We illustrate the method by considering the disappearance of isolated vertices. Since a connected graph has no isolated vertices, a threshold for connectedness must be at least as large as a threshold for disappearance of isolated vertices. The computations for the latter are simpler, because we can express this condition using a sum of identically distributed indicator variables with easily computed expectations. In fact, both properties have the same threshold, since it happens that at the threshold almost every graph consists of one huge component plus isolated vertices.

8.5.22. Theorem. In Model A, $\ln n/n$ (natural logarithm) is a threshold probability function for the disappearance of isolated vertices (that is, $\delta(G) \geq 1$). (The corresponding threshold in Model B is $\frac{1}{2}n\ln n$.)

Proof: Let X be the number of isolated vertices, with X_i indicating whether vertex i is isolated. Then $E(X) = \sum E(X_i) = n(1-p)^{n-1}$. We study the asymptotic behavior of $E(X)$ in terms of $p(n)$. Since

$$(1-p)^n = e^{n\ln(1-p)} = e^{-np}e^{-np^2[1/2+p/3+\cdots]},$$

our expression for $E(X)$ simplifies asymptotically if $np^2 \to 0$. This is equivalent to $p \in o(1/\sqrt{n})$ and implies $(1-p)^n \sim e^{-np}$ and $(1-p)^{-1} \sim 1$, yielding $E(X) \sim ne^{-np}$. To simplify further, set $p = c\ln n/n$ to obtain $ne^{-np} = n^{1-c}$, where c may depend on n. Constant c yields $p \in o(1/\sqrt{n})$, as we needed earlier. When $c > 1$, we have $E(X) \sim n^{1-c} \to 0$, which proves one side of the threshold.

When $c < 1$, we have $E(X) \to \infty$ and use the second moment method. We need only show that $E(X^2) \sim E(X)^2$. This uses another helpful property of indicator variables: $X_i^2 = X_i$. Thus,

$$E(X^2) = \textstyle\sum_{i=1}^n E(X_i^2) + \sum_{i\neq j} E(X_iX_j) = E(X) + n(n-1)E(X_iX_j).$$

The indicator variable X_iX_j has value 1 only when v_i and v_j are both isolated, which forbids $2(n-2)+1$ edges. Thus $E(X_iX_j) = (1-p)^{2n-3}$. Again $(1-p)^n \sim e^{-np}$, so $E(X_iX_j) \sim e^{-2np}$, and

$$E(X^2) \sim E(X) + n(n-1)e^{-2np} \sim E(X) + E(X)^2.$$

Since $E(X) \to \infty$, this yields $E(X^2) \sim E(X)^2$. ∎

Theorem 8.5.22 is stronger than required by the definition of threshold function. The threshold is sharper: we guarantee or forbid isolated vertices when the ratio of $p(n)$ to $\ln n/n$ approaches a nonzero constant, not 0 or ∞.

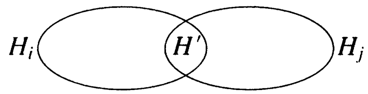
In fact, yet sharper information is known about the threshold for isolated vertices. When $p = \lg n/n + x/n$ and X counts the isolated vertices, $P(X = k) \sim e^{-\mu}\mu^k/k!$, where $\mu = e^{-x}$. (Readers may recognize this limiting distribution as the **Poisson distribution**.) For $k = 0$, we have $P(X = 0) \sim e^{-\mu}$. Thus this additive term in p describes the movement through the threshold from almost always isolated vertices to almost never isolated vertices. Many such sharp thresholds are known, but the techniques for deriving the asymptotic Poisson distribution are beyond our scope here.

Next we derive a threshold function for the appearance of fixed subgraphs. A graph is **balanced** if the average vertex degree in every induced subgraph is no larger than the average degree of the entire graph. All regular graphs and all forests are balanced.

8.5.23. Theorem. If H is a balanced graph with k vertices and l edges, then $p = n^{-k/l}$ is a threshold function in Model A for the appearance of H as a subgraph of almost every G^p.

Proof: Let X be the number of copies of H in G^p; X is the sum of indicator variables for the possible copies of H in K_n. There are $n(n-1)\cdots(n-k+1)$ ways to map $V(H)$ into $[n]$. Each copy of H arises A times, where A is the number of automorphisms of H. We thus have $\frac{1}{A}\prod_{j=0}^{k-1}(n-j)$ variables X_i. Since a copy of H occurs when its edges occur, $P(X_i = 1) = p^l$. Because k is fixed, $E(X) \sim n^k p^l/A$.

Setting $p(n) = c_n n^{-k/l}$ yields $E(X) \sim c_n^l/A$. Hence $c_n \to 0$ yields $E(X) \to 0$, and $c_n \to \infty$ yields $E(X) \to \infty$. It remains only to obtain $E(X^2) \sim E(X)^2$ when $c_n \to \infty$. Again $E(X^2) = E(X) + \sum_{i \neq j} E(X_i X_j)$. The summands are not equal; $E(X_i X_j)$ depends on $H' = H_i \cap H_j$. We group the terms by the choice of $H' \subseteq H$. When H' has r vertices and s edges, the number of edges needed to create H_i and H_j is $2l - s$, so $E(X_i X_j) = p^{2l-s}$.



To specify pairs i, j such that $H' = H_i \cap H_j$, we choose r vertices for H', $k - r$ vertices for each of $H_i - H'$ and $H_j - H'$, and an extension of H' to each of those sets. The number of ways to choose the vertex sets is $\frac{n!}{r!(k-r)!(k-r)!(n-2k+r)!}$, which is asymptotic to $n^{2k-r}/[r!(k-r)!^2]$. The number M of ways to extend H' to obtain copies of H in both specified k-sets depends only on H and H'; it is independent of n and p. Let $\alpha_{H'}$ be the constant $M/[r!(k-r)!^2]$. The contribution to $\sum E(X_i X_j)$ from pairs i, j such that $H_i \cap H_j = H'$ is asymptotic to $\alpha_{H'} n^{2k-r} p^{2l-s}$; we call this $E_{H'}$.

When $r = s = 0$, we have $M = (k!/A)^2$. Hence $\alpha_{H'} \sim n^{2k} p^{2l}/A^2 \sim E(X)^2$ when H' is the "null graph". This is the total contribution to $\sum E(X_i X_j)$ for all i, j with H_i, H_j disjoint and is asymptotic to $E(X)^2$. The proof is completed by showing that the total contribution from all other choices of H' has lower order. We have $E_{H'} \sim \alpha_{H'} A^2 E(X)^2 n^{-r} p^{-s}$. Since $2s/r$ is the average degree of H', the hypothesis that H is balanced yields $2r/s \geq 2k/l$, or $pn^{r/s} \geq pn^{k/l} \to \infty$ when $c_n \to \infty$. Since $pn^{r/s} \to \infty$ is equivalent to $n^{-r}p^{-s} \to 0$, we obtain $E_{H'} \in o(E(X)^2)$ for $H' \neq \emptyset$. Since the number of possible subgraphs H' is bounded (by an expression involving the constants k and l), this implies that $E(X^2) \sim E(X) + E_{\emptyset} \sim E(X)^2$. ∎

This result generalizes for all H. The ratio $d(H) = e(H)/n(H)$ is the **density** of H, and $\rho(H) = \max_{F \subseteq H} d(F)$ is the **maximum density**. These are equal precisely when H is balanced, and then $p = n^{-1/\rho(H)}$ is the threshold for appearance of H. Every graph H has a balanced subgraph F such that $d(F) = \rho(H)$. When $pn^{\rho(H)} \to 0$, almost every G^p has no copy of F; hence it also has no copy of H. In fact, $p = n^{-1/\rho(H)}$ is always a threshold function for the appearance of H (Exercise 25).

EVOLUTION AND GRAPH PARAMETERS

In the subtitle to his book, Palmer [1985] tells us that random graphs involve the study of

> "THRESHOLD FUNCTIONS, which facilitate the careful study of the structure of a graph as it grows, and specifically reveal the mysterious circumstances surrounding the abrupt appearance of the UNIQUE GIANT COMPONENT, which systematically absorbs its neighbors, devouring the larger first and ruthlessly continuing until the last ISOLATED VERTICES have been sucked up, whereupon the Giant is suddenly brought under control by a SPANNING CYCLE."

The evolutionary viewpoint generates random graphs with m edges in a way that yields the same probability space as Model B but makes intuitive reasoning easier. Almost everything suggested about random graphs by intuition or experimentation is true. The evolutionary viewpoint develops this intuition.

Generating m edges simultaneously or one-by-one yields the same probability distribution, making the graphs with m edges equally likely. By studying the likely effect of a new edge on the present structure, we can make intuitive hypotheses about the properties of the graph at any stage. A *stage* of evolution is a range of values for $m(n)$ (or $p(n)$) in which the structural description of the typical graph doesn't change much. We have studied the basic techniques for verifying these descriptions, but the computations can be difficult. Hence we will only describe the stages using the evolutionary intuition.

We remark first that a constant multiple of almost nothing is almost nothing. Therefore, when each of $A_1, \dots, A_r$ happens almost always (r is fixed), it follows that almost always they all happen.

Beginning with many vertices and no edges, each new edge is likely to be isolated. The random graph is a matching until a substantial fraction of the vertices are involved in edges. The thresholds $p \sim cn^{-k/(k-1)}$ for appearance of fixed subtrees generalize this. Let $t_k(n) = n^{-k/(k-1)}$. If $p/t_k \to \infty$ but $p/t_{k+1} \to 0$, then every fixed subtree on k vertices appears, but none on $k+1$ vertices appears. (The statements about individual trees become statements about all trees of that order.) Furthermore, this p is also below the threshold for appearance of fixed cycles (density 1, length bounded by k), so G^p is a forest of trees of order at most k, and every tree on k vertices appears as a component.

Intuitively, the random graph has no cycles in this stage of evolution because when there is no large component a random added edge is much more likely to join two components than to lie in one component. To make the intuition precise, we let X be the number of cycles in G^p and compute

$$E(X) = \sum_{k=3}^{n} \binom{n}{k} \frac{1}{2}(k-1)!p^k < \sum_{k=3}^{n} (np)^k/2k.$$

If $pn \to 0$, then $E(X) \to 0$.

The next major stage of evolution is $p = c/n$ with $0 < c < 1$. With X counting cycles, we can no longer say that $E(X) \sim \sum_{k=3}^{n} (np)^k/2k$, because when

k is a substantial fraction of n the ratio $n^k/(n)_k$ does not approach 1. We must break $E(X)$ into two sums, and the arguments become more difficult. When $pn \to c$, we find that $E(X)$ approaches a constant c', and the number of cycles in G^p is asymptotically Poisson distributed. With cycles in a few components and all components small, we still expect the next edge to join two components or create a cycle in a component that doesn't have one. In this range, the size of the largest component is about $\log n$, there are many components, each having at most one cycle. Most vertices still belong to acyclic components.

When c reaches and passes 1, the structure of G^p changes radically. This is called the **double jump** because the structure of G^p is significantly different for $c < 1$, $c \sim 1$, and $c > 1$. At $pn = 1$, the second moment method guarantees that almost every G^p has a cycle. Also, the order of the largest component jumps from $\log n$ to $n^{2/3}$. For $pn = c > 1$, the number of vertices outside the "giant component" becomes $o(n)$. Also G^p is likely to have some cycle with three crossing chords and be nonplanar.

Next, let p approach $c \ln n/n$. With $c < 1$, we have proved that almost every G^p has isolated vertices. With $c > 1$, these disappear. As we add edges to a disconnected graph, the edges may go within a component or connect two components. When the components are all small, added edges will almost surely join components. Eventually, this results in the creation of a giant component. At this point, added edges are likely to lie within the giant component or to join it to one of the small components. Of the small components, those most likely to receive such edges are the larger ones. In other words, as c passes through 1 the last remaining small components swallowed by the giant component are isolated vertices. This explains intuitively why the threshold for connectedness is the same as the threshold for the disappearance of isolated vertices. With $c > 1$, suddenly almost every G^p also has a spanning cycle. Minimum degree k (and the appearance of the Hamiltonian cycle when $k = 2$) has a threshold that involves a lower-order term: $\ln n/n + (k-1)\ln\ln n/n$.

The last stages of evolution are those where $pn/\ln n \to \infty$ but $p = o(1)$, and then finally $p = c$; this brings us back to where we began our study.

When $p = c\log n/n$ with $c \to \infty$, we leave the domain of sparse graphs. The evolutionary viewpoint becomes less valuable, and we study properties of the random graph. We pay less attention to probability threshold functions and concentrate on the likely value of graph parameters, especially when p is constant. Given a parameter μ, we want to show that $\mu(G^p) \sim f(n)$ for almost every G^p. We can view this as a threshold when $\mu(G^p)$ is almost always between $(1-\epsilon)f(n)$ and $(1+\epsilon)f(n)$, for each $\epsilon > 0$. If $\mu(G^p)$ is almost always between $f(n) - \epsilon g(n)$ and $f(n) + \epsilon g(n)$, where $g(n) = o(f(n))$, then we have a stronger statement, written as $\mu(G^p) \in f(n)(1 + o(1))$.

Some properties that are true for almost all graphs occur in no known examples! For the known lower bound on Ramsey numbers, there is still no construction of an infinite class of graphs such that $\alpha(G) < \log_{\sqrt{2}}(n(G))$ and $\omega(G) < \log_{\sqrt{2}}(n(G))$, even though almost all graphs have this property.

Properties of the random graph can lead to a fast algorithm that solves a difficult problem on almost all inputs. For example, after stating two results about vertex degrees in random graphs, we show how to use properties of the degree sequence to design a fast algorithm to test isomorphism "almost always". In the literature of random graphs, ω_n denotes a function that is unbounded but grows arbitrarily slowly.

8.5.24. Theorem. (Erdős–Rényi [1966]) If $p = \omega_n \log n/n$ and $\epsilon > 0$ is fixed, then almost every G^p satisfies

$$(1-\epsilon)pn < \delta(G^p) \le \Delta(G^p) \le (1+\epsilon)pn. \qquad \blacksquare$$

Most vertices have degree near the average, but there is still considerable variation. Bollobás [1982] showed that for $p \le 1/2$, the vertex of maximum degree is unique in almost every G^p if and only if $pn/\log n \to \infty$. When we complete evolution by returning to the realm of constant edge probability, more detailed results are known about the degree distribution. There will almost always be some vertices with isolated high degrees before the degrees begin to bunch up. Bollobás determined how many distinct degrees can be guaranteed.

8.5.25. Theorem. (Bollobás [1981b]) In Model A with p fixed and $t \in o(n/\log n)^{1/4}$, almost every G^p has different degrees for its t vertices of highest degree. If $t \notin o(n/\log n)^{1/4}$, then almost every G^p has $d_i = d_{i+1}$ for some $i < t$. $\blacksquare$

We apply this result to isomorphism testing. No polynomial-time algorithm is known for this problem, but Babai–Erdős–Selkow [1980] used the degree results for the random graph to develop a fast algorithm that almost always works. We define a set **H** that contains almost all graphs and show that isomorphism with a graph in **H** can be tested quickly.

The testing is done by a *canonical labeling algorithm*, which accepts and labels a graph in a canonical way if it belongs to **H**. The desired property is that when vertices are labeled as $v_1, \ldots, v_n$ in one graph and $w_1, \ldots, w_n$ in another, only the bijection mapping v_i to w_i is a possible isomorphism. Isomorphism can then be tested by comparing the adjacency matrices under this labeling.

8.5.26. Corollary. (Babai–Erdős–Selkow [1980]) There is a quadratic algorithm that tests isomorphism for almost all pairs of graphs.

Proof: Given a graph G on n vertices, presented by its adjacency matrix, compute and sort the vertex degrees, labeling the vertices in decreasing order of degree. Fix $r = \lfloor 3 \lg n \rfloor$. If $d(v_i) = d(v_{i+1})$ for any $i < r$, reject G. Using $p = 1/2$ in Theorem 8.5.25 implies that almost every graph successfully passes this test.

Let $U = \{v_1, \ldots, v_r\}$. With $r = \lfloor 3 \lg n \rfloor$, there are about n^3 distinct subsets of the vertices of U. Since only $n-r$ vertices remain outside U, there is a chance that they can be distinguished by their neighborhoods in U. The set **H** will be all the graphs reaching this stage for which this holds: the vertices of $V - U$ have distinct neighborhoods in U. To test this in $O(n^2)$ time and complete the labeling, for each $x \in V - U$ encode $N(x) \cap U$ as a binary r-tuple. Evaluate these as binary integers, and sort them! These steps take $O(n \log n)$ time. Relabel the vertices v_{r+1} to v_n as $w_{r+1}, \ldots, w_n$ in decreasing order of these values. If two consecutive values are the same, reject G.

If G has passed this far, then G has no nontrivial automorphisms. A graph isomorphic to G has only one isomorphism to G, given by applying the canonical labeling algorithm to it. The last stage, if both graphs pass canonical labeling, is to compare the adjacency matrices with rows and columns indexed by the canonical labeling. The graphs are isomorphic if and only if the matrices are now identical. This comparison takes $O(n^2)$ time.

We must show that for almost every G^p, the adjacency vectors within a specified set of r vertices are distinct for the remaining vertices. If $p \le 1/2$, then the probability for any pair x, y that x, y have the same adjacencies in U is bounded approximately by $(1-p)^r$. We say approximately because U is not chosen at random; choosing U as the set of vertices of highest degree impairs randomness, increasing the probability of a specified edge incident to these vertices. Nevertheless, it doesn't change by much, and the expected number of pairs of vertices outside U with identical adjacencies in U is bounded by $O(\binom{n-r}{2}(1-p)^r)$. Given our choice of r, we can bound the base 2 logarithm of this by $2\lg n - 3\lg b \lg n$, where $b = 1/(1-p) \ge 2$ (if $p \le 1/2$). This tends to $-\infty$, so almost all graphs have distinct adjacency vectors in this set. ∎

The probability of rejection in this labeling algorithm is bounded by $n^{-1/7}$ for sufficiently large n. Later improvements led to an algorithm running in time $O(n^2)$ with rejection probability c^{-n} (Babai–Kučera [1979]).

CONNECTIVITY, CLIQUES, AND COLORING

Studying the "typical behavior" of a random structure often involves studying probability distributions of its parameters. Here we consider connectivity, cliques, and colorings for random graphs.

For random graphs, naive algorithms may become good. For example, finding a maximum clique is NP-hard. If we know that almost every graph has clique number about $2\lg n$, then we can test all vertex subsets up to size $3\lg n$ for being cliques. If $\omega(G) < 3\lg n$, then this computes $\omega(G)$, since every set of size $\omega(G)+1$ is not a clique. If $\omega(G) \ge 3\lg n$, then the algorithm fails to compute $\omega(G)$, but this rarely happens. There are too many subsets of size $2\lg n$ for this to be a polynomial-time algorithm, but it's close, and it illustrates one way in which the properties of random graphs can be used algorithmically.

Some NP-hard problems are trivial for random graphs. Although $\Delta(G) \le \chi'(G) \le \Delta(G)+1$ for every simple graph G (Vizing [1964]), deciding between these values is NP-hard (Holyer [1981]). Vizing proved that $\chi'(G) = \Delta(G)+1$ only when G has at least 3 vertices of maximum degree. Thus Erdős and Wilson [1977], who noted the uniqueness of the vertex of maximum degree when $p = 1/2$, also observed that $\chi'(G) = \Delta(G)$ for the random graph.

For sparse graphs and constant k, the thresholds for connectivity k and minimum degree k are the same. Does this also hold for constant edge probability? Theorem 8.5.18 can be generalized and strengthened to show that if $k \in o(n/\log n)$ and p is fixed, then almost every G^p has k common neighbors for every vertex pair and hence is k-connected (Exercise 33). Improving this requires other methods; Bollobás [1981b] showed for constant p that almost every G^p has connectivity equal to minimum degree.

What about clique number? For fixed k, Theorem 8.5.23 yields a probability threshold for the appearance of a k-clique, but for constant p the clique number grows with n. Determining the clique number is NP-complete, but for a random graph we can guess the correct value with high probability without looking at the graph! Amazingly, for fixed p almost every G^p has one of two possible values for the clique number (as a function of n), and for each $k \in \mathbb{N}$ there is a range of n where the clique number almost always equals k. The approach is to find bounds on $r(n)$ such that almost every G^p has an r-clique and almost none has an $r+1$-clique.

8.5.27. Theorem. (Matula [1972]) For fixed $p = 1/b$ and fixed $\epsilon > 0$, almost every G^p has clique number between $\lfloor d - \epsilon \rfloor$ and $\lfloor d + \epsilon \rfloor$, where $d = 2\log_b n - 2\log_b \log_b n + 1 + 2\log_b(e/2)$.

Proof: (sketch) If X_r is the number of r-cliques, then $E(X_r) = \binom{n}{r}p^{\binom{r}{2}}$. Since $r! \sim (r/e)^r\sqrt{2\pi r}$ (Stirling's approximation), also $E(X_r) \sim (2\pi r)^{-1/2}(enr^{-1}p^{(r-1)/2})^r$. If $r \to \infty$ and $(enr^{-1}p^{(r-1)/2}) \le 1$, then we expect that $E(X_r) \to 0$. To determine $r(n)$ such that this holds, take logarithms (base b) in the inequality and solve for r to find

$$r \ge 2\log_b n - 2\log_b r + 1 + 2\log_b e.$$

This is approximately equivalent to $r \ge d(n)$ as defined above. More precisely, if $r > d + \epsilon$, then almost every G^p has no clique of size r.

The lower bound comes from careful application of the second moment method, as in Theorem 8.5.23, but the dependence of r on n makes the analysis more difficult. The expectation of X_r^2 sums the probability of common occurrence for all ordered pairs of r-cliques. This probability depends only on the number of common vertices, so

$$E(X_r^2) = \binom{n}{r}\sum_{k=0}^{r}\binom{r}{k}\binom{n-r}{r-k}p^{2\binom{r}{2}-\binom{k}{2}}.$$

We want to show that the term for $k = 0$ (disjoint cliques) dominates. Let $E(X_r^2)/E(X_r)^2 = \alpha_n + \beta_n$, where $\alpha_n = \binom{n}{r}^{-1}\binom{n-r}{r}$ and $\beta_n = \binom{n}{r}^{-1}\sum_{k=1}^{r}\binom{r}{k}\binom{n-r}{r-k}b^{\binom{k}{2}}$. We seek $\alpha_n \sim 1$ and $\beta_n \to 0$. When $r \sim 2\log_b n$, an asymptotic formula for $\binom{a}{k}/\binom{b}{k}$ leads to $\alpha_n \sim e^{-r^2/(n-r)} \to 1$. The discussion of β_n is more difficult; see Palmer [1985, p75-80]. ■

Our study of graph parameters can be applied to measure the strength of conditions for Hamiltonian cycles (Palmer [1985, p81-85]). A theorem proves nothing if its hypotheses are never satisfied; this suggests saying that such a theorem has strength 0. A theorem is strong if the conclusion is satisfied only when the hypothesis is satisfied; then the hypotheses cannot be weakened. Define the **strength** of a theorem to be the probability that its hypotheses are satisfied divided by the probability that its conclusion is satisfied.

Consider sufficient conditions for Hamiltonian cycles. Since $p = \log n/n$ is a threshold for a Hamiltonian cycle, almost every G^p is Hamiltonian when p is fixed. Dirac [1952b] showed that G is Hamiltonian when every vertex degree is at least $n/2$ (Theorem 7.2.8). When $p > 1/2$, this condition holds for almost every G^p; when $p \le 1/2$, it almost never holds. Hence the asymptotic strength of Dirac's Theorem is 0 when p is a constant at most $1/2$. The same fate befalls the other degree conditions of Section 7.2.

Meanwhile, Chvátal and Erdős [1972] proved that G is Hamiltonian whenever its connectivity exceeds its independence number (Theorem 7.2.19). Our thresholds for these parameters imply that this result is strong for every constant $p > 0$. We know that $\alpha(G^p) < 2(1+\epsilon)\log_b n$ almost always, and we know that $\kappa(G^p) \ge k$ almost always (when $k = o(n/\log n)$). Hence $\kappa > \alpha$ for almost every G^p, and the asymptotic strength of the theorem is 1.

Finally, we consider chromatic number for constant p. Since $1 - p$ is also constant, we can apply the results on clique number: Almost every G^p has no stable set with more than $(1+o(1))2\log_b n$ vertices, where $b = 1/(1-p)$. Hence $\chi(G^p) \ge (1/2 + o(1))n/\log_b n$ almost always. Achieving this bound requires finding many disjoint stable sets with near-maximum sizes. For a decade, the best result was an algorithmic guarantee of a coloring with at most twice the number of colors in the lower bound.

Bollobás [1988] proved that the lower bound is achievable, by using another probabilistic technique that guarantees finding enough large stable sets. He proved that, in almost every G^p, *every* set having at least $n/(\log_b n)^2$ vertices contains a clique of order at least $2\log_b n - 5\log_b \log_b n$. This allows stable sets of near-maximum size to be extracted until too few vertices remain to cause trouble; the remainder can be given distinct colors.

Before developing Bollobás' approach, we present the earlier result for its algorithmic interest; the greedy algorithm uses at most $(1+\epsilon)n/\log_b n$ colors on almost every G^p. Thus it "almost always works" as an approximation algorithm in the same sense that our earlier isomorphism algorithm almost always works. Garey and Johnson [1976] showed there is no fast algorithm that uses at most twice the optimum number of colors on *every* graph unless P = NP. Bollobás' proof does not yield a fast algorithm for coloring almost every graph with an asymptotically optimal number of colors; it is an existence proof only.

8.5.28. Theorem. (Grimmett–McDiarmid [1975]) Given edge probability p, let $b = 1/(1-p)$. For constant p and constant $\epsilon > 0$, almost every G^p satisfies

$$(1/2 - \epsilon)n/\log_b n \le \chi(G^p) \le (1+\epsilon)n/\log_b n.$$

Proof: The lower bound follows using stable sets as suggested above. For the upper bound, we show that the greedy coloring of $v_1, \ldots, v_n$ in order uses at most $f(n) = (1+\epsilon)n/\log_b n$ colors on almost every G^p (for simplicity, choose ϵ so that $f(n)$ is an integer). Within the set of n-vertex graphs using more colors, let $\mathbf{B}_m$ be the set such that v_m is the first vertex to use color $f_n + 1$. We prove that $\sum_{m=1}^{n} P(\mathbf{B}_m) \to 0$ as $n \to \infty$.

Given G, let $G_m = G[\{v_1, \ldots, v_{m-1}\}]$. Before color $f_n + 1$ is used, color f_n must be used, so for each $G \in \mathbf{B}_m$ the greedy coloring of G_m uses f_n colors. Let k_i be the number of times color i appears in this coloring. To require use of color $f_n + 1$, v_{m+1} must have at least one neighbor of each color $1, \ldots, f_n$. Given the numbers $\{k_i\}$, the probability of this is $\prod_{i=1}^{f(n)}[1-(1-p)^{k_i}]$.

Bollobás and Erdős [1976] simplified the subsequent computations involving this bound by observing that the bound is maximized when the k_i's are all equal (Exercise 8.3.37). Thus

$$\prod_{i=1}^{f(n)}[1-(1-p)^{k_i}] \le [1-(1-p)^{(m-1)/f}]^f < [1-(1-p)^{n/f}]^f.$$

Given G_m, we have $b_n = [1-(1-p)^{n/f(n)}]^{f(n)}$ as a bound on the probability that the full graph G belongs to $\mathbf{B}_m$. Since this holds for each G_m, we conclude that $P(\mathbf{B}_m) < b_n$. This holds for all m, so $\sum_{m=1}^{n} P(\mathbf{B}_m) < nb_n$.

Using $(1-p)^{-x} < e^{-x}$, we obtain $nb_n < ne^{-f(1-p)^{n/f}}$. Substituting $f_n = cn/\log_b n$ yields $(1-p)^{n/f} = n^{-1/c}$. The logarithm of the bound becomes $\log n - cn^{1-1/c}/\log_b n$. This tends to $-\infty$ for $c > 1$, so the probability that the greedy algorithm uses more than $f(n)$ colors is bounded by a function tending to 0. ∎

The order of growth of $\chi(G)$ sheds light on other famous problems in graph theory. Hajós conjectured that every r-chromatic graph contains a subdivision of K_r (see Remark 5.2.21). This was disproved by Catlin [1979] (Exercise 5.2.40). Erdős and Fajtlowicz [1981] observed that the chromatic number of G^p almost always grows like $\Theta(n/\log n)$). On the other hand, the largest r such that G^p contains a subdivision of K_r grows like $\Theta(\sqrt{n})$). Thus the chromatic number is almost always much larger, and Hajós' Conjecture is almost always very false.

In contrast, almost every G^p has a subgraph contractible to K_r when $r \in \Theta(n/\sqrt{\log n})$. Thus almost every graph satisfies the weaker conjecture of Hadwiger (Remark 5.2.21), which states that every r-chromatic graph has a subgraph contractible to K_r.

MARTINGALES

Advanced techniques in probability lead to elegant results on combinatorial structures without the drudgery involved in second moment and higher moment computations. The theory aims to develop paradigms that can be applied without repeating computational details.

Some of these methods employ lists of related random variables. The resulting stochastic process displays more consistent and predictable global behavior than the individual random variables do.

In the classical random walk on a line, at each step there is probability p of moving one unit to the left, probability p of moving one unit to the right, and probability $1-2p$ of not moving. No matter what the earlier history of the walk has been, the expected position after t steps equals the actual position after $t-1$ steps. This is the defining property of a martingale.

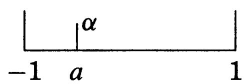
8.5.29. Definition. A **martingale** is a list of random variables $X_0, \dots, X_n$ such that the expectation of X_i, given the values of $X_0, \dots, X_{i-1}$, equals X_{i-1}.

The expected position of the random walk after n steps is at the origin. Less obvious is that the walk is highly unlikely to be very far from the origin, as a function of n. We shall see that this follows from its inability to move more than one unit in each step.

Martingales can make it easy to show that a random variable is highly concentrated around its expected value. When the technique applies, it makes the detailed computation in the Second Moment Method unnecessary. The hard work is accomplished by Azuma's Inequality, also called the Martingale Tail Inequality. This inequality states that if successive random variables in a martingale always differ by at most 1, then the probability that $X_n - X_0$ exceeds $\lambda\sqrt{n}$ is bounded by $e^{-\lambda^2/2}$. We first prove two lemmas. These statements hold for continuous random variables, but again we consider only discrete variables.

8.5.30. Lemma. Let Y be a random variable such that $E(Y) = 0$ and $|Y| \le 1$. If f is a convex function on $[-1, 1]$, then $E(f(Y)) \le \frac{1}{2}[f(-1) + f(1)]$. In particular, $E(e^{tY}) \le \frac{1}{2}(e^t + e^{-t})$ for all $t > 0$.

Proof: When Y takes only the values ± 1, each with probability .5, we have $E(f(Y)) = \frac{1}{2}[f(-1) + f(1)]$. For other distributions, pushing probability "out to the edges" increases $E(f(Y))$. For discrete variables, we can use induction on the number of values with nonzero probability. Convexity implies that $f(a) \le \frac{1-a}{2} f(-1) + \frac{a+1}{2} f(1)$. If $P(Y = a) = \alpha$, then we can decrease the probability at a to 0, increase $P(Y = -1)$ by $\alpha\frac{1-a}{2}$ and increase $P(Y = 1)$ by $\alpha\frac{a+1}{2}$ to obtain a new variable Y' with the same expectation. By the convexity inequality and the induction hypothesis, $E(f(Y)) \le E(f(Y')) \le \frac{1}{2}[f(-1) + f(1)]$. ∎



8.5.31. Definition. For events A and B, the **conditional probability** of A *given* B is obtained by treating the event B as the full probability space, which means normalizing by $P(B)$. Thus we define $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$.

When Y, X are random variables, we write $Y|X$ for "Y *given* X". This defines a random variables for each value of X; we treat X as a constant i and normalize the resulting distribution for Y by $P(X = i)$.

For Azuma's Inequality, we use expectation of conditional variables. For each i, we compute the expected value of Y when restricted to the sample points where $X = i$. The expectation $E(E(Y|X))$ is the expectation of $E(Y|X = i)$ over the choices for i, which occur with probability $P(X = i)$. The result is an expectation over the entire sample space. It removes the effect of conditioning, and we obtain $E(E(Y|X)) = E(Y)$.

8.5.32. Lemma. $E(E(Y|X)) = E(Y)$.

Proof: Let $p_{i,j} = P(X = i \text{ and } Y = j)$. Since $E(Y|X = i) = \frac{\sum_j j p_{i,j}}{P(X=i)}$,

$$E(E(Y|X)) = \sum_i E(Y|X = i)P(X = i) = \sum_i \sum_j j p_{i,j} = E(Y). \qquad \blacksquare$$

8.5.33. Theorem. (Azuma's Inequality) If $X_0, \dots, X_n$ is a martingale with $|X_i - X_{i-1}| \le 1$, then $P(X_n - X_0 \ge \lambda\sqrt{n}) \le e^{-\lambda^2/2}$.

Proof: By translation, we may assume that $X_0 = 0$. For $t > 0$, we have $X_n \ge \lambda\sqrt{n}$ if and only if $e^{tX_n} \ge e^{t\lambda\sqrt{n}}$, and hence $P(X_n \ge \lambda\sqrt{n}) = P(e^{tX_n} \ge e^{t\lambda\sqrt{n}})$. Applied to e^{tX_n}, Markov's Inequality yields $P(e^{tX_n} \ge e^{t\lambda\sqrt{n}}) \le E(e^{tX_n})/e^{\lambda t\sqrt{n}}$. This bound holds for each $t > 0$, and later we will choose t to minimize the bound.

First we prove by induction on n that $E(e^{tX_n}) \le \frac{1}{2}(e^t + e^{-t})$. We introduce X_{n-1} to condition on it. Lemma 8.5.32 yields

$$E(e^{tX_n}) = E(e^{tX_{n-1}}e^{t(X_n - X_{n-1})}) = E(E(e^{tX_{n-1}}e^{t(X_n - X_{n-1})}|X_{n-1})).$$

When we condition on X_{n-1}, the value of X_{n-1} is constant for the inner expectation. Hence we can remove $e^{tX_{n-1}}$ from the inner expectation to obtain $E(e^{tX_n}) = E(e^{tX_{n-1}}E(e^{tY}|X_{n-1}))$, where $Y = X_n - X_{n-1}$. Because $\{X_n\}$ is a martingale, $E(Y) = 0$, and by hypothesis $|Y| \le 1$. Hence Lemma 8.5.30 applies, yielding $E(e^{tY}|X_{n-1}) \le \frac{1}{2}(e^t + e^{-t})$. This itself is now a constant, yielding $E(e^{tX_n}) = \frac{1}{2}(e^t + e^{-t})E(e^{tX_{n-1}})$. The induction hypothesis completes the proof.

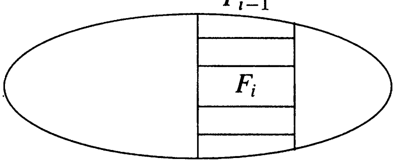
We weaken the bound to a more useful form by observing that $\frac{1}{2}(e^t + e^{-t}) \le e^{t^2/2}$. This holds because the left side is $\sum t^{2k}/(2k)!$ and the right side is $\sum t^{2k}/(2^k k!)$. Hence our original probability is bounded by $e^{nt^2/2 - \lambda t\sqrt{n}}$ for each $t > 0$. We obtain the best bound by minimizing over t. The exponent is quadratic; we minimize it by choosing t to solve $tn - \lambda\sqrt{n} = 0$, or $t = \lambda/\sqrt{n}$. The resulting bound is $e^{-\lambda^2/2}$. $\blacksquare$

Azuma's Inequality is one-sided; it bounds the probability that X_n is much larger than X_0. Since the conditions are symmetric in sign, applying the inequality to $\{-X_i\}$ yields the same inequality for the other tail, in which X_n is much smaller than X_0.

8.5.34. Example. *The pragmatic gambler.* A gambler can bet up to n times, where n is fixed. Each time he bets, he wins or loses 1 with equal probability.

His goal is winning $\lambda\sqrt{n}$, so he stops if he reaches that value. Letting X_i be his winnings after i games, we have $X_i = X_{i-1}$ if $X_{i-1} \geq \lambda\sqrt{n}$, and otherwise $X_i = X_{i-1} \pm 1$, each with probability .5. Hence $\{X_i\}$ is a martingale that changes by at most 1 at each step, and Azuma's Inequality applies. The probability that the gambler will earn $\lambda\sqrt{n}$ is bounded by $e^{-\lambda^2/2}$. If $\lambda = 1$, then there may be a reasonable chance of success, but if $\lambda = 10$, then there is little hope. ■

In combinatorial applications, we consider a special type of martingale. We have an underlying probability space, and X_0 is the expectation of a random variable X. The variable X_n is the value of X at one sample point. We define a martingale $X_0, \ldots, X_n$ that describes a gradual process of learning more about the final value $X_n = X$.



8.5.35. Lemma. Let X be a random variable defined on a probability space. Let $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n$ be a chain of subsets of the space, where F_0 is the full space, F_n is a single outcome, and F_i is a random variable that is a block in a partition of F_{i-1}. The probability of choosing F_i within F_{i-1} is proportional to its probability in the underlying space. If $X_i = E(X|F_i)$, then the list $X_0, \ldots, X_n$ is a martingale.

Proof: We must prove that $E(X_i|X_0, \ldots, X_{i-1}) = X_{i-1}$. In a particular instance of the process, the list of values is the outcome of a particular sequence of restrictions. Each sequence of restrictions that generates the given values $X_0, \ldots, X_{i-1}$ reaches some F_{i-1} such that $E(X|F_{i-1})$ has the given value of X_{i-1}. For every such F_{i-1}, we can take the expectation of X_i over the possible values of F_i. In each case, we obtain X_{i-1}, so the desired formula holds regardless of which F_{i-1} generated the list $X_0, \ldots, X_{i-1}$.

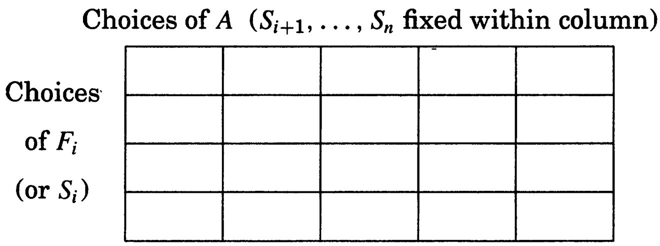
We thus condition on a fixed choice of F_{i-1} to compute $E(X_i|X_0, \ldots, X_{i-1})$. Within F_{i-1}, Lemma 8.5.32 yields $E(X_i) = E(E(X|F_i)) = E(X)$. This is the expectation within the event F_{i-1} (treated as a probability space), so all of these expresssions are conditioned on F_{i-1}, and the final expression is actually $E(X|F_{i-1}) = X_{i-1}$. ■

Such martingales, which we call **restriction martingales**, arise when we gradually discover a randomly generated object. Here F_i is the subset of the probability space where the object is confined after i steps (F for "inFormation"). In coin-flipping, the sample points are list of length n, and F_i may be the knowledge of the first i values. In random graphs, F_i may be the subgraph induced by the vertices $\{v_1, \ldots, v_i\}$, or F_i may be the knowledge of which among the first i edges are present.

To apply Azuma's Inequality, we need to bound $|X_i - X_{i-1}|$. The knowledge of which edges arise incident to a fixed vertex v_i can change the chromatic number by at most 1, since $\chi(G - v_i)$ equals $\chi(G)$ or $\chi(G) - 1$. From this we can conclude that $|X_i - X_{i-1}| \le 1$ in the restriction martingale defined by revealing vertices one by one.

8.5.36. Lemma. Consider a random structure specified by independent steps $S_1, \ldots, S_n$. Let F_i be the knowledge of $S_1, \ldots, S_i$, and let $X_0, \ldots, X_n$ be the corresponding restriction martingale for a random variable X. Let A be the knowledge of S_j for all $j \ne i$, with S_i unknown. If for each such A the values of X on points in A differ by at most 1, then $|X_i - X_{i-1}| \le 1$ for all i (and hence $P(X - E(X)) > \lambda\sqrt{n}) \le e^{-\lambda^2/2}$).

Proof: Consider a particular instance of F_{i-1}, with $X_{i-1} = E(X|F_{i-1})$ given. We arrange the points of F_{i-1} in the cells of a grid. For all these points, the outcomes of $S_1, \ldots, S_{i-1}$ are the same. Each row is a choice for F_i: a block in the partition of F_{i-1}. Each column is an A in which $S_{i+1}, \ldots, S_n$ are fixed and only S_i varies. By hypothesis, in each column the maximum and minimum values of X differ by at most 1. Let m_s, M_s be the minimum and maximum of X in column s.



Because S_i and $S_{i+1}, \ldots, S_n$ are specified independently, the probability of the outcome in row r and column s is $q_r p_s$, where q_r is the probability that S_i yields this row and p_s is the probability that $S_{i+1}, \ldots, S_n$ yields this column. The computation of X_i is the expectation across a single row:

$$\sum m_s p_s \;\le\; E(X|F_i) \;\le\; \sum M_s p_s \;\le\; 1 + \sum m_s p_s .$$

Since these upper and lower bounds are independent of the row index, taking the expectation over the entire grid to compute X_{i-1} yields the same inequalities. Hence X_{i-1} and X_i are confined to a single interval of length 1 and differ by at most 1. Therefore, Azuma's Inequality applies. ∎

When the conditions of Lemma 8.5.36 hold, we conclude immediately that the value of X is highly concentrated around its mean.

8.5.37. Example. *Chromatic number of random graphs.* Fix n, and consider Model A with edge probability p. Suppose we reveal the random n-vertex graph one vertex at a time. At stage i, we learn the edges from v_i to the previous vertices; this is S_i, and Model A specifies the outcomes of the S_i's independently. The event A in which all but S_i are specified is the subgraph $G - v_i$ of the random graph G plus the knowledge of edges from v_i to *later* vertices. Since $\chi(G - v_i) \le \chi(G) \le \chi(G - v_i) + 1$, the value of X differs by at most one over all possibilities in A. The hypotheses of Lemma 8.5.36 hold. Using both tails, we conclude that

$$P(|\chi(G) - E(\chi(G))|) \ge \lambda\sqrt{n}) \le 2e^{-\lambda^2/2}. \qquad \blacksquare$$

The result of Example 8.5.37 says nothing about the value of $E(\chi(G))$. To approximate this we again use Azuma's Inequality. With constant edge probability p, we know that the clique number of G^p is almost always within 1 of $d = 2\log_b n - 2\log_b \log_b n + 1 + 2\log_b(e/2)$, where $b = 1/p$. The same result holds for stable sets using the base $c = 1/(1-p)$ for the logarithm. To show that the chromatic number of G^p is close to $n/(2\log_c n)$, Bollobás showed that it is possible to extract stable sets of almost the maximum size until the number of vertices remaining is too small to matter.

8.5.38. Theorem. (Bollobás [1988]) For almost every G^p with constant $p = 1 - 1/c$, every induced subgraph of order at least $m = \lceil n/\log_c^2 n \rceil$ has a stable set of size at least $r = 2\log_c n - 5\log_c \log_c n$.

Proof: (sketch) We use r**-staset**, by analogy with r-clique, to mean a stable set of size r. Let S be a set of m vertices. We bound the probability that S has no r-staset by $e^{-dm^{1+\epsilon}}$ for some d, ϵ. This in turn bounds the probability that there exists an m-set with no r-staset by $\binom{n}{m}e^{-dm^{1+\epsilon}} < 2^n e^{-dm^{1+\epsilon}}$. Since $n = m^{1+o(1)}$, this bound goes to 0, and the first moment method implies that almost every G^p has no bad m-set.

It suffices to study the subgraph G induced by $[m]$. Let X be the maximum number of pairwise pair-disjoint r-stasets in this subgraph, where *pair-disjoint* means they share at most one vertex. We will show that $X \ge 1$ almost always. To do this, it suffices to show that (1) X is highly concentrated around its mean, and (2) $E(X)$ is bigger than something large (and growing).

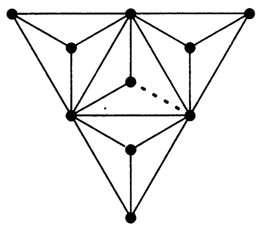
We invoke Azuma's Inequality for (1). Consider the restriction martingale for X that results from revealing G *one edge-slot at a time*. At each step, we learn whether one additional pair of vertices induces an edge. We have $X_0 = E(X)$ and $X_{\binom{m}{2}} = X$. The status of one edge slot changes the value of X by at most 1, so Lemma 8.5.36 applies, and $P(X - E(X)) \le -\lambda\binom{m}{2}^{1/2}) \le e^{-\lambda^2/2}$. With $\lambda = E(X)/\binom{m}{2}^{1/2}$, we have

$$P(X = 0) = P(X - E(X) \le -E(X)) \le e^{-E(X)^2/(m^2-m)}.$$

Hence it suffices to show that $E(X)/m \to \infty$.

To prove this, we consider another random variable $\hat{X}$, the number of r-stasets in G that have no pair in common with *any* other r-staset. Such a collection forms a pairwise pair-disjoint collection of r-stasets, so $X \ge \hat{X}$. We introduced X because the restriction martingale for $\hat{X}$ does not satisfy

$|\hat{X}_i - \hat{X}_{i-1}| \le 1$. In the drawing of $\overline{G}$ in the figure below, for example, we have $r = 4$ and seek 4-cliques; if the last (dotted) edge is present in $\overline{G}$ (absent in G), then $\hat{X} = 0$, but if it is absent from $\overline{G}$ (present in G), then $\hat{X} = 3$.



It is easier to compute $E(\hat{X})$ than $E(X)$. Expressing $\hat{X}$ as the sum of $\binom{m}{r}$ indicator variables, we obtain $E(\hat{X})$ as $\binom{m}{r}$ times the probability that $[r]$ induces an r-staset that is pair-disjoint from all others. This is $(1-p)^{\binom{r}{2}}$ times the conditional probability that $[r]$ does not conflict with other r-stasets, given the event Z that $[r]$ is in fact independent. Let Y be the number of other r-stasets overlapping $[r]$ in at least two elements. By Markov's Inequality, $E(Y|Z) \to 0$ implies $P(Y = 0|Z) \to 1$. Since each set counted shares at least two vertices with $[r]$, we have

$$E(Y|Z) = \sum_{i \ge 2, r-1} \binom{r}{i}\binom{m-r}{r-i}(1-p)^{\binom{r}{2}-\binom{i}{2}}.$$

As $m \to \infty$, this tends to 0; this follows from the expression for r in terms of m. Hence $E(\hat{X})$ is asymptotic to $\binom{m}{r}(1-p)^{\binom{r}{2}}$. The expression for r in terms of m yields $E(\hat{X}) \in \Omega(m^{5/3})$. Thus $E(X)/m \to \infty$, which completes the proof. ■

8.5.39. Corollary. (Bollobás [1988]) For constant edge probability $p = 1 - 1/c$, almost every G^p satisfies

$$(1+\epsilon)n/(2\log_c n) \le \chi(G^p) \le (1+\epsilon')n/(2\log_c n),$$

where $\epsilon = \log_c \log_c n / \log_c n$ and $\epsilon' = 5\log_c \log_c n / \log_c n$.

Proof: The lower bound holds because almost every G^p has no stable set larger than $2\log_c n - 2\log_c \log_c n$. The upper bound follows from Theorem 8.5.38, because we can almost always select stable sets of size $2\log_c n - 5\log_c \log_c n$ until we have only $n/\lg_c^2 n$ vertices left. Since $n/\lg_c^2 n \in o(n/\log_c n)$, we can complete the coloring by using distinct new colors on the remaining vertices. ■

EXERCISES

8.5.1. (−) *Expectation.*

a) Compute the expected number of fixed points in a random permutation of $[n]$.

b) Determine the expected number of vertices of degree k in a random n-vertex graph with edge probability p.

8.5.2. (−) Prove that $1 - p < e^{-p}$ for $p > 0$.

8.5.3. (−) Determine the expected number of monochromatic triangles in a random 2-coloring of $E(K_6)$.

8.5.4. (−) Prove that some 2-coloring of the edges of $K_{m,n}$ has at least $\binom{m}{r}\binom{n}{s}2^{1-rs}$ monochromatic copies of $K_{r,s}$.

8.5.5. (−) The statement "$f(G_n) \le (1+\epsilon)n$" means that for all $\epsilon > 0$, the inequality holds for sufficiently large n. The statement "$f(G_n) \le n + o(n)$" means that $f(G_n)/n \to 1$ as $n \to \infty$. Prove that these two statements are equivalent.

8.5.6. Compute explicitly the probability that the Hamiltonian closure of a random graph with vertex set [5] is complete.

8.5.7. Let G be a graph with p vertices, q edges, and automorphism group of size s. Let $n = (sk^{q-1})^{1/p}$. Prove that some k-coloring of $E(K_n)$ has no monochromatic copy of G. (Chvátal–Harary [1973])

8.5.8. (!) a) Use a random partition of the vertices to prove that every graph has a bipartite subgraph with at least half its edges.

b) Use equipartitions of the vertices to improve part (a): if G has m edges and n vertices, then G has a bipartite subgraph with at least $m\frac{\lceil n/2\rceil}{2\lceil n/2\rceil-1}$ edges.

8.5.9. An army of computers is configured as a complete k-ary tree with leaves at distance l from the root. At a fixed time, each node is working with probability p, independently of other nodes. When a node is not working, the entire subtree below it is inaccessible. What is the expected number of nodes accessible from the root?

8.5.10. Let G be a matching of size n. Select a set of k vertices at random. Compute the expected number of edges induced by the selected vertices.

8.5.11. Consider a drawing in the plane of a simple graph G with n vertices and m edges, where $m \ge 4n$. Let H be a random induced subdrawing, generated by letting each vertex be retained with probability p, independently. Let Y be the number of edge crossings in H. Let $X = Y - [e(H) - (3n(H) - 6)]$. Use expectations to prove that $3n + p^3\nu(G) - pm > 0$, and conclude that $\nu(G) \ge m^3/[64n^2]$, where $\nu(G)$ is the minimum number of crossings in a drawing of G. (Comment: This is an alternative proof of Theorem 6.3.16.)

8.5.12. Given a random permutation of the vertices of a simple graph G, orient each edge toward the vertex with higher index in the permutation. Compute the expected number of sink vertices (outdegree 0) in the resulting orientation. In terms of $n(G)$, determine the minimum and maximum values of this expectation. Prove that the probably of having only one sink is at most $e(G)/\binom{n(G)}{2}$. (Jeurissen [1997])

8.5.13. (!) A **hypergraph** consists of a collection of vertices and a collection of edges; if the vertex set is V, then the edges are subsets of V. The **chromatic number** $\chi(H)$ of a hypergraph H is the minimum number of colors needed to label the vertices so that no edge is monochromatic. A hypergraph is k-**uniform** if its edges all have size k.

a) Prove that every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colorable. (Erdős [1963])

b) Use part (a) to prove that if each vertex of an n-vertex bipartite graph has a list of more than $1 + \lg n$ usable colors, then a proper coloring can be chosen from the lists.

8.5.14. (!) Use the deletion method to prove that a graph with n vertices and average degree $d \ge 1$ has an independent set with at least $n/(2d)$ vertices. (Hint: Choose a random subset by including each vertex independently with a probability p to be chosen later. Compute the expected number of edges induced.)

8.5.15. The maximum size of an n-vertex graph not containing H is $ex(n; H)$. Use the deletion method to prove that $ex(n; C_k) \in \Omega(n^{1+1/(k-1)})$. (Comment: One can also show that $ex(n; C_k) \in O(n^{1+2/k})$ by considering the average degree.) (Bondy–Simonovits)

8.5.16. (!) For $n \in \mathbb{N}$, prove that $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$. Use this to conclude that $R(k,k) > (1/e)(1 - o(1))k2^{k/2}$.

8.5.17. For natural numbers n, t, let $m = n - \binom{n}{t}^2 2^{1-t^2}$. Prove that there is a 2-coloring of the edges of $K_{m,m}$ with no monochromatic copy of $K_{t,t}$.

8.5.18. (+) *Off-diagonal Ramsey numbers.* Suppose that $0 < p < 1$.

a) Prove that if $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1$, then $R(k,l) > n$.

b) Prove that $R(k,l) > n - \binom{n}{k}p^{\binom{k}{2}} - \binom{n}{l}(1-p)^{\binom{l}{2}}$ for all $n \in \mathbb{N}$.

c) Choose n and p in part (b) to prove that $R(3,k) > k^{3/2-o(1)}$. What lower bound on $R(3,k)$ can be obtained from part (a)? (Spencer [1977])

8.5.19. Let H be a graph. For constant p, prove that almost every G^p contains H as an induced subgraph.

8.5.20. a) Fix k, s, t, p. Prove that almost every G^p has the following property: for every choice of disjoint vertex sets S, T of sizes s, t, there are at least k vertices that are adjacent to every vertex of S and no vertex of T. (Blass–Harary [1979])

b) Conclude that almost every G^p is k-connected.

c) Apply the same argument to random tournaments: almost every one has the property that for every choice of disjoint vertex sets S, T of sizes s, t, there are at least k vertices with edges to every vertex of S and from every vertex of T.

8.5.21. A random labeled tournament is generated by orienting each edge v_iv_j as $v_i \to v_j$ or $v_j \to v_i$ independently with probability $1/2$.

a) Prove that almost every tournament is strongly connected.

b) In a tournament, a "king" is a vertex such that every other vertex can be reached from it by a path of length at most 2. It is known that every tournament contains a king. Is it true that in almost every tournament every vertex is a king? (Palmer [1985])

8.5.22. Find a threshold probability function for the property that at least half the possible edges of a graph are present. How sharp is the threshold?

8.5.23. For $p = 1/n$ and fixed $\epsilon > 0$, show that almost every G^p has no component with more than $(1+\epsilon)n/2$ vertices. (Hint: Instead of trying to bound the probability directly, show that it is bounded by the probability of another event, which tends to 0.)

8.5.24. Determine the smallest connected simple graph that is not balanced.

8.5.25. Extend the second moment argument of Theorem 8.5.23 to prove that $n^{-1/\rho(H)}$ is a threshold function for the appearance of H as a subgraph of G^p, where $\rho(G) = \max_{G\subseteq H} e(G)/n(G)$. (Bollobás [1981a], Ruciński–Vince [1985])

8.5.26. Let Q be the following graph property: for every choice of disjoint vertex sets S, T of size $c \lg n$, there is an edge with endpoints in S and T. Prove that almost every graph has property Q if $c > 2$. (Comment: This implies that the random graph has bandwidth at least $n - 2\log n$.)

8.5.27. Prove that if $k = \lg n - (2+\epsilon)\lg\lg n$, then almost every n-vertex tournament has the property that every set of k vertices has a common successor.

8.5.28. A tournament is **transitive** if it has a vertex ordering $u_1, \ldots, u_n$ such that $u_i \to u_j$ if and only if $i < j$. Prove that every tournament has a transitive subtournament with $\lg n$ vertices, and almost every tournament has no transitive subtournament with more than $2\lg n + c$ vertices if c is a constant greater than 1.

8.5.29. (!) *The Coupon Collector.*

a) Consider repetitions of an experiment with independent success probability p. Prove that the expected number of the trial on which the first success occurs is $1/p$.

b) Every box of a certain type of candy contains one of n prizes, each with probability $1/n$. Receiving the grand prize requiries obtaining each of these prizes at least once. Prove that the expected number of the box on which the last prize is obtained is $n\sum_{i=1}^{n} 1/i$.

c) Prove that $m(n) = n\ln n + (k-1)n\ln\ln n$ is a threshold function for the number of boxes needed to obtain at least k copies of each prize. (Hint: Prove that when $p = o(1)$ and k is constant, the probability of at most k successes in m trials with success probability p is asymptotic to the probability of exactly k successes.)

8.5.30. Prove that the length of the longest run in a list of n random heads and tails is $(1+o(1))\lg n$. In other words, for $\epsilon > 0$, almost no list has at least $(1+\epsilon)\lg n$ consecutive identical flips, and almost every list has at least $(1-\epsilon)\lg n$ consecutive identical flips.

8.5.31. With $p = (1-\epsilon)\log n/n$, find a large m such that almost every graph has at least m isolated vertices. What $m(n)$ results from Chebyshev's Inequality?

8.5.32. Given a graph G, say that a k-set S is *bad* if G has no vertex v such that $S \subseteq N(v)$. For fixed p, how large can k be so that almost every G^p has no bad k-set? How slowly can k grow so that almost every G^p has a bad k-set?

8.5.33. By examining common neighbors, prove that if p is fixed and $k = o(n/\log n)$, then almost every G^p is k-connected.

8.5.34. (!) With $p = (1-\epsilon)\log n/n$, how large can m be such that almost every graph has at least m isolated vertices? (Hint: Use Chebyshev's Inequality.)

8.5.35. A **t-interval** is a subset of $\mathbb{R}$ that is the union of at most t intervals. The **interval number** of a graph G is the minimum t such that G is an intersection graph of t-intervals (each vertex is assigned a set that is the union of at most t intervals). Prove that almost all graphs (edge probability 1/2) have interval number at least $(1-o(1))n/(4\lg n)$. (Hint: Compare the number of representations with the number of simple graphs. Comment: Scheinerman [1990] showed that almost all graphs have interval number $(1+o(1))n/(2\lg n)$.) (Erdős–West [1985])

8.5.36. (!) *Threshold for perfect matching in a random bipartite graph.* Let G be a random subgraph of $K_{n,n}$ with partite sets A, B, generated by independent edge probability $p = (1+\epsilon)\ln n/n$, where ϵ is a nonzero constant. Call S a *violated set* if $|N(S)| < |S|$.

a) Prove that if $\epsilon < 0$, then almost every G has no perfect matching.

b) Let S be a minimal violated set. Prove that $|N(S)| = |S| - 1$ and that $G[S \cup N(S)]$ is connected.

c) Suppose that G has no perfect matching. Prove that A or B contains a violated set with at most $\lceil n/2 \rceil$ elements.

d) For $r, s \geq 1$, the number of spanning trees of $K_{r,s}$ is $r^{s-1}s^{r-1}$. Use this, part (b), part (c), and Markov's Inequality to prove that if $\epsilon > 0$, then G almost surely has a perfect matching. (Hint: A summation in the bound on the expected number of minimal violated sets can be bounded by a geometric series.)

8.5.37. Suppose that $0 < p < 1$ and that $k_1, \ldots, k_r$ are nonnegative integers summing to m. Prove that $\prod_{i=1}^{r}[1-(1-p)^{k_i}] \le [1-(1-p)^{m/r}]^r$.

8.5.38. *Tail inequality for binomial distribution.* Let $X = \sum X_i'$, where each X_i' is an indicator variable with success probability $P(X_i' = 1) = .5$, so $E(X) = n/2$. Applying Markov's Inequality to the random variable $Z = (X - E(X))^2$ yields $P(|Z| \ge t) \le Var(X)/t^2$. Setting $t = \alpha\sqrt{n}$ yields a bound on the tail probability: $P(|X - np| \ge \alpha\sqrt{n}) \le 1/(2\alpha^2)$. Use Azuma's Inequality to prove the stronger bound that $P(|X - np| > \alpha\sqrt{n}) < 2e^{-2\alpha^2}$. (Hint: Let $Y_i' = X_i' - .5$. Let F_i be the knowledge of $Y_1', \ldots, Y_i'$, and let $Y_i = E(Y|F_i)$.)

8.5.39. *Bin-packing.* Let the numbers $S = \{a_1, \ldots, a_n\}$ be drawn uniformly and independently from the interval $[0, 1]$. The numbers must be placed in bins, each having capacity 1. Let X be the number of bins needed. Use Lemma 8.5.36 to prove that $P(|X - E(X)|) \ge \lambda\sqrt{n}) \le 2e^{-\lambda^2/2}$.

8.5.40. (!) *Azuma's Inequality and the Traveling Salesman Problem.*

a) Prove the generalization of Azuma's Inequality to general martingales: If $E(X_i) = X_{i-1}$ and $|X_i - X_{i-1}| \le c_i$, then $P(X_n - X_0) \ge \lambda\sqrt{\sum c_i^2}) \le e^{-\lambda^2/2}$.

b) Let Y be the distance from a given point z in the unit square to the nearest of n points chosen uniformly and independently in the unit square. Prove that $E(Y) < c/\sqrt{n}$, for some constant c. (Hint: For a nonnegative continuous random variable Y, $E(Y) = \int_0^\infty P(Y \ge y)dy$, which can be verified using integration by parts. In order to bound this integral, use (somewhere) the inequality $1 - a < e^{-a}$ and the definite integral $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2$.)

c) Apply parts (a) and (b) to prove that the smallest length of a polygon bounding a random set of n points in the unit square is highly concentrated around its expectation. In particular, the probability that this deviates from the expected tour length by more than $\lambda c\sqrt{\ln n}$ is bounded by $2e^{-\lambda^2/2}$, for some appropriate c. (Hint: For the martingale in which X_i is the expected length of the tour when the first i points are known, prove that $|X_i - X_{i-1}| < c(n-i)^{-1/2}$. Lemma 8.5.36 does not apply directly.)

8.6. Eigenvalues of Graphs

Techniques from group theory and linear algebra assist in studying the structure and enumeration of graphs.

From linear algebra, we have seen hints of vector spaces and determinants. In a graph G with edges $e_1, \ldots, e_m$, the **incidence vector** for a set $F \subseteq E(G)$ has coordinates $a_i = 1$ when $e_i \in F$ and $a_i = 0$ when $e_i \notin F$. Let **C** be the set of incidence vectors of even subgraphs (those with all vertex degrees even), and let **B** be the set of incidence vectors of edge cuts. Because these sets are closed under binary vector addition, **C** and **B** are vector spaces (Exercises 1–2), called the **cycle space** and **bond space** of G. Since an even subgraph and an edge cut share an even number of edges, **C** and **B** are orthogonal. This is closely related to the duality between cycles and bonds in Theorem 6.1.14 and Corollary 8.2.42 and to the use of determinants in the Matrix Tree Theorem (Theorem 2.2.12). For further discussion of these vector spaces, see Biggs [1993, Part 1].

Groups arise in studying graph isomorphism, embeddings, and enumeration. The automorphisms of a graph form a group of permutations of its vertices. Group-theoretic ideas lead to algorithms for testing isomorphism and to constructions for embedding on surfaces. Conversely, every group can be modeled using graphs. An introduction to this interplay appears in White [1973]; see also Gross–Yellen [1999, Chapters 13–15].

We restrict our attention to eigenvalues of adjacency matrices. We interpret the characteristic polynomial in terms of subgraphs, relate the eigenvalues to other graph parameters, and characterize the sets of eigenvalues for bipartite graphs and regular graphs. We close with applications to expander graphs and the "Friendship Theorem". An encyclopedic discussion of graph eigenvalues appears in Cvetković–Doob–Sachs [1979]. Chung [1997] presents the modern approach, modifying the adjacency matrix in a way that normalizes the eigenvalues and yields analogous results that hold more generally. For our brief presentation, we use the classical version.

THE CHARACTERISTIC POLYNOMIAL

8.6.1. Definition. The **eigenvalues** of a matrix A are the numbers λ such that $Ax = \lambda x$ has a nonzero solution vector; each such solution is an **eigenvector** associated with λ. The **eigenvalues** of a graph are the eigenvalues of its adjacency matrix A. These are the roots $\lambda_1, \ldots, \lambda_n$ of the **characteristic polynomial** $\phi(G;\lambda) = \det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. The **spectrum** is the list of distinct eigenvalues with their multiplicities $m_1, \ldots, m_t$; we write Spec $(G) = \binom{\lambda_1 \cdots \lambda_t}{m_1 \cdots m_t}$.

8.6.2. Remark. *Elementary properties of eigenvalues.*

0) The eigenvalues are the values λ such that the square matrix $\lambda I - A$ is singular, which is equivalent to $\det(\lambda I - A) = 0$.

1) $\sum \lambda_i = \operatorname{Trace} A$. The **trace** is the sum of the diagonal elements and is the coefficient of λ^{n-1} in $\det(\lambda I - A)$. Since $\det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - \lambda_i)$, that coefficient is also $\sum \lambda_i$. For simple graphs, it is 0.

2) $\prod \lambda_i = (-1)^n \phi(G; 0) = \det A = \sum_\sigma \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$, where the sum runs over permutations σ of $[n]$.

3) For a symmetric real n-by-n matrix A and $\lambda \in \mathbb{R}$, the multiplicity of λ as an eigenvalue of A is $n - \operatorname{rank}(\lambda I - A)$.

4) Adding c to the diagonal shifts the eigenvalues by c, since $\alpha + c$ is a root of $\det(\lambda I - (cI + A))$ if and only if α is a root of $\det(\lambda I - A)$.

8.6.3. Example. *Spectra of cliques and bicliques.* The adjacency matrix of K_n is $J - I$, where J is the matrix of all 1s. Hence the eigenvalues of K_n are 1 less than those of J. Since Spec $J = \binom{n \quad 0}{1 \quad n-1}$, we have Spec $K_n = \binom{n-1 \quad -1}{1 \quad n-1}$.

The adjacency matrix of $K_{m,n}$ has rank 2, so it has two nonzero eigenvalues λ_1, λ_2. The trace is 0, so $\lambda_1 = -\lambda_2$; call this constant b. Hence $\phi(K_{m,n}, \lambda) = \lambda^n - b^2\lambda^{n-2}$. We compute b using $\phi(G;\lambda) = \det(\lambda I - A)$. Since λ appears only on the diagonal, contributions in the permutation expansion to the coefficient of λ^{n-2} arise only from permutations that use $n-2$ positions on the diagonal. The remaining two positions must be $-a_{i,j}$ and $-a_{j,i}$ for some i, j. There are mn nonzero contributions of this form, all negative. Hence $b^2 = mn$, and Spec $(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$. ■

We index the coefficients of the characteristic polynomial so that $\phi(G;\lambda) = \sum_{i=0}^{n} c_i\lambda^{n-i}$. Since $\phi(G;\lambda) = \det(\lambda I - A)$, always $c_0 = 1$ and $c_1 = -\text{Trace}\, A = 0$. Our computation of c_2 for $K_{m,n}$ extends to all graphs.

8.6.4. Definition. A **principal submatrix** of a square matrix A is a submatrix selecting rows and columns with the same indices.

Since contributions to $c_2\lambda^{n-2}$ involve $n-2$ factors of λ from the diagonal, the coefficient c_2 is the sum of the principal 2×2 subdeterminants of $-A$. For a simple graph, $-a_{i,j}$ is -1 when $v_i \leftrightarrow v_j$ and 0 otherwise, so $c_2 = -e(G)$.

Similarly, c_3 is the sum of the principal 3×3 subdeterminants of $-A$. For triple i, j, k, the determinant depends only on the number of edges among v_i, v_j, v_k. The determinant is 0 unless they form a triangle, and then it is -2. Hence c_3 is -2 times the number of 3-cycles in G.

Since principal submatrices are the adjacency matrices of induced subgraphs, in general we have $c_i = (-1)^i \sum_{|S|=i} \det A(G[S])$.

8.6.5. Theorem. (Harary [1962b]) Given a simple graph G, let $\mathbf{H}$ be the set of spanning subgraphs in which every component is an edge or a cycle. If $k(H)$ and $s(H)$ denote the number of components of H and the number of components that are cycles, respectively, then $\det A(G) = \sum_{H \in \mathbf{H}} (-1)^{n(H)-k(H)} 2^{s(H)}$

Proof: The determinant formula is $\det A = \sum_\sigma (-1)^{t(\sigma)} \prod a_{i,\sigma(i)}$, where the sum is over permutations of $[n]$ and $t(\sigma)$ is the number of row exchanges (transpositions) needed to put the positions $i, \sigma(i)$ on the diagonal. When A is a 0,1-matrix, the contribution from σ is nonzero if and only if these entries all equal 1.

We view such a σ as a vertex permutation mapping each v_i to $v_{\sigma(i)}$. This partitions $V(G)$ into orbits. Since $a_{i,\sigma(i)} = 1$ means $v_i \leftrightarrow v_{\sigma(i)}$, there are no orbits of size 1, orbits of size 2 correspond to edges, and longer orbits correspond to cycles. Thus the permutation makes a nonzero contribution when it describes a spanning subgraph H of G in which the components are edges and cycles.

The sign of the contribution is determined by the number of transpositions needed to move the entries to the diagonal. Row exchanges move one element of an orbit at a time to the diagonal, but the last switch moves the last two elements to the diagonal. Hence $t(\sigma) = n(H) - k(H)$. Finally, each cycle of length at least 3 in H can appear in one of two ways in the permutation matrix, since we can follow the cycle in one of two directions. Hence the number of permutations that give rise to H is $2^{s(H)}$. ■

8.6.6. Corollary. (Sachs [1967]) Let $\mathbf{H}_i$ denote the collection of i-vertex subgraphs of a simple graph G whose components are edges or cycles. The characteristic polynomial of G is $\sum c_i\lambda^{n-i}$, where $c_i = \sum_{H\in\mathbf{H}_i}(-1)^{k(H)}2^{s(H)}$.

Proof: This follows from Theorem 8.6.5 and the earlier observation that $c_i = (-1)^i\sum_{|S|=i}\det A(G[S])$. ∎

This formula leads to a recursive expression for the characteristic polynomial (Exercise 5). The formula can be used to construct nonisomorphic trees with the same characteristic polynomial (and only eight vertices) (Exercise 7).

We next discuss the properties of eigenvalues for bipartite graphs.

8.6.7. Proposition. The (i, j)th entry of A^k counts the v_i, v_j-walks of length k. The eigenvalues of A^k are the kth powers of the eigenvalues of A.

Proof: The statement about walks holds easily by induction on k (Exercise 1.2.30). For the second statement, $Ax = \lambda x$ implies $A^kx = \lambda^kx$, by repeated multiplication. Using an arbitrary eigenvector x ensures that the multiplicities of the eigenvalues don't change. ∎

8.6.8. Lemma. If G is bipartite and λ is an eigenvalue of G with multiplicity m, then $-\lambda$ is also an eigenvalue with multiplicity m.

Proof: Adding isolated vertices to give the partite sets equal size merely adds rows and columns of 0's to the adjacency matrix, which does not change the rank and hence changes the spectrum only by including one extra 0 for each vertex added. Hence we may assume that the partite sets have equal sizes.

Since G is bipartite, we can permute the rows and columns of A to obtain the form $A = \binom{0\ \ B}{B'\ \ 0}$, where B is square. If λ is an eigenvalue associated with eigenvector $v = \binom{x}{y}$ (partitioned according to the bipartition of G), then $\lambda v = Av = \binom{0\ \ B}{B^T\ \ 0}\binom{x}{y} = \binom{By}{B^Tx}$. Hence $By = \lambda x$ and $B^Tx = \lambda y$.

Let $v' = \binom{x}{-y}$. We compute $Av' = \binom{B(-y)}{B^Tx} = \binom{-\lambda x}{\lambda y} = -\lambda v'$. Hence v' is an eigenvector of A for the eigenvalue $-\lambda$. Furthermore, m independent eigenvectors for λ yield m independent eigenvectors for $-\lambda$ in this way. Hence $-\lambda$ is an eigenvector of A with the same multiplicity as λ. ∎

8.6.9. Theorem. The following are equivalent statements about a graph G.

A) G is bipartite.

B) The eigenvalues of G occur in pairs λ_i, λ_j such that $\lambda_i = -\lambda_j$.

C) $\phi(G; \lambda)$ is a polynomial in λ^2.

D) $\sum_{i=1}^n \lambda_i^{2t-1} = 0$ for any positive integer t.

Proof: We proved A $\Rightarrow$ B in the lemma.

B $\Leftrightarrow$ C: $(\lambda - \lambda_i)(\lambda - \lambda_j) = (\lambda^2 - a)$ if and only if $\lambda_j = -\lambda_i$. Hence the roots occur in such pairs if and only if $\phi(G; \lambda)$ is a product of linear factors in λ^2.

B $\Rightarrow$ D: If $\lambda_j = -\lambda_i$, then $\lambda_j^{2t-1} = -\lambda_i^{2t-1}$.

D $\Rightarrow$ A: Because $\sum \lambda_i^k$ counts the closed k-walks in the graph (from each starting vertex), condition D forbids closed walks of odd length. This forbids odd cycles, since an odd cycle is an odd closed walk, and hence G is bipartite. ∎

LINEAR ALGEBRA OF REAL SYMMETRIC MATRICES

Relating eigenvalues to other parameters requires several results from linear algebra, including the Spectral Theorem and Cayley–Hamilton Theorem for real symmetric matrices. These are usually stated in more generality, but adjacency matrices are real and symmetric, and here the theorems have shorter proofs. We begin with a lemma that follows *from* the Spectral Theorem when the latter is proved using complex matrices. The proofs of these results may be skipped, especially by readers well-versed in linear algebra.

8.6.10. Lemma. If $f(x) = x^T Ax$, where A is a real symmetric matrix, then f attains its maximum and minimum over unit vectors x at eigenvectors of A, where it equals the corresponding eigenvalues.

Proof: The function f is continuous in $x_1, \ldots, x_n$. For constrained optimization, we use Lagrangian multipliers. Given the constraint $x^T x = 1$, we let $g(x) = x^T x - 1$. Forming $L(x, \lambda) = f(x) - \lambda g(x)$, the extreme values occur where all partial derivatives of L are 0. With respect to λ, this yields $x^T x = 1$.

Let ∇ denote the vector of partial derivatives with respect to $x_1, \ldots, x_n$. We compute $\nabla L(x, \lambda) = \nabla f(x) - \lambda \nabla g(x) = 2Ax - 2\lambda x$. The statement $\nabla f(x) = 2Ax$ uses the symmetry of A. We have $\nabla L = 0$ precisely when $Ax = \lambda x$, which requires x to be an eigenvector of A for eigenvalue λ. This yields $f(x) = x^T Ax = \lambda x^T x = \lambda$. ∎

Since our variables in the optimization are real, we have found at least one real eigenvector and eigenvalue. We can use this inductively to show that all eigenvectors have this property.

8.6.11. Theorem. (Spectral Theorem) A real symmetric $n \times n$ matrix has real eigenvalues and n orthonormal eigenvectors.

Proof: We use induction on n. The claim is trivial for $n = 1$; consider $n > 1$. Let v_n be the eigenvector maximizing $x^T Ax$. Let W be the orthogonal complement of the space spanned by v_n; it has dimension $n - 1$. If $w \in W$, then $v_n^T Aw = w^T Av_n = \lambda_n w^T v_n = 0$. Hence $Aw \in W$. Viewing multiplication by A as a mapping f_A, we have $f_A \colon W \to W$.

Let S be a matrix whose columns are the vectors of an orthonormal basis of $\mathbb{R}^n$ with v_n as the last column. Since the basis is orthonormal, $S^{-1} = S^T$. The matrix for f_A with respect to this basis is $S^T AS$. Since the basis is orthonormal and v_n is an eigenvector, the last column of $S^T AS$ is 0, except for λ_n in the last position. Furthermore, the matrix is symmetric. Hence its first $n - 1$ rows and columns form the matrix A' for f_A on W with respect to this basis.

By the induction hypothesis, A' has orthonormal eigenvectors $v_1, \ldots, v_{n-1}$, with real eigenvalues. Using S, we convert these back into real eigenvectors for A. They have the same real eigenvalues, and they form an orthonormal set. ∎

Next we consider polynomial functions of a matrix. Viewed as members of $\mathbb{R}^{n^2}$, the matrices $I, A, A^2, \ldots, A^{n^2}$ cannot be independent, since there are $n^2 + 1$

of them. Using an equation of linear dependence, we obtain a polynomial p such that $p(A)$ is the zero matrix. The characteristic polynomial itself suffices. This holds for all A, but again we consider only real symmetric matrices.

8.6.12. Theorem. (Cayley–Hamilton Theorem) If $\phi(\lambda)$ is the characteristic polynomial of a real symmetric matrix A, then $\phi(A)$ is the zero matrix (A "satisfies" its own characteristic polynomial).

Proof: Let the eigenvalues of A be $\lambda_1, \dots, \lambda_n$, so $\phi(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. Since powers of A commute, the matrix polynomial obtained by using A for λ factors as $\phi(A) = \prod_{i=1}^{n}(A - \lambda_i I)$. To prove that $\phi(A) = 0$, we need only show that the matrix $\phi(A)$ maps every vector to 0. Write an arbitrary vector x as a linear combination of the basis of eigenvectors guaranteed by the Spectral Theorem. Applying $A - \lambda_i I$ kills the coefficient of v_i. Successively multiplying by all the factors $A - \lambda_i I$ produces the zero vector. ∎

8.6.13. Definition. The **minimum polynomial** ψ of a matrix A is the polynomial of minimum degree satisfied by A and having leading coefficient 1. When A is the adjacency matrix of G, we call this the **minimum polynomial** $\psi(G; \lambda)$ of G.

The minimum polynomial is unique: if A satisfies two such polynomials of the same degree, then A satisfies their difference, which has lower degree.

8.6.14. Theorem. The minimum polynomial of A is $\psi(A) = \prod_{i=1}^{t}(\lambda - \lambda_i)$, where $\{\lambda_1, \dots, \lambda_t\}$ are the distinct eigenvalues of A.

Proof: The minimum polynomial divides every polynomial satisfied by A, since otherwise the remainder would be a polynomial of lower degree satisfied by A. The Cayley–Hamilton Theorem now implies that ψ divides ϕ and must be the product of some of its factors. Killing the vectors in the subspace of eigenvectors for eigenvalue λ_i requires a factor of the form $A - \lambda_i I$. This factor kills all vectors in that subspace, so we only need one copy of each such factor. ∎

8.6.15. Lemma. (Sylvester's Law of Inertia) Let A be a real symmetric matrix. If $x^T Ax$ can be written as a sum of N products of linear expressions, that is $x^T Ax = \sum_{m=1}^{N}(\sum_{i\in S_m} a_{i,m}x_i)(\sum_{j\in T_m} b_{j,m}x_j)$, then N is at least the maximum of the number of positive and the number of negative eigenvalues of A.

Proof: (Tverberg [1982]) Write the linear expressions as $u_m(x)$ and $v_m(x)$. For each m, we have $u_m(x)v_m(x) = L_m^2(x) - M_m^2(x)$, where $L = \frac{1}{2}(u + v)$ and $M = \frac{1}{2}(u - v)$ are also linear combinations of $x_1, \dots, x_n$. This expresses the quadratic form as $x^T Ax = \sum_{m=1}^{N}[L_m^2(x) - M_m^2(x)]$.

On the other hand, A is a real symmetric matrix and thus has orthonormal eigenvectors $w^1, \dots, w^n$. Using this, we write $x^T Ax = x^T S\Lambda S^T x$, where Λ is the diagonal matrix of eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and S has columns $w^1, \dots, w^n$. If S has p positive and q negative eigenvalues, then this becomes $x^T Ax = \sum_{i=1}^{p}(y^i \cdot x)^2 - \sum_{i=n-q+1}^{n}(z^i \cdot x)^2$, where each y^i or z^i is $|\lambda_i|^{1/2}\, w^i$.

Now we consider a homogeneous system of linear equations. We require $L_m(x) = 0$ for $1 \le m \le N$, also $z^i \cdot x = 0$ for $n-q < i \le n$, and $w^i \cdot x = 0$ for $p < i \le n-q$. This places $N + n - p$ homogeneous linear constraints on n variables. If $N < p$, then these equations have a nonzero simultaneous solution x'. Setting x to x' in the two expressions for $x^T Ax$ yields $\sum_{i=1}^{p}(y^i \cdot x')^2 = -\sum_{m=1}^{N} M_m^2(x')$. Since x' is orthogonal to all eigenvectors with nonpositive eigenvalues, the left side is positive, while the right is nonpositive. The contradiction yields $N \ge p$; an analogous argument yields $N \ge q$. ■

EIGENVALUES AND GRAPH PARAMETERS

Eigenvalues provide bounds on various parameters, or alternatively graph parameters yield bounds on the eigenvalues. Our first result uses only the minimum polynomial.

8.6.16. Theorem. The diameter of a graph G is less than the number of distinct eigenvalues of G.

Proof: Let A be the adjacency matrix; A satisfies a polynomial of degree r if and only if some linear combination of $A^0, \dots, A^r$ is 0. Since the number of distinct eigenvalues is the degree of the minimum polynomial, we need only show that $A^0, \dots, A^k$ are linearly independent when $k \le \operatorname{diam}(G)$.

It suffices to show for $k \le \operatorname{diam}(G)$ that A^k is not a linear combination of $A^0, \dots, A^{k-1}$. Choose $v_i, v_j \in V(G)$ such that $d(v_i, v_j) = k$. By counting walks, we have $A^k_{i,j} \ne 0$ but $A^t_{i,j} = 0$ for $t < k$. Therefore, A^k is not a linear combination of the smaller powers. ■

Since the Spectral Theorem guarantees real eigenvalues, we can index our eigenvalues as $\lambda_1 \ge \cdots \ge \lambda_n$. We also refer to λ_1 and λ_n as $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$.

8.6.17. Lemma. If G' is an induced subgraph of G, then

$$\lambda_{\min}(G) \le \lambda_{\min}(G') \le \lambda_{\max}(G') \le \lambda_{\max}(G) .$$

Proof: Since A is a real symmetric matrix, Lemma 8.6.10 yields $\lambda_{\min}(A) \le x^T Ax \le \lambda_{\max}(A)$ for every unit vector x. Consider the adjacency matrix A' of G'. By permuting the vertices of G, we can view A' as an upper left principal submatrix of $A = A(G)$. Let z' be the unit eigenvector of A' such that $A'z' = \lambda_{\max}(G')z'$. Let z be the unit vector in R_n obtained by appending zeros to z'. Then $\lambda_{\max}(G') = z'^T A'z' = z^T Az \le \lambda_{\max}(G)$. Similarly, $\lambda_{\min}(G') \ge \lambda_{\min}(G)$. ■

The behavior of the extreme eigenvalues under vertex deletion is a special case of the "Interlacing Theorem": If G has eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$ and $G - x$ has eigenvalues $\mu_1 \ge \cdots \ge \mu_{n-1}$, then $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \cdots \ge \mu_{n-1} \ge \lambda_n$. We will not need this and hence omit the proof, which involves only linear algebra.

8.6.18. Lemma. For every graph G, $\delta(G) \le \frac{2e(G)}{n(G)} \le \lambda_{\max}(G) \le \Delta(G)$.

Proof: Let x be an eigenvector for eigenvalue λ, and let $x_j = \max_i x_i$ be the largest coordinate value in x. Then $\lambda \le \Delta(G)$ follows from

$$\lambda x_j = (Ax)_j = \sum_{v_i \in N(v_j)} x_i \le d(v_j)x_j \le \Delta(G)x_j\,.$$

For the lower bound, we apply Lemma 8.6.10 to the unit vector with equal coordinates. Since the sum of the entries in the adjacency matrix is twice the number of edges of G, we have

$$\lambda_{\max} \ge \frac{\mathbf{1}_n^T}{\sqrt{n}} A \frac{\mathbf{1}_n}{\sqrt{n}} = \frac{1}{n}\sum\sum a_{ij} = \frac{2e(G)}{n}. \qquad \blacksquare$$

Lemma 8.6.18 enables us to improve the trivial bound $\chi(G) \le 1 + \Delta(G)$ given by the greedy coloring algorithm. Replacing $\Delta(G)$ with the average degree is too small; $K_n + K_1$ has chromatic number n and average degree less than $n-1$. Since $\lambda_{\max}$ is always at least the average degree, $1 + \lambda_{\max}(G)$ has a chance to work and can't be much improved.

8.6.19. Theorem. (Wilf [1967]) For every graph G, $\chi(G) \le 1 + \lambda_{\max}(G)$.

Proof: If $\chi(G) = k$, then we can successively delete vertices without reducing the chromatic number until we obtain a subgraph H such that $\chi(H - v) = k - 1$ for all $v \in V(H)$. As observed in Lemma 5.1.18, $\delta(H) \ge k - 1$. Since H is an induced subgraph of G, Lemma 8.6.18 and then Lemma 8.6.17 yield

$$k \le 1 + \delta(H) \le 1 + \lambda_{\max}(H) \le 1 + \lambda_{\max}(G)\,. \qquad \blacksquare$$

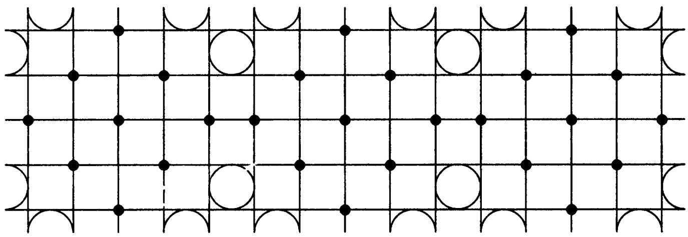
Sylvester's Law of Inertia yields a lower bound on the number of bicliques needed to decompose a graph. Because stars are bicliques and every subgraph of a star is a star, the number of bicliques needed is at most the vertex cover number $\beta(G) = n(G) - \alpha(G)$. Erdős conjectured that equality almost always holds, but this remains open. Graphs with special structure may have efficient partitions using other bicliques. The general lower bound using eigenvalues appears explicitly in Reznick–Tiwari–West [1985], but it is implicit in earlier work decomposing the complete graph (Tverberg [1982], Peck [1984]).

8.6.20. Theorem. For a simple graph G, the number of bicliques needed to decompose G is at least the maximum of the number of positive and number of negative eigenvalues of the adjacency matrix $A(G)$.

Proof: When G decomposes into subgraphs $G_1, \ldots, G_t$, we may write $A(G) = \sum_{i=1}^t B_i$, where B_i is the adjacency matrix of the spanning subgraph of G with edge set $E(G_i)$. When G_i is the biclique with bipartition S_i, T_i, we have $x^T B_i x = 2\sum_{j \in S_i} x_j \sum_{k \in T_i} x_k$. Writing these linear expressions as $u_i(x) = \sqrt{2}\sum_{j \in S_i} x_j$ and $v_i(x) = \sum_{k \in T_i} x_k$, we have $x^T A x = \sum_{i=1}^t x^T B_i x = \sum_{i=1}^t u_i(x) v_i(x)$. Sylvester's Law of Inertia (Lemma 8.6.15) now yields the claimed lower bound. $\blacksquare$

8.6.21. Example. *Biclique decomposition of $C_{(2t+1)n} \square C_n$.* There are simple formulas for the eigenvalues of a cycle (Exercise 6) and for computing the eigenvalues of a cartesian product from the eigenvalues of the factors (Exercise 10). These yield simple formulas for the numbers of positive and negative eigenvalues of $C_m \square C_n$ when m is an odd multiple of n. In particular, $C_{(2t+1)n} \square C_n$ has $(2t+1)(n^2+1)/2$ positive eigenvalues and $(2t+1)(n^2-1)/2$ negative eigenvalues when n is odd (0 is not an eigenvalue).

Furthermore, such a product decomposes into $(2t+1)(n^2+1)/2$ bicliques, consisting of $(2t+1)(n-1)/2$ 4-cycles and $(2t+1)(n+1)/2$ stars (Kratzke–West). Note that 4-cycles and stars are the only subgraphs of $C_m \square C_n$ that are bicliques. The optimal decomposition of $C_{15} \square C_5$ appears below. Edges wrap around from top to bottom and right to left, and all grid points indicate vertices. The heavy dots indicate vertices that are centers of stars in the decomposition, and the circles indicate 4-cycles in the decomposition. ■



EIGENVALUES OF REGULAR GRAPHS

Like bipartite graphs, regular graphs can be characterized using spectra. The n-vector $\mathbf{1}_n$ with all coordinates 1 plays a special role in this and many other arguments involving eigenvalues, as does the matrix J of all 1s.

8.6.22. Theorem. The eigenvalue of G with largest absolute value is $\Delta(G)$ if and only if some component of G is $\Delta(G)$-regular. The multiplicity of $\Delta(G)$ as an eigenvalue is the number of $\Delta(G)$-regular components.

Proof: Let A be the adjacency matrix. The ith entry of $A\mathbf{1}_n$ is $d(v_i)$. When G is k-regular, we obtain $A\mathbf{1}_n = k\mathbf{1}_n$, and thus k is an eigenvalue with eigenvector $\mathbf{1}_n$. In general, let x be an eigenvector for eigenvalue λ, and let x_j be a coordinate of largest absolute value among coordinates of x corresponding to the vertices of some component H of G. For the jth coordinate of Ax, we have

$$|\lambda|\,|x_j| = |(Ax)_j| = \left|\sum_{v_i \in N(v_j)} x_i\right| \le d(v_j)\,|x_j| \le \Delta(G)\,|x_j| .$$

Hence $|\lambda| \le \Delta(G)$. Equality requires $d(v_j) = \Delta(G)$ and $x_i = x_j$ for all $v_i \in N(v_j)$. We can iterate this argument to reach all coordinates for vertices in H. Hence the eigenvalue associated with x has absolute value as large as $\Delta(G)$ only if H is $\Delta(G)$-regular.

Thus the eigenvalue associated with an eigenvector x has absolute value as large as $\Delta(G)$ if and only if (1) each component of G containing a vertex where x is nonzero is $\Delta(G)$-regular, and (2) x is constant on the coordinates corresponding to each such component. We can choose the constant independently for each $\Delta(G)$-regular component, so the dimension of the space of eigenvectors associated with $\Delta(G)$ is the number of $\Delta(G)$-regular components. ■

When G is connected and not regular, it remains true that eigenvalues of largest absolute value have multiplicity 1 and that coordinates of the associated eigenvector have the same sign. This is related to the Perron–Frobenius Theorem of linear algebra and uses arguments like those above; we omit the proof.

Powers of the adjacency matrix yield another characterization.

8.6.23. Theorem. (Hoffman [1963]) A graph G is regular and connected if and only if J is a linear combination of powers of $A(G)$.

Proof: *Sufficiency*. If J can be so expressed, then for each i, j we have $(A^k)_{ij} \neq 0$ for some $k \geq 0$, which requires a v_i, v_j-walk of length k. Hence G is connected. For regularity, consider the matrices JA and AJ. The i, jth position of AJ is $d(v_i)$ (constant on rows), and the i, jth position of JA is $d(v_j)$ (constant on columns). Since J is a linear combination of powers of A, each of which commutes with A, we have $JA = AJ$. Thus the i, jth position is both $d(v_i)$ and $d(v_j)$ and the graph is regular.

Necessity. Since G is k-regular, k is an eigenvalue, and the minimum polynomial is $\psi(G; \lambda) = (\lambda - k)g(\lambda)$ for some polynomial g. Since $\psi(G; A) = 0$, we have $Ag(A) = kg(A)$. Hence each column of $g(A)$ is an eigenvector of A with eigenvalue k. Since G is regular and connected, each such eigenvector is a multiple of $\mathbf{1}_n$. Hence the columns of $g(A)$ are constant. However, $g(A)$ is a linear combination of powers of a symmetric matrix and therefore must itself be symmetric. Hence the columns are equal and $g(A)$ is a multiple of J. ■

When G is simple and regular, $\overline{G}$ is also regular, and the eigenvalues of $\overline{G}$ can be obtained from the eigenvalues of G. This rests on the matrix expression for complementation: $A(\overline{G}) = J - I - A(G)$.

8.6.24. Lemma. $\phi(\overline{G}; \lambda) = (-1)^n \det[(-\lambda - 1)I - A(G) + J]$.

Proof: Direct computation yields $\det(\lambda I - A(\overline{G})) = \det(\lambda I - (J - I - A)) = \det[(\lambda + 1)I - J + A] = (-1)^n \det[(-\lambda - 1)I - A + J]$. ■

8.6.25. Theorem. If a simple graph G is k-regular, then G and $\overline{G}$ have the same eigenvectors. The eigenvalue associated with $\mathbf{1}_n$ is k in G and $n - k - 1$ in $\overline{G}$. If $x \neq \mathbf{1}_n$ is an eigenvector of G for eigenvalue λ of G, then its associated eigenvalue in $\overline{G}$ is $-1 - \lambda$.

Proof: Since $\overline{G}$ is $n - k - 1$-regular, $\mathbf{1}_n$ is an eigenvector for both G and $\overline{G}$, with eigenvalue k for G and $n - k - 1$ for $\overline{G}$. Let x be another eigenvector of G in an orthonormal basis of eigenvectors, and let $\overline{A} = A(\overline{G})$. Since $\mathbf{1}_n \cdot x = 0$, $\sum x_i = 0$. We compute $\overline{A}x = Jx - x - Ax = 0 - x - Ax = (-1 - \lambda)x$. ■

This yields a lower bound on the smallest eigenvalue of a regular graph and another derivation of the spectrum of K_n.

8.6.26. Corollary. For a k-regular simple graph, $\lambda_n \ge k - n$.

Proof: If G is k-regular and $\lambda_1 \ge \cdots \ge \lambda_n$, then the eigenvalues of $\overline{G}$ are $(n-k-1, -1-\lambda_n, \ldots, -1-\lambda_2)$, by Theorems 8.6.22–8.6.25. In particular, $n-k-1 \ge -\lambda_n - 1$. ∎

The eigenvalues of a connected regular simple graph G can be used to count its spanning trees. The eigenvalues need not be rational, yet the result $\tau(G)$ is an integer. The Matrix Tree Theorem (Theorem 2.2.12) says that $\tau(G)$ equals each minor of $Q = D - A$, where A is the adjacency matrix and D is the diagonal matrix of degrees. When G is k-regular, $D = kI$. Letting Adj Q denote the adjugate matrix of Q (the matrix of signed cofactors), the Matrix Tree Theorem is the statement that $\operatorname{Adj} Q = \tau(G)J$. Using Cayley's Formula (Theorem 2.2.3) for spanning trees in K_n, we have $\operatorname{Adj}(nI - J) = n^{n-2}J$.

8.6.27. Lemma. Let D be the diagonal matrix of vertex degrees in a simple graph G, let $A = A(G)$, and let $Q = D - A$. The number of spanning trees of G is $\tau(G) = \det(J + Q)/n^2$.

Proof: Observe that $J^2 = nJ$, $JQ = 0$, and $\operatorname{Adj}(AB) = \operatorname{Adj}(A)\operatorname{Adj}(B)$. We apply this using $J + Q$ and the matrix $nI - J$ that arises from K_n. We have

$$\operatorname{Adj}(nI - J)\operatorname{Adj}(J + Q) = \operatorname{Adj}[(nI - J)(J + Q)] = \operatorname{Adj}(nQ),$$

since $J^2 = nJ$ and $JQ = 0$. We have computed that $\operatorname{Adj}(nI - J) = n^{n-2}J$. Also, $\operatorname{Adj}(nQ) = n^{n-1}\operatorname{Adj} Q$ for any matrix Q. Canceling common factors of n yields $J\operatorname{Adj}(J + Q) = n\tau(G)J$. Multiplying both sides of this on the right by $(J+Q)^T$ yields $J(\det(J + Q)I) = n\tau(G)nJ$. Both sides are multiples of J, so the desired equality holds. ∎

We can now compute $\tau(G)$ from the eigenvalues if G is regular. (This analysis extends to all graphs when the modified system of eigenvalues is used.)

8.6.28. Theorem. If G is a k-regular connected simple n-vertex graph with spectrum $\binom{k\ \ \lambda_2 \cdots \lambda_t}{1\ \ m_2 \cdots m_t}$, then $\tau(G) = n^{-1}\phi'(G;k) = n^{-1}\prod_{j=2}^{t}(k - \lambda_j)^{mj}$.

Proof: Since $J + Q = J + kI - A$, the determinant of $J + Q$ is the value at k of the characteristic polynomial of $A - J$. Since G is k-regular and connected, it has $\mathbf{1}_n$ as an eigenvector with eigenvalue k, and the other eigenvectors are orthogonal to $\mathbf{1}_n$. Every such eigenvector of A is also an eigenvector of $A - J$, with the same eigenvalue, since $(A - J)x = Ax - Jx = Ax = \lambda x$.

Also, $\mathbf{1}_n$ is an eigenvector of $A - J$ with eigenvalue $k - n$. This produces a full set of eigenvalues for $A - J$. Evaluating the characteristic polynomial at k yields $\det(J + Q) = n\prod_{j=2}^{t}(k - \lambda_j)$. The product is $\phi'(G;k)$, since $\phi(G;\lambda)$ has $\lambda - k$ as a non-repeated factor when G is k-regular and connected. By Lemma 8.6.27, we obtain $\tau(G)$ upon dividing by n^2. ∎

The results in Lemma 8.6.24–Theorem 8.6.28 were extended to arbitrary (non-regular) graphs in Kelmans [1967b] (also Kelmans–Chelnokov [1974]) using the eigenvalues of the Laplacian matrix of the graph. This is the matrix Q used above. Another method for counting spanning trees appears in Kelmans [1965, 1966], and another variation on the Matrix Tree Theorem appears in Hartsfield–Kelmans–Shen [1996].

EIGENVALUES AND EXPANDERS

Many applications in computer science require "expander graphs". Walters [1996] collects many definitions that have been used for such graphs. The basic notion of expansion is that all small sets should have large neighborhoods. The aim is to establish good connectivity properties without having many edges.

8.6.29. Definition. An (n, k, c)-**expander** is an X, Y-bigraph G with $|X| = |Y| = n$ such that $\Delta(G) \leq k$ and that $|N(S)| \geq (1 + c(1 - |S|/n)) \cdot |S|$ for every $S \subseteq X$ with $|S| \leq n/2$. An (n, k, c)-**magnifier** is an n-vertex graph G such that $\Delta(G) \leq k$ and that $\left|N(S) \cap \overline{S}\right| \geq c \cdot |S|$ for every $S \subseteq V(G)$ with $|S| \leq n/2$. An n-**superconcentrator** is an acyclic digraph with n sources and n sinks such that for every set A of sources and every set B of $|A|$ sinks, there are $|A|$ disjoint A, B-paths.

Expanders appear in the parallel sorting network of Ajtai, Komlós, and Szemerédi [1983]. The condition for expansion strengthens Hall's Condition; we have not one matching but many. This facilitates using expanders to construct superconcentrators. Applications of superconcentrators are discussed in Alon [1986a]. The bound on maximum degree makes the number of edges linear in n, thereby limiting the cost of constructing the network.

Probabilistic methods (Exercise 22) yield the *existence* of expanders (and superconcentrators) with large n and bounded average degree (Pinsker [1973]), Pippenger [1977], Chung [1978b]). Margulis [1973] used algebraic ideas to construct an explicit example (see also Gabber–Galil [1981]).

Although an appropriately generated random graph will almost always have good expansion properties, it is hard to measure expansion. Tanner [1984] and Alon–Milman [1984, 1985] independently used eigenvalues to remedy this. They proved that graphs have good expansion properties when the two largest eigenvalues are far apart. Since eigenvalues are easy to compute (or approximate), we can generate a graph randomly and then compute its eigenvalues to check the amount of expasion.

We consider only the special case of regular graphs. Expanders are more useful than magnifiers in applications, but it is easy to obtain an $(n, (k + 1), c)$-expander from an (n, k, c)-magnifier (Exercise 21). Hence we consider the relationship between eigenvalues and magnification. Our presentation follows that of Alon–Spencer [1992, p119ff], which discusses additional properties of the eigenvalues of regular (and random) graphs.

8.6.30. Theorem. If G is a k-regular n-vertex graph with second-largest eigenvalue λ, and S is a nonempty proper subset of $V(G)$, then

$$\left|[S,\overline{S}]\right| \ge (k-\lambda)\,|S|\,\left|\overline{S}\right|/n.$$

Proof: Since G is k-regular, $\lambda_{\max}(G) = k$. The claim is trivial if $k - \lambda = 0$, so we may assume that G is connected. We compute

$$x^T(kI - A)x \;=\; k\sum x_i^2 - 2\sum_{ij\in E(G)} x_i x_j \;=\; \sum_{ij\in E(G)} (x_i - x_j)^2.$$

Now let $s = |S|$ and set $x_i = -(n-s)$ for $i \in S$ and $x_i = s$ for $i \notin S$. The sum on the right above becomes $n^2\left|[S,\overline{S}]\right|$.

Because $|S| = s$ implies $\sum x_i = 0$, the vector x is orthogonal to the eigenvector $\mathbf{1}_n$ of A with eigenvalue k. The eigenvector $\mathbf{1}_n$ is also the eigenvector of $kI - A$ for its smallest eigenvalue 0. Using Lemma 8.6.10 and Theorem 8.6.11, the minimum of $\frac{x^T(kI-A)x}{x^Tx}$ over vectors orthogonal to $\mathbf{1}_n$ is the next smallest eigenvalue of $kI - A$, which is $k - \lambda$. Hence

$$x^T(kI - A)x \;\ge\; (k-\lambda)x^Tx \;=\; (k-\lambda)\left(s(n-s)^2 + (n-s)s^2\right) \;=\; (k-\lambda)s(n-s)n.$$

Since $x^T(kI - A)x = n^2\left|[S,\overline{S}]\right|$, we have $\left|[S,\overline{S}]\right| \ge (k-\lambda)s(n-s)/n$. ■

8.6.31. Corollary. If G is a k-regular n-vertex graph with second-largest eigenvalue λ, then G is an (n,k,c)-magnifier, where $c = (k-\lambda)/2k$.

Proof: If S is a set of $s \le n/2$ vertices in G, then Theorem 8.6.30 yields $\left|[S,\overline{S}]\right| \ge k-\lambda)s(n-s)/n$. Each vertex of $\overline{S}$ receives at most k of these edges, so S must have at least $(k-\lambda)s(n-s)/(nk)$ neighbors in $\overline{S}$. Since $(n-s)/n \ge 1/2$, the result follows. ■

Greater separation between the two largest eigenvalues yields greater magnification. Alon and Milman [1984] improved the lower bound to $c \ge (2k-2\lambda)/(3k-2\lambda)$. Alon [1986b] proved a partial converse: If a k-regular graph G is an (n,k,c)-magnifier, then the separation $k-\lambda$ is at least $c^2/(4+2c^2)$.

Explicit constructions of regular graphs are known with separation between λ_1 and λ_2 nearly as large as possible. The second largest eigenvalue of a k-regular graph with diameter d is at least $2\sqrt{k}-1(1-O(1/d))$ (see Nilli [1991]). Lubotzky–Phillips–Sarnak [1986] and Margulis [1988] constructed infinite families of regular graphs where the degree k is 1 more than a prime congruent to 1 mod 4 and the second largest eigenvalue is at most $2\sqrt{k}-1$.

STRONGLY REGULAR GRAPHS

We close with an application to a special class of regular graphs.

8.6.32. Definition. A simple n-vertex graph G is **strongly regular** if there are parameters k, λ, μ such that G is k-regular, every adjacent pair of vertices have λ common neighbors, and every nonadjacent pair of vertices have μ common neighbors.

Properties of eigenvalues of strongly regular graphs provide a short proof of a curious result called the "Friendship Theorem". This theorem says that at any party at which every pair of people have exactly one common acquaintance, there is one person who knows everyone (presumably the host). The resulting graph of the acquaintance relation consists of some number of triangles sharing a vertex. Another motivation for studying strongly regular graphs is their connection with the theory of designs. Strongly regular graphs with $\lambda = \mu$ correspond to symmetric balanced incomplete block designs. Other regular graphs with rich algebraic structure appear in Biggs [1993, part 3].

8.6.33. Theorem. If G is a strongly regular graph with n vertices and parameters k, λ, μ, then $\overline{G}$ is strongly regular with parameters $k' = n - k - 1$, $\lambda' = n - 2 - 2k + \mu$, and $\mu' = n - 2k + \lambda$.

Proof: For each adjacent pair $v \leftrightarrow w$ in G, there are $2(k-1) - \lambda$ other vertices in $N(v) \cup N(w)$, so v and w have $n - 2 - 2(k-1) + \lambda$ common nonneighbors. When $v \not\leftrightarrow w$ there are $2k - \mu$ vertices in $N(v) \cup N(w)$ and thus $n - 2k + \mu$ common nonneighbors. ∎

8.6.34. Theorem. If G is a strongly regular graph with n vertices and parameters k, λ, μ, then $k(k - \lambda - 1) = \mu(n - k - 1)$.

Proof: We count induced copies of P_3 with a fixed vertex v as an endpoint. The middle vertex w can be picked in k ways. For each such w, the third vertex can be any neighbor of w not adjacent to v. With v unavailable, there are always $k - \lambda - 1$ ways to pick the third vertex. On the other hand, the third vertex can be picked in $n - k - 1$ ways as a nonneighbor of v, and for each such choice there are μ common neighbors with v that can serve as w. ∎

8.6.35. Example. *Degenerate cases:* $\mu = 0$ *or* $\lambda = k - 1$ *or* $k = n - 1$. We show that such a strongly regular graph is a disjoint union of $k + 1$-cliques. By Theorem 8.6.34, $\lambda = k - 1$ if and only if $\mu = 0$ or $k = n - 1$. Hence we may assume that $\lambda = k - 1$. Now every neighbor of v is adjacent to every other, which forbids an induced P_3 and forces G to be a disjoint union of cliques. ∎

Henceforth, we assume that $\mu > 0$ and $\lambda < k - 1$. Theorem 8.6.34 states a necessary condition on the set of parameters for a strongly regular graph. Another necessary condition arises from the eigenvalues.

8.6.36. Theorem. (Integrality Condition) If G is strongly regular with n vertices and parameters k, λ, μ, then the two numbers below are nonnegative integers.

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}}\right)$$

Proof: These are nonnegative integers because they are multiplicities of eigenvalues. Consider A^2. The ijth entry of A^2 is k if $i = j$, is λ if $v_i \leftrightarrow v_j$, and is μ if $v_i \not\leftrightarrow v_j$. Since $v_i \leftrightarrow v_j$ marks the 1s in the adjacency matrix and $v_i \not\leftrightarrow v_j$ marks the 1s in the adjacency matrix of the complement, we have $A^2 = kI + \lambda A + \mu(J - I - A)$. Rearranging terms yields $A^2 = (k-\mu)I + (\lambda-\mu)A + \mu J$.

Multiplying $\mathbf{1}_n$ by both expressions for A^2 yields

$$k^2\mathbf{1}_n = (k-\mu)\mathbf{1}_n + (\lambda-\mu)k\mathbf{1}_n + \mu n\mathbf{1}_n,$$

which yields another proof of $k(k-\lambda-1) = \mu(n-k-1)$. Let x be an eigenvector for another eigenvalue $\theta \neq k$. Since x is orthogonal to $\mathbf{1}_n$, we have $Jx = \mathbf{0}_n$. Multiplying x by both expressions for A^2 produces $\theta^2 - (\lambda-\mu)\theta - (k-\mu) = 0$. This quadratic equation for θ has two roots r, s, which must be the values of all the other eigenvalues. The values are $\frac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right)$.

Now let a and b be the multiplicities of the eigenvalues r and s. Example 8.6.35 describes all cases when $\mu = 0$. Hence we may assume that $\mu > 0$, and thus G is connected. Because G is connected, eigenvalue k has multiplicity 1, and we have $1+a+b = n$. Since the eigenvalues sum to 0, we have $k+ra+sb = 0$. The solution to these two linear equations for a and b is $a = -\frac{k+s(n-1)}{r-s}$ and $b = \frac{k+r(n-1)}{r-s}$. These are the values claimed above to be nonnegative integers. ■

The argument above can also be traced in the opposite direction.

8.6.37. Theorem. A k-regular connected graph G is strongly regular with parameters k, λ, μ if and only if it has exactly three eigenvalues $k > r > s$ and these satisfy $r + s = \lambda - \mu$ and $rs = -(k - \mu)$. ■

8.6.38. Example. *Classes of strongly regular graphs.* We consider two cases: $(n-1)(\mu-\lambda) = 2k$ and $(n-1)(\mu-\lambda) \neq 2k$. Excluding the trivial values, the first case requires $\mu = \lambda + 1$, because $0 < 2k < 2n - 2$. By Theorem 8.6.33, G and $\overline{G}$ are thus strongly regular graphs with the same parameters. In this case, we also know that $n = 4\mu + 1$ and that n is the sum of two perfect squares. Furthermore, the eigenvalues r and s have the same multiplicity.

In the second case, rationality requires $(\mu-\lambda)^2 + 4(k-\mu) = d^2$ for some positive integer d, and d must divide $(n-1)(\mu-\lambda) - 2k$. Here the eigenvalues must be integers. Various such examples are known. In the special case $\lambda = 0$ and $\mu = 2$, three such graphs are known, but it is not known whether the list is finite! The known examples, listing the parameters (n, k, λ, μ), are the square $(4, 2, 0, 2)$, the Clebsch graph $(16, 5, 0, 2)$, and the Gewirtz graph $(56, 10, 0, 2)$ (see Cameron–van Lint [1991], p43). The Clebsch graph arises in Exercise 23. Other strongly regular graphs appear in Exercises 24–26. ■

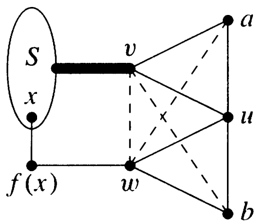
Finally, we prove the Friendship Theorem. It is startling that such a combinatorial-sounding result seems to have no short combinatorial proof. There do exist proofs avoiding eigenvalues (see Hammersley [1983]), but they require complicated numerical arguments to eliminate regular graphs.

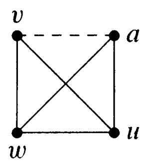
8.6.39. Theorem. (Friendship Theorem—Wilf [1971]). If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others.

Proof: The symmetry of the condition suggests that G might be regular. If G is regular, then it is strongly regular with $\lambda = \mu = 1$. By Theorem 8.6.36, $\frac{1}{2}(n-1 \pm k/\sqrt{k-1})$ now must be an integer. Hence $k/\sqrt{k-1}$ is an integer, which happens only when $k = 2$. However, K_3 is the only 2-regular graph satisfying the condition, and it does have vertices of degree $n - 1$.

Now suppose that G is not regular. We show that $v \not\leftrightarrow w$ requires $d(v) = d(w)$. Insistence on unique common neighbors forbids 4-cycles. Let u be the common neighbor of $\{v, w\}$. Let a be the common neighbor of $\{u, v\}$, and let b be the common neighbor of $\{u, w\}$. Every $x \in S = N(v) - \{u, a\}$ has a common neighbor $f(x)$ with w. If $f(x) = b$ for some $x \in S$, then x, b, u, v is a 4-cycle. If $f(x) = f(x')$ for distinct $x, x' \in S$, then $x, v, x', f(x)$ is a 4-cycle. We have thus shown that $d(w) \geq d(v)$. By symmetry, $d(v) \geq d(w)$.

Since G is not regular, it has two vertices v, w with $d(w) \neq d(v)$. By the preceding paragraph, $v \leftrightarrow w$. Let u be their common neighbor. Since u cannot have the same degree as each of them, we may assume that $d(u) \neq d(v)$. If G has a vertex $x \notin N(v)$, then $d(x) = d(v)$, but this requires $x \leftrightarrow w$ and $x \leftrightarrow u$. This creates the 4-cycle v, u, x, w. Hence $d(v) = n - 1$. ■





EXERCISES

8.6.1. *Interpretation of cycle space and bond space.* Given a graph G, prove that
a) The symmetric difference of two even subgraphs is an even subgraph.
b) The symmetric difference of two edge cuts is an edge cut, and
c) Every edge cut shares an even number of edges with every even subgraph.

8.6.2. *Dimension of cycle space and bond space.* By parts (a) and (b) of Exercise 8.6.1, the cycle space **C** and bond space **B** of a graph G are binary vector spaces. Prove that when G is connected, **C** has dimension $e(G) - n(G) + 1$ and **B** has dimension $n(G) - 1$. (Hint: Show that the cycles created by adding one edge to a particular spanning tree form a basis for the cycle space. Show that $n(G) - 1$ bonds that isolate single vertices form a basis for the bond space, or use orthogonality.)

8.6.3. Recall that the *closed neighborhood* of a vertex v is $N(v) \cup \{v\}$.

a) Let S be a set of vertices in a simple graph G whose neighborhoods are identical. Prove that some eigenvalue of G has multiplicity at least $|S| - 1$. What is it?

b) Let S be a set of vertices in a simple graph G whose closed neighborhoods are identical. Prove that some eigenvalue of G has multiplicity at least $|S| - 1$. What is it?

8.6.4. Let σ_k be the number of subgraphs of a graph G that are k-cycles. Let $L_k = \sum \lambda_i^k$ and $D_k = \sum d_i^k$ be the sums of the kth powers of the eigenvalues and the vertex degrees. Obtain formulas for σ_3 and σ_4 in terms of $\{L_k\}$ and $\{D_k\}$.

8.6.5. *Deletion formulas for the characteristic polynomial.* For clarity in this problem, we write $\phi(G;\lambda)$ as ϕ_G. Let v $[xy]$ be an arbitrary vertex [edge] of G, and let $Z(v)$ $[Z(xy)]$ be the collection of cycles containing v $[xy]$. Prove that the characteristic polynomial satisfies the following recurrences.

a) $\phi_G = \lambda\phi_{G-v} - \sum_{u\in N(v)} \phi_{G-v-u} - 2\sum_{C\in Z(v)} \phi_{G-V(C)}$.

b) $\phi_G = \phi_{G-xy} - \phi_{G-x-y} - 2\sum_{C\in Z(xy)} \phi_{G-V(C)}$.

(Hint: Induction or Sach's formula can be used. Also, the edge-deletion formula can be proved from the vertex-deletion formula. Comment: When G is a forest and v is a leaf with neighbor u, the formulas reduce to $\phi_G = \lambda\phi_{G-v} - \phi_{G-v-u}$ and $\phi_G = \phi_{G-xy} - \phi_{G-x-y}$.)

8.6.6. *Characteristic polynomial for paths and cycles.*

a) Use Exercise 8.6.5 to find recurrences for $\phi(P_n;\lambda)$ and for $\phi(C_n;\lambda)$.

b) Without solving the recurrence, prove that $\{2\cos(2\pi j/n)\colon 0 \le j \le n-1\}$ are the eigenvalues of C_n.

c) Given Spec (C_n), compute Spec G, where G is the graph obtained from C_n by adding edges joining vertices at distance 2 in C_n.

8.6.7. For a tree, prove that the coefficient of λ^{n-2k} in the characteristic polynomial is $(-1)^k\mu_k(G)$, where $\mu_k(G)$ is the number of matchings of size k. Use this to construct a pair of nonisomorphic "co-spectral" 8-vertex trees; both have characteristic polynomial $\lambda^8 - 7\lambda^6 + 9\lambda^4$. (Comment: As $n \to \infty$, almost no trees are uniquely determined by their spectra.) (Schwenk [1973])

8.6.8. (+) Let T be a tree. Prove that $\alpha(T)$ is the number of nonnegative eigenvalues of T. (Hint: See Theorem 8.6.20.) (Cvetković–Doob–Sachs [1979, p233])

8.6.9. Let λ be an eigenvalue of a graph G with n vertices and m edges. Prove that $|\lambda| \le \sqrt{2m(n-1)/n}$.

8.6.10. Let $\lambda_1, \dots, \lambda_m$ and $\mu_1, \dots, \mu_n$ be the eigenvalues of G and H, respectively. Show that the mn eigenvalues of $G \square H$ are $\{\lambda_i + \mu_j\}$. Use this to derive the spectrum of the k-cube. (Hint: Given an eigenvector of $A(G)$ associated with λ_i and an eigenvector of $A(H)$ associated with μ_j, construct an eigenvector for $A(G \square H)$ associated with $\lambda_i + \mu_j$.)

8.6.11. Compute the spectrum of the complete p-partite graph $K_{m,\dots,m}$. (Hint: Use the expression $A(\overline{G}) = J - I - A(G)$ for the adjacency matrix of the complement.)

8.6.12. Given $\phi(G;x) = x^8 - 24x^6 - 64x^5 - 48s^4$, determine G.

8.6.13. (!) Prove that G is bipartite if G is connected and $\lambda_{\max}(G) = -\lambda_{\min}(G)$.

8.6.14. (!) Given a graph G, let $R(G)$ be the matrix whose i,jth entry is $d_G(v_i, v_j)$. Prove that the squashed-cube dimension of a graph (Definition 8.4.12) is at least the maximum of the number of positive eigenvalues and the number of negative eigenvalues of $R(G)$. Conclude that the squashed cube dimension of K_n is $n-1$. (Hint: Rewrite the quadratic form $x^T Rx$ as a sum of squares of linear functions, and apply Sylvester's Law of Inertia.)

8.6.15. (!) The **Laplacian matrix** Q of a graph G is $D - A$, where D is the diagonal matrix of degrees and A is the adjacency matrix. The **Laplacian spectrum** is the list of eigenvalues of Q.

a) Prove that the smallest eigenvalue of Q is 0.

b) Prove that if G is connected, then eigenvalue 0 has multiplicity 1.

c) Prove that if G is k-regular, then $k - \lambda$ is a Laplacian eigenvalue if and only if λ is an ordinary eigenvalue of G, with the same multiplicity.

8.6.16. Given a real symmetric matrix partitioned as $M = \binom{P\ Q}{Q^T\ R}$ with P, R square, a lemma in linear algebra yields $\lambda_{\max}(M) + \lambda_{\min}(M) \le \lambda_{\max}(P) + \lambda_{\max}(R)$.

a) Let A be a real symmetric matrix partitioned into t^2 submatrices $A_{i,j}$ such that the diagonal submatrices A_{ii} are square. Prove that

$$\lambda_{\max}(A) + (t-1)\lambda_{\min}(A) \le \sum_{i=1}^{m} \lambda_{\max}(A_{ii}).$$

b) Prove that $\chi(G) \ge 1 + \lambda_{\max}(G)/(-\lambda_{\min}(G))$ when G is nontrivial. (Wilf)

c) Use the Four Color Theorem to prove that $\lambda_1(G) + 3\lambda_n(G) \le 0$ for planar graphs.

8.6.17. (!) Use Theorem 8.6.28 to count the spanning trees in $K_{m,m}$. (Comment: See Exercise 2.2.11.)

8.6.18. (+) Given a matrix A, let $b_{i,j}$ equal $(-1)^{i+j}$ times the matrix obtained by deleting row i and column j of A. Let Adj A be the matrix whose entry in position i, j is $b_{j,i}$. The definition of the determinant by expansion along rows of A yields $A(\text{Adj } A) = (\det A)I$. Use this formula to prove that if the sum of the columns of A is the vector 0, then $b_{i,j}$ is independent of j. (Comment: With the next exercise, this completes the proof of the Matrix Tree Theorem (Theorem 2.2.12).)

8.6.19. (+) Let $C = AB$, where A and B are $n \times m$ and $m \times n$ matrices. Given $S \subseteq [m]$, let A_S be the $n \times n$ matrix whose columns are the columns of A indexed by S, and let B_S be the $n \times n$ matrix whose rows are the rows of B indexed by S. Prove the Binet–Cauchy Formula: $\det C = \sum_S \det A_S \det B_S$, where the summation extends over all n-element subsets of $[m]$. (Hint: Consider the matrix equation $\begin{pmatrix} I_m & 0 \\ A & I_n \end{pmatrix}\begin{pmatrix} -I_m & B \\ A & 0 \end{pmatrix} = \begin{pmatrix} -I_m & B \\ 0 & AB \end{pmatrix}$.)

8.6.20. A matrix is **totally unimodular** if every square submatrix has determinant in $\{0, 1, -1\}$. Prove that the incidence matrix of a simple graph is totally unimodular if and only if the graph is bipartite. (Reminder: The incidence matrix of a simple graph has two $+1$'s in each column).

8.6.21. $(-)$ Let G be an (n, k, c)-magnifier with vertices $v_1, \ldots, v_n$. Let H be the bipartite graph with partite sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $x_i y_j \in E(H)$ if and only if $i = j$ or $v_i v_j \in E(G)$. Prove that H is an $(n, k+1, c)$-expander.

8.6.22. *Existence of expanders of linear size.*

a) Let X be a random variable giving the size of the union of k s-subsets of $[n]$ chosen at random from $\binom{[n]}{s}$. Prove that $P(X \le l) \le \binom{n}{l}(l/n)^{ks}$.

b) (+) For $\alpha\beta < 1$, prove that there is a constant k such that, when n is sufficiently large, there exists a subgraph of $K_{n,n}$ with maximum degree at most k such that $|N(S)| \ge \beta\,|S|$ whenever $|S| \le \alpha n$. (Hint: Generate bipartite subgraphs of $K_{n,n}$ by taking the union of k random perfect matchings.)

c) Conclude the existence of k such that n, k, c-expanders exist for all sufficiently large n. An (n, α, β, d)**-expander** is a bipartite graph $G \subseteq K_{A,B}$ with $|A| = |B| = n$, $\Delta(G) \le d$, and $|N(S)| \ge \beta\,|S|$ whenever $|S| \le \alpha n$.

8.6.23. Let G be a triangle-free graph on n vertices in which every pair of nonadjacent vertices has exactly two common neighbors. Prove that G is regular and that $n = 1 +$

$\binom{k+1}{2}$, where k is the degree of the vertices in G. Prove that G is strongly regular. What constraints on k are implied by the integrality conditions? Construct examples for all $k \in \{1, 2, 5\}$. A realization for $k = 10$ is known using combinatorial designs.)

8.6.24. (+) Prove that the Petersen graph is strongly regular, and determine its spectrum (the spectrum is easy with properties of strongly regular graphs and not hard without them). Apply the spectrum to show that edges of the complete graph K_{10} cannot be partitioned into three disjoint copies of the Petersen graph. (Hint: Use the spectrum to prove that two copies of the Petersen matrix have a common eigenvector other than the constant vector.) (Schwenk [1983])

8.6.25. Let $F = G \square H$, where G and H are simple graphs. Prove that if every two nonadjacent vertices in F have exactly two common neighbors, then G and H are complete graphs.

8.6.26. The **subconstituents** of a graph are the induced subgraphs of the form $G[U]$, where $v \in V(G)$ and $U = N(v)$ or $U = \overline{N[v]}$. Vince [1989] defined G to be **superregular** if G has no vertices or if G is regular and every subconstituent of G is superregular. Let **S** be the class consisting of $\{aK_b\colon a, b \ge 0\}$ (disjoint unions of isomorphic cliques), $\{K_m \square K_m\colon m \ge 0\}$, C_5, and the complements of these graphs.

a) Prove that every graph in **S** is superregular and that every disconnected superregular graph is in **S**. (Comment: In fact, every superregular graph is in **S**, but the complete inductive proof of this requires several pages (Maddox [1996], West [1996])

b) Prove that every superregular graph is strongly regular.

8.6.27. (+) *Automorphisms and eigenvalues.*

a) Prove that σ is an automorphism of G if and only if the permutation matrix corresponding to σ commutes with the adjacency matrix of G; that is, $PA = AP$.

b) Let x be an eigenvector of G for an eigenvalue of multiplicity 1, and let P be the permutation matrix for an automorphism of G. Prove that $Px = \pm x$.

c) Conclude that when every eigenvalue of G has multiplicity 1, every automorphism of G is an involution, meaning that repeating it yields the identity. (Mowshowitz [1969], Petersdorf–Sachs [1969])

8.6.28. (+) Light bulbs $l_1, \ldots, l_n$ are controlled by switches $s_1, \ldots, s_n$. The ith switch changes the on/off status of the ith light and possibly others, but s_i changes the status of l_j if and only if s_j changes the status of l_i. Initially all the lights are off. Prove that it is possible to turn all the lights on. (Peled [1992]) (Hint: This uses vector spaces, not eigenvalues.)

Appendix A

Mathematical Background

This appendix summarizes aspects of language and mathematics that are not directly part of graph theory but provide useful background for learning graph theory. Where appropriate, we mention examples in the context of graphs, so it is best to read this appendix in conjunction with Chapter 1. This presentation is modeled on material in the first half of *Mathematical Thinking*, by John P. D'Angelo and Douglas B. West (Prentice–Hall, second edition, 2000).

SETS

Our most primitive mathematical notion is that of a **set**. It is so fundamental that we cannot define it in terms of simpler concepts. We think of a set as a collection of distinct objects with a precise description that provides a way of deciding (in principle) whether a given object is in it.

A.1. Definition. The objects in a set are its **elements** or **members**. When x is an element of A, we write $x \in A$ and say "x **belongs to** A". When x is not in A, we write $x \notin A$. If every element of a set B belongs to A, then B is a **subset** of A, and A **contains** B; we write $B \subseteq A$ or $A \supseteq B$.

For example, we may speak of the set A of graphs with n vertices. When we impose an additional restriction, such as requiring that the graphs also be connected, we obtain a subset of A.

When we list the elements of a set explicitly, we put braces around the list; "$A = \{-1, 1\}$" specifies the set A consisting of the elements -1 and 1. Writing the elements in a different order does not change a set. We write $x, y \in S$ to mean that both x and y are elements of S.

A.2. Example. We use the characters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ to name the sets of **natural numbers, integers, rational numbers**, and **real numbers**, respectively. Each set in this list is contained in the next, so we write $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

We treat these sets and their elements as familiar objects. By convention, 0 is not a natural number, so $\mathbb{N} = \{1, 2, 3, \ldots\}$. The set of integers is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The set $\mathbb{Q}$ of rational numbers is the set of real numbers expressible as a/b with $a, b \in \mathbb{Z}$ and $b \neq 0$.

We also take as familiar the elementary arithmetic properties of these number systems. These include the rules that permit algebraic manipulation of expressions, equalities, and inequalities. They also include elementary properties about divisibility of integers. ∎

A.3. Definition. Sets A and B are **equal**, written $A = B$, if they have the same elements. The **empty set**, written $\varnothing$, is the unique set with no elements. A **proper subset** of a set A is a subset that is not A itself.

The empty set is a subset of every set, and every set is a subset of itself. The definition of subgraph (Definition 1.1.16) is similar. Every graph is a subgraph of itself, but something must be discarded to obtain a proper subgraph.

"Solving a mathematical problem" often means describing a given set more simply. We must show that the set of objects satisfying the new description is equal to the given set.

A.4. Remark. *Equality of sets.* To prove that $A = B$, we prove that every element of A is in B and that every element of B is in A; in other words, $A \subseteq B$ and $B \subseteq A$. It also suffices to turn the description of one set into the description of the other by operations that do not change membership.

This book proves many characterization theorems for classes of graphs. Such a theorem states that two sets are the same (example: the set of bipartite graphs is equal to the set of graphs without odd cycles—Theorem 1.2.18).

Often, a mathematical model defines a set S of *solutions*; these are the objects that satisfy the conditions of the problem. We want to list or describe the solutions explicitly; this specifies a set T. The problem is to show that $S = T$. Proving $S \subseteq T$ means showing that every solution belongs to T. Proving $T \subseteq S$ means showing that every member of T is a solution. ∎

A.5. Remark. *Specifying a set.* Given a set A, we may want to specify a subset S consisting of the elements of A that satisfy a given condition. To do so, we write "$S = \{x \in A\colon \text{condition}(x)\}$". We read this as "$S$ is the set of elements x in A such that x satisfies 'condition' ". For example, the expression $\{n \in \mathbb{N} : n^2 \leq 25\}$ is another way to name the set $\{1, 2, 3, 4, 5\}$.

In this format, the set A is the **universe** for x; we can drop this part of the notation when the context makes it clear. For example, $\{n^2\colon n \in \mathbb{N}\}$ is the set of positive integers squares. ∎

Many special sets have common names and/or notation.

A.6. Definition. When $a, b \in \mathbb{Z}$, we write $\{a, \ldots, b\}$ for $\{i \in \mathbb{Z}\colon a \leq i \leq b\}$. When $n \in \mathbb{N}$, we write $[n]$ for $\{1, \ldots, n\}$; also $[0] = \varnothing$. The set of **even numbers**

is $\{2k\colon k \in \mathbb{Z}\}$. The set of **odd numbers** is $\{2k+1\colon k \in \mathbb{Z}\}$. The **parity** of an integer states whether it is even or odd.

Note that 0 is an even number. We say "even" and "odd" for numbers *only* when discussing integers. Every integer is even or odd; none is both.

A.7. Definition. A **partition** of a set A is a list $A_1, \ldots, A_k$ of subsets of A such that each element of A appears in exactly one subset in the list.

The set of even numbers and the set of odd numbers partition $\mathbb{Z}$. In a partition of A into $A_1, \ldots, A_k$, the sets $A_1, \ldots, A_k$ in the list are called "blocks" or "classes" or "parts" or "partite sets". The use of "blocks" is common in combinatorics, but graph theory has another definition for the word "block", so we usually use "classes" or "sets". "Partite sets" is used only for the sets in a partition of the vertex set of a graph into independent sets.

A.8. Remark. *Conventions about universes.* When we write "$[n]$", it is understood that n is a nonnegative integer. When we speak of n as the number of vertices in a graph, by context we know that n is a natural number. When we say only that a number is positive without specifying the number system containing it, we mean that it is a positive real number. Thus, "consider $x > 0$" means "let x be a positive real number", but in "For $n \geq 2$, let G be a n-vertex graph" our convention is that $n \in \mathbb{N}$. ∎

A.9. Definition. A set A is **finite** if there is a one-to-one correspondence between A and $[n]$ for some $n \in \mathbb{N} \cup \{0\}$. This n is the **size** of A, written $|A|$.

Another elementary property of number systems is that a set A cannot be in one-to-one correspondence with both $[m]$ and $[n]$ when $m \neq n$. Thus the size of a finite set is a well-defined integer. **Counting** a set means determining its size.

A.10. Remark. *"If" in definitions.* It is a common convention in defining mathematical properties to say that an object has a certain property **if** it satisfies a certain condition. Subsequently, the condition can be substituted for the property and vice versa, so the "if" really means "if and only if". This conventional usage in definitions reflects the notion that the concept being defined does not exist until the definition is complete. ∎

There are several natural ways to obtain new sets from old sets.

A.11. Definition. Let A and B be sets. Their **union** $A \cup B$ consists of all elements in A or in B (or both). Their **intersection** $A \cap B$ consists of all elements in both A and B. Their **difference** $A - B$ consists of the elements of A that are not in B. Their **symmetric difference** $A \triangle B$ is the set of elements belonging to exactly one of A and B.

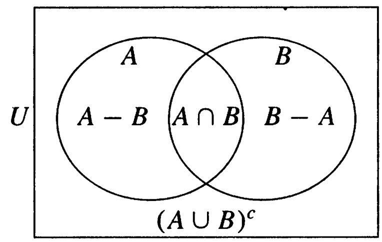
Two sets are **disjoint** if their intersection is the empty set $\varnothing$. If a set A

is contained in some universe U under discussion, then the **complement** $\overline{A}$ of A is the set of elements of U *not* in A.

When we speak of taking the "complement" of a simple graph, we are keeping the vertex set unchanged and taking the complement of the edge set (viewed as pairs of vertices) within the universe of vertex pairs. Other times we speak of the complement $\overline{S}$ of a set of vertices S in G; in this case we mean $\overline{S} = V(G) - S$.

A.12. Remark. In a **Venn diagram**, an outer box represents the universe under consideration, and regions within the box correspond to sets. Nonoverlapping regions correspond to disjoint sets. The four regions in the Venn diagram for two sets A and B represent $A \cap B$, $\overline{(A \cup B)}$, $A - B$, and $B - A$. Note that $A \triangle B = (A - B) \cup (B - A)$,

Since $A - B$ consists of the elements in A and not in B, we have $A - B = A \cap \overline{B}$. Similarly, the diagram suggests that $\overline{B}$ is the union of $A - B$ and $\overline{(A \cup B)}$, which are disjoint. It also suggests that the symmetric difference $A \triangle B$ is obtained from the union by deleting the intersection. ■



A.13. Remark. *When A and B are sets,* $A \triangle B = (A \cup B) - (A \cap B)$. The union starts with all elements in at least one of A and B; we delete those in both.

When A and B are finite sets, $|A \cup B| + |A \cap B| = |A| + |B|$. Each element of the intersection is counted twice on both sides, each element of the symmetric difference is counted once on both sides, and no other elements are counted. ■

A.14. Definition. A **list** with entries in A consists of elements of A in a specified order, with repetition allowed. A **k-tuple** is a list with k entries. We write A^k for the set of k-tuples with entries in A. When $A = \{0, 1\}$, A^k is the set of **binary k-tuples**.

An **ordered pair** (x, y) is a list with two entries. The **cartesian product** of sets S and T, written $S \times T$, is the set $\{(x, y)\colon x \in S, y \in T\}$.

Note that $A^2 = A \times A$ and $A^k = \{(x_1, \ldots, x_k)\colon x_i \in A\}$. We read "$x_i$" as "$x$ sub i". When $S = T = \mathbb{Z}$, the cartesian product $S \times T$ is the **integer lattice**, the set of points in the plane with integer coordinates.

QUANTIFIERS AND PROOFS

Roughly speaking, a mathematical statement is a statement that can be determined to be true or false. This requires correct mathematical grammar, and it requires that variables be "quantified".

For example, the sentence $x^2 - 4 = 0$ cannot be determined to be true or false because we do not know the value of x. It becomes a mathematical statement if we precede it with "When $x = 3$," or "For $x \in \{2, -2\}$," or "For some integer x,".

If a sentence $P(x)$ becomes a mathematical statement whenever the variable x takes a value in the set S, then the two sentences below are mathematical statements.

"For all x in S, the sentence $P(x)$ is true."
"For some x in S, the sentence $P(x)$ is true."

A.15. Definition. In the statement "For all x in S, $P(x)$ is true", the variable x is **universally quantified**. We write this as $(\forall x \in S)P(x)$ and say that $\forall$ is a **universal quantifier**. In "For some x in S, $P(x)$ is true", the variable x is **existentially quantified**. We write this as $(\exists x \in S)P(x)$ and say that $\exists$ is an **existential quantifier**. The set of allowed values for a variable is its **universe**.

A.16. Remark. *English words that express quantification.* Typically, "every" and "for all" represent universal quantifiers, while "some" and "there is" represent existential quantifiers. We can also express universal quantification by referring to an arbitrary element of the universe, as in "Let x be an integer", or "A student failing the exam will fail the course". Below we list common indicators of quantification.

Universal ($\forall$)	(helpers)	Existential ($\exists$)	(helpers)
for [all], for every		for some	
if	then	there exists	such that
whenever, for, given		at least one	for which
every, any	satisfies	some	satisfies
a, arbitrary	must, is	has a	such that
let	be		

The "helpers" may be absent. Consider "The square of a real number is nonnegative". This means $x^2 \geq 0$ for *every* $x \in \mathbb{R}$; it is not a statement about one real number and cannot be verified by an example. When we write "A bipartite graph has no odd cycle", we mean "in every bipartite graph there is no odd cycle". When we write "Let G be a bipartite graph", we mean that every bipartite graph is under consideration. When we take an "arbitrary" vertex in a graph, we are considering each one individually. When we discus an "arbitrary" pair of vertices in a graph, we are considering each pair, one at a time.

The difference between "for every G" and "for every graph G" is that the latter specifies the universe for the universally quantified variable G. ∎

Existential quantifiers state lower bounds; "there is a" and "there are two" mean "at least one" and "at least two". Phrases like "there is a unique" and "there are exactly two" indicate equality. Sometimes equality is clear from context, but it does not hurt to make it explicit when it is intended.

A statement may have more than one quantifier. Consider the sentence "There are triangle-free graphs with arbitrarily large chromatic number". Phrased using explicit quantifiers, this means "For every $n \in \mathbb{N}$, there exists a triangle-free graph with chromatic number at least n". The expression "arbitrarily large" often conveys an implicit universal quantifier in this way.

In contrast, the expression "sufficiently large" imposes an implicit existential quantifier. The statement "$2^n > n^{1000}$ when n is sufficiently large" means "There exists $N \in \mathbb{N}$ such that for all $n \geq N$, the inequality $2^n > n^{1000}$ holds".

A.17. Remark. The meaning of a statement with more than one quantifier depends on their order. Compare these two sentences:

"For every graph G, there exists $m \in \mathbb{N}$ such that every $v \in V(G)$ has degree at most m"
"There exists $m \in \mathbb{N}$ such that for every graph G, every $v \in V(G)$ has degree at most m"

The first statement is true; the second is false. Every (finite) graph has a maximum degree, but there is no maximum over all graphs. We write the two sentences in logical notation as

$$(\forall G)(\exists m \in \mathbb{N})(\forall v \in V(G))(d_G(v) \leq m).$$
$$(\exists m \in \mathbb{N})(\forall G)(\forall v \in V(G))(d_G(v) \leq m).$$

In English, quantifiers often appear at the ends of sentence to enhance readability, as in "I feel happy every time I learn something new." In sentences with abstract concepts and more than one quantifier, we adopt conventions about order to avoid confusion. Quantifiers apply in the order in which they are stated. In particular, a variable is chosen in terms of the preceding variables.

For example, in $(\forall G)(\exists m \in \mathbb{N})P(G, m)$, we have the freedom to choose m after knowing what G is. In $(\exists m \in \mathbb{N})(\forall G)P(G, m)$, we must choose a single m that works for all G. ∎

A.18. Remark. *Negation of quantified statements.* The logical symbol for negation is $\neg$. If it is false that all $x \in S$ make $P(x)$ true, then there must be some $x \in S$ such that $P(x)$ is false. Similarly, negating an existentially quantified statement yields a universally quantified negation. In notation,

$\neg[(\forall x \in S)P(x)]$ has the same meaning as $(\exists x \in S)(\neg P(x))$.
$\neg[(\exists x \in S)P(x)]$ has the same meaning as $(\forall x \in S)(\neg P(x))$.

The universe of quantification *does not change* when the statement is negated.

For example, the false statement in Remark A.17 was

$$(\exists m \in \mathbb{N})(\forall G)(\forall v \in V(G))(d_G(v) \leq m).$$

Its negation is the same as $(\forall m \in \mathbb{N})(\exists G)[\neg((\forall v \in V(G))(d_G(v) \leq m))]$, which we further simplify to $(\forall m \in \mathbb{N})(\exists G)(\exists v \in V(G))(d_G(v) > m)$. This statement is "for every natural number m, there is some graph having a vertex with degree greater than m", which is true. ■

Logical connectives permit us to build compound statements.

A.19. Definition. *Logical connectives.* In the following table, we define the operations named in the first column by the truth values specified in the last column.

Name	Symbol	Meaning	Condition for truth
Negation	$\neg P$	not P	P false
Conjunction	$P \wedge Q$	P and Q	both true
Disjunction	$P \vee Q$	P or Q	at least one true
Biconditional	$P \Leftrightarrow Q$	P if & only if Q	same truth value
Conditional	$P \Rightarrow Q$	P implies Q	Q true whenever P true

A.20. Remark. Conjunction and disjunction are quantifiers over the truth of their component statements. A conjunction ("and") is true precisely when all of its component statements are true. A disjunction ("or") is true precisely when there exists a true statement among its components. Our understanding of negation thus yields logical equivalence between $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ and between $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$. ■

A.21. Definition. In the conditional statement $P \Rightarrow Q$, we call P the **hypothesis** and Q the **conclusion**. The statement $Q \Rightarrow P$ is the **converse** of $P \Rightarrow Q$.

A.22. Remark. *Conditionals.* Conditional statements are the only type in Definition A.19 whose meaning changes when P and Q are interchanged. There is no general implication between $P \Rightarrow Q$ and its converse $Q \Rightarrow P$. Consider these three statements about a graph G: P is "G is a path", Q is "G is bipartite", and R is "G has no odd cycles". Here $P \Rightarrow Q$ is true but $Q \Rightarrow P$ is false. On the other hand, both $Q \Rightarrow R$ and $R \Rightarrow Q$ are true.

Note that here G is a variable. We have dropped G from the notation for the statements because the context is clear. The precise meaning of $P \Rightarrow Q$ using G is $(\forall G)(P(G) \Rightarrow Q(G))$.

A conditional statement is false when and only when the hypothesis is true and the conclusion is false. Thus the meaning of $P \Rightarrow Q$ is $(\neg P) \vee Q$; the two are logically equivalent. Every conditional statement with a false hypothesis is true, regardless of whether the conclusion is true. The meaning of $\neg(P \Rightarrow Q)$ is $P \wedge (\neg Q)$.

Below we list ways to say $P \Rightarrow Q$ in English. ■

If P (is true), then Q (is true).	P is true only if Q is true.
Q is true whenever P is true.	P is a sufficient condition for Q.
Q is true if P is true.	Q is a necessary condition for P.

The business of mathematics is proving implications. Note that universally quantified statements can be interpreted as conditional statements. The statement "$(\forall G \in \mathbf{G})(P(G))$" has the same meaning as "If $G \in \mathbf{G}$, then $P(G)$" (consider the two statements when $\mathbf{G}$ is the family of bipartite graphs and $P(G)$ is the assertion that G has no odd cycles).

The basic proof methods come from the meaning of conditional statements.

A.23. Remark. *Proving implications.* The **direct method** of proving $P \Rightarrow Q$ is to assume that P is true and then to apply mathematical reasoning to deduce that Q is true. When P is "$x \in A$" and Q is "$Q(x)$", the direct method considers an *arbitrary* $x \in A$ and deduces $Q(x)$. There is no "proof by example". The proof must apply to every member of A as a possible instance of x.

The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. Each of these statements fails only when P is true and Q is false. Thus they are equivalent; we can prove $P \Rightarrow Q$ by proving $\neg Q \Rightarrow \neg P$. This is the **contrapositive method**.

We have observed that $(P \Rightarrow Q) \Leftrightarrow \neg[P \wedge (\neg Q)]$. Hence we can prove $P \Rightarrow Q$ by proving that P and $\neg Q$ cannot both be true. We do this by obtaining a contradiction after assuming both P and $\neg Q$. This is the **method of contradiction**.

The two latter methods are **indirect proof**. When the direct method for $P \Rightarrow Q$ doesn't seem to work, we say "Well, suppose not". At that point we are starting from the assumption $\neg Q$. We need not know in advance whether we are seeking to derive $\neg P$ (contrapositive method) or seeking to use P and $\neg Q$ to obtain a contradiction. ∎

Examples of each of these methods appear in the text. Indirect proof is promising when the negation of the conclusion provides useful information. This approach may be easier than finding a direct proof, because both the hypothesis and the negation of the conclusion can be used. If the contradiction we obtain is the impossibility of our original assumption $\neg Q$, then usually we can rewrite the proof in simpler language as a direct proof. If instead we obtain $\neg P$, then we have proved the contrapositive.

A.24. Remark. *Biconditional statements.* The biconditional statement "$P \Leftrightarrow Q$" has the same meaning as "$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$". We read it as "$P$ if and only if Q", where "$Q \Rightarrow P$" is "P if Q", and "$P \Rightarrow Q$" is "P only if Q".

Although sometimes we can prove a biconditional statement by a chain of equivalences, usually we prove a conditional statement and its converse; the latter is also a conditional statement. For each we have the three fundamental methods above. To prove $P \Leftrightarrow Q$, we must prove one statement in each column in the table below. The lines are the direct method, the contrapositive method, and the method of contradiction, respectively. Proving two statements in the same column would amount to proving the same statement twice. ∎

$P \Rightarrow Q$	$Q \Rightarrow P$
$\neg Q \Rightarrow \neg P$	$\neg P \Rightarrow \neg Q$
$\neg(P \wedge \neg Q)$	$\neg(Q \wedge \neg P)$

Students sometimes wonder about the precise meanings of words like "theorem", "lemma", and "corollary" that are used to designate mathematical results. In Greek, *lemma* means "premise" and *theorema* means "thesis to be proved". Thus a theorem is a major result requiring some effort. A lemma is a lesser statement, usually proved in order to help prove other statements. A proposition is something "proposed" to be proved; typically this takes less effort than a theorem. The word *corollary* comes from Latin, as a modification of a word meaning "gift"; a corollary follows easily from a theorem or proposition, without much additional work.

INDUCTION AND RECURRENCE

Many statements having a natural number as a variable can be proved using the technique of induction. In Theorem 1.2.1, we describe the strong version of induction. Here we review the ordinary version that most students learn when they first encounter induction. It involves the Well Ordering Property for the natural numbers, which states that every nonempty subset of $\mathbb{N}$ has a least element. We take this as an axiom, as part of our intuitive understanding of what $\mathbb{N}$ is. Although we then state the Principle of Induction as a Theorem, in reality it is equivalent to the Well Ordering Property for $\mathbb{N}$.

A.25. Theorem. (Principle of Induction) For each natural number n, let $P(n)$ be a mathematical statement. If properties (a) and (b) below hold, then for each $n \in \mathbb{N}$ the statement $P(n)$ is true.
a) $P(1)$ is true.
b) For $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Proof: If $P(n)$ is not true for all n, then the set of natural numbers where it fails is nonempty. By the Well Ordering Property, there is a least natural number in this set. By (a), this number cannot be 1. By (b), it cannot be bigger than 1. The contradiction implies that $P(n)$ is true for all n. ∎

When applying the method of induction, we prove statement (a) in Theorem A.25 as the **basis step** and statement (b) as the **induction step**. Statement (b) is a conditional statement, and its hypothesis ("$P(k)$" is true) is the **induction hypothesis**. We present one example in rather formal language.

A.26. Proposition. If S is a set of n lines in the plane such that every two have exactly one common point and no three have a common point, then S cuts the plane into $1 + n(n+1)/2$ regions.

Proof: We use induction on n to prove the claim for all $n \in \mathbb{N}$. Let $P(n)$ be the statement that the claim holds for all such sets of n lines.

Basis step ($P(1)$). With one line the number of regions is 2, which equals $1 + 1(1+1)/2$.

Induction step ($P(k) \Rightarrow P(k+1)$). The statement $P(k)$ is the induction hypothesis. Let S be a set of $k+1$ lines meeting the conditions. Select a line L

in S (the dashed line in the figure), and let S' be the set of k lines obtained by deleting L from S.

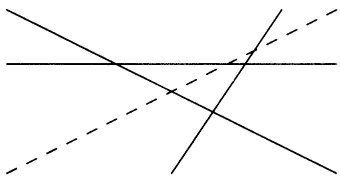
Since S' meets the conditions, the induction hypothesis states that S' cuts the plane into $1 + k(k+1)/2$ regions. When we replace L, some regions are cut. The increase in the number of regions is the number of regions that L cuts. It moves from one of these regions to another each time it crosses a line in S'. Since L crosses each line in S' once, the lines in S' cut L into $k+1$ pieces. Each piece corresponds to a region that L cuts.

Thus the number of regions formed by S is $k+1$ more than the number of regions formed by S'. The number of regions formed by S is

$$1 + k(k+1)/2 + (k+1) = 1 + (k+1)(k+2)/2.$$

We have proved that $P(k)$ implies $P(k+1)$.

By the principle of induction, the claim holds for every $n \in \mathbb{N}$. ∎



A.27. Remark. The discussion of Proposition A.26 suggests several comments about proof by induction. Note first that we could also have used $n = 0$ as the basis step to prove the statement for all nonnegative n.

It is not immediately obvious from the statement of the problem that the number of regions is the same for all sets of n lines, but this follows because we proved a formula for this number that depends only on n.

In the proof of the induction step, we began with L, an instance of the larger-sized problem. This approach ensures that we have considered all such instances; we return to this point shortly.

We proved $P(k+1)$ from $P(k)$ as suggested by statement (b) of Theorem A.25. In most examples in this book, we use a different phrasing that is more consistent with strong induction as introduced in Section 1.2. To prove $P(n)$ for all $n \in \mathbb{N}$, in this example we would write "Basis step: $n = 1$. . . ." and then "Induction step: $n > 1$. . . .". In the proof of the induction step, we would consider an arbitrary set S of n lines and apply the induction hypothesis to the set S' obtained by deleting one line L.

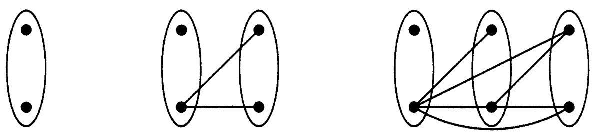
The content of the proof is the same in both phrasings. The phrasing that we have just described emphasizes the item about which the claim is proved. The basis step directly verifies the claim for the smallest value of the induction parameter. When the parameter has a larger value, the claim about the item is proved using the hypothesis that it holds for an earlier item; this is the induction step. Invoking it (repeatedly) yields the claim for each subsequent value of the parameter. ∎

When learning to use induction in graph theory, many students have trouble with two particular aspects. One is when the statement $P(n)$ being proved by induction is itself a conditional statement $A(n) \Rightarrow B(n)$. The induction hypothesis is the statement $A(n-1) \Rightarrow B(n-1)$. A template for the induction step in this situation is presented in Remark 1.3.25, and there are examples of this throughout Chapter 1.

The other pitfall we call the "induction trap", discussed at length in Example 1.3.26. Here we provide another example, using the language of proving $P(n+1)$ from $P(n)$ that sometimes leads students into the trap.

A.28. Example. *The Handshake Problem.* Let a **handshake party** of order n (henceforth "n-party") be a party with n married couples where no spouses shake hands with each other and the $2n-1$ people other than the host shake hands with different numbers of people. We use induction on n to prove that in every n-party, the hostess shakes with exactly $n-1$ people.

We model the party using a simple graph in which the vertices are the people at the party and the edges are the pairs who shake hands. The degree of a vertex is its number of handshaking partners. If no one shakes with his or her spouse, then each degree is between 0 and $2n-2$. The condition that the $2n-1$ numbers other than the host's are distinct implies that the degrees are 0 through $2n-2$. The figure below shows for $n \in \{1, 2, 3\}$ the graph that is forced; each circled pair of vertices indicates a married couple, with host and hostess rightmost in each graph.



Basis step: If $n = 1$, then the hostess shakes with 0 (which equals $n-1$), because the host and hostess don't shake.

Induction step (**INVALID**): The induction hypothesis is that the claim holds for n-parties. Consider such a party. By the induction hypothesis, the degree of the hostess is $n-1$. By our earlier discussion, the degrees of vertices other than the host are $0, \ldots, 2n-2$. We form an $(n+1)$-party by adding one more couple. Let one member of the new couple shake with everyone in the first n couples; the other shakes with no one. This increases the degree of each of the earlier vertices by 1, so those degrees other than the host are now $1, \ldots, 2n-1$, and the new couple have degrees 0 and $2n$. Hence the larger configuration is an $(n+1)$-party. The degree of the hostess has increased by 1, so it is n.

Induction step (**VALID**): The induction hypothesis is that the claim holds for n-parties. Consider an $(n+1)$-party. By our earlier discussion, the degrees other than the host are $0, \ldots, 2n$. Let p_i denote the person of degree i among these. Since p_{2n} shakes with all but one person, the person p_0 who shakes with no one must be the only person missed by p_{2n}. Hence p_0 is the spouse of

p_{2n}. Furthermore, this married couple $S = \{p_0, p_{2n}\}$ is not the host and hostess, since the host is not in $\{p_0, \ldots, p_{2n}\}$.

Everyone not in S shakes with exactly one person in S, namely p_{2n}. If we delete S to obtain a smaller party, then we have n couples remaining (including the host and hostess), no person shakes with a spouse, and each person shakes with one fewer person than in the full party. Hence in the smaller party the people other than the host shake hands with different numbers of people.

By deleting the set S, we thus obtain an n-party (deleting the leftmost couple in the picture for $n = 3$ yields the picture for $n = 2$). Applying the induction hypothesis to this n-party tells us that, outside of the couple S, the hostess shakes with $n - 1$ people. Since she also shakes with $p_{2n} \in S$, in the full $(n + 1)$-party she shakes with n people. ∎

The first argument in Example A.28 falls into the induction trap, because it does not consider all possible $(n + 1)$-parties. It considers only those obtained by adding a couple to an n-party in a certain way, without proving that every $(n + 1)$-party is obtained in this way.

Starting with an arbitrary $(n + 1)$-party forces us to prove that every $(n + 1)$-party arises in this way in order to obtain a configuration where we can apply the induction hypothesis. We cannot discard just any married couple to obtain the smaller party. We must find a couple S such that everyone outside S shakes with exactly one person in S. Only then will the smaller party satisfy the hypotheses needed to be an n-party.

The need to show that our smaller object satisfies the conditions in the induction hypothesis replaces the need to prove that all objects of the larger size were generated by growing from an object of the smaller size.

Sometimes the proof of the induction step uses more than one earlier instance. If we always use both $P(n - 2)$ and $P(n - 1)$ to prove $P(n)$, then we must verify both $P(1)$ and $P(2)$ to get started. The proof of the induction step is not valid for $n = 2$, since there is no $P(0)$ to use.

A.29. Example. Let $a_1, a_2, \ldots$ be defined by $a_1 = 2$, $a_2 = 8$, and $a_n = 4(a_{n-1} - a_{n-2})$ for $n \geq 3$. We seek a formula for a_n in terms of n.

We may try to guess a formula that fits the data. The definition yields $a_3 = 24$, $a_4 = 64$, and $a_5 = 160$. All these satisfy $a_n = n2^n$. Having guessed this as a possible formula for a_n, we can try to use induction to prove it.

When $n = 1$, we have $a_1 = 2 = 1 \cdot 2^1$. When $n = 2$, we have $a_2 = 8 = 2 \cdot 2^2$. In both cases, the formula is correct.

In the induction step, we prove that the desired formula is correct for $n \geq 3$. We use the hypothesis that the formula is correct for the preceding instances $n - 1$ and $n - 2$. This allows us to compute a_n using its expression in terms of earlier values:

$$a_n = 4(a_{n-1} - a_{n-2}) = 4[(n-1)2^{n-1} - (n-2)2^{n-2}] = (2n-2)2^n - (n-2)2^n = n2^n.$$

The validity of the formula for a_n follows from its validity for a_{n-1} and a_{n-2}, which completes the proof. ∎

In this proof, we must verify the formula for $n = 1$ and $n = 2$ in the basis step; the proof of the induction step is not valid when $n = 2$. Example A.29 specifies $a_1, a_2, \ldots$ by a **recurrence relation**. The general term a_n is specified using earlier terms. Similarly, the proof of Proposition A.26 yields a recurrence for the number r_n of regions formed by n lines; $r_n = r_{n-1} + n$, with $r_1 = 2$.

If the recurrence relation uses k earlier terms to compute a_n, then we must provide k initial values in order to specify the terms exactly; this is a recurrence of **order** k. Statements proved by induction about recurrences of order k typically require verification of k instances in the basis step. Standard techniques from enumerative combinatorics yield solutions to many recurrence relations without guessing formulas or directly using induction.

We also sometimes use recursive computation in graph theory. We may have a value for each graph instead of just one for each "size" as in a sequence. If we can express the value for a graph G as a formula in terms of graphs with fewer edges (and specify the values for graphs with no edges), then again we have a recurrence. We use this technique to count spanning trees (Section 2.2) and proper colorings (Section 5.3).

FUNCTIONS

A function transforms elements of one set into elements of another.

A.30. Definition. A **function** f *from* a set A *to* a set B assigns to each $a \in A$ a single element $f(a)$ in B, called the **image** of a under f. For a function f from A to B (written $f\colon A \to B$), the set A is the **domain** and the set B is the **target**. The **image** of a function f with domain A is $\{f(a)\colon a \in A\}$.

We take many elementary functions as familiar, such as the absolute value function and polynomials (both defined on $\mathbb{R}$). "Size" is a function whose domain is the set of finite sets and whose target is $\mathbb{N} \cup \{0\}$.

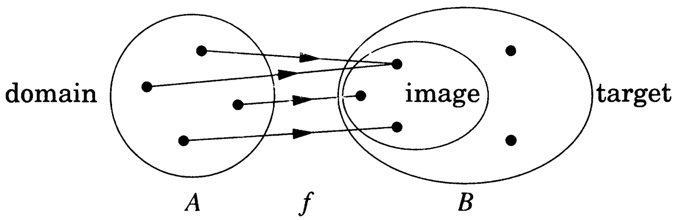
A.31. Definition. For $x \in \mathbb{R}$, the **floor** $\lfloor x \rfloor$ is the greatest integer that is at most x. The **ceiling** $\lceil x \rceil$ is the smallest integer that is at least x. A **sequence** is a function f whose domain is $\mathbb{N}$.

The floor function and ceiling function map $\mathbb{R}$ to $\mathbb{Z}$. When the target of a sequence is A, we have a sequence of elements in A, and we express the sequence as $a_1, a_2, a_3, \ldots$, where $a_n = f(n)$. We have used induction to prove sequences of statements and to prove formulas specifying sequences of numbers.

We may want to know how fast a function from $\mathbb{R}$ to $\mathbb{R}$ grows, particularly when analyzing algorithms. For example, we say that the growth of a function g is (at most) **quadratic** if it is bounded by a quadratic polynomial for all sufficiently large inputs. A more precise discussion of growth rates of functions appears in Appendix B.

A.32. Remark. *Schematic representation.* A function $f \colon A \to B$ is **defined on** A and **maps** A into B. To visualize a function $f \colon A \to B$, we draw a region representing A and a region representing B, and from each $x \in A$ we draw an arrow to $f(x)$ in B. In digraph language, this produces an orientation of a bipartite graph with partite sets A and B in which every element of A is the tail of exactly one edge.

The image of a function is contained in its target. Thus we draw the region for the image inside the region for the target. ■



To describe a function, we must specify $f(a)$ for each $a \in A$. We can list the pairs $(a, f(a))$, provide a formula for computing $f(a)$ from a, or describe the rule for obtaining $f(a)$ from a in words.

A.33. Definition. A function $f \colon A \to B$ is a **bijection** if for every $b \in B$ there is exactly one $a \in A$ such that $f(a) = b$.

Under a bijection, each element of the target is the image of exactly one element of the domain. Thus when a bijection is represented as in Remark A.32, every element of the target is the head of exactly one edge.

A.34. Example. *Pairing spouses.* Let M be the set of men at a party, and let W be the set of women. If the attendees consist entirely of married couples, then we can define a function $f \colon M \to W$ by letting $f(x)$ be the spouse of x. For each woman $w \in W$, there is exactly one $x \in M$ such that $f(x) = w$. Hence f is a bijection from M to W. ■

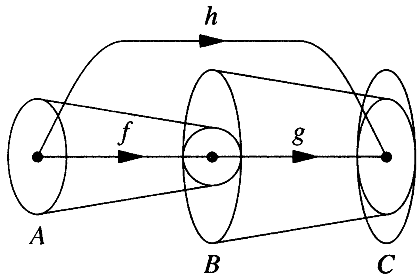
Bijections pair up elements from different sets. Thus we also describe a bijection from A to B as a **one-to-one correspondence** between A and B. Occasionally in the text we say informally that elements of one set "correspond" to elements of another; by this we mean that there is a natural one-to-one correspondence between the two sets.

When A has n elements, listing them as $a_1, \ldots, a_n$ defines a bijection from $[n]$ to A. Viewing the correspondence in the other direction defines a bijection from A to $[n]$. All bijections can be "inverted".

A.35. Definition. If f is a bijection from A to B, then the **inverse** of f is the function $g \colon B \to A$ such that, for each $b \in B$, $g(b)$ is the unique element $x \in A$ such that $f(x) = b$. We write f^{-1} for the function g.

When the target of a function is the domain of a second function, we can create a new function by applying the first and then the second. This yields a function from the domain of the first function into the target of the second.

A.36. Definition. If $f\colon A \to B$ and $g\colon B \to C$, then the **composition** of g with f is a function $h\colon A \to C$ defined by $h(x) = g(f(x))$ for $x \in A$. When h is the composition of g with f, we write $h = g \circ f$.



From the definitions, it is easy to verify that the composition of two bijections is a bijection. We use this in Proposition 1.1.24 in verifying for graphs that a composition of isomorphisms is an isomorphism.

COUNTING AND BINOMIAL COEFFICIENTS

A discussion of counting quickly leads to summations and products. These can be written concisely using appropriate notation.

A.37. Remark. We express summation using $\sum$, the uppercase Greek letter "sigma". When a and b are integers, the value of $\sum_{i=a}^{b} f(i)$ is the sum of the numbers $f(i)$ over the integers i satisfying $a \le i \le b$. Here i is the **index of summation**, and the formula $f(i)$ is the **summand**.

We write $\sum_{j\in S} f(j)$ to sum a real-valued function f over the elements of a set S in its domain. When no subset is specified, as in $\sum_j x_j$, we sum over the entire domain. When the summand has only one symbol that can vary, we may omit the subscript on the summation symbol, as in $\sum x_i$.

Similar comments apply to indexed products using $\prod$, which is the uppercase Greek letter "pi". ∎

Two simple rules help organizing the counting of finite sets by breaking problems into subproblems. These rules follow from the definition of size and properties of bijections.

A.38. Definition. The **rule of sum** states that if A is a finite set and $B_1, \dots, B_m$ is a partition of A, then $|A| = \sum_{i=1}^{m} |B_i|$.

Let T be a set whose elements can be described using a procedure involving steps $S_1, \dots, S_k$ such that step S_i can be performed in r_i ways, regardless of *how* steps $S_1, \dots, S_{i-1}$ are performed. The **rule of product** states that $|T| = \prod_{i=1}^{k} r_i$.

For example, there are q^k lists of length k from a set of size q. There are q choices for each position, regardless of the choices in other positions. By the product rule, there are q^k ways to form the k-tuple.

A.39. Definition. A **permutation** of a finite set S is a bijection from S to S. The **word form** of a permutation f of $[n]$ is the list $f(1), \dots, f(n)$ in that order. An **arrangement** of elements from a set S is a list of elements of S (in order). We write $n!$ (read as "n **factorial**") to mean $\prod_{i=1}^{n} i$, with the convention that $0! = 1$.

The word form of a permutation of $[n]$ includes the full description of the permutation. For counting purposes we refer to the word form *as* the permutation; thus 614325 is a permutation of [6]. With this viewpoint, a permutation of $[n]$ is an arrangement of all the elements of $[n]$.

A.40. Theorem. An n-element set has $n!$ permutations (arrangements without repetition). In general, the number of arrangements of k distinct elements from a set of size n is $n(n-1)\cdots(n-k+1)$.

Proof: We count the lists of k distinct elements from a set S of size n. There is no such list when $k > n$, which agrees with the formula. We construct the lists one element at a time, specifying the element in position $i+1$ after specifying the elements in earlier positions.

There are n ways to choose the image of 1. For each way we do this, there are $n-1$ ways to choose the image of 2. In general, after we have chosen the first i images, avoiding them leaves $n-i$ ways to choose the next image, no matter how we made the first i choices. The rule of product yields $\prod_{i=0}^{k-1}(n-i)$ for the number of arrangements. ■

Often the order of elements in a list is unimportant.

A.41. Definition. A **selection** of k elements from $[n]$ is a k-element subset of $[n]$. The number of such selections is "n choose k", written as $\binom{n}{k}$.

If $k < 0$ or $k > n$, then $\binom{n}{k} = 0$; in these cases there are no selections of k elements from $[n]$. When $0 \le k \le n$, we obtain a simple formula.

A.42. Theorem. For integers n, k with $0 \le k \le n$, $\binom{n}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(n-i)$.

Proof: We relate selections to arrangements. We count the arrangements of k elements from $[n]$ in two ways. Picking elements for positions as in Theorem A.40 yields $n(n-1)\cdot(n-k+1)$ as the number of arrangements.

Alternatively, we can select the k-element subset first and then write it in some order. Since by definition there are $\binom{n}{k}$ selections, the product rule yields $\binom{n}{k}k!$ for the number of arrangements.

In each case, we are counting the set of arrangements, so we conclude that $n(n-1)\cdots(n-k+1) = \binom{n}{k}k!$. Dividing by $k!$ completes the proof. ∎

The formula for $\binom{n}{k}$ can be written as $\frac{n!}{k!(n-k)!}$, but the form in the statement of Theorem A.42 tends to be more useful, especially when k is small. For example, $\binom{n}{2} = n(n-1)/2$ and $\binom{n}{3} = n(n-1)(n-2)/6$, the former being the number of edges in a complete graph with n vertices. This form more directly reflects the counting argument and cancels the $(n-k)!$ appearing in both the numerator and denominator.

The numbers $\binom{n}{k}$ are called the **binomial coefficients** due to their appearance as coefficients in the nth power of a sum of two terms.

A.43. Theorem. (Binomial Theorem) For $n \in \mathbb{N}$, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof: The proof interprets the process of multiplying out the factors in the product $(x+y)(x+y)\cdots(x+y)$. To form a term in the product, we must choose x or y from each factor. The number of factors that contribute x is some integer k in $\{0,\ldots,n\}$, and the remaining $n-k$ factors contribute y. The number of terms of the form $x^k y^{n-k}$ is the number of ways to choose k of the factors to contribute x. Summing over k accounts for all the terms. ∎

Using the definition of size and the composition of bijections, it follows that finite sets A and B have the same size if and only if there is a bijection from A to B. Thus we can compute the size of a set by establishing a bijection from it to a set of known size.

Simple examples include the statements that a complete graph has $\binom{n}{2}$ edges and that therefore there are $2^{\binom{n}{2}}$ simple graphs with vertex set $[n]$. Proposition 1.3.10 uses a bijection to count 6-cycles in the Petersen graph. Exercise 1.3.32 uses a bijection to count graphs with vertex set $[n]$ and even vertex degrees. Theorem 2.2.3 uses a bijection to count trees with vertex set $[n]$.

A.44. Lemma. For $n \in \mathbb{N}$, the number of subsets of $[n]$ with even size equals the number of subsets of $[n]$ with odd size.

Proof: *Proof 1* (bijection). For each subset with even size, delete the element n if it appears, and add n if it does not appear. This always changes the size by 1 and produces a subset with odd size. The map is a bijection, since each odd subset containing n arises only from one even subset omitting n, and each odd subset omitting n arises only from one even subset containing n.

Proof 2 (binomial theorem). Setting $x = -1$ and $y = 1$ in Theorem A.43. yields $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = (-1+1)^n = 0$. (Note that we proved Theorem A.43 using bijections.) ∎

We prove a few identities involving binomial coefficients to illustrate combinatorial arguments involving bijections and the idea of counting a set in two ways. We can prove an equality by showing that both sides count the same set.

A.45. Lemma. $\binom{n}{k} = \binom{n}{n-k}$.

Proof: *Proof 1* (counting two ways). By definition, $[n]$ has $\binom{n}{k}$ subsets of size k. Another way to count selections of k elements is to count selections of $n - k$ elements to omit, and there are $\binom{n}{n-k}$ of these.

Proof 2 (bijections). The left side counts the k-element subsets of $[n]$, the right side counts the $n - k$-element subsets, and the operation of "complementation" establishes a bijection between the two collections. ■

Often, "counting two ways" means grouping the elements in two ways. Sometimes one of the counts only gives a bound on the size of the set. In this case the counting argument proves an inequality; there are several instances of this phenomenon in Chapter 3 (see also Exercise 1.3.31). Here we stick to equalities.

A.46. Lemma. (The Chairperson Identity) $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Proof: Each side counts the k-person committees with a designated chairperson that can be formed from a set of n people. On the left, we select the committee and then select the chair from it; on the right, we select the chair first and then fill out the rest of the committee. ■

Many students see the next formula as the first application of induction, but it also is easily proved by counting a set in two ways.

A.47. Lemma. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof: The right side is $\binom{n+1}{2}$; we can view this as counting the nontrivial intervals with endpoints in the set $\{1, \ldots, n+1\}$. On the other hand, we can group the intervals by length; there is one interval with length n, two with length $n-1$, and so on up to n intervals with length 1. ■

Lemma A.47 generalizes to $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$. To prove this by counting in two ways, partition the set of $k+1$-element subsets of $[n+1]$ into groups so that the size of the ith group will be $\binom{i}{k}$.

Finally, a recursive computation for the binomial coefficients.

A.48. Lemma. (Pascal's Formula) If $n \geq 1$, then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof: We count the k-sets in $[n]$. There are $\binom{n-1}{k}$ such sets not containing n and $\binom{n-1}{k-1}$ such sets containing n. ■

Given the initial conditions for $n = 0$, which are $\binom{0}{0} = 1$ and $\binom{0}{k} = 0$ for $k \neq 0$, Pascal's Formula can be used to give inductive proofs of many statements about binomial coefficients, including Theorems A.42–A.43.

A.49. Remark. *Multinomial coefficients.* Binomial coefficients and the binomial theorem generalize to multinomials. When $\sum n_i = n$, the **multinomial coefficient** $\binom{n}{n_1,\ldots,n_k}$ is the coefficient of $\prod x_i^{n_i}$ in the expansion of $(\sum_{i=1}^k x_i)^n$. It has the value $n!/\prod n_i!$. Terms of the form $\prod x_i^{n_i}$ arise in the expansion only when $\sum n_i = n$. Otherwise, there is nothing to count, and we say that $\binom{n}{n_1,\ldots,n_k} = 0$ when $\sum n_i \neq n$.

The contributions to this coefficient correspond to n-tuples that are arrangements of n objects, using n_i copies of object i for each i. Having a copy of object i in position j corresponds to choosing the term x_i from the jth factor $(x_1 + \cdots + x_k)$.

The formula $n!/\prod n_i!$ is derived by counting these arrangements. There are $n!$ arrangements of n distinct items. If we view these objects as distinct, then we count each arrangement $\prod n_i!$ times, since permuting the copies of a single object does not change the arrangement.

In Corollary 2.2.4, these arrangements correspond to trees with vertex set $[n]$ and specified vertex degrees. When we set $x_i = 1$ for all i, we obtain the total number of n-tuples formed from k types of letters over all multiplicities of repetition; the result is k^n. ■

RELATIONS

Given two objects s and t, not necessarily of the same type, we may ask whether they satisfy a given relationship. Let S denote the set of objects of the first type, and let T denote the set of objects of the second type. Some of the ordered pairs (s, t) may satisfy the relationship, and some may not. The next definition makes this notion precise.

A.50. Definition. When S and T are sets, a **relation** between S and T is a subset of the product $S \times T$. A **relation on** S is a subset of $S \times S$.

We usually specify a relation by a condition on pairs. In Section 1.1, we define several relations associated with a graph G. The *incidence relation* between $S = V(G)$ and $T = E(G)$ is the set of ordered pairs (v, e) such that $v \in V(G)$, $e \in E(G)$, and v is an endpoint of edge e. The *adjacency relation* on the set $V(G)$ is the set of ordered pairs (x, y) of vertices such that x and y are the endpoints of an edge.

A.51. Remark. Let R be a relation defined on a set S. When discussing several items from S, we use the adjective **pairwise** to specify that each pair among these items satisfies R. Thus we can talk about a family of pairwise disjoint sets, or a family of pairwise isomorphic graphs. An independent set in a graph is a set of pairwise nonadjacent vertices. A set of **distinct** objects is a set of pairwise unequal objects.

We need the term "pairwise" because the relation is defined for pairs. For the same reason, we don't use "pairwise" when discussing only two objects.

When two graphs are isomorphic, we don't say they are pairwise isomorphic. Similarly, we say that the endpoints of an edge are adjacent, not pairwise adjacent; the adjacency relation is satisfied by certain pairs of vertices. ∎

To specify a relation between S and T, we can list the ordered pairs satisfying it. Usually it is more convenient to let S index the rows and T the columns of a grid of positions called a **matrix**. We can then specify the relation by recording, in the position for row s and column t, a 1 if (s, t) satisfies the relation and a 0 if (s, t) does not satisfy the relation. Thus the adjacency and incidence matrices of a graph are the matrices recording the adjacency and incidence relations (see Definition 1.1.17).

The condition "have the same parity" defines a relation on $\mathbb{Z}$. If x, y are both even or both odd, then (x, y) satisfies this relation; otherwise it does not. The key properties of parity lead us to an important class of relations.

A.52. Definition. An **equivalence relation** on a set S is a relation R on S such that for all choices of distinct $x, y, z \in S$,
a) $(x, x) \in R$ (**reflexive property**).
b) $(x, y) \in R$ implies $(y, x) \in R$ (**symmetric property**).
c) $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$ (**transitive property**).

For every set S, the **equality relation** $R = \{(x, x)\colon x \in S\}$ is an equivalence relation on S. In Proposition 1.1.24 we show that the isomorphism relation is an equivalence relation on graphs. The notation $G \cong H$ for this relation suggests "equal in some sense".

A.53. Definition. Given an equivalence relation on S, the set of elements equivalent to $x \in S$ is the **equivalence class** containing x.

The equivalence classes of an equivalence relation on S form a partition of S; elements x and y belong to the same class if and only if (x, y) satisfies the relation. The converse assertion also holds. If $A_1, \dots, A_k$ is a partition of S, then the condition "x and y are in the same set in the partition" defines an equivalence relation on S.

Parity partitions the integers into two equivalence classes by their remainder upon division by 2. This notion generalizes to any natural number.

A.54. Definition. Given a natural number n, the integers x and y are **congruent modulo** n if $x - y$ is divisible by n. We write this as $x \equiv y \bmod n$. The number n is the **modulus**.

A.55. Theorem. For $n \in \mathbb{N}$, congruence mod n is an equivalence relation on $\mathbb{Z}$.

Proof: Reflexive property: $x - x$ equals 0, which is divisible by n.

Symmetric property: If $x \equiv y \bmod n$, then by definition $n \mid (x - y)$. Since $y - x = -(x - y)$, and since n divides $-m$ if and only if n divides m, we also have $n \mid (y - x)$, and hence $y \equiv x \bmod n$.

Transitive property: If $n|(x-y)$ and $n|(y-z)$, then integers a, b exist such that $x-y=an$ and $y-z=bn$. Adding these equations yields $x-z=an+bn=(a+b)n$, so $n|(x-z)$. Thus the relation is transitive. ∎

A.56. Definition. The equivalence classes of the relation "congruence modulo n" on $\mathbb{Z}$ are the **remainder classes** or **congruence classes** modulo n. The set of congruence classes is written as $\mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$.

There are n remainder classes modulo n. For $0 \le r < n$, the rth class in $\mathbb{Z}_n$ is $\{kn+r\colon k \in \mathbb{Z}\}$. Numbers a and b lie in the rth class if and only if they both have remainder r upon division by n. Thus "$m \equiv r \bmod n$" has the same meaning as "m is r more than a multiple of n".

THE PIGEONHOLE PRINCIPLE

The pigeonhole principle is a simple notion that leads to elegant proofs and can reduce case analysis. In every set of numbers, the average is between the minimum and the maximum. When dealing with integers, the pigeonhole principle allows us to take the ceiling or floor of the average in the desired direction.

A.57. Lemma. (Pigeonhole Principle) If a set consisting of more than kn objects is partitioned into n classes, then some class receives more than k objects.

Proof: The contrapositive states that if every class receives at most k objects, then in total there are at most kn objects. ∎

The pigeonhole principle can reduce case analysis by allowing us to use additional information about an extreme element of a set. This simple idea can crop up unexpectedly, but its use can be quite effective. When we find that we need the pigeonhole principle, there is no trouble applying it: we need a sufficiently big value in our set, and the pigeonhole principle provides it.

Some applications of the pigeonhole principle are rather subtle. Section 8.3 presents several of these. The subtlety arises when it is unclear how to define the objects and the classes so that the pigeonhole principle will apply.

Proposition 1.3.15 proves the next proposition using Remark A.13. Here we use the pigeonhole principle instead.

A.58. Proposition. If G is a simple n-vertex graph with $\delta(G) \ge (n-1)/2$, then G is connected.

Proof: Choose $u, v \in V(G)$. If $u \not\leftrightarrow v$, then at least $n-1$ edges join $\{u, v\}$ to the remaining vertices, since $\delta(G) \ge (n-1)/2$. There are $n-2$ other vertices, so the pigeonhole principle implies that one of them receives two of these edges. Since G is simple, this vertex is a common neighbor of u and v.

For every two vertices $u, v \in V(G)$, we have proved that u and v are adjacent or have a common neighbor. Thus G is connected. ∎

The pigeonhole principle can also be useful in statements about trees, where the number of vertices is one more than the number of edges. If each vertex selects an edge in some way, then some edge must be selected twice. The idea is to design the selection so that when an edge is selected twice, the desired outcome occurs. Applications of this idea occur in Lemma 8.1.10 and Theorem 8.3.2.

The pigeonhole principle is the discrete version of the statement that the average of a set of numbers is between the minimum and the maximum. This statement is made explicit for vertex degrees in Corollary 1.3.4. Other applications are sprinkled throughout the book.

Appendix B

Optimization and Complexity

A salesman plans to visit $n - 1$ other cities and return home. The natural objective is to minimize the total travel time. If we assign each edge of K_n a weight equal to the travel time between the corresponding cities, then we seek the spanning cycle of minimum total weight. This is the famous **Traveling Salesman Problem (TSP)**. Seemingly analogous to the Minimum Spanning Tree problem, the TSP as yet has no good algorithm.

Similarly, although we have a good algorithm for finding maximum matchings, we have none for finding the maximum size of an independent set of vertices. Since the former is the special case of the latter for line graphs, it is not too surprising that it is easier to solve.

INTRACTABILITY

We defined a *good algorithm* (Definition 3.2.3) to be an algorithm that runs (correctly) in time bounded by a polynomial function of the input size. One algorithm for the TSP considers all spanning cycles and selects the cheapest one. This is not a good algorithm, because K_n has $(n - 1)!/2$ spanning cycles, and this has grows faster than every polynomial function of n. The computation takes too long for graphs of any substantial size. Practical applications require solving TSPs on graphs with hundreds or thousands of vertices.

No one has found a good algorithm, and no one has proved that none exists. The TSP belongs to a large class of problems having the property that a good algorithm for any one of them will yield a good algorithm for every one of them. A good algorithm for B yields a good algorithm for A if we can "reduce" problem A to problem B.

As an easy example of this, we can use a good algorithm for the TSP (problem B) to recognize Hamiltonian graphs (problem A). From a graph G, form an instance of the TSP on vertex set $V(G)$ by assigning weight 0 to vertex pairs that are edges of G and weight 1 to pairs that are not. The graph G has a Hamiltonian cycle if and only if the optimal solution to this instance of the TSP has cost 0. The time for the transformation is polynomial in $n(G)$, so a good algorithm for the TSP produces a good algorithm to test for spanning cycles. We conclude that the TSP is at least as hard as the Hamiltonian cycle problem.

In the formal discussion, we consider only **decision problems**, where the answer is YES or NO. This makes sense for recognizing Hamiltonian graphs, but the TSP is an optimization problem. When formulated as a decision problem (called MINIMUM SPANNING CYCLE), the input for the TSP is a weighted graph G and a number k, and the problem is to test whether G has a spanning cycle with weight at most k. Repeated applications of this decision problem (at most a polynomial number of applications) can be used to find the minimum weight of a spanning cycle. Similarly, MAXIMUM INDEPENDENT SET takes a graph G and an integer k as input and tests $\alpha(G) \geq k$.

We judge a graph algorithm by its maximum (*worst-case*) running time over inputs on n vertices, as a function of n. The **complexity** of a decision problem is the minimum worst-case running time over all solution algorithms, again as a function of the size of the problem.† In describing the growth of a function g, we compare it with a reference function f. We define several sets of functions in terms of f. The sets $O(f)$ and $\Omega(f)$ describe functions bounded above and below by multiples of f. Functions in $\Theta(f)$ grow at about the same rate as f, those in $o(f)$ grow more slowly, and those in $\omega(f)$ grow more quickly.

$$\begin{aligned}
O(f) &= \{g\colon \exists c, a \in \mathbb{R} \text{ such that } |g(x)| \leq c|f(x)| \text{for} x > a\} \\
\Omega(f) &= \{g\colon \exists c, a \in \mathbb{R} \text{ such that } |g(x)| \geq c|f(x)| \text{for} x > a\} \\
\Theta(f) &= O(f) \cap \Omega(f) \\
o(f) &= \{g\colon |g(x)|/|f(x)| \to 0\} \\
\omega(f) &= \{g : |g(x)|/|f(x)| \to \infty\}
\end{aligned}$$

The class of problems with polynomial complexity (solved by a good algorithm) is called "P". We have discussed only deterministic algorithms: each input leads to exactly one polynomial-time computation.

Now we consider nondeterministic algorithms. For many decision problems with no known good algorithm, short proofs exist for YES answers. For example, if we guess the right order of vertices in the HAMILTONIAN CYCLE problem (specified by a sequence of $O(n \log n)$ bits), then we can verify rapidly that this order forms a spanning cycle.

A **nondeterministic polynomial-time algorithm** tries all values of a polynomial-length sequence of bits simultaneously, applying a polynomial-time computation to each guess (polynomial in the length of the input). If any guess demonstrates a YES answer to the decision problem, then the algorithm says YES. Otherwise, the answer is NO. This amounts to saying that when the answer is YES, there is a polynomial-time proof of this. The nondeterminism lies not in the answer but in the choice of the computation path.

†Technically, the **size** of a problem instance is its length in some encoding of the problem. Measuring the size of a graph problem by the number of vertices suffices for our purposes. A polynomial in n is also a polynomial in n^2 or n^3, so the distinction is unimportant unless the problem involves exponentially large edge weights.

The class of problems solvable by nondeterministic polynomial-time algorithms is called "NP". A machine that has the power to follow many computation paths in parallel can also follow one; hence $P \subseteq NP$. It is commonly believed that $P \neq NP$. This has not been proved, so NP cannot be taken to mean "non-polynomial". Instead, we use the informal term **intractable** for the problems in NP that are essentially as hard as all the problems in NP.

A problem is **NP-hard** if a polynomial-time algorithm for it could be used to construct a polynomial-time algorithm for each problem in NP. It is **NP-complete** if it belongs to NP and is NP-hard. If some NP-complete problem belongs to P, then $P = NP$. No polynomial-time algorithm is known for any of the many NP-complete problems. This supports the prevailing belief that $P \neq NP$. Garey–Johnson [1979] presents a thorough introduction to this topic.

Given one NP-complete problem, NP-completeness of other problems follows by reduction arguments as suggested earlier. We present several such arguments in this appendix. Here we list complexities for some of the problems discussed in this book.

Standard style in computer science uses uppercase names for decision problems. For a problem whose name includes an optimization, the decision problem is testing whether the value is as extreme as a number given as part of the input. Parameters in the name, however, are fixed as part of the statement of the problem.

This is an important distinction. For example, k-INDEPENDENT SET for fixed k is in P, since the number of k-sets of vertices is a polynomial in n of degree k when k is fixed, and we can simply test them all for independence. On the other hand, MAXIMUM INDEPENDENT SET is NP-complete; this is the problem of testing whether G has an independent set of size at least k where k is part of the input (and can grow with n).

Problems in P	NP-complete Problems
k-INDEPENDENT SET	MAXIMUM INDEPENDENT SET
GIRTH (SHORTEST CYCLE)	CIRCUMFERENCE (LONGEST CYCLE)
EULERIAN CIRCUIT	HAMILTONIAN CYCLE
DIAMETER	LONGEST PATH, HAMILTONIAN PATH
CONNECTIVITY	
2-COLORABILITY	k-COLORABILITY (for any fixed $k \geq 3$)
MAXIMUM MATCHING	$\Delta(G)$-EDGE-COLORABILITY
PLANARITY	GENUS

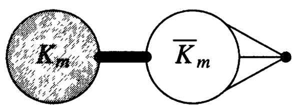
NP-completeness is related to the lack of an easily testable necessary and sufficient condition for YES answers. A **good characterization** is a characterization by a condition checkable in polynomial time. The characterization for Eulerian graphs is good, and GIRTH, DIAMETER, and 2-COLORABILITY can all be solved in polynomial time using Breadth-First Search. Polynomial behavior is less obvious for CONNECTIVITY, but min-max relations like Menger's Theorem generally lead to polynomial-time optimization algorithms, often based on network flow methods (Section 4.3).

HEURISTICS AND BOUNDS

Our traveling salesman still awaits instruction. NP-completeness does not eliminate the need for an answer. We seek heuristic algorithms that find solutions close to optimal. Perhaps we can prove a guarantee about how far from the optimum the result may be. For example, we may be content to have a solution whose cost is at most twice the optimum, if we have an algorithm that can quickly generate such a solution. An **approximation algorithm** always generates a solution whose ratio to the optimum is bounded by a constant.†

Greediness is a simple heuristic. For the minimum spanning tree problem, the result is optimal. On other problems, greedy algorithms may perform very badly. Consider MAXIMUM INDEPENDENT SET. We may generate an independent set iteratively by picking a vertex and deleting it and its neighbors. How should we pick the next vertex? If we always choose right, then the result is a maximum independent set. A greedy heuristic is to pick a vertex of minimum degree in what remains, since this leaves the largest set of candidates for the independent set. The result can be arbitrarily bad.

B.1. Example. *Defeating the greedy algorithm.* Consider $(K_1 + K_m) \vee \overline{K}_m$. This graph has one vertex of degree m, m vertices of degree $m+1$, and m vertices of degree $2m-1$. The greedy heuristic picks the vertex of minimum degree and deletes it and its neighbors, leaving a clique. Hence the greedy algorithm finds an independent set of size 2, when in fact $\alpha(G) = m$. ■



Nevertheless, the greedy algorithm works well on large graphs generated randomly. In this model (see Section 8.5), it almost always finds an independent set of size at least half the maximum. Exercise 12 presents two heuristics for MINIMUM VERTEX COVER; one fails as in Example B.1, but the other yields an approximation algorithm.

Next we consider simple heuristics for the TSP, where $\{v_1, \ldots, v_n\}$ are the vertices and w_{ij} denotes the weight (cost) of edge v_iv_j. From an arbitrary starting vertex, it seems reasonable to move to a new vertex via the least-cost incident edge. We iteratively move to the closest unvisited neighbor of the current vertex. This is a "greedy" algorithm and runs quickly. It is the **nearest-neighbor** heuristic.

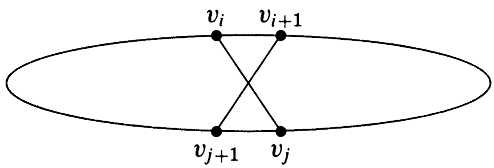
†Better yet: an **approximation scheme** is a family of algorithms indexed by a parameter ϵ, say $\epsilon = 1/k$ for the kth algorithm, such that the kth algorithm has performance ratio bounded by $1+\epsilon$. Each algorithm has polynomial complexity, but the degree of the polynomial grows with k. Improving the performance ratio requires more time.

B.2. Example. *Failure of the Nearest-neighbor heuristic.* Consider a TSP with weight 0 on a Hamiltonian path P, weight n^2 on all other edges incident to the endpoints of P, and weight 1 on all remaining edges. This example has many spanning cycles of weight n, but the nearest-neighbor heuristic yields a cycle of weight at least n^2 from any starting vertex. Thus the cost of the cycle produced by the algorithm is not bounded by a constant multiple of the optimal cost, and it is not an approximation algorithm. ∎

There are many similar heuristics. We could try to grow a cycle one vertex at a time, greedily absorbing the vertex whose insertion in the cycle causes the least increase in cost. This **nearest-insertion** heuristic has a better chance than nearest-neighbor, because at stage i of the nearest-neighbor heuristic we make a choice among $n - i$ alternatives, whereas at stage i in nearest-insertion we choose among $(n - i)i$ alternatives (which to add and where to insert it). Nevertheless, this also is not an approximation algorithm (Exercise 7).

Another approach is to start with a candidate spanning cycle and try to improve it. Maintaining a feasible solution (an actual cycle) and considering small changes to improve it is called **local search**. Allowing changes takes us beyond greedy algorithms and may perform better.

To improve the current cycle, we consider changing a pair of edges. If $(v_1, \ldots, v_n)$ is our cycle, we could substitute the edges $v_i v_j$ and $v_{i+1}v_{j+1}$ for $v_i v_{i+1}$ and $v_j v_{j+1}$ to obtain a new cycle (the other possible switch leads to two disjoint cycles instead of one cycle). The switch is beneficial if $w_{i,j} + w_{i+1,j+1} < w_{i,i+1} + w_{j,j+1}$. The current cycle has $\binom{n}{2} - n = \binom{n-1}{2}$ pairs of nonincident edges to consider switching. The algorithm of Lin–Kernighan [1973], which has proved remarkably difficult to improve in practice, considers switches among three edges at a time.



The following theorem seems to doom efforts to find an approximation algorithm for the general TSP.

B.3. Theorem. (Sahni–Gonzalez [1976]) If there is a constant $c \geq 1$ and a polynomial-time algorithm A such that A produces for each instance of the TSP a spanning cycle with cost at most c times the optimum, then P=NP.

Proof: We show that such an algorithm A could be used to construct a polynomial-time algorithm for HAMILTONIAN CYCLE, which is NP-complete (Corollary B.11). Given an n-vertex graph G, construct an instance of TSP on the same vertex set by letting $w_{ij} = 1$ if $v_i v_j \in E(G)$, and $w_{ij} = cn$ otherwise.

In this instance of TSP, every spanning cycle with cost at most cn has cost exactly n and corresponds to a spanning cycle of the original input G. Since A

produces a solution with cost at most c times the optimum, it produces one of cost n if and only if G has a spanning cycle.

Thus our polynomial algorithm for HAMILTONIAN CYCLE generates an instance of TSP as stated and runs A on this instance. The graph G is Hamiltonian if and only if A produces a cycle of cost n. ∎

Approximation algorithms do exist for some special classes of TSP problems. To prove that an algorithm is an approximation algorithm, we need a lower bound on the optimal solution.

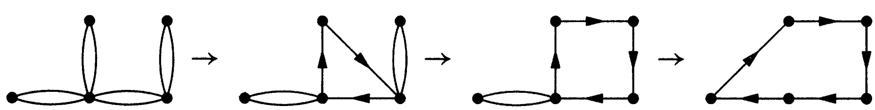
Let M be the cost of a minimum spanning tree in a weighted graph G. If we delete an edge from the optimal cycle in G, we obtain a spanning path. Since this is a spanning tree, its cost is at least M. The cost of the optimal cycle is at least M plus the minimum cost of an edge not in some tree with cost M. We can run Kruskal's Algorithm for minimum spanning trees to compute this bound.

B.4. Theorem. On the class of Traveling Salesman Problems where the input satisfies the triangle inequality, there is an approximation algorithm that finds a spanning cycle with cost at most twice the optimum.

Proof: Satisfying the triangle inequality means that the matrix of costs satisfies $w_{i,j} + w_{j,k} \geq w_{i,k}$ for all i, j, k. We know that the cost of the optimal cycle is at least M, where M is the cost of the minimum spanning tree. We use the triangle inequality and the minimum spanning tree to obtain a spanning cycle with cost at most $2M$.

Begin by duplicating each edge in a minimum spanning tree. As on the left below. This makes all degrees even, so there is an Eulerian circuit; the circuit has $2n$ edges and total cost $2M$. We successively reduce the number of edges without increasing the cost until only n edges remain. By maintaining the property that the circuit visits all vertices, we ensure that the circuit at the end is a spanning cycle with cost at most $2M$.

If a circuit has more than n edges, then it visits some vertex more than once, say via edges $v_i \to v_j \to v_k$ and $v_r \to v_j \to v_s$. Replace the edges v_iv_j and v_jv_k with v_iv_k. The result is still a circuit visiting all the original vertices. Furthermore, the triangle inequality guarantees that the total cost of the edges is no larger than before. In the figure below, we orient edges to suggest a particular succession of circuits. ∎



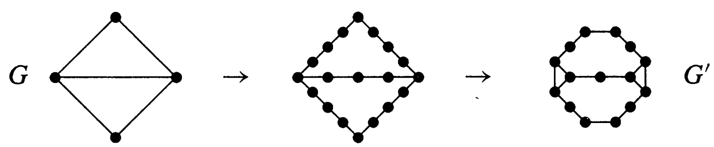
The algorithm of Theorem B.4 was rediscovered many times. Christofides [1976] improved the performance ratio to 3/2. After finding a minimum spanning tree, we needn't double all the edges to obtain an Eulerian circuit. It suffices to add edges to pair vertices that have odd degree in the tree. The resulting graph has an Eulerian circuit, and the subsequent part of the algorithm produces a spanning cycle with cost at most M plus the cost of the matching. To obtain the performance ratio $3/2$, it then suffices to show that there is such a matching with cost at most half the cost of the optimal cycle (Exercise 8). The improvement has value because a minimum weight matching saturating the vertices of odd degree in the tree can be found in polynomial time.

NP-COMPLETENESS PROOFS

A **transformation** of problem A to problem B is a procedure that converts instances of A into instances of B so that the answer to A on the original instance is determined by the answer to B on the transformed instance. If we have an efficient (polynomial-time) transformation of A to B and an efficient algorithm for B, then we have an efficient algorithm for A. We say that A **reduces to** or **transforms to** B.

If A is NP-hard and reduces to B by a polynomial-time transformation, then B is also NP-hard (a polynomial algorithm for B yields a polynomial algorithm for A and hence for all of NP). If also B is in NP, then we say that B is NP-complete by **reduction from** or **transformation from** A.

The direction of the reduction is crucial. For example, EULERIAN CIRCUIT reduces easily to HAMILTONIAN CYCLE. Given an input graph G, replace each edge with a path of four edges through three new vertices, add edges to make the neighbors of each original vertex pairwise adjacent, and delete $V(G)$. The graph G is Eulerian if and only if the resulting graph G' is Hamiltonian. Applying an algorithm for HAMILTONIAN CYCLE to G' would determine whether G is Eulerian. This tells use that HAMILTONIAN CYCLE is at least as hard as EULERIAN CIRCUIT, up to a polynomial factor. Since EULERIAN CIRCUIT is easy (in P), we learn nothing of use about the complexity of HAMILTONIAN CYCLE.



The reduction technique requires an initial NP-complete problem. Cook [1971] provided SATISFIABILITY as such a problem. SATISFIABILITY takes as input a logical formula expressed as a list of clauses; each clause is a collection of literals (variables or their negations). A clause is considered true when at least one of its literals is true. A formula is **satisfiable** if there is an assignment of truth values to the variables that makes every clause true. The question is whether such an assignment exists.

Cook proved that for every problem A in NP, an instance of SATISFIABILITY can be produced in polynomial time from an instance of A such that the answer to the SATISFIABILITY instance is the same as the answer to the instance of A. Thus every problem in NP reduces to SATISFIABILITY.

This effort need not be repeated for each NP-complete problem. To prove that B is NP-hard, we can reduce SATISFIABILITY to B. Any NP-complete problem may be reduced to show that B is NP-hard. With more options, in principle the proofs become easier to find, but in practice a few fundamental NP-complete problems serve as the known NP-complete problem in most NP-completeness proofs.

Starting from SATISFIABILITY, Karp [1972] provided 21 such problems. These include many fundamental problems of graph theory, including HAMILTONIAN CYCLE and MAXIMUM INDEPENDENT SET. It helps to have as restrictive a version of an NP-complete problem as possible while still remaining NP-complete. Since a restricted version has less flexibility, it is easier to reduce it to the problem we are trying to prove NP-complete.

For example, SATISFIABILITY remains NP-complete when restricted by requiring that every clause have three literals. The restricted problem is called 3-SATISFIABILITY or 3-SAT. This is proved NP-complete by considering an arbitrary instance of SATISFIABILITY and replacing each clause by an equivalent collection of clauses with three literals (at the cost of introducing some additional variables). 3-SAT is sufficiently restrictive that many NP-completeness proofs use reduction from 3-SAT. It is easier yet to reduce the more restrictive 2-SAT, but this is so restrictive that 2-SAT is solvable in polynomial time.

We start from 3-SAT because the NP-completeness of SATISFIABILITY and the reduction of it to 3-SAT do not involve graph theory. Our reductions to 3-COLORABILITY and DIRECTED HAMILTONIAN PATH follow the presentation of Gibbons [1985].

B.5. Definition. 3-SATISFIABILITY (3-SAT)

Instance: A set of logical **variables** $U = \{u_j\}$ and a set $\mathbf{C} = \{C_i\}$ of **clauses**, where each clause consists of three literals, a **literal** being a variable u_i or its negation $\overline{u}_i$.

Question: Can the value of each variable be set to True or False so that each clause is "satisfied" (contains at least one true literal)?

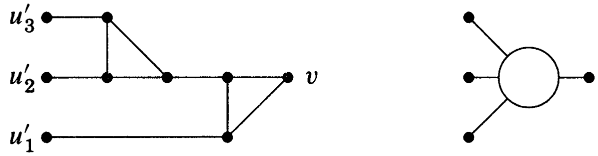
B.6. Theorem. (Karp [1972]) 3-SAT is NP-complete.

Proof: (Exercise 14). ∎

B.7. Theorem. (Stockmeyer [1973]) 3-COLORABILITY is NP-complete.

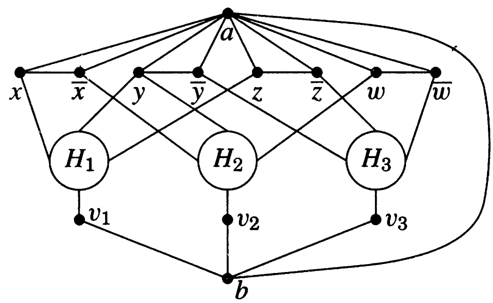
Proof: In this problem, we are given a graph and asked whether is it 3-colorable. If the answer is YES, then there exists a proper 3-coloring, and we can verify in quadratic time that the coloring is proper. Hence 3-COLORABILITY is in NP. To prove that it is NP-hard, we reduce 3-SAT to 3-COLORABILITY.

Consider an instance of 3-SAT with variables $U = \{u_j\}$ and clauses $\mathbf{C} = \{C_i\}$. We transform this into a graph G that is 3-colorable if and only if the instance of 3-SAT is satisfiable. We use the auxiliary graph H below, calling $\{u_1', u_2', u_3'\}$ the **inputs** and v the **output**. When we use H in the transformation, we will attach it to a larger graph at the inputs, as suggested on the right.



We consider 3-colorings using the color set $\{0, 1, 2\}$. Every proper 3-coloring of H in which the inputs all have color 0 also assigns color 0 to v. On the other hand, if the inputs receive colors that are not all 0, then this coloring extends to a proper 3-coloring of H in which v does not have color 0.

From our instance of 3-SAT, we construct a graph G having vertices u_j and $\overline{u}_j$ for each variable in U, a copy H_i of H for each clause C_i, and two special vertices a, b. For each j, the vertices $a, u_j, \overline{u}_j$ form a triangle. For each clause C_i, the subgraph H_i attaches to the graph formed thus far at the vertices for the literals in C_i. The vertex for the jth literal in C_i plays the role of u_j' in H_i. Except for these attachment vertices, the subgraphs H_i are pairwise disjoint. Finally, the vertex b is adjacent to a and to the output vertex v_i for each H_i. Below we draw the graph G resulting from an instance of 3-SAT having four variables and three clauses.



A satisfying truth assignment for $\{c_i\}$ yields a proper 3-coloring f of G. When u_i is true in the assignment, set $f(u_i) = 1$ and $f(\overline{u}_i) = 0$; otherwise, set $f(u_i) = 0$ and $f(\overline{u}_i) = 1$. For each clause, some literal is true; hence for each H_i at least one of $\{u_1', u_2', u_3'\}$ has color 1. By our observation about H, we can extend f so that each v_i has a color other than 0. We complete the proper 3-coloring by setting $f(b) = 0$ and $f(a) = 2$.

Conversely, suppose that G has a proper 3-coloring f. By renaming colors if necessary, we may assume that $f(a) = 2$ and $f(b) = 0$. Since $f(a) = 2$, for each variable we have one literal colored 0 and one colored 1. Consider the truth assignment in which variable u_j is true or false when $f(u_j)$ is 1 or 0, respectively. We claim that this is a satisfying truth assignment. Since $f(b) = 0$, every output vertex v_i has nonzero color. By our observation about H, the vertices corresponding to the inputs in H_i cannot all have color 0. Therefore, each clause contains at least one true literal. ∎

Exercise 2 extends this to k-COLORABILITY for fixed $k \geq 3$. For each k it is a special case of CHROMATIC NUMBER, which thus is also NP-complete.

B.8. Theorem. (Karp [1972]) INDEPENDENT SET, CLIQUE, and VERTEX COVER are NP-complete.

Proof: A YES answer to the question of whether an input graph G has an independent set as large as an input integer k can be verified by exhibiting the set and checking (in quadratic time) that its vertices are independent. Hence INDEPENDENT SET is in NP.

Exercise 5.1.31 states that G is m-colorable if and only if $G \square K_m$ has an independent set of size $n(G)$. This transformation reduces CHROMATIC NUMBER to INDEPENDENT SET. The construction of $G \square K_m$ is quadratic in $n(G)$, since CHROMATIC NUMBER is trivial if $m > n$. We conclude that INDEPENDENT SET is NP-hard.

Since cliques in G are independent in $\overline{G}$, CLIQUE and INDEPENDENT SET are polynomially equivalent. Since $\alpha(G) + \beta(G) = n(G)$ (Lemma 3.1.21), INDEPENDENT SET and VERTEX COVER are polynomially equivalent. ∎

We next consider problems of traversing graphs via spanning paths and cycles. Digraph problems are more general than graph problems, since we can model graphs by using symmetric digraphs. Thus, it may be easiest to prove a digraph version of a problem NP-hard and then obtain the graph version by a simple restriction.

B.9. Definition. Given vertices x, y in a digraph D, the DIRECTED HAMILTONIAN PATH problem asks whether D has a spanning x, y-path.

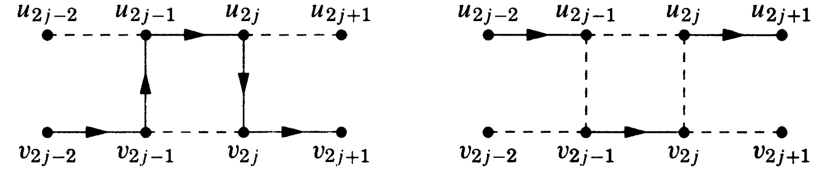
B.10. Theorem. (Karp [1972]) DIRECTED HAMILTONIAN PATH is NP-complete.

Proof: A spanning x, y-path in a digraph D can be verified in linear time. Thus DIRECTED HAMILTONIAN PATH is in NP. To show that it is NP-hard, we reduce VERTEX COVER to DIRECTED HAMILTONIAN PATH.

Consider an instance of VERTEX COVER, consisting of a graph G and an integer k; we want to know whether G has a vertex cover of size (at most) k. We construct a digraph D such that D has a Hamiltonian path if and only if G has a vertex cover of size at most k. We index the edges incident to each vertex arbitrarily. When the edge $e = uv$ is the ith edge incident to u and the jth edge incident to v, we write $e_i(u) = e = e_j(v)$.

To build D, we start with $k+1$ special vertices $z_0, \dots, z_k$. For each $v \in V(G)$, we add a path $P(v) = v_1, \dots, v_{2r}$ to D, where $r = d_G(v)$. We add edges from each of $z_0, \dots, z_{k-1}$ to the start of each $P(v)$ and from the end of each $P(v)$ to each of $z_1, \dots, z_k$. Also, for each edge $e = e_i(u) = e_j(v)$ in $E(G)$, we create the edges $u_{2i-1}v_{2j-1}$, $v_{2j-1}u_{2i-1}$, $u_{2i}v_{2j}$, and $v_{2j}u_{2j}$.

Suppose that G has a vertex cover of size k, consisting of vertices $v^1, \dots, v^k$. We form a z_0, z_k-path in D by following $z_0, P(v^1), z_1, \dots, P(v^k), z_k$. This path omits all of $P(u)$ for each uncovered vertex u. We absorb these vertices in pairs, absorbing each pair $u_{2i-1}u_{2i}$ by making a detour from the path $P(v)$ for the vertex that covers the edge $uv = e_i(u) = e_j(v)$. The detour is $v_{2j-1}u_{2j-1}u_{2j}v_{2j}$, as shown below. Because the vertices v_{2j-1}, v_{2j} are associated with only one edge, each such detour is requested at most once. After implementing all the detours, we have a Hamiltonian z_0, z_k-path in D.



Conversely, suppose that there is such a path Q. Note that every vertex of $P(u)$ except the first and last has indegree and outdegree 2, for each $u \in V(G)$. We show first for $i \geq 1$ and $u \in V(G)$ that $u_{2i}u_{2i+1} \in E(Q)$ if and only if $u_{2i-2}u_{2i-1} \in E(Q)$, where for $i = 1$ we take u_0 to mean some element of $\{z_r\}$. Similarly, when $i = d(u)$ we take u_{2i+1} to mean some element of $\{z_r\}$ Since the ith edge incident to u is well-defined, we can let the vertex v and index j be such that $e_i(u) = e_j(v)$.

If $u_{2i-2}u_{2i-1} \notin E(Q)$, then Q must enter u_{2i-1} from v_{2j-1}. This implies that Q can only leave u_{2i-1} on $u_{2i-1}u_{2i}$ and can only enter v_{2j} on $u_{2i}v_{2j}$. This in turn implies that $u_{2i}u_{2i+1} \notin E(Q)$.

If $u_{2i-2}u_{2i-1} \in E(P)$, then P cannot leave v_{2j-1} on $v_{2j-1}u_{2i-1}$ and must leave v_{2j-1} on $v_{2j-1}v_{2j}$. This implies that Q enters v_{2j} on $v_{2j-1}v_{2j}$ and not on $u_{2i}v_{2j}$. Hence Q does not leave u_{2i} on $u_{2i}v_{2j}$ and must leave Q on $u_{2i}u_{2i+1}$. (In this case, Q may include $\{u_{2i-1}u_{2i}, v_{2i-2}v_{2i-1}, v_{2i}v_{2i+1}\}$ or $\{u_{2i-1}v_{2i-1}, v_{2i}u_{2i}\}$.)

Now let $S = \{v \in V(G)\colon z_i v_1 \in Q\}$; these are the k vertices in G whose initial copies are entered from $z_0, \ldots, z_{k-1}$ by Q. Our argument shows for each edge uv that $u \notin S$ implies $v \in S$. Hence S is a vertex cover, and we have the desired reduction of VERTEX COVER to DIRECTED HAMILTONIAN PATH. ∎

B.11. Corollary. (Karp [1972]) DIRECTED HAMILTONIAN CYCLE, HAMILTONIAN PATH, HAMILTONIAN CYCLE are NP-complete.

Proof: All these problems are in NP. To reduce DIRECTED HAMILTONIAN PATH to DIRECTED HAMILTONIAN CYCLE, add one vertex z and edges vz and zu to an instance requesting a spanning u, v-path in D.

The reduction of HAMILTONIAN PATH (with specified endpoints) to HAMILTONIAN CYCLE is the same.

To reduced DIRECTED HAMILTONIAN PATH to HAMILTONIAN PATH, consider an instance that requests a u, v-path in D. To form an instance G of HAMILTONIAN PATH, first split each vertex x into a path x^-, x^0, x^+. Let x^- inherit all edges with heads at x, and let x^+ inherit all edges with tails at x. A spanning u, v-path in D becomes a spanning u^-, v^+-path in G by replacing each vertex x with the sequence x^-, x^0, x^+.

Conversely, since a spanning u^-, v^+-path in G must visit each x^0, it must visit traverse all sequences of the form x^-, x^0, x^+, forwards or backwards. Since no vertices of the same sign are adjacent, these traversals must all be in the same direction, and then they collapse to the desired u, v-path in D. Thus G has a spanning u^-, v^+-path if and only if D has a spanning u, v-path. ∎

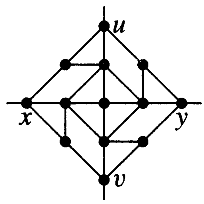
Exploration of the boundary between P and NP-complete seeks more restricted problems that remain NP-complete and larger classes of inputs on which there are polynomial-time solution algorithms. The former provides easier NP-completeness proofs and places limits on the successes of the latter type. The latter aim is that of extending the applicability of good algorithms.

We illustrate this process by proving that 3-COLORABILITY remains NP-complete for planar graphs.

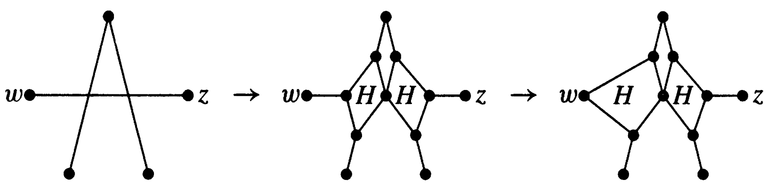
B.12. Theorem. (Stockmeyer [1973], see also Garey–Johnson–Stockmeyer [1976]) PLANAR 3-COLORABILITY is NP-complete.

Proof: As usual, it is easy to verify that a 3-coloring is proper. We reduce 3-COLORABILITY to PLANAR 3-COLORABILITY. Given an arbitrary graph G, we construct a planar graph G' such that G' is 3-colorable if and only if G is 3-colorable.

Consider a drawing of G in the plane. We replace each crossing by a planar "gadget" that has the effect of propagating color across the crossing in a 3-coloring. In every proper 3-coloring of the graph H below, u and v have the same color, as do x and y (Exercise 17). The partial edges indicate where copies of H will be linked to the rest of the graph.



In the drawing of G, each edge involved in k crossings is cut into $k+1$ segments by the crossings. On each segment, add a new vertex. Replace each crossing by a copy of H attached by its terminals to the new vertices on the four segments incident to the crossing. Finally, for each original edge wz, choose one endpoint and contract the edge between it and the vertex on the segment of wz incident to it. An edge involved in no crossings returns to its original state.



In a proper 3-coloring of the new graph G', the propagation of color across the gadgets requires the endpoints of each original edge to have different colors. Hence restricting this coloring to the original vertices yields a 3-coloring of G.

Conversely, given a proper 3-coloring of G, we can start along each edge from the endpoint used in a copy of H and propagate the colors to properly 3-color G'. We can do this because H has both a proper 3-coloring in which u, v, x, y all get the same color and a proper 3-coloring in which x and y have a different color from u and v (Exercise 17). ∎

HAMILTONIAN CYCLE also remains NP-complete for planar graphs. Indeed, it remains NP-complete for graphs that are planar, 3-regular, 3-connected, and have no face of length less than 5 (Garey–Johnson–Tarjan [1976]). Also it is NP-complete for bipartite graphs (Krishnamoorthy [1975]) and for line graphs (Bertossi [1981]; see also Exercise 14). The restriction of 3-COLORABILITY to line graphs is the same as 3-EDGE-COLORABILITY for graphs; this is NP-complete even for 3-regular graphs, by reduction from 3-SAT (Holyer [1981]).

EXERCISES

B.1. (−) Using algorithms developed earlier in this text, describe a good algorithm to compute $\alpha(G)$ when G is bipartite.

B.2. (!) Use Theorem B.7 to prove that k-COLORABILITY is NP-complete for each fixed value of k that is at least 3.

B.3. Give a polynomial-time algorithm for 2-COLORABILITY.

B.4. Prove that HAMILTONIAN CYCLE and HAMILTONIAN PATH are polynomially equivalent. That is, show that a polynomial-time algorithm for either one can be used to obtain a polynomial-time algorithm for the other.

B.5. Testing for a cycle of fixed length k in an input graph with n vertices can be done in time bounded by a multiple of $k!n^k$: look at each of the $\binom{n}{k}$ vertex subsets of size k in turn and test all possible orderings. Since k is a constant, this is polynomial time. For a 4-cycle, this runs in time $O(n^4)$. Devise an algorithm that will test for the presence of a 4-cycle in time $O(n^2)$. (Richards–Liestman [1985])

B.6. Given a graph G and an integer k, the MINIMUM DEGREE SPANNING TREE problem asks whether G has a spanning tree T such that $\Delta(T) \le k$, and the LONGEST PATH problem asks whether G has a path of length at least k. In the k-PATH problem, k is not part of the input, and the question is whether G has a path of length at least k.

a) Prove that MINIMUM DEGREE SPANNING TREE and LONGEST PATH are NP-complete.

b) Prove the k-PATH is in P for each fixed k.

B.7. Construct a family of examples to prove that the performance ratio of the nearest-insertion heuristic for the TSP is not bounded by any constant.

B.8. (!) Consider an instance of TSP where the costs satisfy the triangle inequality. Prove that some matching of the odd-degree vertices in a minimum spanning tree uses edges with total cost at most half the minimum cost of a spanning cycle. Conclude that Christofides' algorithm has performance ratio at most 3/2.

B.9. Prove that in order to solve the TSP exactly, it suffices to have an algorithm that solves the TSP when the edge weights satisfy the triangle inequality. (Hint: Given an arbitrary instance of the TSP, produce in polynomial time an instance of the TSP where the edge weights satisfy the triangle inequality ($w_{ij} + w_{jk} \ge w_{ik}$) and the set of optimal tours is the same as in the original instance.)

B.10. Prove that 2-SAT belongs to P.

B.11. A road system has one snowplow. The narrow city streets must be plowed; the plow can clear such streets completely in one traversal. There are also rural roads that the plow can use to change position, but these don't need to be plowed. Thus we have a weighted graph with edges of two types; those of type 1 must be traversed, those of type 2 need not be. The state wants an algorithm that will find a minimum-length circuit traversing all type 1 edges in such a graph. Prove that this problem is NP-hard, by reduction from the HAMILTONIAN CYCLE problem.

B.12. (!) *Heuristic algorithms for vertex covering.* Algorithm 1: include a vertex of maximum remaining degree, delete, iterate until the remaining graph is a stable set. Algorithm 2: choose an arbitrary edge, include both endpoints, delete them, iterate until the remaining graph is a stable set. The heuristic in Algorithm 1 may seem more powerful, but Algorithm 2 has a better performance guarantee!

a) Prove that algorithm 2 always produces a vertex cover with size at most twice the minimum size $\beta(G)$.

b) Prove that algorithm 1 may produce a vertex cover of size about $\log \beta(G)$ times the minimum. (Hint: Construct a bipartite graph G for which Algorithm 1 chooses about $\beta(G)/i$ vertices of degree i for each $1 \le i \le \beta(G)$.)

B.13. (+) A graph G is **α-critical** if $\alpha(G - e) > \alpha(G)$ for every $e \in E(G)$. Prove that a connected α-critical graph has no cut-vertex. (Hint: If e_1, e_2 are edges incident to a cut-vertex x, use the maximum independent sets in $G - e_1$ and $G - e_2$ to build an independent set in G with more than $\alpha(G)$ vertices.)

B.14. SATISFIABILITY differs from 3-SAT in that clauses may have arbitrary size. Prove that a clause containing more than three literals can be replaced by clauses with three literals (allowing the addition of a few new variables) so that the original clause is satisfiable if and only if the new instance of 3-SAT is satisfiable. Conclude that SATISFIABILITY reduces to 3-SAT. (Karp [1972])

B.15. (!) Given that HAMILTONIAN CYCLE is NP-complete for 3-regular graphs, prove that COVERING CIRCUIT is NP-complete. This is the question of whether the input graph G contains a closed trail that includes at least one endpoint of every edge. Prove that the line graph of G is Hamiltonian if and only if G has a covering circuit. Conclude that HAMILTONIAN CYCLE is NP-complete for line graphs.

B.16. The DOMINATING SET problem considers an input graph H and an input integer k and asks whether H has a dominating set of size at most k (Definition 3.1.26).

a) Given a graph G, let G' be the graph obtained by adding an extra copy of each edge of G and subdividing one copy of each edge (thus each edge is replaced by a triangle involving a new vertex). Prove that G has a vertex cover of size at most k if and only if G' has a dominating set of size at most k.

b) Use the NP-completeness of VERTEX COVER to prove the NP-completeness of DOMINATING SET.

B.17. Prove the claims made in Theorem B.12 about 3-colorings of the gadget H.

B.18. (*) Use HAMILTONIAN PATH in directed graphs to prove that 3-MATROID INTERSECTION is NP-complete.

B.19. (*) Use 3-D MATCHING to prove that 3-MATROID INTERSECTION is NP-complete. Given a collection of triples of the form (x_1, x_2, x_3) with $x_i \in V_i$ and sets V_1, V_2, V_3 disjoint, 3-D MATCHING is the problem of finding the maximum number of triples such that each element appears in at most one of the triples selected (by comparison, bipartite matching is the 2-D matching problem with the sets being pairs).

Appendix C

Hints for Selected Exercises

In this appendix we provide some general guidelines and some specific suggestions for selected exercises. This should help students who have trouble getting started in finding and writing proofs.

GENERAL DISCUSSION

The first step is making sure that one understands exactly what the problem is asking. Some problems request a verification of a mathematical statement. Definitions may provide a road map for what needs to be verified. Sometimes, the desired statement follows from a theorem already proved, and then one needs to verify that its hypotheses hold.

Other problems may involve some experimentation to discover the mathematical statement that needs to be proved. Sometimes one examines small cases to discover a general pattern and then proves that pattern by induction. In other problems, exploration of examples can help one understand why the claim is true.

Understand definitions and use them carefully. A disconnected graph need not have an isolated vertex. Loops and cycles are different concepts. Understand the difference between *maximal* and *maximum*. A vertex cover is a set of vertices, and an edge cover is a set of edges. A graph with connectivity 3 is 2-connected, and a 3-chromatic graph is 17-colorable.

When seeking a direct proof of a conditional statement, one can work from both ends. List statements that follow from the hypothesis. List statements that suffice to imply the conclusion. When some statement appears in both lists, the problem is solved.

When unsuccessful with the direct method, list what would follow if the conclusion were not true. If something in this list contradicts something in the list of statements that follow from the hypothesis (or other known true statements), then again the problem is solved, using the method of contradiction.

Contradiction is particularly appropriate for statements of impossibility. To prove that something exists, often one can construct an example and prove that it has the desired properties; this is the direct method.

Most conditional statements can be interpreted as universally quantified statements. The proof of a universally quantified statement must be valid for every value of the variable in the given universe. Examples can help one understand or explain a proof, but an example by itself does not provide a proof. Explaining why the example has the desired property, in language that applies to all possible examples, can lead to the proof of the desired statement.

Induction often works to prove statements that have a natural number parameter. Beware of the induction trap |—(Example 1.3.26, Example A.28). Remember the template for proving conditional statements by induction (Remark 1.3.25). Exploration of small cases can help one understand either the statement to be proved or the way to go from one value of the parameter to the next, but such discussion is *not* part of the final proof except as needed in the basis step.

Other techniques include extremality and the pigeonhole principle. Sometimes one considers a smallest counterexample to a desired statement and then uses its existence to obtain a smaller counterexample. This can be viewed as induction or contradiction or extremality.

It may not be obvious what technique works in a particular problem. Sometimes many different techniques work and produce different proofs. Mathematicians find proofs by working hard; both stubbornness and flexibility are virtues. One tries all imaginable techniques to solve a problem. Practice increases understanding and speed in finding proofs.

The final step is to produce a careful and complete exposition of the solution. Writing out a proof can reveal hidden subtleties or cases that have been overlooked. It can also expose thoughts that turn out to be irrelevant. Producing a well-written solution often involves repeated revision. It is helpful to write a solution early, put it aside, and read it again before submission to see whether it is still complete, convincing, and comprehensible. The process of writing solutions helps develop a useful skill: the ability to express oneself concisely, clearly, and accurately.

SUPPLEMENTAL SPECIFIC HINTS

1.1.14. Contradiction often helps with a nonexistence conclusion. Suppose that such a decomposition exists; what conclusions can be drawn about the board?

1.1.25. Another nonexistence conclusion. Assume that a 7-cycle exists, and use the properties of the Petersen graph to obtain a contradiction.

1.1.26. Consider an edge in G (one can also start with a vertex).

1.1.27. Start with a vertex in G.

1.1.29. Consider the acquaintances or nonacquaintances (whichever set is larger) of one particular person.

1.1.32. Consider the parity of the sizes of the partite sets.

1.1.34. Since the three subgraphs are to be pairwise isomorphic, it will help to examine a drawing of the graph that has three-fold symmetry.

1.1.37. Compare contributions from the ends of the paths with contributions from the internal vertices.

1.1.38. Get a decomposition from a bipartition and a bipartition from a decomposition.

1.2.15. In this problem the starting vertex is specified, so having the first and last edge be the same is not enough.

1.2.17. Use the transitivity of the connection relation.

1.2.18. Draw G for small values of n to see what the answer should be; then prove it.

1.2.19. For the upper bound, use the fact that when a and b are relatively prime, there are integers p and q such that $pa + qb = 1$.

1.2.26. The characterization of bipartite graphs makes this easy.

1.2.28. The problem does not restrict attention to induced subgraphs.

1.2.38. Use induction and Lemma 1.2.25.

1.2.40. If P and Q are disjoint, then consider a shortest path from $V(P)$ to $V(Q)$, and obtain a contradiction.

1.3.12. It may help to construct the example first in the case $k = 1$ and then generalize.

1.3.15. For part (b), consider the complements.

1.3.18. Assuming that e is a cut-edge and H is a component of $G - e$, count the edges of H from the viewpoint of each partite set. Setting these formulas equal leads to a contradiction.

1.3.19. For the second part, make the desired graphs correspond to 3-regular graphs.

1.3.22. In part (a), what happens if an outside vertex has three neighbors in $V(C)$? For part (b), part (a) provides one bound on the number of edges between $V(C)$ and $V(G) - V(C)$; the hypothesis on minimum degree provides another.

1.3.28. When k is even, exhibit an isomorphism. When k is odd, find an odd cycle in Q'_k.

1.3.31. For part (a), consider Example 1.3.18.

1.3.33. For part (a), establish a one-to-one correspondence between the nonneighbors of x and the pairs of neighbors of x.

1.3.34. Consider two adjacent vertices x, y, and establish a one-to-one correspondence between $N(x)$ and $N(y)$.

1.3.43. For the construction, a regular graph can't work. Vertices of high and low degrees are needed, but both must be adjacent to high-degree vertices.

1.3.50. Construct for each n an example with few edges, and use induction on n to prove that it is optimal. The degree-sum formula implies that an n-vertex simple graph with fewer than $3n/2$ edges has a vertex of degree at most 2.

1.3.53. Define a graph to model pairs of people who can still play together. What is the condition that permits an additional game to be played?

1.3.55. For part (a), show first that $\Delta(G) \geq n(G)/2$. For part (b), show that $\overline{G}$ must be disconnected.

1.3.57. Keep the paradigm of Remark 1.3.25 in mind.

1.3.63. Any inductive proof of sufficiency must verify that the "smaller object" satisfies the condition before the induction hypothesis can be applied.

1.4.16. For part (a), keep the definition of l in mind.

1.4.23. The shortest proof uses a graph transformation.

1.4.25. Given an orientation with more than two vertices of odd outdegree, make an appropriate alteration.

1.4.29. Use the strong connectedness of D and the reference odd cycle of G to build an odd cycle in D.

1.4.34. Show that in the subgraph F of G consisting of edges oriented oppositely in H, indegree equals outdegree at every vertex. Bring G closer to H by finding a 3-cycle reversal involving a vertex of maximum outdegree in F.

1.4.37. Use strong induction on the order of the tournament.

1.4.38. In one direction, show that there is such a tournament with n vertices if there is one with $n-2$ vertices. Be careful about $n=6$.

2.1.2. In part (b), the statement includes the possibility of adding multiple copies of edges already present.

2.1.17. Compare with the proof of A⇒B,C in Theorem 2.1.4.

2.1.25. Use induction on n; for the induction step, delete a leaf.

2.1.27. Keep the quantifier in mind. Prove that the condition on the list of numbers is both necessary and sufficiency for the *existence* of a tree with vertex degrees $d_1, \dots, d_n$. Two implications must be proved.

2.1.29. Count the edges in two ways.

2.1.31. Consider the contrapositive.

2.1.33. Two implications must be proved. Each concerns a connected n-vertex graph.

2.1.34. Use induction on n.

2.1.40. Express G as a union of the right number of paths. What can be done if they are not pairwise intersecting?

2.1.41. Use a spanning tree of some component.

2.1.47. In part (a), remember that the union of a u,v-path and a v,w-path need not be a u,w-path.

2.1.59. In the problem, both n and k are fixed. The answer must be given in terms of these two parameters.

2.1.61. Form G' from $G-x-y$ by adding k disjoint edges joining $N_G(x)$ to $N_G(y)$.

2.2.5. Consider for each 5-cycle the number of edges that will be used.

2.2.7. By symmetry, each edge of K_n lies in the same number of spanning trees of K_n.

2.2.9. Use the Prüfer code.

2.2.19. In a tree with vertex set $[n]$, cut the edge at vertex n on the path from n to 1.

2.2.24. Build the edges in decreasing order of the difference between their endpoints.

2.2.29. If a tree is not a caterpillar, then it contains the tree Y of Example 2.2.18.

2.2.33. Consider the set of vertices reachable by paths from the root.

2.3.11. What happens if a minimum-weight spanning tree contains an edge of weight larger that the maximum weight in a bottleneck spanning tree?

2.3.13. Consider the heaviest edge of T' among those not in T.

2.3.31. In the induction step, partition the set of words in an optimal code into two sets according to the first bit.

3.1.8. Consider the symmetric difference of two perfect matchings.

3.1.9. Use vertex covers, or compare a maximal matching with a maximum matching using symmetric difference.

3.1.16. Use induction on k.

3.1.24. Transform this into a graph problem.

3.1.25. Find one appropriate permutation matrix and then adjust what remains by a constant factor to apply the induction hypothesis.

3.1.26. Use Corollary 3.1.13 for part (a) and induction on n for part (b).

3.1.29. What does the König–Egerváry theorem say about the vertex cover problem when G has no matching of size k.

3.1.30. This requires a bound and an example that achieves the bound.

3.1.39. Consider the edges joining a maximum independent set to its complement.

3.2.11. Suppose that the first occurrence of such a rejection in the Proposal Algorithm is a rejecting x even though xa is a pair in some stable matching M. If a rejects x for y, note that y must be paired with some women b in M. What can be deduced about the preferences of these people?

3.3.2. Vertex cover is not strong enough to prove optimality of the matching.

3.3.7. This is easy when k is even. For odd k, construct an example for $k = 3$ and generalize it.

3.3.11. Use Tutte's Theorem.

3.3.12. To prove that there is a matching whose size is the weight of some generalized cover, let T be a maximal set of vertices maximizing the quantity $o(G - T) - |T|$ (the deficiency).

3.3.14. Show for each $S \subseteq V(G)$ that $o(G - S) - |S|$ is sufficiently small. A different approach is to let X be the k-vertex set inducing the fewest edges and prove that there is a matching from X into $\overline{X}$.

3.3.16. Generalize the argument of Corollary 3.3.8.

3.3.17. Given adjacent vertices x, y, verify Tutte's condition for $G - x - y$.

3.3.18. Think of an appropriate size of a Tutte set S and an appropriate number of odd components in $G - S$.

3.3.19. Use Petersen's Theorem to obtain a 1-factor. Assemble copies of P_4 by considering a consistent orientation of the cycles in the remaining 2-factor.

4.1.5. Show that G' is connected and has no cut-vertex. There is also a short proof using internally disjoint paths:

4.1.10. Theorem 4.1.11 and Corollary 1.3.5 yield a short proof.

4.1.14. Prove the contrapositive.

4.1.17. Show first that if $\left|[S, \overline{S}]\right| = 3$, then S and $\overline{S}$ have odd size.

4.1.18. For the first part, show that the subgraph induced by the smaller side of an edge cut with at most two edges has too many edges to avoid having a triangle.

4.1.23. Verify Tutte's Condition. Keep in mind that the forbidden condition is large stars as induced subgraphs, not just any large stars.

4.1.26. Necessity is proved by following individual cycles. For sufficiency, define an appropriate auxiliary graph whose vertices are the components of $G - F$. Show that this graph is bipartite to obtain the desired partition of the components of $G - F$.

4.1.27. Recall that an edge cut is a bond if and only if on each side of the cut the vertices induce a connected subgraph.

4.2.6. Use ear decomposition.

4.2.14. This is essentially an edge version of Theorem 4.2.2, with the added conclusion that the common points happen in the same order on the two paths.

4.2.21. What do we know about connected graphs having at most two vertices of odd degree?

4.2.23. From a given bipartite graph G, design a graph H so that applying Menger's Theorem to H will yield the needed result on G.

4.2.28. Use the Expansion Lemma and Menger's Theorem.

4.3.13. Design a network so that there is an assignment of participants to groups if and only if the network has a flow of value $\sum m_i$. Use the Ford–Fulkerson Theorem to express the condition for such a flow in terms of cuts. Show that the given condition on the data is equivalent to that condition on the source/sink cuts.

5.1.20. One approach is to prove the contrapositive. Another is to delete an odd cycle.

5.1.22. An ordering is needed so that when each vertex is encountered, it has at most two neighbors among the earlier vertices.

5.1.23. The lower bounds involve counting and/or the pigeonhole principle. The interesting part is providing a construction to show that the graph is $k+2$-colorable with $k+1$ does not divide n.

5.1.26. (also next problem) Obtain a clique and a proper coloring of the same size.

5.1.30. Given a proper r-coloring of G_n, produce a collection of distinct subsets of the colors, each subset associated with an element of n. Conversely, show how to use such a collection to produce a proper coloring.

5.1.31. From a proper m-coloring, build an independent set of the same size. From a big enough independent set, construct a proper m-coloring.

5.1.32. As in Theorem 1.2.23, one can use induction or encode the colors as binary k-tuples and apply the pigeonhole principle.

5.1.39. Obtain a quadratic inequality for k in terms of m by using upper and lower bounds on $e(G)$.

5.1.41. Using the induction hypothesis, a new graph can't violate the bound unless we delete a vertex and find that the chromatic numbers of both the graph and its complement decrease. Is it possible for this to happen when these chromatic numbers already sum to the maximum?

5.1.44. Obtain the upper bound from part (a). For the lower bound, use the same orientation as in Theorem 5.1.21.

5.1.51. Modify some proper k-coloring so that it becomes a proper $k+1$-coloring that has the pre-specified values on the vertices of P.

5.2.2. Consider the complement.

5.2.9. Since G' is connected, it suffices to consider deletion of edges from G'; see Remark 5.2.12.

5.2.15. Use large neighborhoods as color classes while there remain vertices of high degree; then apply Brooks' Theorem.

5.2.17. For part (b), consider complements.

5.2.19. Modify $\overline{K}_a + K_{n-a}$ to obtain $T_{n,r}$, counting the changes in the number of edges.

5.2.21. The proof of Theorem 5.2.9 transforms a graph with no $r+1$-clique into an r-partite graph with at least at many edges. The number of edges strictly increases unless equality holds in each step of the computation. What is needed for equality to hold?

5.2.27. For the upper bound, deleting the edges of a cycle from a counterexample leaves a forest, which restricts the girth. Reducing to the case $\delta(G) \geq 3$ now requires $n \leq 8$, which contradicts Exercise 5.2.26.

5.2.28. For the upper bound, reduce to the case $\delta(G) \geq 3$, and consider a shortest cycle C. Deleting $V(C)$ leaves a forest, and its leaves have at least three neighbors on C.

5.2.29. For part (b), if the largest and smallest color classes differ in size by more than 1, use part (a) to alter the coloring appropriately.

5.2.32. Part (a) uses the properties of k-critical graphs for G and H. In part (c), one can give explicit examples with orders 4, 6, 8 and then apply part (a).

5.2.40. To compute the chromatic number, consider independent sets. To forbid subdivisions of complete graphs, consider vertex cuts; a subdivision of K_k must have $k-1$ pairwise internally disjoint paths joining two of its vertices of degree $k-1$.

5.2.43. Use induction on k. In the induction step, discard an appropriate subset of $V(G)$.

5.2.44. Reduce to the case $\delta(G) \geq k-1$ and then use induction on k.

5.3.3. If $\chi(G;k) = k^4 - 4k^3 + 3k^2$, then how many edges and vertices does G have?

5.3.4. For part (a), use the chromatic recurrence or Theorem 5.3.10.

5.3.6. Explain how contributions to the coefficient of $k^{n(G)-1}$ arise in computing $\sum_{r=1}^{n} p_r(G)k_{(r)}$.

5.3.12. For part (a), use the chromatic recurrence. For part (b), consult Exercise 1.3.40 for the maximum number of edges in an n-vertex graph with r components.

5.3.18. Part (a) implies that part (b) needs only one computation. Expressing a chromatic polynomial as the sum of two chromatic polynomials involves addition of an edge as in Example 5.3.9.

5.3.23. Use a simplicial elimination ordering.

5.3.26. For part (a), use a simplicial elimination ordering; the simplicial vertex may or may not be in $G \cap H$. For part (b), $N(x)$ may or may not equal $V(G) - x$.

5.3.28. Build a simplicial elimination ordering of G and a transitive orientation of $\overline{G}$.

6.1.20. Which plane graphs have only one face? When a plane graph has more than one face, what kind of edge can be deleted to reduce the number of faces?

6.1.24. In the induction step, delete an edge of a cycle.

6.1.25. Use Euler's Formula.

6.1.28. This can be proved by applying Euler's Formula to an appropriate planar graph. Also it can be proved using induction.

6.1.30. Mimic the proof of Theorem 6.1.23.

6.2.6. In order to apply the claim and the induction hypothesis, find an appropriate vertex to delete from a larger graph.

6.2.7. Given the graph G to be tested, construct a graph H such that G is outerplanar if and only if H is planar. Kuratowski's Theorem then applies to H.

6.2.8. Several cases need to be considered concerning how a subdivision of K_5 might be arranged in the graph.

6.2.9. For the construction, start with $n = 5$. Reduce the upper bound to considering planar graphs with $2n$ edges that have minimum degree at least three and contain a triangle. In each case, obtain disjoint cycles or a subdivision of $K_{3,3}$.

6.2.11. Let H' be a subgraph contractible to H. Prove that if H is not all of H', then H' has an edge incident to a vertex of degree 2. Use this in an inductive proof.

6.3.5. Use the Four Color Theorem.

6.3.9. When C has length 4, replacing the inside (or outside) with a single edge between opposite vertices of C allows one to obtain a 4-coloring of a C-lobe that uses distinct colors on two opposite vertices of C.

6.3.12. For the construction, use groups of three vertices to build "alcoves" such that no guard can see into more than one alcove.

6.3.20. Show that the lower bound resulting from Proposition 6.3.10 is at least $r/2$ when $s > (r-2)^2/2$, and provide a decomposition into $r/2$ planar subgraphs.

6.3.27. Consider the copies of $K_{m,n}$ in a drawing of $K_{m+1,n}$.

6.3.28. (also next problem). Consider what happens when a vertex moves across an edge. For the second part, consider a drawing where the crossings are easy to count.

7.1.11. For necessity in part (b), show first that the average vertex degree must be 2.

7.1.16. Letting $G = L(H)$, use H to show for $S \subseteq V(G)$ that the number of components of $G - S$ is at most $|S| + 1$.

7.1.17. Extract an embedding of G from an embedding of $L(G)$.

7.1.20. Consider Definition 1.4.20.

7.1.24. It may help to consider first the special case $H = K_2$, though this need not be proved separately.

7.1.26. Every edge incident to the cut-vertex must appear in some color.

7.1.33. Show that an improvement can be made when one color appears too often and another not enough.

7.2.17. Reduce the problem to the case where the two Hamiltonian graphs are cycles.

7.2.23. For each k, determine how large S must be so that $G - S$ has k components.

7.2.29. Write the given condition involving the degrees of G and $\overline{G}$ solely in terms of the degree of G; then show that G satisfies Chvátal's condition for spanning paths.

7.2.31. Use the Chvátal–Erdős Theorem

7.2.32. Be careful about how the transformation modifies the degrees; the condition is stated correctly. (Theorem 7.2.19).

7.3.3. The dual graph is 3-regular.

7.3.4. Consider the faces inside and outside a spanning cycle separately.

7.3.5. Translate this problem into a statement proved earlier.

7.3.17. Reduce this to studying the graph analyzed in Example 7.3.6.

7.3.18. Modify the graph to obtain a situation where Grinberg's Theorem applies.

7.3.21. Given the triple of colors on $\{x_{i-1}x_i, y_{i-1}y_i, z_{i-1}z_i\}$ consider the possibilities for the triple of colors on $\{x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}\}$.

Appendix D

Glossary of Terms

In addition to terms used in this book, this glossary also contains other related terms that the reader may encounter in further study. This includes some alternative terminology used by other authors.

The items are informal sketches of definitions. Numbers in brackets are page references for the full definition or the first usage. When used without specification, "G" indicates a graph or possibly also a digraph, "D" indicates a digraph, v or e indicate a vertex or edge, and n indicates the number of vertices.

Absorption property (matroids) [351]: $r(X) = r(X \cup e) = r(X \cup f)$ implies $r(X) = f(X \cup f \cup e)$
Acyclic [67]: without cycles
Acyclic orientation [203, 208]: orientation without cycles
Adjacency matrix A [6]: entry $a_{i,j}$ is number of edges from vertex i to vertex j
Adjacency relation: set of unordered or ordered pairs forming edges in graph or digraph
Adjacency set $N(v)$: the set of vertices adjacent to v
Adjacent [2]: vertices that are endpoints of an edge, sometimes used to describe edges with a common endpoint
Adjoins: is adjacent to
Adjugate: matrix of cofactors
Almost always [430]: having asymptotic probability 1
M-alternating path: a path alternating between edges in M and not in M
Ancestor [100]: in a rooted tree, a vertex along the path to the root
Antichain: family of pairwise incomparable items (under an order relation)
Anticlique: stable set
Antihole: induced subgraph isomorphic to the complement of a cycle
Approximation algorithm [496]: polynomial-time algorithm with bounded performance ratio
Approximation scheme [496]: family of approximation algorithms with arbitrarily good performance ratio
Arborescence: a directed forest in which every vertex has outdegree at most one
Arboricity $\Upsilon(G)$ [372]: minimum number of forests covering the edges
Arc: directed edge (ordered pair of vertices)
k-arc-connected: same as k-edge-connected for digraphs
Articulation point: a vertex whose deletion increases the number of components
Assignment Problem [126]: minimize (or maximize) the sum of the edge weights in a perfect matching of a complete bipartite graph with equal part-sizes
Asteroidal triple [346]: three distinct vertices with each pair connected by a path avoiding the neighborhood of the third

Asymmetric: having no automorphisms other than the identity
Asymptotic [431]: having ratio approaching 1
Augmentation property (matroids) [352]: $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$ implies the existence of $e \in I_2 - I_1$ such that $I_1 \cup e \in \mathbf{I}$
Augmenting path [109]: for a matching, an alternating path that can be used to increase the size of the matching; for a flow, increases the flow value
Automorphism [14]: a permutation of the vertices that preserves the adjacency relation
Automorphism group Γ: the group of automorphisms under composition
Average degree: $\sum d(v)/n(G) = 2e(G)/n(G)$.
Azuma's Inequality: a bound on the probability in the tail of a distribution

Backtracking [156]: depth-first-search
Balanced graph [434]: the full graph is the subgraph with the largest average vertex degree
Balanced k-partite: partite sets differ by at most one in size (see equipartite)
Bandwidth: the minimum, over vertex numberings by distinct integers, of the maximum difference between labels of adjacent vertices
Barycenter [78]: vertex minimizing the sum of distances to other vertices
Base (matroids) [349]: maximal independent set
Base exchange property (matroids) [351]: for all $B_1, B_2 \in \mathbf{B}$ and $e \in B_1 - B_2$, there exists an element $f \in B_2 - B_1$ such that $B_1 - e + f$ is a base.
Berge graph [340]: a graph with no odd hole or odd antihole
Best possible: fails to be true when some condition is loosened
Bicentral tree: a tree whose center is an edge
Biclique [9]: complete bipartite graph
Biconnected: 2-connected
Bigraphic [65, 185]: a pair of sequences realizable as the vertex degrees for the partite sets in a simple bipartite graph
X, Y-bigraph [24]: a bipartite graph with bipartition X, Y
Binary matrix (or vector): all entries in $\{0, 1\}$
Binary matroid [357]: representable over the field with two elements
Binary tree [101]: rooted tree in which every non-leaf vertex has at most two children
Binomial coefficient [487] $\binom{n}{k}$: the number of ways to choose a subset of size k from an n-element set, equal to $n!/[k!(n-k)!]$.
Biparticity [422]: number of bipartite subgraphs needed to partition the edges
Bipartite graph [4]: a graph whose vertices can be covered by two independent sets
Bipartite Ramsey number: for a bipartite G, the minimum n such that 2-coloring the edges of $K_{n,n}$ forces a monochromatic G
Bipartition [24]: a partition of the vertex set into two independent sets
Birkhoff diamond [259]: a particular reducible configuration for the Four Color Problem
Block [155]: (1) a maximal subgraph with no cut-vertex; (2) a graph with no cut-vertex; (3) a class in a partition of a set
Block-cutpoint graph 156: simple bipartite graph in which the partite sets are the blocks and the cutvertices of G and the adjacency relation is containment
Block graph: intersection graph of the blocks in G
Blossom [142]: an odd cycle arising in Edmonds' algorithm for general matching
Bond [154]: a minimal edge cut
Bond matroid [362]: dual of the cycle matroid of a graph
Bond space [452]: orthogonal complement to the cycle space; linear combinations of bonds (over field of two elements)
Book embedding: a decomposition of G into outerplanar graphs with a consistent ordering of the vertices (as on the spine of a book)
Bouquet: a graph consisting of one vertex and some number of loops
Branch vertex [249]: a vertex of degree at least 3
Branching: a digraph where each vertex has indegree one except one that has indegree 0
r-branching [404]: branching rooted at r
Breadth-first search [99]: a search exploring vertices in order by distance from root
Breadth-first tree: tree generated by a breadth-first search from a root

Bridge [304]: cut-edge
H-bridge of G: H-fragment (used by other authors)
Bridgeless graph [304]: graph without cut-edges
Brooks' Theorem: $\chi(G) \le \Delta(G)$ for connected graphs, except for cliques and odd cycles

Cactus [160]: a connected graph in which every edge appears in at most one cycle
(k, g)-cage [49]: a $k$regular graph of smallest order among those with girth g
Capacity [176, 178]: a limit on flow (1) through an edge in a network; (2) across a cut
Cartesian product $G_1 \square G_2$ [193]: the graph with vertex set $V(G_1) \times V(G_2)$ and edges given by $(u_1, u_2) \leftrightarrow (v_1, v_2)$ if 1) $u_1 = v_1$ and $u_2 \leftrightarrow v_2$ in G_2 or 2) $u_2 = v_2$ and $u_1 \leftrightarrow v_1$ in G_1
Caterpillar [88]: a tree with a single path containing at least one endpoint of every edge
Cayley's Formula [81]: statement there are n^{n-2} trees with vertex set $[n]$
2-cell [268]: on a surface, a region homeomorphic to a disc, meaning that every closed curve is contractible to a point
2-cell embedding [268]: an embedding in which every region is a 2-cell
Center [72]: subgraph induced by the vertices of minimum eccentricity
Central tree [78]: a tree whose center is one vertex
α, β-chain: a path alternating between colors α and β
Characteristic polynomial $\phi(G; \lambda)$ [453]: characteristic polynomial of the adjacency matrix of the graph, whose roots are the eigenvalues
Children [100]: in a rooted tree, neighbors of the current vertex that are farther from the root
Chinese Postman Problem [99]: problem of finding the cheapest closed walk covering all the edges in an edge-weighted graph
Choice number [408]: choosability
Choosability $\chi_l(G)$ [408]: minimum k such that G is k-choosable
k-choosable [408]: for all lists of size k assigned to vertices of G, there is a proper coloring that selects a color for each vertex from its list
Chord [225]: edge joining two nonconsecutive vertices of a path or cycle
Chordal graph [225]: having no chordless cycle
Chordless cycle [225]: an induced cycle of length at least 4
Chordless path: a path that is an induced subgraph
Chromatic index $\chi'(G)$ [275]: edge-chromatic number
Chromatic number $\chi(G)$ [5, 191]: minimum number of colors in a proper coloring.
Chromatic polynomial $\chi(G; k)$ [220]: a polynomial whose value at k is the number of proper colorings of G using colors from $\{1, \dots, k\}$.
Chromatic recurrence: recurrence relation for chromatic polynomial
k-chromatic [192]: having chromatic number k
Circle graph [341]: an intersection graph of chords of a circle
Circuit [27, 60]: equivalence class of closed trails without specifying starting vertex (an even graph); (caution—used by some authors to mean *cycle*)
Circulant graph: a graph constructed as equally-spaced vertices on a circle with adjacency depending only on distance
Circular-arc graph [341]: an intersection graph of arcs of a circle
Circulation [187]: a flow in a network with net flow 0 at each vertex
Circumference [293]: the length of the longest cycle
Clause [499]: a collection of literals in a logical (Boolean) formula
Claw [12]: the graph $K_{1,3}$
Claw-free: having no induced $K_{1,3}$
Clique [4]: set of pairwise-adjacent vertices (used by many authors to mean *complete graph*)
Clique cover [226]: a set of cliques covering the vertices (minimum size = $\theta(G)$)
Clique decomposition: a partition of the edge set into complete subgraphs
Clique edge cover: a set of complete subgraphs covering the edges
Clique identification: a perfection-preserving operation that merges cliques in two graphs
Clique number $\omega(G)$: maximum order of a clique in G
Clique partition number: minimum size of a clique decomposition
Clique tree [327]: an intersection representation of a chordal graph, consisting of a host tree with a bijection between its vertices and the maximal cliques of G such that the cliques containing each vertex form a subtree of the host

Clique-vertex incidence matrix [328]: 0,1-matrix in which entry (i, j) is 1 if and only if vertex j belongs to maximal clique i
Closed ear [164]: a path between two (possibly equal) old vertices through new vertices
Closed-ear decomposition [164]: construction of a graph from a cycle by addition of closed ears
Closed neigborhood [116]: a vertex and all its neighbors
Closed set (matroids) [360]: a set whose span is itself
Closed walk [20]: a walk whose last vertex is the same as its first
Closure [289, 360]: (1) the graph $C(G)$ obtained from G by iteratively adding edges joining nonadjacent vertices with degree-sum at least $n(G)$; (2) image under a closure operator
Closure operator [360]: an operator that is expansive, order-preserving, and idempotent
Cobase [360]: a base of the dual matroid
Cocircuit [360]: a circuit of the dual matroid
Cocritical pair: two nonadjacent vertices whose addition as an edge increases the clique number
Cocycle matroid [362]: the dual of a cycle matroid
Cocycle space: bond space
Cograph [202]: P_4-free graph (equivalent to *complement reducible* graph)
Color class [191]: in a coloring, a set of objects having the same color
Color-critical [192]: a graph such that every proper subgraph has smaller chromatic number
k-colorable [191]: having a proper coloring with at most k colors
k-coloring [191, 380]: a partition into k sets
P coloring: a vertex partition into subsets inducing graphs with property **P**
Column matroid $M(A)$ [351]: matroid whose independent sets are the linearly independent subsets of columns of the matrix A
Comma-free code: no code word is a prefix of another
Common system of distinct representatives (CSDR) [171]: given families **A** and **B** of sets, a CSDR is a set of elements that is an SDR of **A** and is an SDR of **B**
Comparability graph [228]: graph having a transitive orientation
Complement $\overline{G}$[3]: simple graph or digraph with the same vertex set as G, defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$
Complement reducible [344]: reducible to the trivial graph by iteratively taking complements of components
Complete graph K_n [9]: simple graph in which each two vertices are adjacent
Complete k-partite graph $K_{n_1,\ldots,n_k}$ [207]: k-partite graph in which every pair of vertices not belonging to the same partite set is adjacent (sizes of the partite sets are $n_1, \ldots, n_k$)
Completely labeled cell [388]: simplicial region with distinct labels on corners
Complexity [494]: the worst-case number of operations needed, as a function of the input size
Component [22]: maximal connected subgraph
S-component of G: see S-lobe
Composition $G_1[G_2]$ [332]: a graph whose vertex set is the cartesian product $V(G_1) \times V(G_2)$, defined by $(u_1, u_2) \leftrightarrow (v_1, v_2)$ if and only if $u_1 \leftrightarrow v_1$ in G_1, or $u_1 = v_1$ and $u_2 \leftrightarrow v_2$ in G_2
Conflict graph [252]: graph whose vertices are the bridges of a cycle, with bridges adjacent (conflicting) when they have three common endpoints or four alternating endpoints on the cycle
Conflicting chords: two chords whose endpoints alternate on a specified cycle
Conjugate partition: two partitions of n such that one gives the row sizes and the other the column sizes of a Ferrers diagram
Connected [6]: having a u, v-path for every pair of vertices u, v
k-connected [149, 164]: having connectivity at least k
Connection relation [21]: relation satisfied by vertices x, y if there is an x, y-path
Connectivity $\kappa(G)$ [149, 164]: the minimum number of vertices whose deletion disconnects the graph or reduces it to one vertex (sometimes called "vertex connectivity" for clarity)
Consecutive 1s property (for rows) [328]: having a permutation of columns so 1s appear consecutively in each row
Conservation constraint [176]: for a flow, the condition of net flow 0 at a vertex
Consistent rounding [186]: conversion of the data and the row/column sums in a matrix to nearest integers up or down such that row and column sums remain correct
Construction procedure: a procedure for iteratively building members of a class of graphs from a small base graph or graphs
Contraction [84]: replaces edge uv by a vertex w incident to the edges formerly incident to u or v

Converse D^{-1}: obtained from digraph D by switching the head and tail in each edge
Convex embedding [248]: a plane graph in which every bounded face is a convex set and the outer boundary is a convex polygon
Convex function [443]: satisfies the inequality $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ for all a, b and $0 \leq \lambda \leq 1$
Convex quadrilateral: not no corner in the triangle formed by the other three
Cost [125]: name of the objective function for many weighted minimization problems
Cotree: with respect to a graph, the edges not belonging to a given spanning tree
F-covering: covering of edge set by subgraphs in the family **F**
Critical edge [122, 339]: edge whose deletion increases the independence number
Critical graph: used with respect to many graph properties, indicating that the deletion of any vertex (or edge, depending on context) destroys the property
k-critical graph [192]: usually means color-critical with chromatic number k
Critically 2-connected: deletion of an edge destroys 2-connectedness
Crossing [234]: in a drawing of a graph, an internal intersection of two edges
Crossing number $\nu(G)$ [262]: minimum number of crossings when drawing G in the plane
k-cube Q_k [36]: k-dimensional cube
Cubic graph [304]: a regular graph of degree 3
Cut $[S, \overline{S}]$ [166]: the edges from a vertex subset to its complement (especially in networks)
Cut-edge [23]: an edge whose deletion increases the number of components
Cutset: a separating set of vertices
Cut-vertex [23]: vertex whose deletion increases the number of components
Cycle [5, 55]: a simple graph whose vertices can be placed on a circle so that vertices are adjacent if and only if they appear consecutively on the circle (caution—used by some authors to mean *even graph*)
Cycle double cover [312]: a list of cycles such that each edge appears in two items in the list
k-cycle [9]: a cycle of length k, consisting of k vertices and k edges
Cycle matroid $M(G)$ [350]: the matroid whose circuits are the cycles of G
Cycle rank: dimension of cycle space, equal to #edges − #vertices + #components
Cycle space [452]: the nullspace of the incidence matrix; the elements correspond to the even subgraphs
Cyclic edge-connectivity: number of edges that must be deleted to disconnect a component so that every remaining component contains a cycle
Cyclically k-edge-connected: cyclic edge-connectivity at least k

de Bruijn graph [61]: digraph encoding possible transitions between k-ary n-tuples as additional characters are received
Decision problem [494]: a computational problem with a YES/NO answer
Decomposition [11]: an expression of G as a union of edge-disjoint subgraphs
F-decomposition [397]: decomposition using graphs in the family **F**
F-decomposition number of G: minimum number of graphs in an **F**-decomposition of G
Degree $d(v)$ [6, 34]: (1) for a vertex, the number of times it appears in edges (may be modified by "in-" or "out-" in a digraph); (2) for a regular graph, the degree of each vertex
Degree sequence $d_1 \geq \cdots \geq d_n$ [44]: the list of vertex degrees, usually indexed in nonincreasing order regardless of vertex order
Degree set: the set of vertex degrees (appearing once each)
Degree-sum Formula: $\sum d(v) = 2e(G)$
Deletion method [428] a strengthening of the existence argument in the probabilistic method
Demand [184]: sink constraint in transportation network
Density [435]: ratio of number of edges to number of vertices
Dependent edge [231]: an edge in an acyclic orientation whose reversal creates a cycle
Dependent set (matroids) [349]: a set containing a circuit
Depth-first search [156]: backtracking search from a vertex, exploring from the most recently reached vertex and backing up when it has no new neighbors
Descendants of x [100]: in a rooted tree, members of the subtree rooted at x
Diagonal Ramsey number [385]: Ramsey number for an instance where the thresholds (numbers or graphs) are equal

Diameter [70]: the maximum of the distance $d(u, v)$ over vertex pairs u, v
Digraph [53]: directed graph
Dijkstra's Algorithm [97]: algorithm to compute shortest paths from one vertex
Dilworth's Theorem [413]: maximum number of pairwise incomparable elements equals minimum number of totally ordered subsets needed to cover all elements
k-dimensional cube Q_k [36]: simple graph with vertex set $\{0, 1\}^k$ where vertices are adjacent if and only if their names differ in exactly one coordinate
Dinitz Conjecture [410]: each bipartite graph G is $\Delta(G)$-list-edge-colorable
Directed graph [53]: vertex set, edge set, and specification of head and tail for each edge
Directed walk, trail, path, cycle, etc. [57]: same as without the adjective "directed" (the head of an edge is the tail of the next edge)
Disc: a planar region bounded by a simple closed curve
Disconnected [6]: a graph with more than one component
Disconnecting set [152]: a set of edges whose deletion makes some vertex unreachable from some other vertex
Disjoint union $G_1 + G_2$ [39]: the union of two graphs with disjoint vertex sets
Disjointness graph: complement of intersection graph
Distance $d(u, v)$ [70]: the minimum length of a u, v-path
Distance-preserving embedding [400]: mapping $f\colon V(G) \to V(H)$ so that $d_H(f(u), f(v)) = d_G(u, v)$.
Dodecahedron [243]: planar graph with 20 vertices, 30 edges, and 12 faces of length 5
Dominating set [116]: a set $S \subseteq V$ such that every vertex outside S has a neighbor in S
Domination number [116]: the minimum size of a dominating set of vertices
Double jump [437]: the markedly different structure of the random graph in Model A for probability functions of the form c/n with $c < 1$, $c = 1$, and $c > 1$.
Double star [77]: a tree with at most two vertices of degree more than 1
Double torus [266]: the (orientable) surface with two handles
Double triangle [280]: $K_4 - e$
Doubly stochastic matrix [120]: square matrix having sum 1 in each row and column
Dual augmentation property (matroids) [362]: disjoint sets independent in a matroid and its dual can be enlarged to a complementary base and cobase
Dual edge e^* [236]: the edge of the dual graph G^* corresponding to edge e of a plane graph G
Dual graph G^* [236]: for a plane graph G, the graph with a vertex for each region of G, where vertices are adjacent if the boundaries of their regions in G share an edge (extends to 2-cell embeddings on any surface)
Dual hereditary system (or matroid) M [360]: the hereditary system whose bases are the complements of the bases of M
Dual problem [113]: for a problem $\max c^T x$ such that $Ax \le b$ and $x \ge 0$, the dual is $\min y^T b$ such that $yA \ge c$ and $y \ge 0$
Duality gap: strict inequality between optimal values of a pair of dual integer programs
Duplication of vertex x [321]: adding x' with $N(x') = N(x)$

Ear [163]: path whose internal vertices have degree two (or are "new")
Ear decomposition [163]: construction of G from a cycle by addition of ears
Eccentricity $\epsilon_G(v)$ [70]: for a vertex, the maximum distance to other vertices
Edge [2]: (1) in a graph, a pair of vertices ($E(G)$ denotes the edge set); (2) in a hypergraph, a subset of the vertex set
Edge-choosability $\chi_l'(G)$ [409]: minimum k such that G is k-edge-choosable
k-edge-choosable [409]: for all lists of size k assigned to edges of G, there exists a proper edge-coloring that selects a color for each edge from its list
Edge-chromatic number $\chi'(G)$ [275]: the minimum number of colors in a proper edge-coloring
k-edge-colorable [275]: having a proper edge-coloring with at most k colors
Edge-coloring [274]: an assignment of labels to the edges
k-edge-connected [152, 164]: having edge-connectivity at least k
Edge-connectivity $\kappa'(G)$ [152]: the minimum number of edges whose deletion disconnects G
Edge cover [114]: a set of edges incident to all the vertices
Edge cut $[S, \overline{S}]$ [152, 164]: the set of edges joining a vertex in S to a vertex not in S
Edge-reconstructible: a graph that can be determined (up to isomorphism) by knowing the multiset of subgraphs obtained by deleting single edges

Edge-Reconstruction Conjecture: the conjecture that every graph with at least four edges is edge-reconstructible
Edge-transitive [18]: having for each pair $e, f \in E(G)$ a permutation that maps e to f
Eigenvalue [453]: for a graph, an eigenvalue of the adjacency matrix
Eigenvector of A [453]: a vector x such that $A_x = \lambda x$ for some constant λ
Elementary contraction [84]: contraction
Elementary cycle: boundary of a region in a plane graph (caution - some authors who use "cycle" to mean *circuit* use "elementary cycle" to mean *cycle*)
Elementary subdivision [162]: replacement of an edge by a path of two edges connecting the endpoints of the original edge (see *edge subdivision*)
Embedding [234]: a mapping of a graph into a surface, such that (the images of) its edges do not intersect except for shared endpoints
Empty graph [22]: graph having no edges
Endpoint [2]: (1) each member of an edge; (2) the first or last vertex of a path, trail, or walk
End-vertex: a vertex of degree 1
Equipartite [207]: having partite sets differing in size by at most 1
Equitable coloring: having color classes differing in size by at most 1
Equivalence [399]: as a graph, a union of pairwise disjoint complete graphs
Equivalence relation [490]: reflexive, symmetric, and transitive relation
Erdős number: distance from Erdős in the collaboration graph of mathematicians
Euler characteristic: for a surface of genus γ, $2 - 2\gamma$
Euler tour: Eulerian circuit
Eulerian circuit [26, 60]: a closed trail containing every edge
Eulerian (di)graph [26, 60]: a graph or digraph having an Eulerian circuit
Eulerian trail [26, 60]: a trail containing every edge
Euler's Formula [241]: the formula $n - e + f = 2 - 2\gamma$ for 2-cell embeddings of a connected n-vertex graph with e edges and f faces on a surface of genus γ
Even cycle [24]: cycle with an even number of edges (or vertices)
Even graph [26]: graph with all vertex degrees even
Even pair [348]: vertex pair x, y such that every chordless x, y-path has even length
Even triangle [280]: triangle T such that every vertex has an even number of neighbors in T
Even vertex [26]: vertex of even degree
Evolution: the model of generating random graphs by successively adding random edges
(n, k, c)-expander [463]: bipartite graph with partite sets of size n and vertex degrees at most k such that each set S with at most half the vertices of the first partite set has at least $(1 + c(1 - |S|/n)|S|$ neighbors
Expansion: in 3-regular graph, subdivides two edges and adds one edge joining the new vertices
Expansion Lemma [162]: adding a vertex of degree k to a k-connected graph preserves k-connectedness
Expansive property [358]: for a function σ on the subsets of a set, the requirement that $X \subseteq \sigma(X)$ for all X
Expectation [427]: for a discrete random variable, $\sum k\text{Prob}\ (X = k)$
Exterior region: the unbounded region in a plane graph
Exterior vertex: vertex on the unbounded region

Face 235]: a region of an embedding
Factor [136]: a spanning subgraph
f-factor [140]: a spanning subgraph with $d(v) = f(v)$
k-factor [140]: a spanning k-regular subgraph
k-factorable [276]: having a decomposition into k-factors
Factorization: an expression of G as the edge-disjoint union of spanning subgraphs
k-factorization [276]: a decomposition of a graph into k-factors
x, U-fan [170]: pairwise internally-disjoint paths from x to distinct vertices of U
Fáry's Theorem [246]: a planar graph has a straight-line embedding in the plane
Fat triangle [275]: a 3-vertex graph in which each pair has the same edge multiplicity
Feasible flow [176]: a network flow satisfying edge-constraints and having net flow 0 at each internal vertex

Feasible solution [322]: a choice of values for the variables that satisfies all the constraints in an optimization problem

Ferrers digraph: a digraph (loops allowed) with no x, y, z, w (not necessarily distinct) such that $x \to y$ and $z \to w$ but $z \not\to y$ and $x \not\to w$; equivalently, the successor sets or predecessor sets are ordered by inclusion; equivalently, the adjacency matrix has no 2-by-2 permutation submatrix.

Five Color Theorem [257]: the theorem that planar graphs are 5-colorable

Flat [266]: a closed set in a matroid

Flow [176]: an assignment of weights to edges of a network

k-flow [307]: an assignment of weights in $\{-k+1, \ldots, k-1\}$ to edges of a digraph so that net flow out is zero at each vertex

Flower (in Edmonds' Blossom Algorithm) [142]: consists of a stem (alternating path from an unsaturated vertex) and a blossom (odd cycle with a nearly-perfect matching)

Forcibly Hamiltonian: a degree sequence such that every simple graph with that degree sequence is Hamiltonian

Forest [67]: a disjoint union of trees, an acyclic graph

Four Color Theorem [260]: the theorem that planar graphs are 4-colorable

Fraternal orientation [345]: an orientation such that two vertices are adjacent if they have a common successor

H-fragment of G [252]: a component of $G - H$ together with the edges to its vertices of attachment

H-free [41]: having no copy of H as an induced subgraph

Free matroid [357]: matroid in which every set of elements is independent

Friendship Theorem [467]: if every pair of people in a set have exactly one common friend in the set, then someone is everyone's friend

Fundamental cycle [374]: for a spanning tree, a cycle formed by adding an edge to it

Gammoid [377]: a matroid on E arising from vertex sets F, E in a digraph by letting independent sets be those that are saturated by a set of disjoint paths starting in F

Generalized chromatic number: minimum number of classes needed to partition the vertices so that the subgraph induced by each color class has property **P**

Generalized Petersen graph [316]: the graph with vertices $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ and edges $\{u_iu_{i+1}\}$, $\{u_iv_i\}$, and $\{v_iv_{i+k}\}$, where addition is modulo n

Generalized Ramsey number $r(G_1, \ldots, G_k)$ [386]: the minimum n such that k-coloring the edges of K_n forces a copy of G_i in color i for some i

Genus γ [266]: (1) for a surface, the number of handles in its topological description (2) for a graph, the minimum genus surface on which it embeds

Geodesic: a shortest path between its endpoints

Geodetic: having the property that each pair of vertices u, v are the endpoints of a unique path of length $d(u, v)$

Girth [13]: the length of a shortest cycle in G

k-gon: in an embedding, a k-cycle bounding a region

Good algorithm [124]: algorithm running in polynomial time

Good characterization [495]: a characterization that is checkable in polynomial time

Good coloring: often means proper coloring

Gossip problem [406]: minimize the number of calls so that each vertex transmits to every other by an increasing path

Graceful labeling [87]: an assignment of distinct integers to vertices such that 1) the integers are between 0 and $e(G)$, and 2) the differences between the labels at the endpoints of the edges yield the integers $1, \ldots, e(G)$

Graceful graph [87]: a graph with a graceful labeling

Graceful tree [87]: a tree with a graceful labeling

Graceful tree conjecture [87]: every tree has a graceful labeling

Graph [2]: a set of vertices, a set of edges, and an assignment of a set at most two vertices as endpoints of each edge

Graphic matroid $M(G)$ [350]: matroid whose independent sets are the acyclic subsets of $E(G)$

Graphic sequence [44]: a list of integers realizable as the degree sequence of a simple graph

Greedy algorithm [95, 354]: a fast algorithm to find a good feasible solution by iteratively making a heuristically good choice

Greedy coloring [194]: with respect to some vertex ordering, color each vertex with the least-indexed color not already appearing among the neighbors of the vertex being colored
Grinberg condition [303]: necessary for Hamiltonian cycles in planar graphs, that summing (length−2) over the inside faces or over the outside faces yields the same total
Grötzsch graph [205]: the smallest triangle-free 4-chromatic graph
Grundy number: the maximum number of colors in an application of the greedy coloring algorithm

Hadwiger conjecture [213]: every k-chromatic graph has a subgraph contractible to K_k (true for "almost all" graphs)
Hajós conjecture [213]: every k-chromatic graph contains a K_k-subdivision (false for $k > 5$)
Hall's condition [110]: for every subset S of a partite set X in a bipartite graph, at least $|S|$ vertices have neighbors in S
Hall's theorem [110]: Hall's condition is necessary and sufficient for the existence of a matching that saturates X
Hamilton tour: Hamiltonian cycle
Hamiltonian [286]: having a Hamiltonian cycle
Hamiltonian closure [289]: graph obtained by successively adding edges joining vertices whose degree-sum is as large as the number of vertices
Hamiltonian-connected [297]: having a Hamiltonian path from each vertex to every other
Hamiltonian cycle [286]: a cycle containing each vertex
Hamiltonian path [291]: a path containing each vertex
Harary graphs [150]: a family of k-connected n-vertex graphs with the fewest edges
Head [53]: the second vertex of an edge in a digraph
Heawood's Formula [268]: the chromatic number of a graph embedded on the oriented surface with γ handles is at most $\lfloor 1/2(7 + \sqrt{1 + 48\gamma}) \rfloor$.
Helly property [80]: the property of the real line (or trees) that pairwise intersecting subsets have a common intersection point
Hereditary class [226]: a class F such that all induced subgraphs of graphs in F are also in F
Hereditary family [349]: a family F of sets such that every subset of a member of F is in F
Hereditary system [349]: a system consisting of a hereditary family and the alternative ways of specifying that family
Hole [340]: a chordless cycle in a graph
Homeomorphic: two graphs obtainable from the same graph by subdivision of edges
Homogeneous [380]: in Ramsey theory, a set whose colored pieces have the same color
Homomorphism: a map $f : V(G) \to V(H)$ that preserves adjacency
Huffman code [103]: prefix-free encoding of data to minimize expected search time
Hungarian Algorithm [126]: an algorithm for solving the assignment problem
Hypercube Q_k [36]: k-dimensional cube
Hypergraph [449]: a generalization of graph in which edges may be any subset of the vertices
Hyperplane (matroids) [360]: a maximal closed proper subset of the ground set
Hypohamiltonian: a non-Hamiltonian graph whose vertex-deleted subgraphs are all Hamiltonian
Hypotraceable: a non-traceable graph whose vertex-deleted subgraphs are all traceable

Icosahedron [243]: planar triangulation with 12 faces, 30 edges, and 20 vertices
Idempotence property (matroids) [359]: $\sigma^2(X) = \sigma(X)$ for all X
Identification: an operation replacing two vertices by a single vertex with the combined incidences (same as contraction if the vertices are adjacent)
Imperfect graph [232]: has $\chi(H) > \omega(H)$ for some induced subgraph H
Incidence matrix [6]: (1) for a graph, the 0,1-matrix in which entry (i, j) is 1 if and only if vertex i and edge j are incident; (2) for a digraph, entry (i, j) is 1 if vertex i is the head of edge j, -1 if it is the tail, 0 otherwise; (2) in general, the matrix of a membership relation
Incident [6]: 1) a vertex v and edge e with $v \in e$; 2) two edges with a common endpoint
Inclusion-exclusion principle [223]: number of objects outside $A_1, \ldots, A_n$ is $\sum_{S \in [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$
Incomparability graph: the complement of a comparability graph
Incorporation property (matroids) [359]: $r(\sigma(X)) = r(X)$
Indegree [58]: for a vertex in a directed graph, the number of edges of which it is the head
Independence number $\alpha(G)$ [113]: maximum size of an independent set of vertices

Independent domination number [117]: minimum size of an independent dominating set
Independent set [3]: a set of pairwise nonadjacent vertices
Indicator variable [427]: a random variable taking values in $\{0, 1\}$
Induced circuit property (matroids) [355]: adding an element to an independent set creates at most one circuit
Induced sub(di)graph $G[A]$ [23]: the sub(di)graph on vertex set $A \subseteq V(G)$ obtained by taking A and all edges of G having both endpoints in A
Integer program [323]: linear program plus requirement that variables be integer-valued
Integrality Theorem [181]: in a network with integer edge capacities, there is an optimal flow expressible as units of flow along source/sink paths
Interlacing Theorem [458]: for each vertex x, the eigenvalues $\{\lambda_i\}$ of G and $\{\mu_i\}$ of $G - x$ satisfy $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n$
Internal vertices [20]: (1) for a path, the non-endpoints; (2) for a plane graph, the vertices not on the boundary of the exterior face
Internally disjoint paths [161]: paths intersecting only at endpoints
Intersection graph [324]: for a family of sets, the graph having a vertex for each set and having vertices adjacent when the sets intersect
Intersection number [397]: minimum size of a set U such that G is an intersection graph of subsets of U (equals minimum number of complete subgraphs covering $E(G)$)
Intersection of matroids [366]: the hereditary system whose independent sets are the common independent sets in the matroids
Intersection representation [324]: an assignment of a set S_v to each vertex v such that $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$
Interval graph [195]: a graph having an interval representation
Interval number [451]: minimum t such that G has a t-interval representation
Interval representation of G [195]: a collection of intervals whose intersection graph is G
t-interval [451]: a union of at most t intervals in $\mathbb{R}$
t-interval representation [451]: an intersection representation where each assigned set is a t-interval
In-tree [89]: a directed tree in which each edge is oriented toward the root
Involution [470]: a permutation whose square is the identity
Isolated vertex or edge [22]: incident to no (other) edge
Isometric embedding [400]: a distance-preserving mapping of $V(G)$ into $V(H)$
Isomorphic decomposition: decomposition into isomorphic subgraphs
Isomorphism [7]: a vertex bijection preserving the adjacency relation
Isthmus: a cut-edge
Join $G \vee H$ [138]: the disjoint union $G + H$ plus the edges $\{uv : u \in V(G), v \in V(H)\}$
Joined to: adjacent to
Junction: vertex of degree at least three

Kempe chain [258]: a path between two vertices that alternates between two colors (particularly as used in forbidding minimal 5-chromatic planar graphs)
Kernel [57, 410]: in a digraph, an independent in-dominating set
Kernel perfect [410]: having a kernel in each induced subgraph
Kirchhoff's current law: net flow around a closed walk is 0
Kite [12]: simple 4-vertex graph obtained by deleting one edge from K_4
König-Egerváry Theorem [112]: maximum matching and minimum vertex in a bipartite graph have equal size
König's Other Theorem [115]: maximum independent and minimum edge cover in a bipartite graph with no isolated vertices have equal size
Krausz decomposition [285]: edge covering by complete subgraphs using each vertex at most twice (leads to the graph for which this is the line graph)
Kronecker product: tensor product
Kruskal's algorithm [95]: grows a minimum weighted spanning tree by iteratively adding the cheapest edge in the graph that does not complete a cycle
Kuratowski subgraph [247]: subdivision of K_5 or $K_{3,3}$
Kuratowski's Theorem [246]: a graph is planar if and only if it has no subdivision of K_5 or $K_{3,3}$

Labeling: assignment of integers to vertices
Leaf [67]: vertex of degree 1
Leaf block [156]: a block containing only one cut-vertex
Length [20]: the number of steps (or sum of weights) from start to finish
Lexicographic product $G[H]$ [393]: composition
Line: another name for edge
Line graph $L(G)$ [168, 273]: the intersection graph of the edges of G, where vertices correspond to edges of G and are adjacent if the corresponding edges share a vertex
Linear matroid [351]: matroid whose independent sets are the sets of independent columns of some matrix over some field
Linear program [179]: problem of optimizing a linear function with linear constraints
Link: edge
k-linked: a stronger condition than k-connected, in which for every choice of two k-tuples of vertices $(u_1, \dots, u_k)$ and $(v_1, \dots, v_k)$, there exists a set of k internally disjoint paths connecting corresponding vertices u_i, v_i.
List chromatic index [409]: edge-choosability
List chromatic number [408]: choosability
List Coloring Conjecture [409]: edge-choosability always equals edge-chromatic number
Literal [500]: a logical (true/false) variable or its negation
S-lobe [211]: a subgraph of G induced by $S \cup V_i$, where V_i is the vertex set of a component of $G - S$
Local search: technique for solving optimization problems by successively making small changes in a feasible solution
Loop [2]: an edge whose endpoints are the same
Loopless [6]: having no loops

(n, k, c)-magnifier [463]: n-vertex graph of maximum degree k in which each set S with at most half the vertices has at least $c\,|S|$ neighbors outside S
Markov chain [54]: discrete system with transition probabilities
Markov's inequality [432]: for a nonnegative random variable, Prob $(X \geq t) \leq E(X)/t$
Martingale [443]: sequence of random variables such that $E(X_i | X_0, \dots, X_{i-1}) = X_{i-1}$
Matching [107]: a set of edges sharing no endpoints
b-matching: given a constraint vector b, a subgraph H with $d_H(v) \leq b(v)$ for all v
Matrix rounding [186]: problem of converting the data and row/column sums in a matrix to nearest integers up or down such that row and column sums remain correct
Matrix-Tree Theorem [86]: subtracting the adjacency matrix from the diagonal matrix of degrees, deleting a row and column, and taking the determinant yields the number of spanning trees
Matroid [354]: a hereditary system satisfying any one of a list of many equivalent properties
Matroid basis graph [376]: graph whose vertex set is the collection of bases of a matroid, adjacent when their symmetric difference has two elements
Matroid Covering Theorem [372]: the number of independent sets needed to cover the elements of a matroid is $\max_{X \subseteq E} \lceil |X| / r(X) \rceil$
Matroid Intersection Theorem [367]: the maximum size of a common independent set in two matroids on E equals the minimum over $X \subseteq E$ of the rank of X in the first matroid plus the rank of $\overline{X}$ in the second matroid
Matroid Packing Theorem [372]: the maximum number of pairwise disjoint bases in a matroid is $\min_{r(X) < r(E)} \lfloor (|E| - CA(X))/(r(E) - r(X)) \rfloor$
Matroid Union Theorem [370]: the union of matroids $M_1, \dots, M_k$ is a matroid with rank function $r(X) = \min_{Y \subseteq X} (|X - Y| + \sum r_i(Y))$
Max-flow Min-cut Theorem [180]: maximum flow value equals minimum cut value
Maximal clique [31]: a maximal set of pairwise adjacent vertices
Maximal path or trail [27]: non-extendible path or trail
Maximal planar graph [242]: equivalent to planar triangulation
Maximum Cardinality Search [325]: an algorithm for recognizing chordal graphs
Maximum degree Δ [34]: maximum of the vertex degrees
Maximum flow [176]: a feasible network flow of maximum value, or the value itself
Maximum genus $\gamma_M(G)$: the maximum genus surface on which G has a 2-cell embedding
Maximum (P-object) [31]: for a property P, no larger object of the same type also has property P

Menger's theorems [167–169]: min-max characterizations of connectivity by number of pairwise internally-disjoint or edge-disjoint paths between pairs of vertices
Meyniel graph [330]: any graph in which every odd cycle of length at least 5 has at least two chords
Minimal imperfect graph [320]: imperfect graph where every proper induced subgraph is perfect
Minimally 2-connected [175]: deleting any edge destroys 2-connectedness
Minimum cut [178]: a source/sink cut having minimum value, or the value of such a cut
Minimum degree $\delta(G)$ [34]: minimum of the vertex degrees
Minimum (P-object) [31]: for a property P, no smaller object of the same type also has property P
Minimum Spanning Tree (MST) [95]: spanning tree with minimum sum of edge weights
Minor [251, 362]: graph (or matroid) obtained by deletions and contractions
Mixed graph: a graph model allowing directed and undirected edges
Möbius ladder: the graph obtained by adding to an even cycle the chords between vertex pairs at maximum distance on the cycle (can be drawn as a ladder with a twist)
Möbius strip: the non-orientable surface obtained by identifying two opposite sides of a rectangle using opposite orientation
Model A [430]: probability distribution generating simple graphs with vertex set $[n]$ by letting each pair be an edge with probability $p(n)$, independently
Model B [430]: probability distribution making the simple graphs with vertex set $[n]$ and m edges equally likely
rth-moment [433]: expectation of X^r
Monochromatic [386]: in a coloring, a set having all elements the same color
Monotone graph property [432]: preserved under deletion of edges or vertices
Multigraph: used by many authors to mean graphs that allow (but don't require) multiple edges and loops (some authors forbid loops from multigraphs)
Multinomial coefficient [489]: counts arrangements having fixed multiplicities of items; with k_i items of type i, there are $(\sum k_i)!/\prod(k_i!)$ ways to arrange them in a list
Multiple edges [2]: edges with the same endpoints

Nearest-insertion [497]: TSP heuristic to grow a cycle
Nearest-neighbor [496]: TSP heuristic to grow a path
Neighborhood $N(v)$ [34]: set of neighbors of v (*closed* neighborhood $N[v]$ also includes v)
Neighbors [2]: (noun) the vertices in the neighborhood; (verb) "is adjacent to"
Net outflow [178]: at a vertex, the total exiting flow minus the total entering flow
Network [176]: a directed graph with a distinguished initial vertex (source) and a distinguished terminal vertex (sink), in which each edge is assigned a flow capacity and possibly also a flow demand (lower bound)
Node: vertex, especially in network flow problems
Nondeterministic algorithm [494]: allowed to "guess" by having parallel computation paths
Nondeterministic polynomial algorithm [494]: having a polynomial-time computation path for each guess of a polynomial number of bits
Nonorientable surface: a surface with only one side
Nontrivial graph [22]: having at least one edge
Nonplanar [243]: having no embedding in the plane
Nowhere-zero k-flow [207]: a $k$flow in which all assigned weights are nonzero
NP [495]: the class of problems solvable by nondeterministic polynomial algorithms
NP-complete [495]: NP-hard and in NP
NP-hard [495]: provides a polynomial algorithm for every problem in NP
Null graph [3]: graph having no vertices
Numbering: a bijection from $V(G)$ to $[n(G)]$

Obstruction: forbidden substructure
Odd antihole [340]: complement of an odd hole
Odd component [136]: component with an odd number of vertices
Odd cycle [24]: cycle with an odd number of edges (vertices)
Odd graph: the disjointness graph of the k-subsets of $[2k+1]$
Odd hole: chordless odd cycle
Odd vertex [27]: vertex of odd degree

Odd walk [24]: walk of odd length
Open walk [20]: walk in which the first and last vertex are different
Optimal tour: a solution to the traveling salesman problem or Chinese postman problem
Order of graph [34]: the number of vertices
Ordered graph [406]: a graph with an order relation (usually linear) on the edges
Order-preserving property [358]: for a function σ on the set of subsets of a set, the requirement that $X \subseteq Y$ implies $\sigma(X) \subseteq \sigma(Y)$
Orientable surface: a surface with two distinct sides
Orientation of graph [62]: a digraph obtained by designating a head and tail for each edge
Outdegree [58]: for a vertex, the number of edges of which it is the tail
Outerplanar graph [239]: a planar graph embeddable in the plane so that all the vertices are on the boundary of the exterior region
Outerplane graph [239]: a particular embedding of an outerplanar graph

Parallel elements [351]: non-loops in a matroid that form a set of rank 1
Parent [100]: the neighbor of a vertex along the path to the root in a rooted tree
Parity [473]: odd or even
Parity subgraph of G [312]: subgraph H such that $d_H(v) \equiv d_G(v) \bmod 2$ for all $v \in V(G)$
k-partite [5]: same as *k-colorable*
Partite set [4]: a set in a vertex partition into independent sets (color class)
Partition matroid [357]: a matroid induced by a partition of the ground set in which a set is independent if and only if it has at most one element from each block of the partition
Partitionable graph [335]: a graph with $aw + 1$ vertices where each vertex-deleted subgraph is colorable by w stable sets of size a and coverable by a cliques of size w
Path [5]: a simple graph whose vertices can be listed so that vertices are adjacent if and only if they are consecutive in the list
u, v-path [20]: a path with u and v as endpoints
Path addition [163]: a step in an ear decomposition
Path decomposition [414]: expression of a graph as a union of pairwise edge-disjoint paths
Paw [12]: simple 4-vertex graph obtained by adding one edge to a claw
p-critical graph [334]: an imperfect graph whose proper induced subgraphs are all perfect
Pendant edge [67]: edge incident with a vertex of degree 1
Pendant vertex [67]: a vertex of degree 1
α-perfect [319]: $\alpha(H) = \theta(H)$ for evey induced subgraph H
β-perfect: [335] $\alpha(H)\omega(H) \geq n(H)$ for every induced subgraph H
γ-perfect: [319] $\chi(H) = \omega(H)$ for evey induced subgraph H
Perfect elimination ordering [224]: deletion order such that when each vertex is deleted, its neighborhood in what remains is a clique (same as *simplicial elimination ordering*)
Perfect graph [226]: graph such that $\chi(H) = \omega(H)$ for every induced subgraph H
Perfect Graph Theorem (PGT) [226, 320]: a graph is perfect if and only if its complement is perfect
Perfect order [331]: a vertex order yielding optimal greedy colorings for all subgraphs
Perfectly orderable graph [331]: having a perfect order
Perfect matching [107]: a set of edges such that each vertex belongs to exactly one of them
Peripheral vertex [70]: a vertex of maximum eccentricity
Permutation [486]: a bijection from a finite set to itself
Permutation graph: representable by a permutation σ by $v_i \leftrightarrow v_j$ if and only if σ reverses the order of i and j
Permutation matrix [120]: a 0,1-matrix having exactly one 1 in each row and column
Petersen graph [12]: the disjointness graph of the 2-sets in a 5-element set
Pigeonhole principle [491]: every set of numbers has one at least as large as the average
Pigeonhole property [427]: a finite probability space has an element where the value of a random variable is at least as large as its expectation
Planar graph [5, 235]: a graph embeddable in the plane
Plane graph [235]: a particular planar embedding of a planar graph
Plane tree [101]: tree with fixed cyclic embedding order of edges at each vertex
Planted tree [101]: rooted plane tree
Platonic solid [242]: bounded regular polyhedron

Point: vertex
Polygonal curve [234]: concatenation of segments
Polyhedron [242]: an intersection of half-spaces
Polytope: the convex hull of a set of vertices
Positional game [120]: a game in which the objective is seizing the positions of a winning set
kth-power (G^k): the graph with vertex set $V(G)$ in which $u \leftrightarrow v$ if and only if $d_G(u, v) \leq k$
Predecessor [54]: for v in a digraph, a vertex u with $u \rightarrow v$
Predecessor set [58]: for v in a digraph, the set of precedessors
Prefix-free code [101]: no code word is a prefix of another
Prim's Algorithm [104]: grows a minimum spanning tree by adding a leaf to the current tree in the cheapest way
Principal submatrix: square submatrix using rows and columns with the same indices
Product dimension [398]: minimum number of coordinate in a product representation of G
Product representation [398]: encoding of graph such that vertices are adjacent if and only if their codes differ in every coordinate
Proper coloring [192]: (1) for vertices, a coloring in which no edge is monochromatic; (2) for edges, a coloring in which edges sharing an endpoint have distinct colors
Proper subgraph of G [192]: a subgraph not equal to G
Proper subset of S [472]: a subset not equal to S
Proposal Algorithm [131]: procedure for creating a stable matching
Prüfer code [81]: for a labeled tree, a sequence of length $n-2$ obtained by successively deleting the leaf with smallest label and recording its neighbor's label
Pseudograph: graph model that allows loops and multiple edges, used by authors who define multigraphs not to have loops

Radius [70]: the minimum of the vertex eccentricities
Ramsey number [380]: the minimum number of vertices such that assigning colors to all pairs of those vertices produces a monochromatic clique of specified size (or a specified graph) in one of the colors
Random graph [430]: a graph from a probability space, most often the space in which each labeled pair of vertices independently has probability p of adjacency; typically, $p = 1/2$ or p is a function of n
Random variable [427]: a variable that takes on a value at each point in a probability space
Rank (matroids) [349]: for a set of elements, the largest size of an independent set it contains
Reconstructible [38]: a graph determined (up to isomorphism) by the list of subgraphs obtainable by deleting a single vertex
Reconstruction Conjecture [38]: claim that all graphs with at least 3 vertices are reconstructible
Rectilinear crossing number: the minimum number of crossings in a drawing of the graph in the plane in which all edges appear as straight line segments
Reducible configuration [258]: forbidden from purported minimal 5-chromatic planar graph
Reflexive [490]: (1) a digraph with a loop at every vertex; (2) a binary relation R with xRx for all x
Region [235]: for an embedding of a graph on a surface, a maximal connected subset of the surface that does not contain any part of the graph
Regular [34]: having all vertex degrees equal
Regular matroid [351]: representable over every field
k-regular [34]: having all vertex degrees equal to k
Representable matroid [351]: linear matroid
Restriction martingale [445]: martingale in which the value of successive variables is an expectation over a shrinking subset of the probability space
Rigid circuit graph: chordal graph
Robbins' Theorem [166]: every 2-edge-connected graph has a strong orientation
Root [100]: (1) a distinguished vertex; (2) in a branching, the vertex with indegree 0
Rooted plane tree [100]: a tree with a distinguished root vertex so that children of each non-leaf have a specified left-to-right ordering in the plane
Rotation scheme: a description of a 2-cell embedding; a circular permutation of the edges appearing at each vertex, giving their counter-clockwise order around the vertex

SATISFIABILITY [499]: the problem of finding truth values for variables to make a logical input formula true
Satisfiable [499]: formula having a "yes" answer in the SATISFIABILITY problem
Saturated vertex [107]: for a matching, a matched vertex
Score sequence [62]: the sequence of outdegrees in a tournament
Second moment method [433]: method for obtaining threshold functions
Self-complementary [11]: isomorphic to the complement
Self-converse: isomorphic to the converse
Self-dual: isomorphic to the dual
Semi-strong perfect graph theorem [344]: if $V(G) = V(H)$ and a set of vertices induces P_4 in G if and only if it induces P_4 in H, then G is perfect if and only if H is perfect
Semipath: an semiwalk in which each vertex appears at most once
Semiwalk: a sequence of edges (or adjacent vertices) in a directed graph such that each successive pair of edges are adjacent, without regard to the orientation of the edges
Separable: having a cut-vertex
Separating set: a vertex set whose deletion increases the number of components
k-set [380]: set of size k
Shannon Switching Game [365]: a game played on a matroid by the Spanner and the Cutter, one trying to seize a set of elements spanning a specified element, the other trying to prevent this
Shift graph [202]: graph on the 2-subsets of $[n]$ having $\{i, j\}$ adjacent to $\{j, k\}$ when $i < j < k$
Signed (di)graph: special case of weighted (di)graph, assigning $+$ or $-$ to each edge
Simple [2]: (1) a graph with no loops or multiple edges; (2) a digraph having at most one edge with each ordered pair of endpoints; (3) a matroid having no loops or parallel elements
Simplicial vertex [224]: (1) a vertex whose neighbors induce a clique;
Sink [176]: a distinguished terminal vertex, or any vertex with outdegree 0
Size [35, 473]: (1) the number of edges; (2) the number of elements
Skew partition [347]: a partition X, Y of $V(G)$ such that $G[X]$ and $\overline{G}[Y]$ are disconnected
f-soluble [148]: having an edge weighting so that the sum of the weights incident to v is $f(v)$
Source [176]: a distinguished initial vertex, or any vertex with indegree 0
Source/sink cut [178]: a partition of the vertices of a network into sets S, T such that S contains the source and T contains the sink
Span function [358]: the span of a set X in a hereditary system consists of X and the elements not in X that complete circuits with subsets of X
Spanning subgraph: a subgraph containing each vertex
Spanning set [67]: a set whose span (in a hereditary system on E) is E
Spanning tree [67]: a spanning, connected, acyclic subgraph
Spectrum [453]: the list of eigenvalues with multiplicities
Split graph [345]: a graph whose vertices can be covered by a clique and an independent set
Splittance: minimum number of edges to be added or deleted to obtain a split graph
Square of a graph: the second power
Squashed-cube dimension [401]: minimum length of the vectors in a squashed cube embedding
Squashed-cube embedding [401]: encodes vertices by $0, 1, *$-vectors such that distance between two vertices is the number of coordinates where one has 0 and the other has 1
Stability number [319]: independence number
Stable matching [130]: a matching having no instance of x and y each prefering the other to their current partner in the matching
r-staset [447]: stable set of size r
Stable set [3, 319]: a set of pairwise nonadjacent vertices (same as *independent set*)
Star [67]: the tree $K_{1,n-1}$ with at most one non-leaf
Star-cutset [333]: separating set inducing a subgraph having a vertex adjacent to all others
Star-cutset Lemma [334]: no p-critical graph has a star-cutset
Steinitz exchange property [358]: the property of span functions that if e is in the span of $X \cup f$ but not in the span of X, then f is in the span of $X \cup e$
Steinitz's Theorem: 3-connected planar graphs have only one embedding in the plane (more precisely, only one dual graph)
Strength [440]: of a theorem, the fraction of the time when the conclusion holds that the hypothesis also holds

Strict digraph [294]: a digraph having no loops and at most one edge with each ordered pair of endpoints
Strictly balanced: average vertex degree in subgraphs is maximized only by the full graph
Strong absorption property (matroids) [355]: if $r(X \cup e) = r(X)$ for all $e \in Y$, then $r(X \cup Y) = r(X)$
Strong component [56]: maximal strongly connected subdigraph
Strong orientation [165]: orientation of G in which each vertex is reachable from every other
Strong Perfect Graph Conjecture (SPGC) [320]: the conjecture that a graph is perfect if and only if it has no odd hole or odd antihole
Strong product $G_1 \cdot G_2$: a graph product with vertex set $V(G_1) \times V(G_2)$ and edge set $(u_1, v_1) \leftrightarrow (u_2, v_2)$ if $u_1 = u_2$ or $u_1 \leftrightarrow u_2$ and $v_1 = v_2$ or $v_1 \leftrightarrow v_2$
Strongly connected (or strong) digraph [56]: a digraph with each vertex reachable from all others
Strongly perfect [330]: a graph in which some stable set meets every maximal clique
Strongly regular [464]: a k-regular graph whose adjacent pairs have λ common neighbors, and whose nonadjacent pair have μ common neighbors
Subconstituent [470]: the subgraph induced by a vertex neighborhood or by a vertex non-neighborhood
Subdigraph [56]: a subgraph of a directed graph
Subdivision [212]: (1) the operation of replacing an edge by a path of two edges through a new vertex; (2) a graph obtained by a sequences of subdivisions
H-subdivision [212]: a graph obtained from H by subdivisions
Subgraph [5]: a graph whose vertices and edges all belong to G
Submodular function [354]: a function such that $r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y)$ for all sets X, Y
Submodularity property (matroids) [354]: having a submodular rank function
k-subset [471]: subset with k elements
Subtree representation [324]: assigns subtrees of a host tree to each vertex of a chordal graph so that vertices are adjacent if and only if the corresponding subtrees intersect
Successor [54]: for u in a digraph, a vertex v with $u \to v$
Successor set [58]: for u in a digraph, the set of successors
Sum [39]: (1) for cycles and cocycles, same as symmetric difference; (2) for a graph, the disjoint union; (3) for matroids on disjoint sets, the matroid on their union whose independent sets are all unions of an independent set from each
Supergraph of G: a graph containing G
Superregular [470]: a regular graph that is null or whose subconstituents are all superregular
Supply [184]: source constraint in a transportation network
2-switch [46]: a degree-preserving switch of two disjoint edges for two others not present
Symmetric [490]: (1) for a graph, having a non-trivial automorphism; (2) for a simple digraph, $u \to v \Leftrightarrow v \to u$; (3) for a binary relation R, $xRy \Leftrightarrow yRx$
Symmetric difference $A \triangle B$ [109, 473]: the set of elements in exactly one of A and B
System of distinct representatives (SDR) [119]: from a collection of sets, a choice of one member from each set so that all the representatives are distinct
Szekeres-Wilf Theorem [231]: $\chi(G) \le 1 + \max_{H \subseteq G} \delta(H)$

Tail [53]: the first vertex of an edge in a digraph
Tait coloring [301]: for a planar cubic graph, a proper 3-edge-coloring
Tarry's Algorithm [95]: procedure for exploring a maze
Telegraph problem [423]: directed version of gossip problem with one-way transmissions
Telephone problem [422]: gossip problem
Tensor product: weak product
Ternary matroid [357]: representable over the field with three elements
Thickness [261]: the minimum number of planar graphs whose union is G
Threshold dimension: minimum number of threshold graphs whose union is G
Threshold function for Q [433]: a function t such that Q almost always or almost never occurs, depending on whether the parameter in the model belongs to $o(t)$ or to $\omega(t)$.
Threshold graph: having a threshold t and a vertex weighting w such that $u \not\leftrightarrow v$ iff $w(u)+w(v) \le t$; many other characterizations, including absence of a 2-switch and existence of a construction ordering by adding isolated or dominating vertices
Topological graph theory: the study of drawings of graphs on surfaces

Toroidal [266]: graph having a 2-cell embedding on the torus
Torus [266]: the (orientable) surface with one handle
Total coloring [411]: a labeling of both the vertices and edges so that elements that are adjacent or incident receive different colors
Total Coloring Conjecture [411]: every graph G has a total coloring using at most $\Delta(G) + 2$ colors
Total domination number [117]: minimum number of vertices in a set S such that every vertex has a neighbor in S
Total interval number: minimum of the total number of intervals used to represent G as the intersection graph of unions of intervals on the real line
Totally unimodular [469]: a matrix in which all square submatrices have determinant 0 or ± 1
Toughness [288]: the minimum t such that $|S| \geq t \cdot c(G - S)$ for every separating set S, where $c(G - S)$ is the number of components of the subgraph obtained by deleting S
Tournament [61]: an orientation of the complete graph
Trace [453]: sum of the diagonal elements of a matrix
Traceable: having a Hamiltonian path
Trail [20, 59]: a walk in which no edge appears more than once
Transitive digraph [228]: $u \to v$ and $v \to w$ together imply $u \to w$
Transitive closure: (1) for a digraph D, the digraph with $u \to w$ whenever there is a path from u to w in D; (2) for a relation R, the relation S with xSy whenever there is a sequence $x_0, \ldots, x_k$ with $x = x_0 R x_1 R \cdots R x_k = y$
Transitivity of dependence (matroids) [359]: $e \in \sigma(X)$ and $X \subseteq \sigma(Y)$ imply $e \in \sigma(Y)$
Transportation constraints [184]: supplies and demands
Transportation Problem [185]: generalization of the assignment problem with supplies at each source and demands at each destination
Transversal [125]: a system of distinct representatives (this is the word used when the concept is generalized); also used for a system of representatives not necessarily distinct
Transversal matroid [352]: a matroid whose elements are one partite set of a bipartite graph and whose independent sets are the subsets saturated by matchings
Traveling Salesman Problem (TSP) [493]: problem of finding a minimum-weight spanning cycle
Tree [67]: a connected graph with no cycles
k-ary tree [101]: rooted tree with at most k children at each non-leaf vertex
k-tree [345]: a chordal graph obtained from a k-clique by iteratively adding a vertex whose neighborhood when added is a k-clique
Triangle [12]: a cycle of length 3
Triangle-free [41]: not having K_3 as a subgraph
Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$
Triangular chord: chord of length two along a path or cycle
Triangulated graph [225]: a graph with no chordless cycle
Triangulation [242]: a graph embedding on a surface such that every region is a 3-gon
Trivalent: having degree 3
Trivial graph [22]: graph with no edges (some authors restrict to one vertex)
k-tuple [474]: a list of length k
Turán graph [207]: an equipartite complete multipartite graph
Turán's theorem [208]: charcterization of the complete equipartite r-partite graphs as the largest graph of a given order with no $r + 1$-clique
Tutte polynomial: a generalization of the chromatic polynomial and of other polynomials
Tutte's Theorem [146, 174, 250]: (1) for matchings, characterization of graphs with 1-factors; (2) for connectivity, characterization of 3-connected graphs by contractions to wheels; (3) for planar graphs, 3-connected planar graphs have embeddings with all bounded faces convex.
Twins [348]: vertices having the same neighborhood (false twins are adjacent vertices with the same closed neighborhoods)

Unavoidable set [258]: a collection of configurations such that every graph in a specified class contains some configuration in the collection
Underlying graph [56]: the graph obtained from a digraph by treated edges as unordered pairs
Unicyclic: having exactly one cycle
k-uniform hypergraph [449]: having only edges of size k

Uniform matroid $U_{k,n}$ [357]: matroid on $[n]$ whose independent sets are the sets of size at most k
Uniformity property (matroids) [354]: for all $X \subseteq E$, the maximal independent subsets of X have the same size
Union ($G_1 \cup G_2$) [25]: a graph whose vertex set is the union of the vertices in G_1 and G_2 and whose edge set is the union of the edges in G_1 and G_2 (written $G_1 + G_2$ if the vertex sets are disjoint)
Union of matroids [369]: the union of matroids $M_1, \ldots, M_k$ is the hereditary system whose independent sets are $\{I_1 \cup \cdots \cup I_k\colon I_i \in \mathbf{I}_i\}$
Unit-distance graph [201]: the graph with vertex set $\mathbb{R}^2$ in which points are adjacent if the distance between them is 1
Unlabeled graph [9]: informal term for isomorphism class
M-unsaturated [107]: vertex not belonging to an edge of M
Upper embeddable: having a 2-cell embedding on a surface of genus $\lfloor (e(G) - n(G) + 1)/2 \rfloor$

Valence: vertex degree
Value of a flow [176]: the net flow out of the source or into the sink
Variance [433]: expected squared deviation from the mean
Vectorial matroid [351]: linear matroid
Vertex [2]: element of $V(G)$, the vertex set
Vertex chromatic number [191]: chromatic number
Vertex connectivity [149]: connectivity
Vertex cover [112]: a set of vertices containing at least one endpoint of every edge
Vertex-critical: deletion of any vertex changes the parameter
Vertex cut [149, 164]: a separating set of vertices
Vertex-deleted subgraph [37]: a subgraph obtained by deleting one vertex
Vertex multiplication [320]: a replacement of vertices of G by independent sets such that copies of x and y are adjacent if and only if $xy \in E(G)$
Vertex partition: a partition of the vertex set
Vertex set $V(G)$ [2]: the set of elements on which the graph is defined
Vertex-transitive [14]: for each pair $x, y \in V(G)$, some automorphism of G maps x to y
Vizing's Theorem [275]: upper bound on edge-chromatic number in terms of maximum degree and maximum edge multiplicity

Walk [20, 59]: an alternating list of vertices and edges in a graph such that each vertex belongs to the edge before and after it (in a digraph, must follow arrows)
u, v-walk [20]: a walk from u to v.
Weak elimination property [352]: property of matrices that the union of distinct intersecting circuits contains a circuit that avoids a specified point in the intersection
Weak product $G_1 \otimes G_2$: a graph product with vertices $V(G_1) \times V(G_2)$, and edges $(u_1, v_1) \leftrightarrow (u_2, v_2)$ iff $u_1 \leftrightarrow u_2$ and $v_1 \leftrightarrow v_2$
Weakly chordal [330]: having no chordless cycle of length at least 5 in G or $\overline{G}$
Weakly connected [56]: a directed graph whose underlying graph is connected
Weight: a real number
Weighted: having an assignment of weights (to edges and/or vertices)
Well Ordering Property [19]: every nonempty set (of natural numbers) has a least element
Wheel [174]: a graph obtained by taking the join of a cycle and a single vertex
Whitney's 2-isomorphism Theorem [376]: a characterization of the pairs of graphs whose cycle matroids are isomorphic
Wiener index [72]: the sum of the pairwise distances between vertices

Zero flow: a flow in a network with flow 0 on every edge

Appendix E

Supplemental Reading

Many books have been published about graph theory. Here we list a few for the interested reader who seeks an alternative presentation or more detailed material on special topics. We list several general textbooks grouped approximately into three levels. Specialized texts and monographs follow, listed by the relevant chapter in this book. Finally, we list some books that present additional topics in graph theory.

General / elementary:

Chartrand, G. *Graphs as Mathematical Models*. Prindle–Weber–Schmidt, 1977. Reprinted as *Introductory Graph Theory*, Dover, 1985.

Clark J. and D.A. Holton, *A first look at graph theory*. World Scientific, 1991.

Trudeau R.J., *Introduction to graph theory* (originally *Dots and Lines*, 1976). Dover, 1993.

Wilson R.J. *Introduction to graph theory*. Academic Press, 1979, 1972; Longman, 1985.

Wilson R.J. and J.J. Watkins, *Graphs: An introductory approach*. John Wiley & Sons, 1990.

General / intermediate:

Bondy J.A. and U.S.R. Murty, *Graph Theory with Applications*. Elsevier, 1976.

Chartrand G. and L. Lesniak, *Graphs and Digraphs*. PWS Publishers, 1979; Wadsworth–Brooks/Cole, 1986; Chapman & Hall, 1996.

Gould R., *Graph Theory*. Benjamin/Cummings, 1988.

Gross J. and J. Yellen, *Graph Theory*. CRC Press, 1999.

Harary F., *Graph Theory*. Addison-Wesley, 1969.

Ore O., *Theory of Graphs*. AMS Colloq. **38**, Amer. Math. Soc., 1962.

General / advanced:

Berge, C. *Graphs*. North-Holland 1973, 1976, 1985. (1970, 1983 in French.)

Bollobás B., *Graph Theory: An Introductory Course*. Grad. Texts in Math. **63**; Springer-Verlag, 1979.

Bollobás B., *Modern Graph Theory*. Grad. Texts Math. **184**; Springer, 1998.

Diestel R., *Graph Theory* Grad. Texts Math. **173**; Springer-Verlag, 1996, 2000.

Zykov A.A. *Fundamentals of graph theory* Nauka, 1987 (Russian). Transl. by L. Boron, C. Christenson, and B. Smith, BCS Associates, 1990.

Chapter 1:

Asratian A.S., T.M.J. Denley, and R. Häggkvist, *Bipartite graphs and their applications*. Cambridge Tracts in Math., **131**; Cambridge Univ. Press, 1998.

Fleischner H., *Eulerian Graphs and Related Topics*, Vols 1 & 2. Ann. Discrete Math. **45** & **50**, North-Holland, 1990 & 1991.

Harary F., R.Z. Norman, and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*. John Wiley & Sons, 1965.

Chapter 2:

Buckley F. and F. Harary, *Distance in Graphs*. Addison-Wesley, 1990

Moon J., *Counting Labelled Trees*. Canadian Math. Congress, 1970.

Chapter 3:

Gusfield D. and R.W. Irving, *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, 1989.

Haynes T.W., S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*. Pure and Applied Math. **208**; Marcel Dekker, 1998.

Lovász L. and M.D. Plummer, *Matching Theory*. North-Holland, 1986.

Chapter 4:

Ahuja R.K., T.L. Magnanti, and J. Orlin, *Network Flows*. Prentice-Hall, 1993.

Ford L.R. and D.R. Fulkerson, *Flows in Networks*. Princeton Univ. Press, 1962.

Tutte W.T., Connectivity in Graphs. Univ. Toronto Press, 1966.

Chapter 5:

Jensen T.R. and B. Toft, *Graph coloring problems*. Wiley-Interscience, 1995.

Chapter 6:

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Appendix F

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Author Index

Subject Index

A page number in italics indicates a definition (many page numbers for definitions also appear in Appendix D). A single listing in italics may indicate the definition for a concept so prevalent (such as "graph") that it would not be productive to list its occurrences. Consistent with this, an item that appears on very few pages may have none italicized for the definition.

When there are many pages listed, page numbers in bold indicate material such as the proof of a major result or the main treatment of the concept; this may also include a definition. Pages ranges may include isolated pages where the concept does not appear.

continued from inside front cover

$l(D)$	maximum length of path
$l(F)$	length of a face
$\lg x$	logarithm base 2
$\ln x$	natural logarithm
M	matching
$M(G)$	incidence matrix
$M(G)$	cycle matroid of G
M^*	dual hereditary system
$M.F$	contraction of M to F
$M\|F$	restriction of M to F
$\mathbb{N}$	set of natural numbers
N	network
$N(v) N_G(v)$	(open) neighborhood
$N[v]$	closed neighborhood
$N^+(v), N^-(v)$	out-, in-neighborhood
$n(G)$	order (number of vertices)
$O(f), o(f)$	growth rate
$o(H)$	number of odd components
$P(A)$	probability of an event
P_n	path with n vertices
pdim G	product dimension
qdim G	squashed-cube dimension
Q_k	k-dimensional hypercube
rad G	radius
$R(k, l)$	Ramsey number
$R(G, H)$	graph Ramsey number
$\mathbb{R}$	set of real numbers
$\mathbb{R}^2$	$\mathbb{R} \times \mathbb{R}$
r_M	rank function of matroid
S_γ	surface with γ handles
Spec(G)	spectrum (eigenvalues)
A^T	transpose of matrix
T	tree, tournament
$T_{n,r}$	Turán graph
$t_r(n)$	size of Turán graph
$U_{k,n}$	uniform matroid
$u(e)$	upper bound on flow
val(f)	value of a flow f
$V(G)$	vertex set
W_n	wheel with n vertices
$w(e)$	weight of edge
$\mathbb{Z}$	set of integers
$\mathbb{Z}_p$	integers modulo p

Greek alphabet

$\alpha(G)$	independence number
$\alpha'(G)$	maximum size of matching
$\beta(G)$	vertex cover number
$\beta'(G)$	edge cover number
$\gamma(G)$	genus, domination number
$\Delta(G)$	maximum degree
$\Delta^+(G), \Delta^-(G)$	maximum out-, in-degree
$\delta(G)$	minimum degree
$\delta^+(G), \delta^-(G)$	minimum out-, in-degree
$\partial(v)$	demand at a vertex
$\epsilon_G(u)$	eccentricity of u in G
$\Theta(f)$	growth rate
$\theta(G)$	clique cover number
$\theta'(G)$	intersection number
$\kappa(G)$	(vertex) connectivity
$\kappa'(G)$	edge-connectivity
$\kappa(x, y)$	local connectivity
$\kappa'(x, y)$	local edge-connectivity
$\kappa(r; G)$	local-global connectivity
$\lambda(x, y)$	max # disjoint paths
$\lambda'(x, y)$	max # edge-disjoint paths
$\lambda_1, \dots, \lambda_n$	eigenvalues
$\mu_1, \dots, \mu_n$	eigenvalues
$\mu(e), \mu(G)$	edge multiplicity
$\nu(G)$	crossing number
$\prod$	product
$\rho(G)$	maximum density
$\sum$	summation
σ, π, τ	permutation
$\sigma(v)$	supply at a vertex
σ_M	span function
$\tau(G)$	number of spanning trees
$\Upsilon(G)$	arboricity
$\phi(G; \lambda)$	characteristic polynomial
$\chi(G)$	chromatic number
$\chi'(G)$	edge-chromatic number
$\chi(G; k)$	chromatic polynomial
$\chi_l(G)$	list chromatic number
$\psi(G; \lambda)$	minimum polynomial
$\Omega(f), \omega(f)$	growth rate
$\omega(G)$	clique number

INTRODUCTION TO GRAPH THEORY

SECOND EDITION (2001)

SOLUTION MANUAL

SUMMER 2005 VERSION

MATHEMATICS DEPARTMENT

UNIVERSITY OF ILLINOIS

NOTICE

This is the Summer 2005 version of the Instructor's Solution Manual for *Introduction to Graph Theory*, by Douglas B. West. A few solutions have been added or clarified since last year's version.

Also present is a (slightly edited) annotated syllabus for the one-semester course taught from this book at the University of Illinois.

This version of the Solution Manual contains solutions for 99.4% of the problems in Chapters 1–7 and 93% of the problems in Chapter 8. The author believes that only Problems 4.2.10, 7.1.36, 7.1.37, 7.2.39, 7.2.47, and 7.3.31 in Chapters 1–7 are lacking solutions here. There problems are too long or difficult for this text or use concepts not covered in the text; they will be deleted in the third edition.

The positions of solutions that have not yet been written into the files are occupied by the statements of the corresponding problems. These problems retain the $(-)$, (!), $(+)$, $(*)$ indicators. Also $(\bullet)$ is added to introduce the statements of problems without other indicators. Thus every problem whose solution is not included is marked by one of the indicators, for ease of identification.

The author hopes that the solutions contained herein will be useful to instructors. The level of detail in solutions varies. Instructors should feel free to write up solutions with more or less detail according to the needs of the class. Please do not leave solutions posted on the web.

Due to time limitations, the solutions have not been proofread or edited as carefully as the text, especially in Chapter 8. Please send corrections to *west@math.uiuc.edu*. The author thanks Fred Galvin in particular for contributing improvements or alternative solutions for many of the problems in the earlier chapters.

This will be the last version of the Solution Manual for the second edition of the text. The third edition will have many new problems, such as those posted at http://www.math.uiuc.edu/ west/igt/newprob.html . The effort to include all solutions will resume for the third edition. Possibly other pedagogical features may also be added later.

Inquiries may be sent to *west@math.uiuc.edu*. Meanwhile, the author apologizes for any inconvenience caused by the absence of some solutions.

Douglas B. West

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MATH 412

SYLLABUS FOR INSTRUCTORS

Text: West, *Introduction to Graph Theory,* second edition,
Prentice Hall, 2001.

Many students in this course see graph algorithms repeatedly in courses in computer science. Hence this course aims primarily to improve students' writing of proofs in discrete mathematics while learning about the structure of graphs. Some algorithms are presented along the way, and many of the proofs are constructive. The aspect of algorithms emphasized in CS courses is running time; in a mathematics course in graph theory from this book the algorithmic focus is on proving that the algorithms work.

Math 412 is intended as a rigorous course that challenges students to think. Homework and tests should require proofs, and most of the exercises in the text do so. The material is interesting, accessible, and applicable; most students who stick with the course will give it a fair amount of time and thought.

An important aspect of the course is the clear presentation of solutions, which involves careful writing. Many of the problems in the text have hints, either where the problem is posed or in Appendix C (or both). Producing a solution involves two main steps: finding a proof and properly writing it. It is generally beneficial to the learning process to provide "collaborative study sessions" in which students can discuss homework problems in small groups and an instructor or teaching assistant is available to answer questions and provide direction. Students should then write up clear and complete solutions on their own.

This course works best when students have had prior exposure to writing proofs, as in a "transition" course. Some students may need further explicit discussions of the structure of proofs. Such discussion appear in many texts, such as

D'Angelo and West, *Mathematical Thinking: Problem-Solving and Proofs*;
Eisenberg, *The Mathematical Method: A Transition to Advanced Mathematics*;
Fletcher/Patty, *Foundations of Higher Mathematics*;
Galovich, *Introduction to Mathematical Structures*;
Galovich, *Doing Mathematics: An Introduction to Proofs and Problem Solving*;
Solow, *How to Read and Do Proofs*.

Suggested Schedule

The subject matter for the course is the first seven chapters of the text, skipping most optional material. Modifications to this are discussed below. The 22 sections are allotted an average of slightly under two lectures each.

In the exercises, problems designated by $(-)$ are easier or shorter than most, often good for tests or for "warmup" before doing homework problems. Problems designated by $(+)$ are harder than most. Those designated by (!) are particularly instructive, entertaining, or important. Those designated by $(*)$ make use of optional material.

The semester at the University of Illinois has 43 fifty-minute lectures. The final two lectures are for optional topics, usually chosen by the students from topics in Chapter 8.

Chapter 1	Fundamental Concepts	8
Chapter 2	Trees and Distance	5.5
Chapter 3	Matchings and Factors	5.5
Chapter 4	Connectivity and Paths	6
Chapter 5	Graph Coloring	6
Chapter 6	Planar Graphs	5
Chapter 7	Edges and Cycles	5
*	Total	41

Optional Material

No later material requires material marked optional. The "optional" marking also suggests to students that the final examination will not cover that material.

The optional subsections on Disjoint Spanning Trees (Bridg-It) in Section 2.1 and Stable Matchings in Section 3.2 are always quite popular with the students. The planarity algorithm (without proof) in 6.2 is appealing to students, as is the notion of embedding graphs on the torus through Example 6.3.21. Our course usually includes these four items.

The discussion of f-factors in Section 3.3 is also very instructive and is covered when the class is proceeding on schedule. Other potential additions include the applications of Menger's Theorem at 4.2.24 or 4.2.25.

Other items marked optional generally should not be covered in class.

Additional text items not marked optional that can be skipped when behind schedule:

1.1: 31, 35	1.2: 16, 21–23	1.3: 24, 31–32	1.4: 1, 4, 7, 25–26
2.1: 8, 14–16	2.2: 13–19	2.3: 7–8	3.2: 4
4.1: 4–6	4.2: 20–21	5.1: 11, 22(proof)	5.3: 10–11, 16(proof)
6.1: 18–20, 28	6.3: 9–10, 13–15		7.2: 17

Comments

There are several underlying themes in the course, and mentioning these at appropriate moments helps establish continuity. These include
1) TONCAS (The Obvious Necessary Condition(s) are Also Sufficient).
2) Weak duality in dual maximation and minimization problems.
3) Proof techniques such as the use of extremality, the paradigm for inductive proofs of conditional statements, and the technique of transforming a problem into a previously solved problem.

Students sometimes find it strange that so many exercises concern the Petersen graph. This is not so much because of the importance of the Petersen graph itself, but rather because it is a small graph and yet has complex enough structure to permit many interesting exercises to be asked.

Chapter 1. In recent years, most students enter the course having been exposed to proof techniques, so reviewing these in the first five sections has become less necessary; remarks in class can emphasis techniques as reminders. To minimize confusion, digraphs should not be mentioned until section 1.4; students absorb the additional model more easily after becoming comfortable with the first.

1.1: p3-6 contain motivational examples as an overview of the course; this discussion should not extend past the first day no matter where it ends (the definitions are later repeated where needed). The material on the Petersen graph establishes its basic properties for use in later examples and exercises.

1.2: The definitions of path and cycle are intended to be intuitive; one shouldn't dwell on the heaviness of the notation for walks.

1.3: Although characterization of graphic sequences is a classical topic, some reviewers have questioned its importance. Nevertheless, here is a computation that students enjoy and can perform.

1.4: The examples are presented to motivate the model; these can be skipped to save time. The de Bruijn graph is a classical application. It is desirable to present it, but it takes a while to explain.

Chapter 2.

2.1: Characterization of trees is a good place to ask for input from the class, both in listing properties and in proving equivalence.

2.2: The inductive proof for the Prüfer correspondence seems to be easier for most students to grasp than the full bijective proof; it also illustrates the usual type of induction with trees. Most students in the class have seen determinants, but most have considerable difficulty understanding the proof of the Matrix Tree Theorem; given the time involved, it is best

just to state the result and give an example (the next edition will include a purely inductive proof that uses only determinant expansion, not the Cauchy-Binet Formula). Students find the material on graceful labelings enjoyable and illuminating; it doesn't take long, but also it isn't required. The material on branchings should certaily be skipped in this course.

2.3: Many students have seen rooted trees in computer science and find ordinary trees unnatural; Kruskal's algorithm should clarify the distinction. Many CS courses now cover the algorithms of Kruskal, Dijkstra, and Huffman; here cover Kruskal and perhaps Dijkstra (many students have seen the algorithm but not a proof of correctness), and skip Huffman.

Chapter 3.

3.1: Skip "Dominating Sets", but present the rest of the section.

3.2: Students find the Hungarian algorithm difficult; explicit examples need to be worked along with the theoretical discussion of the equality subgraph. "Stable Matchings" is very popular with students and should be presented unless far behind in schedule. Skip "Faster Bipartite Matching".

3.3: Present all of the subsection on Tutte's 1-factor Theorem. The material on f-factors is intellectually beautiful and leads to one proof of the Erdős-Gallai conditions, but it is not used again in the course and is an "aside". Skip everything on Edmonds' Blossom Algorithm: matching algorithms in general graphs are important algorithmically but would require too much time in this course; this is "recommended reading".

Chapter 4.

4.1: Students have trouble distinguishing "k-connected" from "connectivity k" and "bonds" from "edge cuts". Bonds are dual to cycles in the matroidal sense; there are hints of this in exercises and in Chapter 7, but the full duality cannot be explored before Chapter 8.

4.2: Students find this section a bit difficult. The proof of 4.2.10 is similar to that of 4.2.7, making it omittable, but the application in 4.2.14 is appealing. The details of 4.2.20-21 can be skipped or treated lightly, since the main issue is the local version of Menger's theorem. 4.2.24-25 are appealing applications that can be added; 4.2.5 (CSDR) is a fundamental result but takes a fair amount of effort.

4.3: The material on network flow is quite easy but can take a long time to present due to the overhead of defining new concepts. The basic idea of 4.3.13-15 should be presented without belaboring the details too much. 4.3.16 is a more appealing application that perhaps makes the point more effectively. Skip "Supplies and Demands".

Chapter 5.

5.1: If time is short, the proof of 5.1.22 (Brooks' Theorem) can be merely sketched.

5.2: Be sure to cover Turán's Theorem. Presentation of Dirac's Theorem in 5.2.20 is valuable as an application of the Fan Lemma (Menger's Theorem). The subsequent material has limited appeal to undergraduates.

5.3: The inclusion-exclusion formula for the chromatic polynomial is derived here (5.3.10) without using inclusion-exclusion, making it accessible to this class without prerequisite. However, this proof is difficult for students to follow in favor of the simple inclusion-exclusion proof, which will be optional since that formula is not prerequisite for the course. Hence this formula should be omitted unless students know inclusion-exclusion. Chordal graphs and perfect graphs are more important, but these can also be treated lightly if short of time. Skip "Counting Acyclic Orientations".

Chapter 6.

6.1: The technical definitions of objects in the plane should be treated very lightly. It is better to be informal here, without writing out formal definitions unless explicitly requested by students. Outerplanar graphs are useful as a much easier class on which to solve problems (exercises!) like those on planar graphs; 6.18-20 are fundamental observations about outerplanar graphs, but other items are more important if time is short. 6.1.28 (polyhedra) is an appealing application but can be skipped.

6.2: The preparatory material 6.2.4-7 on Kuratowski's Theorem can be presented lightly, leaving the annoying details as reading; the subsequent material on convex embedding of 3-connected graphs is much more interesting. Presentation of the planarity algorithm is appealing but optional; skip the proof that it works.

6.3: The four color problem is popular; for undergraduates, show the flaw in Kempe's proof (p271), but don't present the discharging rule unless ahead of schedule. Students find the concept of crossing number easy to grasp, but the results are fairly difficult; try to go as far as the recursive quartic lower bound for the complete graph. The edge bound and its geometric application are impressive but take too much time for undergraduates. The idea of embeddings on surfaces can be conveyed through the examples in 6.3.21 on the torus. Interested students can be advised to read the rest of this section.

Chapter 7.

7.1: The proof of Vizing's Theorem is one of the more difficult in the course, but undergraduates can gain follow it when it is presented with sufficient colored chalk. The proof can be skipped if short of time. Skip

"Characterization of Line Graphs", although if time and interest is plentiful the necessity of Krausz's condition can be explained.

7.2: Chvátal's theorem (7.2.13) is not as hard to present as it looks if the instructor has the statement and proof clearly in mind. Nevertheless, the proof is somewhat technical and can be skipped (the same can be said of 7.2.17). More appealing is the Chvátal–Erdős Theorem (7.2.19), which certainly should be presented. Skip "Cycles in Directed Graphs".

7.3: The theorems of Tait and Grinberg make a nice culmination to the required material of the course. Skip "Snarks" and "Flows and Cycle Covers". Nevertheless, these are lively topics that can be recommended for advanced students.

Chapter 8. If time permits, material from the first part of sections of Chapter 8 can be presented to give the students a glimpse of other topics. The best choices for conveying some understanding in a brief treatment are Section 8.3 (Ramsey Theory or Sperner's Lemma) and Section 8.5 (Random Graphs). Also possible are the Gossip Problem (or other items) from Section 8.4 and some of the optional material from earlier chapters. The first part of Section 8.1 (Perfect Graphs) may also be usable for this purpose if perfect graphs have been discussed in Section 5.3. Sections 8.2 and 8.6 require more investment in preliminary material and thus are less suitable for giving a "glimpse".

1.FUNDAMENTAL CONCEPTS

1.1. WHAT IS A GRAPH?

1.1.1. *Complete bipartite graphs and complete graphs.* The complete bipartite graph $K_{m,n}$ is a complete graph if and only if $m = n = 1$ or $\{m, n\} = \{1, 0\}$.

1.1.2. *Adjacency matrices and incidence matrices for a 3-vertex path.*

$$\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix} \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&0\\0&1\end{pmatrix} \begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix} \begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix} \begin{pmatrix}0&1\\1&1\\1&0\end{pmatrix} \begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix} \begin{pmatrix}0&1\\1&0\\1&1\end{pmatrix}$$

Adjacency matrices for a path and a cycle with six vertices.

$$\begin{pmatrix}0&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\0&0&0&0&1&0\end{pmatrix} \qquad \begin{pmatrix}0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\end{pmatrix}$$

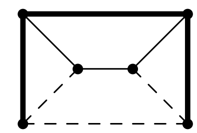
1.1.3. *Adjacency matrix for $K_{m,n}$.*

	m	n
m	0	1
n	1	0

1.1.4. $G \cong H$ *if and only if* $\overline{G} \cong \overline{H}$. If f is an isomorphism from G to H, then f is a vertex bijection preserving adjacency and nonadjacency, and hence f preserves non-adjacency and adjacency in $\overline{G}$ and is an isomorphism from $\overline{G}$ to $\overline{H}$. The same argument applies for the converse, since the complement of $\overline{G}$ is G.

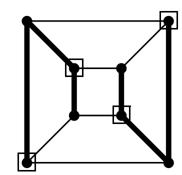
1.1.5. *If every vertex of a graph G has degree 2, then G is a cycle—FALSE.* Such a graph can be a disconnected graph with each component a cycle. (If infinite graphs are allowed, then the graph can be an infinite path.)

1.1.6. *The graph below decomposes into copies of P_4.*

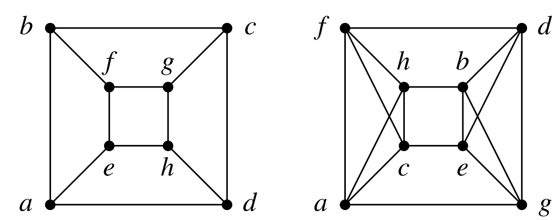


1.1.7. *A graph with more than six vertices of odd degree cannot be decomposed into three paths.* Every vertex of odd degree must be the endpoint of some path in a decomposition into paths. Three paths have only six endpoints.

1.1.8. *Decompositions of a graph.* The graph below decomposes into copies of $K_{1,3}$ with centers at the marked vertices. It decomposes into bold and solid copies of P_4 as shown.



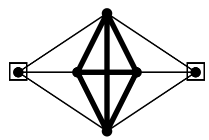
1.1.9. *A graph and its complement.* With vertices labeled as shown, two vertices are adjacent in the graph on the right if and only if they are not adjacent in the graph on the left.



1.1.10. *The complement of a simple disconnected graph must be connected—TRUE.* A disconnected graph G has vertices x, y that do not belong to a path. Hence$x$ and y are adjacent in $\overline{G}$. Furthermore, x and y have no common neighbor in G, since that would yield a path connecting them. Hence every vertex not in $\{x, y\}$ is adjacent in $\overline{G}$ to at least one of $\{x, y\}$. Hence every vertex can reach every other vertex in $\overline{G}$ using paths through $\{x, y\}$.

1.1.11. *Maximum clique and maximum independent set.* Since two vertices have degree 3 and there are only four other vertices, there is no clique of size 5. A complete subgraph with four vertices is shown in bold.

Since two vertices are adjacent to all others, an independent set containing either of them has only one vertex. Deleting them leaves P_4, in which the maximum size of an independent set is two, as marked.



1.1.12. *The Petersen graph.* The Petersen graph contains odd cycles, so it is not bipartite; for example, the vertices 12, 34, 51, 23, 45 form a 5-cycle.

The vertices 12, 13, 14, 15 form an independent set of size 4, since any two of these vertices have 1 as a common element and hence are nonadjacent. Visually, there is an independent set of size 4 marked on the drawing of the Petersen graph on the cover of the book. There are many ways to show that the graph has no larger independent set.

Proof 1. Two consecutive vertices on a cycle cannot both appear in an independent set, so every cycle contributes at most half its vertices. Since the vertex set is covered by two disjoint 5-cycles, every independent set has size at most 4.

Proof 2. Let ab be a vertex in an independent set S, where $a, b \in [5]$. We show that S has at most three additional vertices. The vertices not adjacent to ab are ac, bd, ae, bc, ad, be, and they form a cycle in that order. Hence at most half of them can be added to S.

1.1.13. *The graph with vertex set $\{0, 1\}^k$ and $x \leftrightarrow y$ when x and y differ in one place is bipartite.* The partite sets are determined by the parity of the number of 1's. Adjacent vertices have opposite parity. (This graph is the k-dimensional hypercube; see Section 1.3.)

1.1.14. *Cutting opposite corner squares from an eight by eight checkerboard leaves a subboard that cannot be partitioned into rectangles consisting of two adjacent unit squares.* 2-coloring the squares of a checkerboard so that adjacent squares have opposite colors shows that the graph having a vertex for each square and an edge for each pair of adjacent squares is bipartite. The squares at opposite corners have the same color; when these are deleted, there are 30 squares of that color and 32 of the other

color. Each pair of adjacent squares has one of each color, so the remaining squares cannot be partitioned into sets of this type.

Generalization: the two partite sets of a bipartite graph cannot be matched up using pairwise-disjoint edges if the two partite sets have unequal sizes.

1.1.15. *Common graphs in four families: $A = \{paths\}$, $B = \{cycles\}$, $C = \{complete\ graphs\}$, $D = \{bipartite\ graphs\}$.*

$A \cap B = \varnothing$: In a cycle, the numbers of vertices and edges are equal, but this is false for a path.

$A \cap C = \{K_1, K_2\}$: To be a path, a graph must contain no cycle.

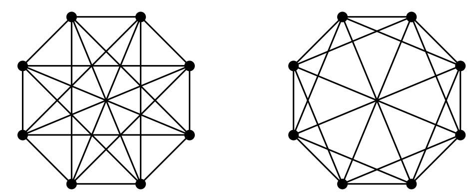
$A \cap D = A$: non-bipartite graphs have odd cycles.

$B \cap C = K_3$: Only when $n = 3$ does $\binom{n}{2} = n$.

$B \cap D = \{C_{2k}\colon k \geq 2\}$: odd cycles are not bipartite.

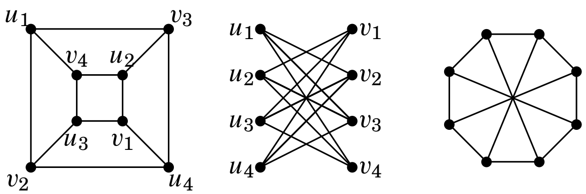
$C \cap D = \{K_1, K_2\}$: bipartite graphs cannot have triangles.

1.1.16. *The graphs below are not isomorphic.* The graph on the left has four cliques of size 4, and the graph on the right has only two. Alternatively, the complement of the graph on the left is disconnected (two 4-cycles), while the complement of the graph on the right is connected (one 8-cycle).



1.1.17. *There are exactly two isomorphism classes of 4-regular simple graphs with 7 vertices.* Simple graphs G and H are isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are isomorphic, because an isomorphism $\phi\colon V(G) \to V(H)$ is also an isomorphism from $\overline{G}$ to $\overline{H}$, and vice versa. Hence it suffices to count the isomorphism classes of 2-regular simple graphs with 7 vertices. Every component of a finite 2-regular graph is a cycle. In a simple graph, each cycle has at least three vertices. Hence each class it determined by partitioning 7 into integers of size at least 3 to be the sizes of the cycles. The only two graphs that result are C_7 and $C_3 + C_4$ – a single cycle or two cycles of lengths three and four.

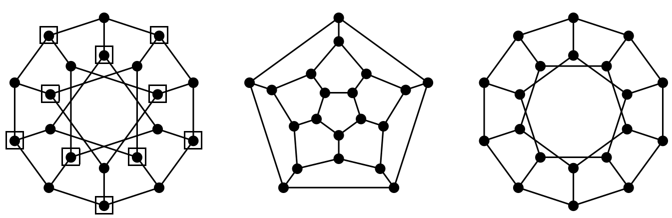
1.1.18. *Isomorphism.* Using the correspondence indicated below, the first two graphs are isomorphic; the graphs are bipartite, with $u_i \leftrightarrow v_j$ if and only if $i \neq j$. The third graph contains odd cycles and hence is not isomorphic to the others.



Visually, the first two graphs are Q_3 and the graph obtained by deleting four disjoint edges from $K_{4,4}$. In Q_3, each vertex is adjacent to the vertices whose names have opposite parity of the number of 1s, except for the complementary vertex. Hence Q_3 also has the structure of $K_{4,4}$ with four disjoint edges deleted; this proves isomorphism without specifying an explicit bijection.

1.1.19. *Isomorphism of graphs.* The rightmost two graphs below are isomorphic. The outside 10-cycle in the rightmost graph corresponds to the intermediate ring in the second graph. Pulling one of the inner 5-cycles of the rightmost graph out to the outside transforms the graph into the same drawing as the second graph.

The graph on the left is bipartite, as shown by marking one partite set. It cannot be isomorphic to the others, since they contain 5-cycles.



1.1.20. *Among the graphs below, the first (F) and third (H) are isomorphic, and the middle graph (G) is not isomorphic to either of these.*

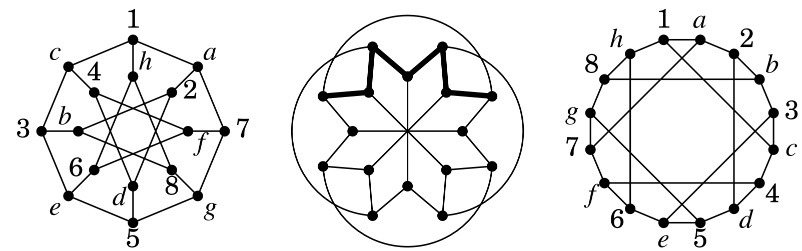
F and H are isomorphic. We exhibit an isomorphism (a bijection from $V(F)$ to $V(H)$ that preserves the adjacency relation). To do this, we name the vertices of F, write the name of each vertex of F on the corresponding vertex in H, and show that the names of the edges are the same in H and F. This proves that H is a way to redraw F. We have done this below using the first eight letters and the first eight natural numbers as names.

To prove quickly that the adjacency relation is preserved, observe that $1, a, 2, b, 3, c, 4, d, 5, e, 6, f, 7, g, 8, h$ is a cycle in both drawings, and the additional edges $1c$, $2d$, $3e$, $4f$, $5g$, $6h$, $7a$, $8b$ are also the same in both drawings. Thus the two graphs have the same edges under this vertex correspondence. Equivalently, if we list the vertices in this specified order for

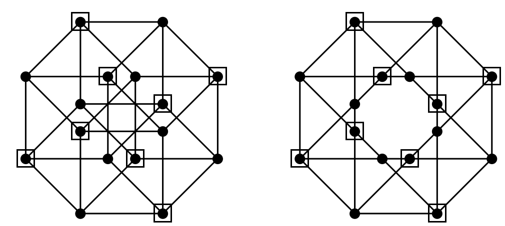
the two drawings, the two adjacency matrices are the same, but that is harder to verify.

G is not isomorphic to F or to H. In F and in H, the numbers form an independent set, as do the letters. Thus F and H are bipartite. The graph G cannot be bipartite, since it contains an odd cycle. The vertices above the horizontal axis of the picture induce a cycle of length 7.

It is also true that the middle graph contains a 4-cycle and the others do not, but it is harder to prove the absence of a 4-cycle than to prove the absence of an odd cycle.



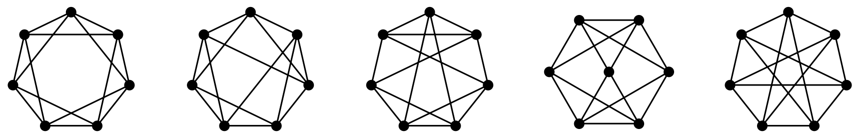
1.1.21. *Isomorphism.* Both graphs are bipartite, as shown below by marking one partite set. In the graph on the right, every vertex appears in eight 4-cycles. In the graph on the left, every vertex appears in only six 4-cycles (it is enough just to check one). Thus they are not isomorphic. Alternatively, for every vertex in the right graph there are five vertices having common neighbors with it, while in the left graph there are six such vertices.



1.1.22. *Isomorphism of explicit graphs.* Among the graphs below, $\{G_1, G_2, G_5\}$ are pairwise isomorphic. Also $G_3 \cong G_4$, and these are not isomorphic to any of the others. Thus there are exactly two isomorphism classes represented among these graphs.

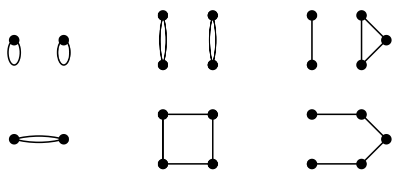
To prove these statements, one can present explicit bijections between vertex sets and verify that these preserve the adjacency relation (such as by displaying the adjacency matrix, for example). One can also make other structural arguments. For example, one can move the highest vertex in G_3 down into the middle of the picture to obtain G_4; from this one can list the desired bijection.

One can also recall that two graphs are isomorphic if and only if their complements are isomorphic. The complements of G_1, G_2, and G_5 are cycles of length 7, which are pairwise isomorphic. Each of $\overline{G}_3$ and $\overline{G}_4$ consists of two components that are cycles of lengths 3 and 4; these graphs are isomorphic to each other but not to a 7-cycle.



G_1 G_2 G_3 G_4 G_5

1.1.23. *Smallest pairs of nonisomorphic graphs with the same vertex degrees.* For multigraphs, loopless multigraphs, and simple graphs, the required numbers of vertices are 2,4,5; constructions for the upper bounds appear below. We must prove that these constructions are smallest.



a) general b) loopless c) simple

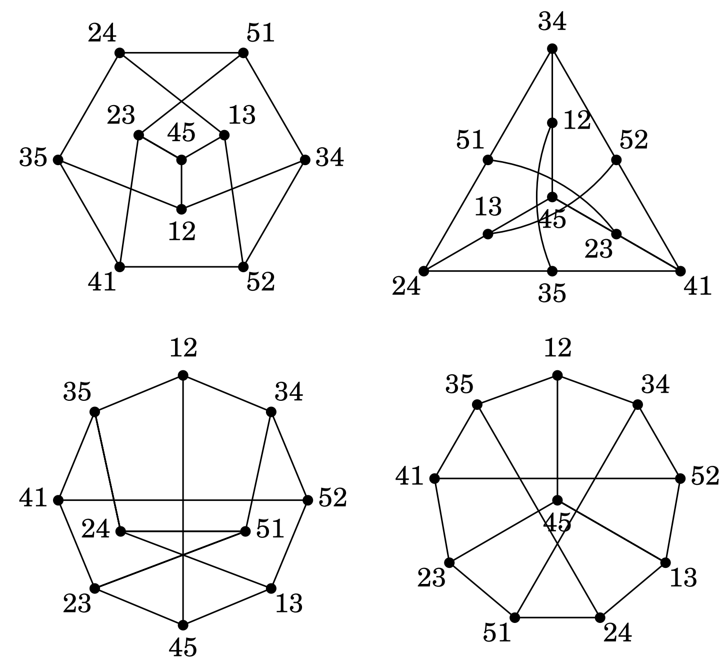
a) With 1 vertex, every edge is a loop, and the isomorphism class is determined by the number of edges, which is determined by the vertex degree. Hence nonisomorphic graphs with the same vertex degrees have at least two vertices.

b) Every loopless graph is a graph, so the answer for loopless graphs is at least 2. The isomorphism class of a loopless graph with two vertices is determined by the number of copies of the edge, which is determined by the vertex degrees. The isomorphism class of a loopless graph with three vertices is determined by the edge multiplicities. Let the three vertex degrees be a, b, c, and let the multiplicities of the opposite edges be x, y, z, where Since $a = y + z$, $b = x + z$, and $c = x + y$, we can solve for the multiplicities in terms of the degrees by $x = (b+c-a)/2$, $y = (a+c-b)/2$, and $z = (a + b - c)/2$. Hence the multiplicities are determined by the degrees, and all loopless graphs with vertex degrees a, b, c are pairwise isomorphic. Hence nonisomorphic loopless graphs with the same vertex degrees have at least four vertices.

c) Since a simple graph is a loopless graph, the answer for simple graphs is at least 4. There are 11 isomorphism classes of simple graphs with four vertices. For each of 0,1,5, or 6 edges, there is only one isomorphism class. For 2 edges, there are two isomorphism classes, but they have

different lists of vertex degrees (similarly for 4 edges). For 3 edges, the three isomorphism classes have degree lists 3111, 2220, and 2211, all different. Hence nonisomorphic simple graphs with the same vertex degrees must have at least five vertices.

1.1.24. *Isomorphisms for the Petersen graph.* Isomorphism is proved by giving an adjacency-preserving bijection between the vertex sets. For pictorial representations of graphs, this is equivalent to labeling the two graphs with the same vertex labels so that the adjacency relation is the same in both pictures; the labels correspond to a permutation of the rows and columns of the adjacency matrices to make them identical. The various drawings of the Petersen graph below illustrate its symmetries; the labelings indicate that these are all "the same" (unlabeled) graph. The number of isomorphisms from one graph to another is the same as the number of isomorphisms from the graph to itself.



1.1.25. *The Petersen graph has no cycle of length 7.* Suppose that the Petersen graph has a cycle C of length 7. Since any two vertices of C are connected by a path of length at most 3 on C, any additional edge with endpoints on C would create a cycle of length at most 4. Hence the third neighbor of each vertex on C is not on C.

Thus there are seven edges from $V(C)$ to the remaining three vertices. By the pigeonhole principle, one of the remaining vertices receives at least three of these edges. This vertex x not on C has three neighbors on C. For any three vertices on C, either two are adjacent or two have a common neighbor on C (again the pigeonhole principle applies). Using x, this completes a cycle of length at most 4. We have shown that the assumption of a 7-cycle leads to a contradiction.

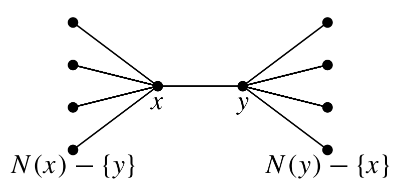
Alternative completion of proof. Let u be a vertex on C, and let v, w be the two vertices farthest from u on C. As argued earlier, no edges join vertices of C that are not consecutive on C. Thus u is not adjacent to v or w. Hence u, v have a common neighbor off C, as do u, w. Since u has only one neighbor off C, these two common neighbors are the same. The resulting vertex x is adjacent to all of u, v, w, and now x, v, w is a 3-cycle.

1.1.26. *A k-regular graph of girth four has at least $2k$ vertices, with equality only for $K_{k,k}$.* Let G be k-regular of girth four, and chose $xy \in E(G)$. Girth 4 implies that G is simple and that x and y have no common neighbors. Thus the neighborhoods $N(x)$ and $N(y)$ are disjoint sets of size k, which forces at least $2k$ vertices into G. Possibly there are others.

Note also that $N(x)$ and $N(y)$ are independent sets, since G has no triangle. If G has no vertices other than these, then the vertices in $N(x)$ can have neighbors only in $N(y)$. Since G is k-regular, every vertex of $N(x)$ must be adjacent to every vertex of $N(y)$. Thus G is isomorphic to $K_{k,k}$, with partite sets $N(x)$ and $N(y)$. In other words, there is only one such isomorphism class for each value of k.

Comment. One can also start with a vertex x, choose y from among the k vertices in $N(x)$, and observe that $N(y)$ must have $k-1$ more vertices not in $N(x) \cup \{x\}$. The proof then proceeds as above.

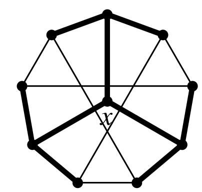
(An alternative proof uses the methods of Section 1.3. A triangle-free simple graph with n vertices has at most $n^2/4$ edges. Since G is k-regular, this yields $n^2/4 \geq nk/2$, and hence $n \geq 2k$. Furthermore, equality holds in the edge bound only for $K_{n/2,n/2}$, so this is the only such graph with $2k$ vertices. (C. Pikscher))



1.1.27. *A simple graph of girth 5 in which every vertex has degree at least k has at least k^2+1 vertices, with equality achieveable when $k \in \{2, 3\}$.* Let G be k-regular of girth five. Let S be the set consisting of a vertex x and

its neighbors. Since G has no cycle of length less than five, G is simple, and any two neighbors of x are nonadjacent and have no common neighbor other than x. Hence each $y \in S - \{x\}$ has at least $k-1$ neighbors that are not in S and not neighbors of any vertex in S. Hence G has at least $k(k-1)$ vertices outside S and at least $k+1$ vertices in S for at least k^2+1 altogether.

The 5-cycle achieves equality when $k = 2$. For $k = 3$, growing the graph symmetrically from x permits completing the graph by adding edges among the non-neighbors of x. The result is the Petersen graph. (Comment: For $k > 3$, it is known that girth 5 with minimum degree k and exactly k^2+1 vertices is impossible, except for $k = 7$ and possibly for $k = 57$.)



1.1.28. *The Odd Graph has girth 6.* The Odd Graph O_k is the disjointness graph of the set of k-element subsets of $[2k+1]$.

Vertices with a common neighbor correspond to k-sets with $k-1$ common elements. Thus they have exactly one common neighbor, and O_k has no 4-cycle. Two vertices at distance 2 from a single vertex have at least $k-2$ common neighbors. For $k > 2$, such vertices cannot be adjacent, and thus O_k has no 5-cycle when $k > 2$. To form a 6-cycle when $k \geq 2$, let $A = \{2, \ldots, k\}$, $B = \{k+2, \ldots, 2k\}$, $a = 1$, $b = k+1$, $c = 2k+1$. A 6-cycle is $A \cup \{a\}$, $B \cup \{b\}$, $A \cup \{c\}$, $B \cup \{a\}$, $A \cup \{b\}$, $B \cup \{c\}$.

The Odd Graph also is not bipartite. The successive elements $\{1, \ldots, k\}$, $\{k+1, \ldots, 2k\}$, $\{2k+1, 1, \ldots, k-1\}$, . . ., $\{k+2, \ldots, 2k+1\}$ form an odd cycle.

1.1.29. *Among any 6 people, there are 3 mutual acquaintances or 3 mutual strangers.* Let G be the graph of the acquaintance relation, and let x be one of the people. Since x has 5 potential neighbors, x has at least 3 neighbors or at least 3 nonneighbors. By symmetry (if we complement G, we still have to prove the same statement), we may assume that x has at least 3 neighbors. If any pair of these people are acquainted, then with x we have 3 mutual acquaintances, but if no pair of neighbors of x is acquainted, then the neighbors of x are three mutual strangers.

1.1.30. *The number of edges incident to v_i is the ith diagonal entry in MM^T and in A^2.* In both MM^T and A^2 this is the sum of the squares of the entries in the ith row. For MM^T, this follows immediately from the definition of matrix multiplication and transposition, but for A^2 this uses the graph-theoretic fact that $A = A^T$; i.e. column i is the same as row i. Because G is simple, the entries of the matrix are all 0 or 1, so the sum of the squares in a row equals the number of 1s in the row. In M, the 1s in a row denote incident edges; in A they denote vertex neighbors. In either case, the number of 1s is the degree of the vertex.

If $i \neq j$, then the entry in position (i, j) of A^2 is the number of common neighbors of v_i and v_j. The matrix multiplication puts into position (i, j) the "product" of row i and column j; that is $\sum_{k=1}^{n} a_{i,k}a_{k,j}$. When G is simple, entries in A are 1 or 0, depending on whether the corresponding vertices are adjacent. Hence $a_{i,k}a_{k,j} = 1$ if v_k is a common neighbor of v_i and v_j; otherwise, the contribution is 0. Thus the number of contributions of 1 to entry (i, j) is the number of common neighbos of v_i and v_j.

If $i \neq j$, then the entry in position (i, j) of MM^T is the number of edges joining v_i and v_j (0 or 1 when G has no multiple edges). The ith row of M has 1s corresponding to the edges incident to v_i. The jth column of M^T is the same as the jth row of M, which has 1s corresponding to the edges incident to v_j. Summing the products of corresponding entries will contribute 1 for each edge incident to both v_i and v_j; 0 otherwise.

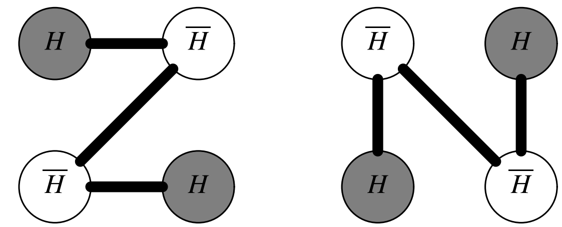
Comment. For graphs without loops, both arguments for (i, j) in general apply when $i = j$ to explain the diagonal entries.

1.1.31. *K_n decomposes into two isomorphic ("self-complementary") subgraphs if and only if n or $n-1$ is divisible by 4.*

a) The number of vertices in a self-complementary graph is congruent to 0 or 1 $(\bmod 4)$. If G and $\overline{G}$ are isomorphic, then they have the same number of edges, but together they have $\binom{n}{2}$ edges (with none repeated), so the number of edges in each must be $n(n-1)/4$. Since this is an integer and the numbers n and $n-1$ are not both even, one of $\{n, n-1\}$ must be divisible by 4.

b) Construction of self-complementary graphs for all such n.

Proof 1 (explicit construction). We generalize the structure of the self-complementary graphs on 4 and 5 vertices, which are P_4 and C_5. For $n = 4k$, take four vertex sets of size k, say X_1, X_2, X_3, X_4, and join all vertices of X_i to those of X_{i+1}, for $i = 1, 2, 3$. To specify the rest of G, within these sets let X_1 and X_4 induce copies of a graph H with k vertices, and let X_2 and X_3 induce $\overline{H}$. (For example, H may be K_k.) In $\overline{G}$, both X_2 and X_3 induce H, while X_1 and X_4 induce $\overline{H}$, and the connections between sets are $X_2 \leftrightarrow X_4 \leftrightarrow X_1 \leftrightarrow X_3$. Thus relabeling the subsets defines an isomorphism between G and $\overline{G}$. (There are still other constructions for G.)



For $n = 4k+1$, we add a vertex x to the graph constructed above. Join x to the $2k$ vertices in X_1 and X_4 to form G. The isomorphism showing that $G - x$ is self-complementary also works for G (with x mapped to itself), since this isomorphism maps $N_G(x) = X_1 \cup X_4$ to $N_{\overline{G}}(x) = X_2 \cup X_3$.

Proof 2 (inductive construction). If G is self-complementary, then let H_1 be the graph obtained from G and P_4 by joining the two ends of P_4 to all vertices of G. Let H_2 be the graph obtained from G and P_4 by joining the two center vertices of P_4 to all vertices of G. Both H_1 and H_2 are self-complementary. Using this with $G = K_1$ produces the two self-complementary graphs of order 5, namely C_5 and the bull.

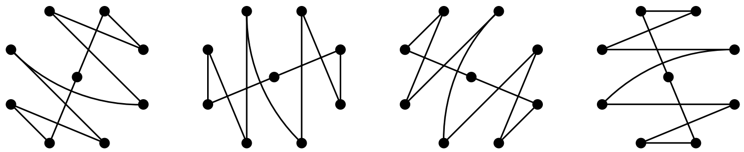
Self-complementary graphs with order divisible by 4 arise from repeated use of the above using $G = P_4$ as a starting point, and self-complementary graphs of order congruent to 1 modulo 4 arise from repeated use of the above using $G = K_1$ as a starting point. This construction produces many more self-complementary graphs than the explicit construction in Proof 1.

1.1.32. *$K_{m,n}$ decomposes into two isomorphic subgraphs if and only if m and n are not both odd.* The condition is necessary because the number of edges must be even. It is sufficient because $K_{m,n}$ decomposes into two copies of $K_{m,n/2}$ when n is even.

1.1.33. *Decomposition of complete graphs into cycles through all vertices.* View the vertex set of K_n as $\mathbb{Z}_n$, the values $0, \ldots, n-1$ in cyclic order. Since each vertex has degree $n-1$ and each cycle uses two edges at each vertex, the decomposition has $(n-1)/2$ cycles.

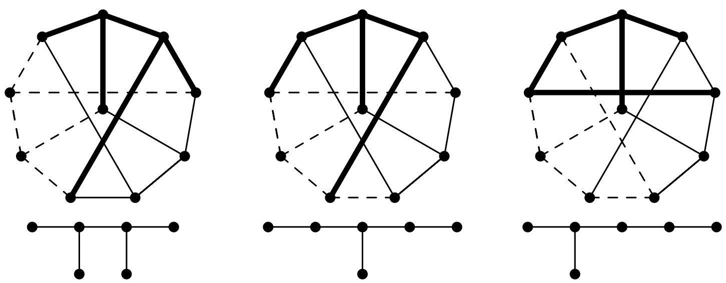
For $n = 5$ and $n = 7$, it suffices to use cycles formed by traversing the vertices with constant difference: $(0, 1, 2, 3, 4)$ and $(0, 2, 4, 1, 3)$ for $n = 5$ and $(0, 1, 2, 3, 4, 5, 6)$, $(0, 2, 4, 6, 1, 3, 5)$, and $(0, 3, 6, 2, 5, 1, 4)$ for $n = 7$.

This construction fails for $n = 9$ since the edges with difference 3 form three 3-cycles. The cyclically symmetric construction below treats the vertex set as $\mathbb{Z}_8$ together with one special vertex.



1.1.34. *Decomposition of the Petersen graph into copies of P_4.* Consider the drawing of the Petersen graph with an inner 5-cycle and outer 5-cycle. Each P_4 consists of one edge from each cycle and one cross edge joining them. Extend each cross edge e to a copy of P_4 by taking the edge on each of the two 5-cycles that goes in a clockwise direction from e. In this way, the edges on the outside 5-cycle are used in distinct copies of P_4, and the same holds for the edges on the inside 5-cycle.

Decomposition of the Petersen graph into three pairwise-isomorphic connected subgraphs. Three such decompositions are shown below. We restricted the search by seeking a decomposition that is unchanged by 120° rotations in a drawing of the Petersen graph with 3-fold rotational symmetry. The edges in this drawing form classes of size 3 that are unchanged under rotations of 120°; each subgraph in the decomposition uses exactly one edge from each class.



1.1.35. *K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n+1$ is not divisible by 3.* The number of edges is $n(n-1)/2$. If $n+1$ is divisible by 3, then n and $n-1$ are not divisible by 3. Thus decomposition into three subgraphs of equal size is impossible in this case.

If $n+1$ is not divisible by 3, then $e(K_n)$ is divisible by 3, since n or $n-1$ is divisible by 3. We construct a decomposition into three subgraphs that are pairwise isomorphic (there are many such decompositions).

When n is a multiple of 3, we partition the vertex set into three subsets V_1, V_2, V_3 of equal size. Edges now have two types: within a set or joining two sets. Let the ith subgraph G_i consist of all the edges within V_i and all the edges joining the two other subsets. Each edge of K_n appears in exactly

one of these subgraphs, and each G_i is isomorphic to the disjoint union of $K_{n/3}$ and $K_{n/3,n/3}$.

When $n \equiv 1 \pmod 3$, consider one vertex w. Since $n-1$ is a multiple of 3, we can form the subgraphs G_i as above on the remaining $n-1$ vertices. Modify G_i to form H_i by joining w to every vertex of V_i. Each edge involving w has been added to exactly one of the three subgraphs. Each H_i is isomorphic to the disjoint union of $K_{1+(n-1)/3}$ and $K_{(n-1)/3,(n-1)/3}$.

1.1.36. *If K_n decomposes into triangles, then $n-1$ or $n-3$ is divisible by 6.* Such a decomposition requires that the degree of each vertex is even and the number of edges is divisible by 3. To have even degree, n must be odd. Also $n(n-1)/2$ is a multiple of 3, so 3 divides n or $n-1$. If 3 divides n and n is odd, then $n-3$ is divisible by 6. If 3 divides $n-1$ and n is odd, then $n-1$ is divisible by 6.

1.1.37. *A graph in which every vertex has degree 3 has no decomposition into paths with at least five vertices each.* Suppose that G has such a decomposition. Since every vertex has degree 3, each vertex is an endpoint of at least one of the paths and is an internal vertex on at most one of them. Since every path in the decomposition has two endpoints and has at least three internal vertices, we conclude that the number of paths in the decomposition is at least $n(G)/2$ and is at most $n(G)/3$, which is impossible.

Alternatively, let k be the number of paths. There are $2k$ endpoints of paths. On the other hand, since each internal vertex on a path in the decomposition must be an endpoint of some other path in the decomposition, there must be at least $3k$ endpoints of paths. The contradiction implies that there cannot be such a decomposition.

1.1.38. *A 3-regular graph G has a decomposition into claws if and only if G is bipartite.* When G is bipartite, we produce a decomposition into claws. We use all claws obtained by taking the three edges incident with a single vertex in the first partite set. Each claw uses all the edges incident to its central vertex. Since each edge has exactly one endpoint in the first partite set, each edge appears in exactly one of these claws.

When G has a decomposition into claws, we partition $V(G)$ into two independent sets. Let X be the set of centers of the claws in the decomposition. Since every vertex has degree 3, each claw in the decomposition uses all edges incident to its center. Since each edge is in at most one claw, this makes X an independent set. The remaining vertices also form an independent set, because every edge is in some claw in the decomposition, which means that one of its endpoints must be the center of that claw.

1.1.39. *Graphs that decompose K_6.*

Triangle—No. A graph decomposing into triangles must have even degree at each vertex. (This excludes all decompositions into cycles.)

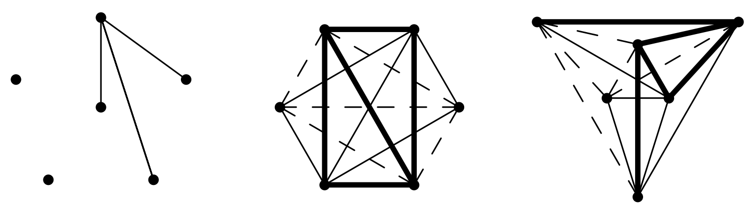
Paw, P_5—No. K_6 has 15 edges, but each paw or P_5 has four edges.

House, Bowtie, Dart–No. K_6 has 15 edges, but each house, bowtie, or dart has six edges.

Claw—Yes. Put five vertices 0, 1, 2, 3, 4 on a circle and the other vertex z in the center. For $i \in \{0, 1, 2, 3, 4\}$, use a claw with edges from i to $i+1$, $i+2$, and z. Each edge appears in exactly one of these claws.

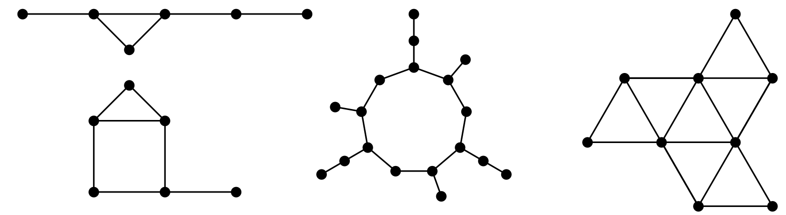
Kite—Yes. Put all six vertices on a circle. Each kite uses two opposite edges on the outside, one diagonal, and two opposite edges of "length" 2. Three rotations of the picture complete the decomposition.

Bull—Yes. The bull has five edges, so we need three bulls. Each bull uses degrees 3, 3, 2, 1, 1, 0 at the six vertices. Each bull misses one vertex, and each vertex of K_6 has five incident edges, so three of the vertices will receive degrees 3, 2, 0 from the three bulls, and the other three will receive degrees 3, 1, 1. Thus we use vertices of two types, which leads us to position them on the inside and outside as on the right below. The bold, solid, and dashed bulls obtained by rotation complete the decomposition.



1.1.40. *Automorphisms of P_n, C_n, and K_n.* A path can be left alone or flipped, a cycle can be rotated or flipped, and a complete graph can be permuted arbtrarily. The numbers of automorphisms are 2, $2n$, $n!$, respectively. Correspondingly, the numbers of distinct labelings using vertex set $[n]$ are $n!/2$, $(n-1)!/2$, 1, respectively. For P_n, these formulas require $n > 1$.

1.1.41. *Graphs with one and three automorphisms.* The two graphs on the left have six vertices and only the identity automorphism. The two graphs on the right have three automorphisms.



1.1.42. *The set of automorphisms of a graph G satisfies the following:*

a) The composition of two automorphisms is an automorphism.

b) The identity permutation is an automorphism.

c) The inverse of an automorphism is also an automorphism.

d) Composition of automorphisms satisfies the associative property.

The first three properties are essentially the same as the transitive, reflexive, and symmetric properties for the isomorphism relation; see the discussion of these in the text. The fourth property holds because composition of functions always satisfies the associative property (see the discussion of composition in Appendix A).

1.1.43. *Every automorphism of the Petersen graph maps the 5-cycle (12,34,51,23,45) into a 5-cycle with vertices ab, cd, ea, bc, de by a permutation of* $[5]$ *taking 1,2,3,4,5 to a, b, c, d, e, respectively.* Let σ denote the automorphism, and let the vertex ab be the image of the vertex 12 under σ. The image of 34 must be a pair disjoint from ab, so we may let $cd = \sigma(34)$. The third vertex must be disjoint from the second and share an element with the first. We may select a to be the common element in the first and third vertices. Similarly, we may select c to be the common element in the second and fourth vertices. Since nonadjacent vertices correspond to sets with a common element, the other element of the fourth vertex must be b, and the fifth vertex can't have a or b and must have d and e. Thus every 5-cycle must have this form and is the image of (12,34,51,23,45) under the specified permutation σ.

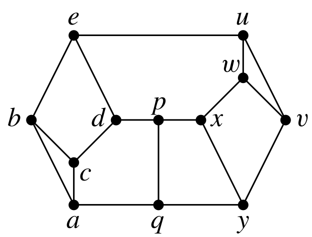
The Peterson graph has 120 automorphisms. Every permutation of $[5]$ preserves the disjointness relation on 2-element subsets and therefore defines an automorphism of the Petersen graph. Thus there are at least 120 automorphism. To show that there are no others, consider an arbitrary automorphism σ. By the preceding paragraph, the 5-cycle C maps to some 5-cycle (ab, cd, ea, bc, de). This defines a permutation f mapping $1, 2, 3, 4, 5$ to a, b, c, d, e, respectively. It suffices to show that the other vertices must also have images under σ that are described by f.

The remaining vertices are pairs consisting of two nonconsecutive values modulo 5. By symmetry, it suffices to consider just one of them, say 24. The only vertex of C that 24 is adjacent to (disjoint from) is 51. Since

$\sigma(51) = ea$, and the only vertex not on (ab, cd, ea, bc, de) that is adjacent to ea is bd, we must have $\sigma(24) = bd$, as desired.

1.1.44. *For each pair of 3-edge paths $P = (u_0, u_1, u_2, u_3)$ and $Q = (v_0, v_1, v_2, v_3)$ in the Petersen graph, there is an automorphism of the Petersen graph that turns P into Q.* In the disjointness representation of the Petersen graph, suppose the pairs corresponding to the vertices of P are ab, cd, ef, g_h, respectively. Since consecutive pairs are disjoint and the edges are unordered pairs, we may write the pairs so that a, b, c, d, e are distinct, $f = a$, $g = b$, and $h = c$. Putting the vertex names of Q in the same format $AB, CD, EF, G \square H$, we chose the isomorphism generated by the permutation of [5] that turns a, b, c, d, e into A, B, C, D, E, respectively.

1.1.45. *A graph with 12 vertices in which every vertex has degree 3 and the only automorphism is the identity.*



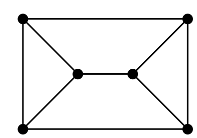
There are many ways to prove that an automorphism must fix all the vertices. The graph has only two triangles (abc and uvw). Now an automorphism must fix p, since is the only vertex having no neighbor on a triangle, and also e, since it is the only vertex with neighbors on both triangles. Now d is the unique common neighbor of p and e. The remaining vertices can be fixed iteratively in the same way, by finding each as the only unlabeled vertex with a specified neighborhood among the vertices already fixed. (This construction was provided by Luis Dissett, and the argument forbidding nontrivial automorphisms was shortened by Fred Galvin. Another such graph with three triangles was found by a student of Fred Galvin.)

1.1.46. *Vertex-transitivity and edge-transitivity.* The graph on the left in Exercise 1.1.21 is isomorphic to the 4-dimensional hypercube (see Section 1.3), which is vertex-transitive and edge-transitive via the permutation of coordinates. For the graph on the right, rotation and inside-out exchange takes care of vertex-transitivity. One further generating operation is needed to get edge-transitivity; the two bottom outside vertices can be switched with the two bottom inside vertices.

1.1.47. *Edge-transitive versus vertex-transitive. a) If G is obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2, then G is edge-transitive but not vertex-transitive.* Every edge consists of an old vertex and a new vertex. The $n!$ permutations of old vertices yield automorphism. Let $x\&y$ denote the new vertex on the path replacing the old edge xy; note that $x\&y = y\&x$. The edge joining x and $x\&y$ is mapped to the edge joining u and $u\&v$ by any automorphism that maps x to u and y to v. The graph is not vertex-transitive, since $x\&y$ has degree 2, while x has degree $n - 1$.

b) If G is edge-transitive but not vertex-transitive and has no isolated vertices, then G is bipartite. Let uv be an arbitrary edge of G. Let S be the set of vertices to which u is mapped by automorphisms of G, and let T be the set of vertices to which v is mapped. Since G is edge-transitive and has no isolated vertex, $S \cup T = V(G)$. Since G is not vertex-transitive, $S \neq V(G)$. Together, these statements yield $S \cap T = \varnothing$, since the composition of two automorphisms is an automorphism. By edge-transitivity, every edge of G contains one vertex of S and one vertex of T. Since $S \cap T = \varnothing$, this implies that G is bipartite with vertex bipartition S, T.

c) The graph below is vertex-transitive but not edge-transitive. A composition of left-right reflections and vertical rotations can take each vertex to any other. The graph has some edges on triangles and some edges not on triangles, so it cannot be edge-transitive.



1.2. PATHS, CYCLES, AND TRAILS

1.2.1. *Statements about connection.*

a) Every disconnected graph has an isolated vertex—FALSE. A simple 4-vertex graph in which every vertex has degree 1 is disconnected and has no isolated vertex.

b) A graph is connected if and only if some vertex is connected to all other vertices—TRUE. A vertex is "connected to" another if they lie in a common path. If G is connected, then by definition each vertex is connected to every other. If some vertex x is connected to every other, then because a u, x-path and x, v-path together contain a u, v-path, every vertex is connected to every other, and G is connected.

c) The edge set of every closed trail can be partitioned into edge sets of cycles—TRUE. The vertices and edges of a closed trail form an even graph, and Proposition 1.2.27 applies.

d) If a maximal trail in a graph is not closed, then its endpoints have odd degree. If an endpoint v is different from the other endpoint, then the trail uses an odd number of edges incident to v. If v has even degree, then there remains an incident edge at v on which to extend the trail.

1.2.2. *Walks in K_4.*

a) K_4 has a walk that is not a trail; repeat an edge.

b) K_4 has a trail that is not closed and is not a path; traverse a triangle and then one additional edge.

c) The closed trails in K_4 that are not cycles are single vertices. A closed trail has even vertex degrees; in K_4 this requires degrees 2 or 0, which forbids connected nontrivial graphs that are not cycles. By convention, a single vertex forms a closed trail that is not a cycle.

1.2.3. *The non-coprimality graph with vertex set $\{1, \ldots, 15\}$.* Vertices 1,11,13 are isolated. The remainder induce a single component. It has a spanning path 7,14,10,5,15,3,9,12,8,6,4,2. Thus there are four components, and the maximal path length is 11.

1.2.4. *Effect on the adjacency and incidence matrices of deleting a vertex or edge.* Assume that the graph has no loops.

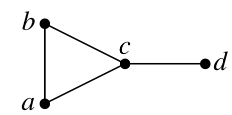
Consider the vertex ordering $v_1, \ldots, v_n$. Deleting edge $v_i v_j$ simply deletes the corresponding column of the incidence matrix; in the adjacency matrix it reduces positions i, j and j, i by one.

Deleting a vertex v_i eliminates the ith row of the incidence matrix, and it also deletes the column for each edge incident to v_i. In the adjacency matrix, the ith row and ith column vanish, and there is no effect on the rest of the matrix.

1.2.5. *If v is a vertex in a connected graph G, then v has a neighbor in every component of $G - v$.* Since G is connected, the vertices in one component of $G - v$ must have paths in G to every other component of $G - v$, and a path can only leave a component of $G - v$ via v. Hence v has a neighbor in each component.

No cut-vertex has degree 1. If G is connected and $G - v$ has k components, then having a neighbor in each such component yields $d_G(v) \ge k$. If v is a cut-vertex, then $k \ge 2$, and hence $d_G(v) \ge 2$.

1.2.6. *The paw.* Maximal paths: acb, $abcd$, $bacd$ (two are maximum paths). Maximal cliques: abc, cd (one is a maximum clique). Maximal independent sets: c, bd, ad (two are maximum independent sets).



1.2.7. *A bipartite graph has a unique bipartition (except for interchanging the two partite sets) if and only if it is connected.* Let G be a bipartite graph. If u and v are vertices in distinct components, then there is a bipartition in which u and v are in the same partite set and another in which they are in opposite partite sets.

If G is connected, then from a fixed vertex u we can walk to all other vertices. A vertex v must be in the same partite set as u if there is a u, v-walk of even length, and it must be in the opposite set if there is a u, v-walk of odd length.

1.2.8. *The biclique $K_{m,n}$ is Eulerian if and only if m and n are both even or one of them is 0.* The graph is connected. It vertices have degrees m and n (if both are nonzero), which are all even if and only if m and n are both even. When m or n is 0, the graph has no edges and is Eulerian.

1.2.9. *The minimum number of trails that decompose the Petersen graph is 5.* The Petersen graph has exactly 10 vertices of odd degree, so it needs at least 5 trails, and Theorem 1.2.33 implies that five trails suffice.

The Petersen graph does have a decomposition into five paths. Given the drawing of the Petersen graph consisting of two disjoint 5-cycles and edges between them, form paths consisting of one edge from each cycle and one edge joining them.

1.2.10. *Statements about Eulerian graphs.*

a) Every Eulerian bipartite graph has an even number of edges—TRUE.

Proof 1. Every vertex has even degree. We can count the edges by summing the degrees of the vertices in one partite set; this counts every edge exactly once. Since the summands are all even, the total is also even.

Proof 2. Since every walk alternates between the partite sets, following an Eulerian circuit and ending at the initial vertex requires taking an even number of steps.

Proof 3. Every Eulerian graph has even vertex degrees and decomposes into cycles. In a bipartite graph, every cycle has even length. Hence the number of edges is a sum of even numbers.

b) Every Eulerian simple graph with an even number of vertices has an even number of edges—FALSE. The union of an even cycle and an odd cycle that share one vertex is an Eulerian graph with an even number of vertices and an odd number of edges.

1.2.11. *If G is an Eulerian graph with edges e, f that share a vertex, then G need not have an Eulerian circuit in which e, f appear consecutively.* If G consists of two edge-disjoint cycles sharing one common vertex v, then edges incident to v that belong to the same cycle cannot appear consecutively on an Eulerian circuit.

1.2.12. *Algorithm for Eulerian circuit.* We convert the proof by extremality to an iterative algorithm. Assume that G is a connected even graph. Initialize T to be a closed trail of length 0; a single vertex.

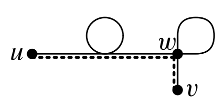
If T is not all of G, we traverse T to reach a vertex v on T that is incident to an edge e not in T. Beginning from v along e, traversing an arbitrary trail T' not using edges of T; eventually the trail cannot be extended. Since $G - E(T)$ is an even graph, this can only happen upon a return to the original vertex v, completing a closed trail. Splice T' into T by traversing T up to v, then following T', then the rest of T.

If this new trail includes all of $E(G)$, then it is an Eulerian circuit, and we stop. Otherwise, let this new trail be T and repeat the iterative step.

Since each successive trail is longer and G has finitely many edges, the procedure must terminate. It can only terminate when an Eulerian circuit has been found.

1.2.13. *Each u, v-walk contains a u, v-path.*

a) (induction). We use ordinary induction on the length l of the walk, proving the statement for all pairs of vertices. A u, v-walk of length 1 is a u, v-path of length 1; this provides the basis. For the induction step, suppose $l > 1$, and let W be a u, v-walk of length l; the induction hypothesis is that walks of length less than l contain paths linking their endpoints. If $u = v$, the desired path has length 0. If $u \neq v$, let wv be the last edge of W, and let W' be the u, w-walk obtained by deleting wv from W. Since W' has length $l - 1$, the induction hypothesis guarantees a u, w-path P in W'. If $w = v$, then P is the desired u, v-path. If $w \neq v$ and v is not on P, then we extend P by the edge wv to obtain a u, v-path. If $w \neq v$ and v is on P, then P contains a u, v-path. In each case, the edges of the u, v-path we construct all belong to W.



b) (extremality) Given a u, v-walk W, consider a shortest u, v-walk W' contained in W. If this is not a path, then it has a repeated vertex, and the portion between the instances of one vertex can be removed to obtain a shorter u, v-walk in W than W'.

1.2.14. *The union of the edge sets of distinct u, v-paths contains a cycle.*

Proof 1 (extremality). Let P and Q be distinct u, v-paths. Since a path in a simple graph is determined by its set of edges, we may assume (by symmetry) that P has an edge e not belonging to Q. Within the portion of P before P traverses e, let y be the last vertex that belongs to Q. Within the portion of P after P traverses e, let z be the first vertex that belongs to Q. The vertices y, z exist, because $u, v \in V(Q)$. The y, z-subpath of P combines with the y, z- or z, y-subpath of Q to form a cycle, since this subpath of Q contains no vertex of P between y and z.

Proof 2 (induction). We use induction on the sum l of the lengths of the two paths, for all vertex pairs simultaneously. If P and Q are u, v-paths, then $l \geq 2$. If $l = 2$, then we have distinct edges consisting of u and v, and together they form a cycle of length 2. For the induction step, suppose $l > 2$. If P and Q have no common internal vertices, then their union is a cycle. If P and Q have a common internal vertex w, then let P', P'' be the u, w-subpath of P and the w, v-subpath of P. Similarly define Q', Q''. Then P', Q' are u, w-paths with total length less than l. Similarly, P'', Q'' are w, v-paths with total length less than l. Since P, Q are distinct, we must have P', Q' distinct or P'', Q'' distinct. We can apply the induction hypothesis to the pair that is a pair of distinct paths joining the same endpoints. This pair contains the edges of a cycle, by the induction hypothesis, which in turn is contained in the union of P and Q.

The union of distinct u, v-walks need not contain a cycle. Let $G = P_3$, with vertices u, x, v in order. The distinct u, v-walks with vertex lists u, x, u, x, v and u, x, v, x, v do not contain a cycle in their union.

1.2.15. *If W is a nontrivial closed walk that does not contain a cycle, then some edge of W occurs twice in succession (once in each direction).*

Proof 1 (induction on the length l of W). We are given $l \geq 1$. A closed walk of length 1 is a loop, which is a cycle. Thus we may assume $l \geq 2$.

Basis step: $l = 2$. Since it contains no cycle, the walk must take a step and return immediately on the same edge.

Induction step: $l > 2$. If there is no vertex repetition other than first vertex = last vertex, then W traverses a cycle, which is forbidden. Hence there is some other vertex repetition. Let W' be the portion of W between the instances of such a repetition. The walk W' is a shorter closed walk than W and contains no cycle, since W has none. By the induction hypothesis, W' has an edge repeating twice in succession, and this repetition also appears in W.

Proof 2. Let w be the first repetition of a vertex along W, arriving from v on edge e. From the first occurrence of w to the visit to v is a w, v-walk, which is a cycle if $v = w$ or contains a nontrivial w, v-path P. This completes a cycle with e unless in fact P is the path of length 1 with edge e, in which case e repeats immediately in opposite directions.

1.2.16. *If edge e appears an odd number of times in a closed walk W, then W contains the edges of a cycle through e.*

Proof 1 (induction on the length of W, as in Lemma 1.2.7). The shortest closed walk has length 1. Basis step ($l = 1$): The edge e in a closed walk of length 1 is a loop and thus a cycle. Induction step ($l > 1$): If there is no vertex repetition, then W is a cycle. If there is a vertex repetition, choose two appearances of some vertex (other than the beginning and end of the walk). This splits the walk into two closed walks shorter than W. Since each step is in exactly one of these subwalks, one of them uses e an odd number of times. By the induction hypothesis, that subwalk contains the edges of a cycle through e, and this is contained in W.

Proof 2 (parity first, plus Lemma 1.2.6). Let x and y be the endpoints of e. As we traverse the walk, every trip through e is x, e, y or y, e, x. Since the number of trips is odd, the two types cannot alternate. Hence some two successive trips through e have the same direction. By symmetry, we may assume that this is $x, e, y, \ldots, x, e, y$.

The portion of the walk between these two trips through e is a y, x-walk that does not contain e. By Lemma 1.2.6, it contains a y, x-path (that does not contain e. Adding e to this path completes a cycle with e consisting of edges in W.

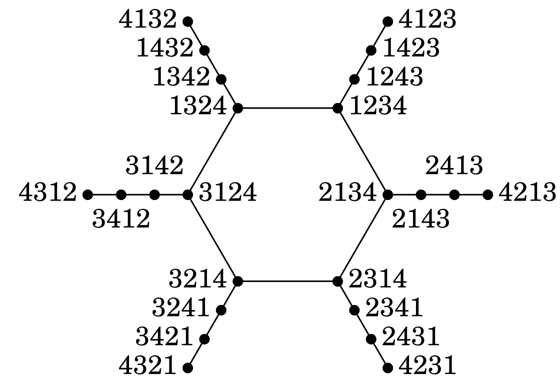
Proof 3 (contrapositive). If edge e in walk W does not lie on a cycle consisting of edges in W, then by our characterization of cut-edges, e is a cut-edge of the subgraph H consisting of the vertices and edges in W. This means that the walk can only return to e at the endpoint from which it most recently left e. This requires the traversals of e to alternate directions along e. Since a closed walk ends where it starts (that is, in the same component of $H - e$), the number of traversals of e by W must be even.

1.2.17. *The "adjacent-transposition graph" G_n on permutations of $[n]$ is connected.* Note that since every permutation of $[n]$ has $n - 1$ adjacent pairs that can be transposed, G_n is $(n-1)$-regular. Therefore, showing that G_n is connected shows that it is Eulerian if and only if n is odd.

Proof 1 (path to fixed vertex). We show that every permutation has a path to the identity permutation $I = 1, \ldots, n$. By the transitivity of the connection relation, this yields for all $u, v \in V(G)$ a u, v-path in G. To create a v, I-path, move element 1 to the front by adjacent interchanges, then move 2 forward to position 2, and so on. This builds a walk to I, which contains a path to I. (Actually, this builds a path.)

Proof 2 (direct u, v-path). Each vertex is a permutation of $[n]$. Let $u = a_1, \ldots, a_n$ and $v = b_1, \ldots, b_n$; we construct at u, v-path. The element b_1 appears in u as some a_i; move it to the front by adjacent transpositions, beginning a walk from u. Next find b_2 among $a_2, \ldots, a_n$ and move it to position 2. Iterating this procedure brings the elements of v toward the front, in order, while following a walk. It reaches v when all positions have been "corrected". (Actually, the walk is a u, v-path.) Note that since we always bring the desired element forward, we never disturb the position of the elements that were already moved to their desired positions.

Proof 3 (induction on n). If $n = 1$, then $G_n \cong K_1$ and G is connected (we can also start with $n = 2$). For $n > 1$, assume that G_{n-1} is connected. In G_n, the subgraph H induced by the vertices having n at the end is isomorphic to G_{n-1}. Every vertex of G is connected to a vertex of H by a path formed by moving element n to the end, one step at a time. For $u, v \in V(G)$, we thus have a path from u to a vertex $u' \in V(H)$, a path from v to a vertex $v' \in V(H)$, and a u', v'-path in H that exists by the induction hypothesis. By the transitivity of the connection relation, there is a u, v-path in G. This completes the proof of the induction step. (The part of G_4 used in the induction step appears below.)



Proof 4 (induction on n). The basis is as in Proof 3. For $n > 1$, note that for each $i \in [n]$, the vertices with i at the end induce a copy H_i of G_{n-1}. By the induction hypothesis, each such subgraph is connected. Also, H_n has vertices with i in position $n - 1$ whenever $1 \le i \le n - 1$. We can interchange the last two positions to obtain a neighbor in H_i. Hence there is an edge from each H_i to H_n, and transitivity of the connection relation again completes the proof.

1.2.18. *For $k \ge 1$, there are two components in the graph G_k whose vertex set is the set of binary k-tuples and whose edge set consists of the pairs that differ in exactly two places.* Changing two coordinates changes the number of 1s in the name of the vertex by zero or by ± 2. Thus the parity of the number of 1s remains the same along every edge. This implies that G_k has at least two components, because there is no edge from an k-tuple with an even number of 1s to an k-tuple with an odd number of 1s.

To show that G_k has at most two components, there are several approaches. In each, we prove that any two vertices with the same parity lie on a path, where "parity" means parity of the number of 1s.

Proof 1. If u and v are vertices with the same parity, then they differ in an even number of places. This is true because each change of a bit in obtaining one label from the other switches the parity. Since they differ in an even number of places, we can change two places at a time to travel from u to v along a path in G_k.

Proof 2. We use induction on k. Basis step ($k = 1$): G_1 has two components, each an isolated vertex. Induction step ($k > 1$): when $k > 1$, G_k consists of two copies of G_{k-1} plus additional edges. The two copies are obtained by appending 0 to all the vertex names in G_{k-1} or appending 1 to them all. Within a copy, the edges don't change, since these vertices all agree in the new place. By the induction hypothesis, each subgraph has two components. The even piece in the 0-copy has $0\cdots000$, which is adjacent to $0\cdots011$ in the odd piece of the 1-copy. The odd piece in the 0-copy has $0\cdots010$, which is adjacent to $0\cdots001$ in the even piece of the 1-copy. Thus the four pieces reduce to (at most) two components in G_k.

1.2.19. *For $n, r, s \in \mathbb{N}$, the simple graph G with vertex set $\mathbb{Z}_n$ and edge set $\{ij\colon |j-i| \in \{r,s\}\}$ has $\gcd(n,r,s)$ components.* **Note**: The text gives the vertex set incorrectly. When $r = s = 2$ and n is odd, it is necessary to go up to $n \equiv 0$ to switch from odd to even.

Let $k = \gcd(n,r,s)$. Since k divides n, the congruence classes modulo n fall into congruence classes modulo k in a well-defined way. All neighbors of vertex i differ from i by a multiple of k. Thus all vertices in a component lie in the same congruence class modulo k, which makes at least k components.

To show that there are only k components, we show that all vertices with indices congruent to $i \pmod k$ lie in one component (for each i). It suffices to build a path from i to $i+k$. Let $l = \gcd(r,s)$, and let $a = r/l$ and $b = s/l$. Since there are integers (one positive and one negative) such that $pa + qb = 1$, moving p edges with difference $+r$ and q edges with difference $+s$ achieves a change of $+l$.

We thus have a path from i to $i+l$, for each i. Now, $k = \gcd(l,n)$. As above, there exist integers p', q' such that $p'(l/k) + q'(n/k) = 1$. Rewriting this as $p'l = k - q'n$ means that if we use p' of the paths that add l, then we will have moved from i to $i+k \pmod n$.

1.2.20. *If v is a cut-vertex of a simple graph G, then v is not a cut-vertex of $\overline{G}$.* Let $V_1, \ldots, V_k$ be the vertex sets of the components of $G - v$; note that $k \ge 2$. Then $\overline{G}$ contains the complete multipartite graph with partite sets $V_1, \ldots, V_k$. Since this includes all vertices of $\overline{G} - v$, the graph $\overline{G} - v$ is connected. Hence v is not a cut-vertex of $\overline{G}$.

1.2.21. *A self-complementary graph has a cut-vertex if and only if it has a vertex of degree 1.* If there is a vertex of degree 1, then its neighbor is a cut-vertex (K_2 is not self-complementary).

For the converse, let v be a cut-vertex in a self-complementary graph G. The graph $\overline{G - v}$ has a *spanning biclique*, meaning a complete bipartite subgraph that contains all its vertices. Since G is self-complementary, also G must have a vertex u such that $G - u$ has a spanning biclique.

Since each vertex of $G - v$ is nonadjacent to all vertices in the other components of $G - v$, a vertex other than u must be in the same partite set of the spanning biclique of $G - u$ as the vertices not in the same component as u in $G - v$. Hence only v can be in the other partite set, and v has degree at least $n - 2$. We conclude that v has degree at most 1 in $\overline{G}$, so G has a vertex of degree at most 1. Since a graph and its complement cannot both be disconnected, G has a vertex of degree 1.

1.2.22. *A graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.*

Necessity. Let G be a connected graph. Given a partition of $V(G)$ into nonempty sets S, T, choose $u \in S$ and $v \in T$. Since G is connected, G has a u,v-path P. After its last vertex in S, P has an edge from S to T.

Sufficiency.

Proof 1 (contrapositive). We show that if G is not connected, then for some partition there is no edge across. In particular, if G is disconnected, then let H be a component of G. Since H is a maximal connected subgraph of G and the connection relation is transitive, there cannot be an edges with one endpoint in $V(H)$ and the other endpoint outside. Thus for the partition of $V(G)$ into $V(H)$ and $V(G) - V(H)$ there is no edge with endpoints in both sets.

Proof 2 (algorithmic approach). We grow a set of vertices that lie in the same equivalence class of the connection relation, eventually accumulating all vertices. Start with one vertex in S. While S does not include all vertices, there is an edge with endpoints $x \in S$ and $y \notin S$. Adding y to S produces a larger set within the same equivalence class, using the transitivity of the connection relation. This procedure ends only when there are no more vertices outside S, in which case all of G is in the same equivalence class, so G has only one component.

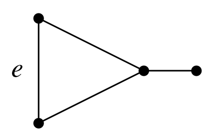
Proof 3 (extremality). Given a vertex $x \in V(G)$, let S be the set of all vertices that can be reached from x via paths. If $S \ne V(G)$, consider the partition into S and $V(G) - S$. By hypothesis, G has an edge with endpoints

$u \in S$ and $v \notin S$. Now there is an x,v-path formed by extending an x,u-path along the edge uv. This contradicts the choice of S, so in fact S is all of $V(G)$. Since there are paths from x to all other vertices, the transitivity of the connection relation implies that G is connected.

1.2.23. *a) If a connected simple graph G is not a complete graph, then every vertex of G belongs to some induced subgraph isomorphic to P_3.* Let v be a vertex of G. If the neighborhood of v is not a clique, then v has a pair x, y of nonadjacent neighbors; $\{x, v, y\}$ induces P_3. If the neighborhood of v is a clique, then since G is not complete there is some vertex y outside the set S consisting of v and its neighbors. Since G is connected, there is some edge between a neighbor w of v and a vertex x that is not a neighbor of v. Now the set $\{v, w, x\}$ induces P_3, since x is not a neighbor of v.

One can also use cases according to whether v is adjacent to all other vertices or not. The two cases are similar to those above.

b) When a connected simple graph G is not a complete graph, G may have edges that belong to no induced subgraph isomorphic to P_3. In the graph below, e lies in no such subgraph.



1.2.24. *If a simple graph with no isolated vertices has no induced subgraph with exactly two edges, then it is a complete graph.* Let G be such a graph. If G is disconnected, then edges from two components yield four vertices that induce a subgraph with two edges. If G is connected and not complete, then G has nonadjacent vertices x and y. Let Q be a shortest x,y-path; it has length at least 2. Any three successive vertices on Q induce P_3, with two edges.

Alternatively, one can use proof by contradiction. If G is not complete, then G has two nonadjacent vertices. Considering several cases (common neighbor or not, etc.) always yields an induced subgraph with two edges.

1.2.25. *Inductive proof that every graph G with no odd cycles is bipartite.*

Proof 1 (induction on $e(G)$). Basis step ($e(G) = 0$): Every graph with no edges is bipartite, using any two sets covering $V(G)$.

Induction step ($e(G) > 0$): Discarding an edge e introduces no odd cycles. Thus the induction hypothesis implies that $G - e$ is bipartite.

If e is a cut-edge, then combining bipartitions of the components of $G - e$ so that the endpoints of e are in opposite sets produces a bipartition of G. If e is not a cut-edge of G, then let u and v be its endpoints, and let X, Y be a bipartition of $G - e$. Adding e completes a cycle with a u,v-path in $G - e$; by hypothesis, this cycle has even length. This forces u and v to be in opposite sets in the bipartition X, Y. Hence the bipartition X, Y of $G - e$ is also a bipartition of G.

Proof 2 (induction on $n(G)$). Basis step ($n(G) = 1$): A graph with one vertex and no odd cycles has no loop and hence no edge and is bipartite.

Induction step ($n(G) > 1$): When we discard a vertex v, we introduce no odd cycles. Thus the induction hypothesis implies that $G - v$ is bipartite. Let $G_1, \ldots, G_k$ be the components of $G - v$; each has a bipartition. If v has neighbors u, w in both parts of the bipartition of G_i, then the edges uv and vw and a shortest u,w-path in G_i form a cycle of odd length. Hence we can specify the bipartition X_i, Y_i of G_i so that X_i contains all neighbors of v in G_i. We now have a bipartition of G by letting $X = \bigcup X_i$ and $Y = \{v\} \cup (\bigcup Y_i)$.

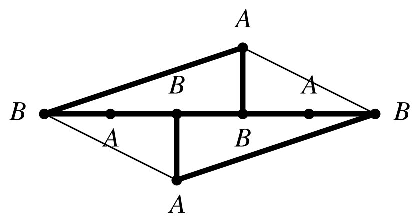
1.2.26. *A graph G is bipartite if and only if for every subgraph H of G, there is an independent set containing at least half of the vertices of H.* Every bipartite graph has a vertex partition into two independent sets, one of which must contain at least half the vertices (though it need not be a maximum independent set). Since every subgraph of a bipartite graph is bipartite, the argument applies to all subgraphs of a bipartite graph, and the condition is necessary.

For the converse, suppose that G is not bipartite. By the characterization of bipartite graphs, G contains an odd cycle H. This subgraph H has no independent set containing at least half its vertices, because every set consisting of at least half the vertices in an odd cycle must have two consecutive vertices on the cycle.

1.2.27. *The "transposition graph" on permutations of $[n]$ is bipartite.* The partite sets are determined by the parity of the number of pairs i, j such that $i < j$ and $a_i > a_j$ (these are called **inversions**). We claim that each transposition changes the parity of the number of inversions, and therefore each edge in the graph joins vertices with opposite parity. Thus the permutations with an even number of inversions form an independent set, as do those with an odd number of inversions. This is a bipartition, and thus the graph is bipartite.

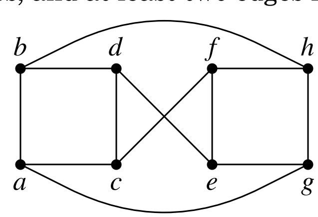
Consider the transposition that interchanges the elements in position r and position s, with $r < s$. No pairs involving elements that are before r or after s have their order changed. If $r < k < s$, then interchanging a_r and a_s changes the order of a_r and a_k, and also it changes the order of a_k and a_s. Thus for each such k the number of inversions changes twice and retains the same parity. This describes all changes in order except for the switch of a_r and a_s itself. Thus the total number of changes is odd, and the parity of the number of inversions changes.

1.2.28. *a) The graph below has a unique largest bipartite subgraph, obtained by deleting the central edge.* Deleting the central edge leaves a bipartite subgraph, since the indicated sets A and B are independent in that subgraph. If deleting one edge makes a graph bipartite, then that edge must belong to all odd cycles in the graph, since a bipartite subgraph has no odd cycles. The two odd cycles in bold have only the central edge in common, so no other edge belongs to all odd cycles.



b) In the graph below, the largest bipartite subgraph has 10 edges, and it is not unique. Deleting edges bh and ag yields an X,Y-bigraph with $X=\{b,c,e,h\}$ and $Y=\{a,d,f,g\}$. Another bipartite subgraph with 10 edges is obtained by deleting edges de and cf; the bipartition is $X=\{b,c,f,g\}$ and $Y=\{a,d,e,h\}$. (Although these two subgraphs are isomorphic, they are two subgraphs, just as the Petersen graph has ten claws, not one.)

It remains to show that we must delete at least two edges to obtain a bipartite subgraph. By the characterization of bipartite graphs, we must delete enough edges to break all odd cycles. We can do this with (at most) one edge if and only if all the odd cycles have a common edge. The 5-cycles (b,a,c,f,h) and (b,d,e,g,h) have only the edge bh in common. Therefore, if there is a single edge lying in all odd cycles, it must be bh. However, (a,c,f,h,g) is another 5-cycle that does not contain this. Therefore no edge lies in all odd cycles, and at least two edges must be deleted.



1.2.29. *A connected simple graph not having P_4 or C_3 as an induced subgraph is a biclique.* Choose a vertex x. Since G has no C_3, $N(x)$ is independent. Let $S=V(G)-N(X)-\{x\}$. Every $v\in S$ has a neighbor in $N(x)$; otherwise, a shortest v,x-path contains an induced P_4. If $v\in S$ is adjacent to w but not z in $N(x)$, then v,w,x,z is an induced P_4. Hence all of S is adjacent to all of $N(x)$. Now $S\cup\{x\}$ is an independent set, since G has no C_3. We have proved that G is a biclique with bipartition $N(x)$, $S\cup\{x\}$.

1.2.30. *Powers of the adjacency matrix.*

a) In a simple graph G, the (i,j)th entry in the kth power of the adjacency matrix $\mathbf{A}$ is the number of (v_i,v_j)-walks of length k in G. We use induction on k. When $k=1$, $a_{i,j}$ counts the edges (walks of length 1) from i to j. When $k>1$, every (v_i,v_j)-walk of length k has a unique vertex v_r reached one step before the end at v_j. By the induction hypothesis, the number of (v_i,v_r)-walks of length $k-1$ is entry (i,r) in $\mathbf{A}^{k-1}$, which we write as $a_{i,r}^{(k-1)}$. The number of (v_i,v_j)-paths of length k that arrive via v_r on the last step is $a_{i,r}^{(k-1)}a_{r,j}$, since $a_{r,j}$ is the number of edges from v_r to v_j that can complete the walk. Counting the (v_i,v_j)-walks of length k by which vertex appears one step before v_j yields $\sum_{r=1}^n a_{i,r}^{(k-1)}a_{r,j}$. By the definition of matrix multiplication, this is the (i,j)th entry in $\mathbf{A}^k$. (The proof allows loops and multiple edges and applies without change for digraphs. When loops are present, note that there is no choice of "direction" on a loop; a walk is a list of edge traversals).

b) A simple graph G with adjacency matrix A is bipartite if and only if, for each odd integer r, the diagonal entries of the matrix A^r are all 0. By part (a), $A_{i,i}^r$ counts the closed walks of length r beginning at v_i. If this is always 0, then G has no closed walks of odd length through any vertex; in particular, G has no odd cycle and is bipartite. Conversely, if G is bipartite, then G has no odd cycle and hence no closed odd walk, since every closed odd walk contains an odd cycle.

1.2.31. *K_n is the union of k bipartite graphs if and only if $n\le 2^k$ (without using induction).*

a) Construction when $n\le 2^k$. Given $n\le 2^k$, encode the vertices of K_n as distinct binary k-tuples. Let G_i be the complete bipartite subgraph with bipartition X_i,Y_i, where X_i is the set of vertices whose codes have 0 in position i, and Y_i is the set of vertices whose codes have 1 in position i. Since every two vertex codes differ in some position, $G_1\cup\cdots\cup G_k=K_n$.

b) Upper bound. Given that K_n is a union of bipartite graphs $G_1,\ldots,G_k$, we define a code for each vertex. For $1\le i\le k$, let X_i,Y_i be a bipartition of G_i. Assign vertex v the code $(a_1,\ldots,a_k)$, where $a_i=0$ if $v\in X_i$, and $a_i=1$ if $v\in Y_i$ or $v\notin X_i\cup Y_i$. Since every two vertices are adjacent and the edge joining them must be covered in the union, they lie in opposite partite sets in some G_i. Therefore the codes assigned to the vertices are distinct. Since the codes are binary k-tuples, there are at most 2^k of them, so $n\le 2^k$.

1.2.32. *"Every maximal trail in an even graph is an Eulerian circuit"—FALSE.* When an even graph has more than one component, each component has a maximal trail, and it will not be an Eulerian circuit unless the other components have no edges. The added hypothesis needed is that the graph is connected.

The proof of the corrected statement is essentially that of Theorem 1.2.32. If a maximal trail T is not an Eulerian circuit, then it is incident to a missing edge e, and a maximal trail in the even graph $G - E(T)$ that starts at e can be inserted to enlarge T, which contradicts the hypothesis that T is a maximal trail.

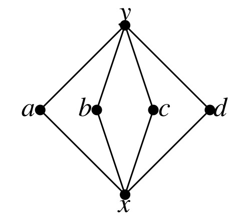
1.2.33. *The edges of a connected graph with $2k$ odd vertices can be partitioned into k trails if $k > 0$.* The assumption of connectedness is necessary, because the conclusion is not true for $G = H_1 + H_2$ when H_1 has some odd vertices and H_2 is Eulerian.

Proof 1 (induction on k). When $k = 1$, we add an edge between the two odd vertices, obtain an Eulerian circuit, and delete the added edge. When $k > 1$, let P be a path connecting two odd vertices. The graph $G' = G - E(P)$ has $2k-2$ odd vertices, since deleting $E(P)$ changes degree parity only at the ends of P. The induction hypothesis applies to each component of G' that has odd vertices. Any component not having odd vertices has an Eulerian circuit that contains a vertex of P; we splice it into P to avoid having an additional trail. In total, we have used the desired number of trails to partition $E(G)$.

Proof 2 (induction on $e(G)$). If $e(G) = 1$, then $G = K_2$, and we have one trail. If G has an even vertex x adjacent to an odd vertex y, then $G' = G - xy$ has the same number of odd vertices as G. The trail decomposition of G' guaranteed by the induction hypothesis has one trail ending at x and no trail ending at y. Add xy to the trail ending at x to obtain the desired decomposition of G. If G has no even vertex adjacent to an odd vertex, then G is Eulerian or every vertex of G is odd. In this case, deleting an edge xy reduces k, and we can add xy as a trail of length one to the decomposition of $G - xy$ guaranteed by the induction hypothesis.

1.2.34. *The graph below has 6 equivalence classes of Eulerian circuits.* If two Eulerian circuits follow the same circular arrangement of edges, differing only in the starting edges or the direction, then we consider them equivalent. An equivalence class of circuits is characterized by the pairing of edges at each vertex corresponding to visits through that vertex.

A 2-valent vertex has exactly one such pairing; a 4-valent vertex has three possible pairings. The only restriction is that the pairings must yield a single closed trail. Given a pairing at one 4-valent vertex below, there is a forbidden pairing at the other, because it would produce two edge-disjoint 4-cycles instead of a single trail. The other two choices are okay. Thus the answer is $3 \cdot 2 = 6$.



Alternatively, think of making choices while following a circuit. Because each circuit uses each edge, and because the reversal of a circuit C is in the same class as C, we may follow a canonical representative of the class from a along ax. We now count the choices made to determine the circuit. After x we can follow one of 3 choices. This leads us through another neighbor of x to y. Now we cannot use the edge ya or the edge just used, so two choices remain. This determines the rest of the circuit. For each of the three ways to make the initial choice, there was a choice of two later, so there are $3 \cdot 2 = 6$ ways to specify distinct classes of circuits. (Distinct ways of making the choices yields a distinct pairing at some vertex.)

1.2.35. *Algorithm for Eulerian circuits.* Let G be a connected even graph. At each vertex partition the incident edges into pairs (each edge appears in a pair at each endpoint). Start along some edge. At each arrival at a vertex, there is an edge paired with the entering edge; use it to exit. This can end only by arriving at the initial vertex along the edge paired with the initial edge, and it must end since the graph is finite. At the point where the first edge would be repeated, stop; this completes a closed trail. Furthermore, there is no choice in assembling this trail, so every edge appears in exactly one such trail. Therefore, the pairing decomposes G into closed trails.

If there is more than one trail in the decomposition, then there are two trails with a common vertex, since G is connected. (A shortest path connecting vertices in two of the trails first leaves the first trail at some vertex v, and at v we have edges from two different trails.) Given edges from trails A and B at v, change the pairing by taking a pair in A and a pair in B and switching them to make two pairs that pair an edge of A with an edge of B. Now when A is followed from v, the return to A does not end the trail, but rather the trail continues and follows B before returning to the original edge. Thus changing the pairing at v combines these two trails into one trail and leaves the other trails unchanged.

We have shown that if the number of trails in the decomposition exceeds one, then we can obtain a decomposition with fewer trails be changing the pairing. Repeating the argument produces a decomposition using one closed trail. This trail is an Eulerian circuit.

1.2.36. *Alternative characterization of Eulerian graphs.*

a) If G is loopless and Eulerian and $G' = G - uv$, then G' has an odd number of u,v-trails that visit v only at the end.

Proof 1 (exhaustive counting and parity). Every extension of every trail from u in G' eventually reaches v, because a maximal trail ends only at a vertex of odd degree. We maintain a list of trails from u. The number of choices for the first edge is odd. For a trail T that has not yet reached v, there are an odd number of ways to extend T by one edge. We replace T in the list by these extensions. This changes the number of trails in the list by an even number. The process ends when all trails in the list end at v. Since the list always has odd size, the total number of these trails is odd.

Proof 2 (induction and stronger result). We prove that the same conclusion holds whenever u and v are the only vertices of odd degree in a graph H, regardless of whether they are adjacent. This is immediate if H has only the edge uv. For larger graphs, we show that there are an odd number of such trails starting with each edge e incident to u, so the sum is odd. If $e = uv$, then there is one such trail. Otherwise, when $e = uw$ with $w \neq v$, we apply the induction hypothesis to $H - e$, in which w and v are the only vertices of odd degree.

The number of non-paths in this list of trails is even. If T is such a trail that is not a path, then let w be the first instance of a vertex repetition on T. By traversing the edges between the first two occurrences of w in the opposite order, we obtain another trail T' in the list. For T', the first instance of a vertex repetition is again w, and thus $T'' = T$. This defines an involution under which the fixed points are the u,v-paths. The trails we wish to delete thus come in pairs, so there are an even number of them.

b) If v is a vertex of odd degree in a graph G, then some edge incident to v lies in an even number of cycles. Let $c(e)$ denote the number of cycles containing e. Summing $c(e)$ over edges incident to v counts each cycle through v exactly twice, so the sum is even. Since there are an odd number of terms in the sum, $c(e)$ must be even for some e incident to v.

c) A nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles. Necessity: By part (a), the number of u,v-paths in $G - uv$ is odd. The cycles through uv in G correspond to the u,v-paths in $G - uv$, so the number of these cycles is odd.

Sufficiency: We observe the contrapositive. If G is not Eulerian, then G has a vertex v of odd degree. By part (b), some edge incident to v lies in an even number of cycles.

1.2.37. *The connection relation is transitive.* It suffices to show that if P is a u,v-path and P' is a v,w-path, then P and P' together contain a u,w-path. At least one vertex of P is in P', since both contain v. Let x be the first vertex of P that is in P'. Following P from u to x and then P' from x to w yields a u,w path, since no vertex of P before x belongs to P'.

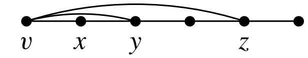
1.2.38. *Every n-vertex graph with at least n edges contains a cycle.*

Proof 1 (induction on n). A graph with one vertex that has an edge has a loop, which is a cycle. For the induction step, suppose that $n > 1$. If our graph G has a vertex v with degree at most 1, then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges. By the induction hypothesis, $G - v$ contains a cycle, and this cycle appears also in G. If G has no vertex of degree at most 1, then every vertex of G has degree at least 2. Now Lemma 1.2.25 guarantees that G contains a cycle.

Proof 2 (use of cut-edges). If G has no cycle, then by Theorem 1.2.14 every edge is a cut-edge, and this remains true as edges are deleted. Deleting all the edges thus produces at least $n + 1$ components, which is impossible.

1.2.39. *If G is a loopless graph and $\delta(G) \geq 3$, then G has a cycle of even length.* An endpoint v of a maximal path P has at least three neighbors on P. Let x, y, z be three such neighbors of v in order on P. Consider three v,y-paths: the edge vy, the edge vx followed by the x,y-path in P, and the edge vz followed by the z,y-path in P.

These paths share only their endpoints, so the union of any two is a cycle. By the pigeonhole principle, two of these paths have lengths with the same parity. The union of these two paths is an even cycle.



1.2.40. *If P and Q are two paths of maximum length in a connected graph G, then P and Q have a common vertex.* Let m be the common length of P and Q. Since G is connected, it has a shortest path R between $V(P)$ and $V(Q)$. Let l be the length of R. Let the endpoints of R be $r \in V(P)$ and $r' \in V(Q)$. The portion P' of P from r to the farther endpoint has length at least $m/2$. The portion Q' of Q from r to the farther endpoint has length at least $m/2$. Since R is a shortest path, R has no internal vertices in P or Q.

If P and Q are disjoint, then P' and Q' are disjoint, and the union of P', Q', and R is a path of length at least $m/2 + m/2 + l = m + l$. Since the maximum path length is m, we have $l = 0$. Thus $r = r'$, and P and Q have a common vertex.

The graph consisting of two edge-disjoint paths of length $2k$ sharing their midpoint is connected and hence shows that P and Q need not have a common edge.

1.2.41. *A connected graph with at least three vertices has two vertices x, y such that 1) $G - \{x, y\}$ is connected and 2) x, y are adjacent or have a common neighbor.* Let x be a endpoint of a longest path P in G, and let v be its neighbor on P. Note that P has at least three vertices. If $G-x-v$ is connected, let $y=v$. Otherwise, a component cut off from $P-x-v$ in $G-x-v$ has at most one vertex; call it w. The vertex w must be adjacent to v, since otherwise we could build a longer path. In this case, let $y=w$.

1.2.42. *A connected simple graph having no 4-vertex induced subgraph that is a path or a cycle has a vertex adjacent to every other vertex.* Consider a vertex x of maximum degree. If x has a nonneighbor y, let x, v, w be the begining of a shortest path to y (w may equal y). Since $d(v) \le d(x)$, some neighbor z of x is not adjacent to v. If $z \leftrightarrow w$, then $\{z, x, v, w\}$ induce C_4; otherwise, $\{z, x, v, w\}$ induce P_4. Thus x must have no nonneighbor.

1.2.43. *The edges of a connected simple graph with $2k$ edges can be partitioned into paths of length 2.* The assumption of connectedness is necessary, since the conclusion does not hold for a graph having components with an odd number of edges.

We use induction on $e(G)$; there is a single such path when $e(G)=2$. For $e(G)>2$, let $P=(x,y,z)$ be an arbitrary path of length two in G, and let $G'=G-\{xy, yz\}$. If we can partition $E(G)$ into smaller connected subgraphs of even size, then we can apply the induction hypothesis to each piece and combine the resulting decompositions. One way to do this is to partition $E(G')$ into connected subgraphs of even size and use P.

Hence we are finished unless G' has two components of odd size (G' cannot have more than three components, since an edge deletion increases the number of components by at most one). Each odd component contains at least one of $\{x, y, z\}$. Hence it is possible to add one of xy to one odd component and yz to the other odd component to obtain a partition of G into smaller connected subgraphs.

1.3. VERTEX DEGREES & COUNTING

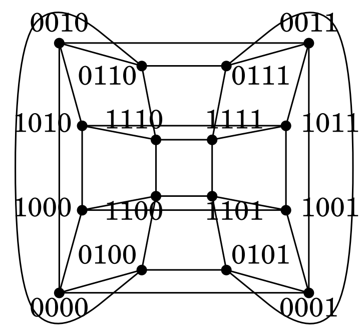
1.3.1. *A graph having exactly two vertices of odd degree must contain a path from one to the other.* The degree of a vertex in a component of G is the same as its degree in G. If the vertices of odd degree are in different components, then those components are graphs with odd degree sum.

1.3.2. *In a class with nine students where each student sends valentine cards to three others, it is not possible that each student sends to and receives cards from the same people.* The sending of a valentine can be represented as a directed edge from the sender to the receiver. If each student sends to and receives cards from the same people, then the graph has $x \to y$ if and

only if $y \to x$. Modeling each opposed pair of edges by a single unoriented edge yields a 3-regular graph with 9 vertices. This is impossible, since every graph has an even number of vertices of odd degree.

1.3.3. *If $d(u)+d(v)=n+k$ for an edge uv in a simple graph on n vertices, then uv belongs to at least k triangles.* This is the same as showing that u and v have at least k common neighbors. Let S be the neighbors of u and T the neighbors of v, and suppose $|S \cap T| = j$. Every vertex of G appears in S or T or none or both. Common neighbors are counted twice, so $n \ge |S|+|T|-j = n+k-j$. Hence $j \ge k$. (Almost every proof of this using induction or contradiction does not need it, and is essentially just this counting argument.)

1.3.4. *The graph below is isomorphic to Q_4.* It suffices to label the vertices with the names of the vertices in Q_4 so that vertices are adjacent if and only if their labels differ in exactly one place.



1.3.5. *The k-dimensional cube Q_k has $\binom{k}{2}2^k$ copies of P_3.*

Proof 1. To specify a particular subgraph isomorphic to P_3, the 3-vertex path, we can specify the middle vertex and its two neighbors. For each vertex of Q_k, there are $\binom{k}{2}$ ways to choose two distinct neighbors, since Q_k is a simple k-regular graph. Thus the total number of P_3's is $\binom{k}{2}2^k$.

Proof 2. We can alternatively choose the starting vertex and the next two. There are 2^k ways to pick the first vertex. For each vertex, there are k ways to pick a neighbor. For each way to pick these vertices, there are $k-1$ ways to pick a third vertex completing P_3, since Q_k has no multiple edges. The product of these factors counts each P_3 twice, since we build it from each end. Thus the total number of them is $2^k k(k-1)/2$.

Q_k has $\binom{k}{2}2^{k-2}$ copies of C_4.

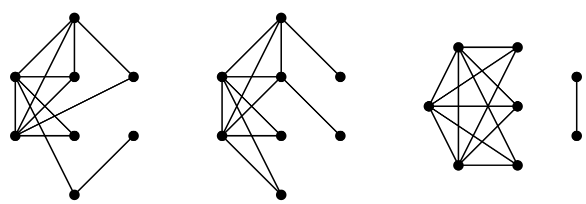
Proof 1 (direct counting). The vertices two apart on a 4-cycle must differ in two coordinates. Their two common neighbors each differ from each in exactly one of these coordinates. Hence the vertices of a 4-cycle must use all 2-tuples in two coordinates while keeping the remaining coordinates fixed. All such choices yield 4-cycles. There are $\binom{k}{2}$ ways to choose the two coordinates that vary and 2^{k-2} ways to set a fixed value in the remaining coordinates.

Proof 2 (prior result). Every 4-cycle contains four copies of P_3, and every P_3 contains two vertices at distance 2 in the cube and hence extends to exactly one 4-cycle. Hence the number of 4-cycles is one-fourth the number of copies of P_3.

1.3.6. *Counting components.* If G has k components and H has l components, then $G + H$ has $k + l$ components. The maximum degree of $G + H$ is $\max\{\Delta(G), \Delta(H)\}$.

1.3.7. *Largest bipartite subgraphs.* P_n is already bipartite. C_n loses one edge if n is odd, none if n is even. The largest bipartite subgraph of K_n is $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, which has $\lfloor n^2/4\rfloor$ edges.

1.3.8. *The lists (5,5,4,3,2,2,2,1), (5,5,4,4,2,2,1,1), and (5,5,5,3,2,2,1,1) are graphic, but (5,5,5,4,2,1,1,1) is not.* The answers can be obtained from the Havel-Hakimi test; a list is graphic if and only if the list obtained by deleting the largest element and deleting that many next-largest elements is graphic. Below are graphs realizing the first three lists, found by the Havel-Hakimi algorithm.



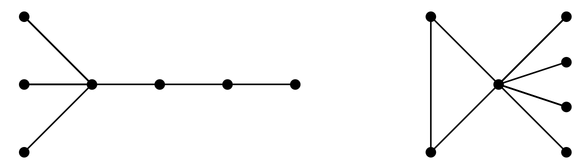
From the last list, we test $(4, 4, 3, 1, 0, 1, 1)$, reordered to $(4, 4, 3, 1, 1, 1, 0)$, then $(3, 2, 0, 0, 1, 0)$. This is not the degree list of a simple graph, since a vertex of degree 3 requires three other vertices with nonzero degree.

1.3.9. *In a league with two divisions of 13 teams each, no schedule has each team playing exactly nine games against teams in its own division and four games against teams in the other division.* If this were possible, then we could form a graph with the teams as vertices, making two vertices adjacent if those teams play a game in the schedule. We are asking for the subgraph induced by the 13 teams in a single division to be 9-regular. However, there is no regular graph of odd degree with an odd number of vertices, since for every graph the sum of the degrees is even.

1.3.10. *If l, m, n are nonnegative integers with $l + m = n \geq 1$, then there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree if and only if m is even, except for $(l, m, n) = (2, 0, 2)$.* Since every graph has an even number of vertices of odd degree, and the only simple connected graph with two vertices has both degrees odd, the condition is necessary.

To prove sufficiency, we construct such a graph G. If $m = 0$, let $G = C_l$ (except $G = K_1$ if $l = 1$). For $m > 0$, we can begin with $K_{1,m-1}$, which has m vertices of odd degree, and then add a path of length l beyond one of the leaves. (Illustration shows $l = 3$, $m = 4$.)

Alternatively, start with a cycle of length l, and add m vertices of degree one with a common neighbor on the cycle. That vertex of the cycle has even degree because m is even. Many other constructions also work. It is also possible to prove sufficiency by induction on n for $n \geq 3$, but this approach is longer and harder to get right than an explicit general construction.



1.3.11. *If C is a closed walk in a simple graph G, then the subgraph consisting of the edges appearing an odd number of times in C is an even graph.* Consider an arbitrary vertex $v \in V(G)$. Let S be the set of edges incident to v, and let $f(e)$ be the number of times an edge e is traversed by C. Each time C passes through v it enters and leaves. Therefore, $\sum_{e\in S} f(e)$ must be even, since it equals twice the number of times that C visits v. Hence there must an even number of odd contributions to the sum, which means there are an even number of edges incident to v that appear an odd number of times in C. Since we can start a closed walk at any of its vertices, this argument holds for every $v \in V(G)$.

1.3.12. *If every vertex of G has even degree, then G has no cut-edge.*

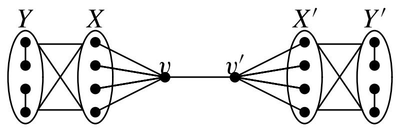
Proof 1 (contradiction). If G has a cut-edge, deleting it leaves two induced subgraphs whose degree sum is odd. This is impossible, since the degree sum in every graph is even.

Proof 2 (construction/extremality). For an edge uv, a maximal trail in $G - uv$ starting at u can only end at v, since whenever we reach a vertex we have use an odd number of edges there. Hence a maximal such trail is a (u, v)-trail. Every (u, v)-trail is a (u, v)-walk and contains a (u, v)-path. Hence there is still a (u, v)-path after deletion of uv, so uv is not a cut-edge.

Proof 3 (prior results). Let G be an even graph. By Proposition 1.2.27, G decomposes into cycles. By the meaning of "decomposition", every edge of G is in a cycle. By Theorem 1.2.14, every edge in a cycle is not a cut-edge. Hence every edge of G is not a cut-edge.

For $k \in \mathbb{N}$, some $(2k+1)$-regular simple graph has a cut-edge.

Construction 1. Let H, H' be copies of $K_{2k,2k}$ with partite sets X, Y for H and X', Y' for H'. Add an isolated edge vv' disjoint from these sets. To $H + H' + vv'$, add edges from v to all of X and from v' to all of X', and add k disjoint edges within Y and k disjoint edges within Y'. The resulting graph G_k is $(2k+1)$-regular with $8k+2$ vertices and has vv' as a cut-edge. Below we sketch G_2; the graph G_1 is the graph in Example 1.3.26.



Construction 2a (inductive). Let G_1 be the graph at the end of Example 1.3.26 (or in Construction 1). This graph is 3-regular with 10 vertices and cut-edge xy; note that $10 = 4 \cdot 1 + 6$. From a $(2k-1)$-regular graph G_{k-1} with $4k+2$ vertices such that $G_{k-1} - xy$ has two components of order $2k+1$, we form G_k. Add two vertices for each component of $G_{k-1} - xy$, adjacent to all the vertices of that component. This adds degree two to each old vertex, gives degree $2k+1$ to each new vertex, and leaves xy as a cut-edge. The result is a $(2k+1)$-regular graph G_k of order $4k+6$ with cut-edge xy.

Construction 2b (explicit). Form H_k from K_{2k+2} by removing k pairwise disjoint edges and adding one vertex that is adjacent to all vertices that lost an incident edge. Now H_k has $2k+2$ vertices of degree $2k+1$ and one of degree $2k$. Form G_k by taking two disjoint copies of H_k and adding an edge joining the vertices of degree $2k$. The graphs produced in Constructions 2a and 2b are identical.

1.3.13. *Meeting on a mountain range.* A **mountain range** is a polygonal curve from $(a,0)$ to $(b,0)$ in the upper half-plane; we start A and B at opposite endpoints. Let P be a highest peak; A and B will meet there. Let the segments from P to $(a,0)$ be $x_1, \ldots, x_r$, and let the segments from P to $(b,0)$ be $y_1, \ldots, y_s$. We define a graph to describe the positions; when A is on x_i and B is on y_j, the corresponding vertex is (i,j). We start at the vertex (r,s) and must reach $(1,1)$. We introduce edges for the possible transitions. We can move from (i,j) to $(i,j+1)$ if the common endpoint of y_j and y_{j+1} has height between the heights of the endpoints of x_i. Similarly, (i,j) is adjacent to $(i+1,j)$ if the common endpoint of x_i and x_{i+1} has height between the heights of the endpoints of x_j. To avoid triviality, we may assume that $r+s>2$.

We prove that (r,s) and $(1,1)$ are the only vertices of odd degree in G. This suffices, because every graph has an even number of vertices of odd degree, which implies that (r,s) and $(1,1)$ are in the same component, connected by a path.

The possible neighbors of (i,j) are the pairs obtained by changing i or j by 1. Let X and Y be the intervals of heights attained by x_i and y_j, and let $I = X \cap Y$. If the high end of I is the high end of exactly one of X and Y, then exactly one neighboring vertex can be reached by moving past the end of the corresponding segment. If it is the high end of both, then usually one or three neighboring vertices can be reached, the latter when both segments reach "peaks" at their high ends. However, if $(i,j) = (1,1)$, then the high end of both segments is P and there is no neighbor of this type. Similarly, the low end of I generates one or three neighbors, except that when $(i,j) = (r,s)$ there is no neighbor of this type.

No neighbor of (i,j) is generated from both the low end and the high end of I. Since the contributions from the high and low end of I to the degree of (i,j) are both odd, each degree is even, except for (r,s) and $(1,1)$, where exactly one of the contributions is odd.

1.3.14. *Every simple graph with at least two vertices has two vertices of equal degree.* The degree of a vertex in an n-vertex simple graph is in $\{0, \ldots, n-1\}$. These are n distinct values, so if no two are equal then all appear. However, a graph cannot have both an isolated vertex and a vertex adjacent to all others.

This does not hold for graphs allowing loops. In the 2-vertex graph with one loop edge and one non-loop edge, the vertex degrees are 1 and 3.

This does not hold for loopless graphs. In the 3-vertex loopless graph with pairs having multiplicity $0, 1, 2$, the vertex degrees are $1, 3, 2$.

1.3.15. *Smallest k-regular graphs.* A simple k-regular graph has at least $k+1$ vertices, so K_{k+1} is the smallest. This is the only isomorphism class of k-regular graphs with $k+1$ vertices. With $k+2$ vertices, the complement of a k-regular graph must be 1-regular. There is one such class when k is even ($(k+2)/2$ isolated edges), none when k is odd. (Two graphs are isomorphic if and only if their complements are isomorphic.)

With $k+3$ vertices, the complement is 2-regular. For $k \ge 3$, there are distinct choices for such a graph: a $(k+3)$-cycle or the disjoint union of a 3-cycle and a k-cycle. Since these two 2-regular graphs are nonisomorphic, their complements are nonisomorphic k-regular graphs with $k+3$ vertices.

1.3.16. *For $k \ge 2$ and $g \ge 2$, there exists a k-regular graph with girth g.* We use strong induction on g. For $g = 2$, take the graph consisting of two vertices and k edges joining them.

For the induction step, consider $g > 2$. Here we use induction on k. For $k = 2$, a cycle of length g suffices. For $k > 2$, the induction hypothesis provides a $(k-1)$-regular graph H with girth g. Since $\lceil g/2 \rceil < g$, the global induction hypothesis also provides a graph G with girth $\lceil g/2 \rceil$ that is $n(H)$-regular. Replace each vertex v in G with a copy of H; each vertex in the copy of H is made incident to one of the edges incident to v in G.

Each vertex in the resulting graph inherits $k-1$ incident edges from H and one from G, so the graph is k-regular. It has cycles of length g in copies of H. A cycle C in G is confined to a single copy of H or visits more than one such copy. In the first case, its length is at least g, since H has girth g. In the second case, the copies of H that C visits correspond to a cycle in G, so C visits at least $\lceil g/2 \rceil$ such copies. For each copy, C must enter on one edge and then move to another vertex before leaving, since the copy is entered by only one edge at each vertex. Hence the length of such a cycle is at least $2\lceil g/2 \rceil$.

1.3.17. *Deleting a vertex of maximum degree cannot increase the average degree, but deleting a vertex of minimum degree can reduce the average degree.* Deleting any vertex of a nontrivial regular graph reduces the average degree, which proves the second claim. For the first claim, suppose that G has n vertices and m edges, and let a and a' be the average degrees of G and $G-x$, respectively. Since $G-x$ has $m-d(x)$ edges and degree sum $2m-2d(x)$, we have $a' = \frac{na-2d(x)}{n-1} \le \frac{(n-2)a}{n-1} < a$ if $d(x) \ge a > 0$. Hence deleting a vertex of maximum degree in nontrivial graph reduces the average degree and cannot increase it.

1.3.18. *If $k \ge 2$, then a k-regular bipartite graph has no cut-edge.* Since components of k-regular graphs are k-regular, it suffices to consider a connected k-regular X,Y-bigraph. Let uv be a cut-edge, and let G and H be the components formed by deleting uv. Let $m = |V(G) \cap X|$ and $n = |V(G) \cap Y|$. By symmetry, we may assume that $u \in V(G) \cap Y$ and $v \in V(H) \cap X$.

We count the edges of G. The degree of each vertex of G in X is k, so G has mk edges. The degree of each vertex of G in Y is k except for $d_G(u) = k-1$, so G has $nk-1$ edges. Hence $m_k = nk-1$, which is impossible because one side is divisible by k and the other is not. The proof doesn't work if $k=1$, and the claim is false then.

If vertex degrees k and $k+1$ are allowed, then a cut-edge may exist. Consider the example of $2K_{k,k}$ plus one edge joining the two components.

1.3.19. *A claw-free simple graph with maximum degree at least 5 has a 4-cycle.* Consider five edges incident to a vertex v of maximum degree in such a graph G. Since G has no induced claw, the neighbors of v must induce at least three edges. Since these three edges have six endpoints among the five neighbors of v, two of them must be incident, say xy and yz. Adding the edges xv and zv to these two completes a 4-cycle.

There are arbitrarily large 4-regular claw-free graphs with no 4-cycles.

Consider a vertex v in such a graph G. Since v has degree 4 and is not the center of an induced claw and does not lie on a 4-cycle, the subgraph induced by v and its neighbors consists of two edge-disjoint triangles sharing v (a bowtie). Since this happens at each vertex, G consists of pairwise edge-disjoint triangles, with each vertex lying in two of them. Hence each triangle has three neighboring triangles. Furthermore, two triangles that neighbor a given triangle in this way cannot neighbor each other; that would create a 4-cycle in the graph.

Define a graph H with one vertex for each triangle in G; let vertices be adjacent in H if the corresponding triangles share a vertex in G. Now H is a 3-regular graph with no 3-cycles; a 3-cycle in H would yield a 4-cycle in G using two edges from one of the corresponding triangles. Also H must have no 4-cycles, because a 4-cycle in G could be built using one edge from each of the four triangles corresponding to the vertices of a 4-cycle in H. Note that $e(G) = 2n(G)$ and $n(H) = e(G)/3 = 2n(G)/3$.

On the other hand, given any 3-regular graph H with girth at least 5, reversing the construction yields G with the desired properties and $3n(H)/2$ vertices. Hence it suffices to show that there are arbitrarily large 3-regular graphs with girth at least 5. Disconnected such examples can be formed by taking many copies of the Petersen graph as components. The graph G is connected if and only if H is connected. Connected instances of H can be obtained from multiple copies of the Petersen graph by applying 2-switches (Definition 1.3.32).

Alternatively, arbitrarily large connected examples can be constructed by taking two odd cycles (say length $2m+1$) and joining the ith vertex on the first cycle to the $2i$th vertex (modulo $2m+1$) on the second cycle (this generalizes the Petersen graph). We have constructed a connected 3-regular graph. Since we add disjoint edges between the cycles, there is no triangle. A 4-cycle would have to alternate edges between the two odd cycles with one edge of each, but the neighbors of adjacent vertices on the first cycle are two apart on the second cycle.

1.3.20. *K_n has $(n-1)!/2$ cycles of length n, and $K_{n,n}$ has $n!(n-1)!/2$ cycles of length $2n$.* Each cycle in K_n is a listing of the vertices. These can be listed in $n!$ orders, but we obtain the same subgraph no matter where we start the cycle and no matter which direction we follow, so each cycle is listed $2n$ times. In $K_{n,n}$, we can list the vertices in order on a cycle (alternating between the partite sets), in $2(n!)^2$ ways, but by the same reasoning each cycle appears $(2n)\cdot 2$ times.

1.3.21. *$K_{m,n}$ has $6\binom{m}{3}\binom{n}{3}$ 6-cycles.* To extend an edge in $K_{m,n}$ to a 6-cycle, we choose two more vertices from each side to be visited in order as we follow the cycle. Hence each edge in $K_{n,n}$ appears in $(m-1)(n-1)(m-2)(n-2)$

6-cycles. Since each 6-cycle contains 6 edges, we conclude that $K_{n,n}$ has $mn(m-1)(n-1)(m-2)(n-2)/6$ 6-cycles.

Alternatively, each 6-cycle uses three vertices from each partite set, which we can choose in $\binom{m}{3}\binom{n}{3}$ ways. Each such choice of vertices induces a copy of $K_{3,3}$ with 9 edges. There are $3! = 6$ ways to pick three disjoint edges to be omitted by a 6-cycle, so each $K_{3,3}$ contains 6 6-cycles.

1.3.22. *Odd girth and minimum degree in nonbipartite triangle-free n-vertex graphs.* Let $k = \delta(G)$, and let l be the minimum length of an odd cycle in G. Let C be a cycle of length l in G.

a) Every vertex not in $V(C)$ has at most two neighbors in $V(C)$. It suffices to show that any two neighbors of such a vertex v on C must have distance 2 on C, since having three neighbors would then require $l = 6$.

Since G is triangle-free, v does not have consecutive neighbors on C. If v has neighbors x and y on C separated by distance more than 2 on C, then the detour through v can replace the x,y-path of even length on C to form a shorter odd cycle.

b) $n \ge kl/2$ (and thus $l \le 2n/k$). Since C is a shortest odd cycle, it has no chords (it is an induced cycle). Since $\delta(G) = k$, each vertex of C thus has at least $k-2$ edges to vertices outside C. However, each vertex outside C has at most two neighbors on C. Letting m be the number of edges from $V(C)$ to $V(G) - V(C)$, we thus have $l(k-2) \le m \le 2(n-l)$. Simplifying the inequality yields $n \ge kl/2$.

c) The inequality of part (b) is sharp when k is even. Form G from the cycle C_l by replacing each vertex of C_l with an independent set of size $k/2$ such that two vertices are adjacent if and only if the vertices they replaced were adjacent. Each vertex is now adjacent to the vertices arising from the two neighboring classes, so G is k-regular and has $lk/2$ vertices. Deleting the copies of any one vertex of C_l leaves a bipartite graph, since the partite sets can be labeled alternately around the classes arising from the rest of C_l. Hence every odd cycle uses a copy of each vertex of C_l and has length at least l, and taking one vertex from each class forms such a cycle.

1.3.23. *Equivalent definitions of the k-dimensional cube.* In the direct definition of Q_k, the vertices are the binary k-tuples, with edges consisting of pairs differing in one place. The inductive definition gives the same graph. For $k = 0$ both definitions specify K_1. For the induction step, suppose $k \ge 1$. The inductive definition uses two copies of Q_{k-1}, which by the induction hypothesis is the "1-place difference" graph of the binary $(k-1)$-tuples. If we append 0 to the $(k-1)$-tuples in one copy of Q_{k-1} and 1 to the $(k-1)$-tuples in the other copy, then within each set we still have edges between the labels differing in exactly one place. The inductive construction now adds edges consisting of corresponding vertices in the two copies. This is also what the direction definition does, since k-tuples chosen from the two copies differ in the last position and therefore differ in exactly one position if and only if they are the same in all other positions.

$e(Q_k) = k2^{k-1}$. By the inductive definition, $e(Q_k) = 2e(Q_{k-1}) + 2^{k-1}$ for $k \ge 1$, with $e(Q_0) = 0$. Thus the inductive step for a proof of the formula is $e(Q_k) = 2(k-1)2^{k-2} + 2^{k-1} = k_2^{k-1}$.

1.3.24. *$K_{2,3}$ is the smallest simple bipartite graph that is not a subgraph of the k-dimensional cube for any k.* Suppose the vectors x, y, a, b, c are the vertices of a copy of $K_{2,3}$ in Q_k. Any one of a, b, c differs from x in exactly one coordinate and from y in another (it can't be the same coordinate, because then $x = y$). This implies that x and y differ in two coordinate i, j. Paths from x to y in two steps can be formed by changing i and then j or changing j and then i; these are the only ways. In a cube two vertices have at most two common neighbors. Hence $K_{2,3}$ is forbidden. Any bipartite graph with fewer vertices or edges is contained in $K_{2,3} - e$ or $K_{1,5}$, but $K_{2,3} - e$ is a subgraph of Q_3, and $K_{1,5}$ is a subgraph of Q_5, so $K_{2,3}$ is the smallest forbidden subgraph.

1.3.25. *Every cycle of length $2r$ in a hypercube belongs to a subcube of dimension at most r, uniquely if $r \le 3$.* Let C be a cycle of length $2r$ in Q_k; $V(C)$ is a collection of binary vectors of length k. Let S be the set of coordinates that change at some step while traversing the vectors in $V(C)$. In order to return to the first vector, each position must flip between 0 and 1 an even number of times. Thus traversing C changes each coordinate in S at least twice, but only one coordinate changes with each edge. Hence $2|S| \le 2r$, or $|S| \le r$. Outside the coordinates of S, the vectors of $V(C)$ all agree. Hence $V(C)$ is contained in a $|S|$-dimensional subcube.

As argued above, at most two coordinates vary among the vertices of a 4-cycle; at least two coordinates vary, because otherwise there are not enough vectors available to have four distinct vertices. By the same reasoning, exactly three three coordinates vary among the vertices of any 6-cycle; we cannot find six vertices in a 2-dimensional subcube. Thus the r-dimensional subcube containing a particular cycle is unique when $r \le 3$.

Some 8-cycles are contained in 3-dimensional subcubes, such as $000x$, $001x$, $011x$, $010x$, $110x$, $111x$, $101x$, $100x$, where x is a fixed vector of length $n-3$. Such an 8-cycle is contained in $n-3$ 4-dimensional subcubes, obtained by letting some position in x vary.

1.3.26. *A 3-dimensional cube contains 16 6-cycles, and the k-dimensional cube Q_k contains $16\binom{k}{3}2^{k-3}$ 6-cycles.* If we show that every 6-cycle appears in exactly one 3-dimensional subcube, then multiplying the number of 3-dimensional subcubes by the number of 6-cycles in each subcube counts each 6-cycle exactly once.

For any set S of vertices not contained in a 3-dimensional subcube, there must be four coordinates in the corresponding k-tuples that are not constant within S. A cycle through S makes changes in four coordinates. Completing the cycle requires returning to the original vertex, so any coordinate that changes must change back. Hence at least eight changes are needed, and each edge changes exactly one coordinate. The cycle has length at least 8; hence 6-cycles are contained in 3-dimensional subcubes.

Furthermore, there are only four vertices possible when $k-2$ coordinates are fixed, so every 6-cycle involves changes in three coordinates. Hence the only 3-dimensional subcube containing the 6-cycle is the one that varies in the same three coordinates as the 6-cycle.

By Example 1.3.8, there are $\binom{k}{3}2^{k-3}$ 3-dimensional subcubes, so it remains only to show that Q_3 has 16 cycles of length 6. We group them by the two omitted vertices. The two omitted vertices may differ in 1, 2, or 3 coordinates. If they differ in one place (they are adjacent), then deleting them leaves a 6-cycle plus one edge joining a pair of opposite vertices. Since Q_3 has 12 edges, there are 12 6-cycles of this type. Deleting two complementary vertices (differing in every coordinate) leaves only a 6-cycle. Since Q_3 has four such pairs, there are four such 6-cycles. The remaining pairs differ in two positions. Deleting such a pair leaves a 4-cycle plus two pendant edges, containing no 6-cycle. This considers all choices for the omitted vertices, so the number of 6-cycles in Q_3 is $12+4$.

1.3.27. *Properties of the "middle-levels" graph.* Let G be the subgraph of Q_{2k+1} induced by vertices in which the numbers of 1s and 0s differs by 1. These are the $(2k+1)$-tuples of weight k and weight $k+1$, where **weight** denotes the number of 1s.

Each vertex of weight k has $k+1$ neighbors of weight $k+1$, and each vertex of weight $k+1$ has $k+1$ neighbors of weight k. There are $\binom{2k+1}{k}$ vertices of each weight. Counting edges by the Degree-Sum Formula,

$$e(G) = (k+1)\frac{n(G)}{2} = (k+1)\binom{2k+1}{k+1} = (2k+1)\binom{2k}{k}.$$

The graph is bipartite and has no odd cycle. The 1s in two vertices of weight k must be covered by the 1s of any common neighbor of weight $k+1$. Since the union of distinct k-sets has size at least $k+1$, there can only be one common neighbor, and hence G has no 4-cycle. On the other hand, G does have a 6-cycle. Given any arbitary fixed vector of weight $k-1$ for the last $2k-2$ positions, we can form a cycle of length six by using 110, 100, 101, 001, 011, 010 successively in the first three positions.

1.3.28. *Alternative description of even-dimensional hypercubes.* The simple graph Q_k' has vertex set $\{0,1\}^k$, with $u \leftrightarrow v$ if and only if u and v *agree* in exactly one coordinate. Let the *odd vertices* be the vertices whose name has an odd number of 1s; the rest are *even vertices*.

When k is even, $Q_k' \cong Q_k$. To show this, rename all odd vertices by changing 1s into 0s and 0s into 1s. Since k is even, the resulting labels are still odd. Since k is even, every edge in Q_k' joins an even vertex to an odd vertex. Under the new naming, it joins the even vertex to an odd vertex that differs from it in one coordinate. Hence the adjacency relation becomes precisely the adjacency relation of Q_k.

When k is odd, $Q_k' \ncong Q_k$, because Q_k' contains an odd cycle and hence is not bipartite. Starting from one vertex, form a closed walk by successively following k edges where each coordinate is the coordinate of agreement along exactly one of these edges. Hence each coordinate changes exactly $k-1$ times and therefore ends with the value it had at the start. Thus this is a closed walk of odd length and contains an odd cycle.

1.3.29. *Automorphisms of Q_k.*

a) A subgraph H of Q_k is isomorphic to Q_l if and only if it is the subgraph induced by a set of vertices agreeing in some set of $k-l$ coordinates. Let f be an isomorphism from H to Q_l, and let v be the vertex mapped to the vertex $\mathbf{0}$ of Q_l whose coordinates are all 0. Let $u_1, \ldots, u_l$ be the neighbors of v in H mapped to neighbors of $\mathbf{0}$ in Q_l by f. Each u_i differs from v in one coordinate; let S be the set of l coordinates where these vertices differ from v. It suffices to show that vertices of H differ from v only on the coordinates of S. This is immediate for $l \le 1$.

For $l \ge 2$, we prove that each vertex mapped by f to a vertex of Q_l having weight j differs from v in j positions of S, by induction on j. Let x be a vertex mapped to a vertex of weight j in Q_l. For $j \le 1$, we have already argued that x differs from v in j positions of S. For $j \ge 2$, let y and z be two neighbors of x whose images under f have weight $j-1$ in Q_l. By the induction hypothesis, y and z differ from v in j positions of S. Since $f(y)$ and $f(z)$ differ in two places, they have two common neighbors in Q_l, which are x and another vertex w. Since w has weight $j-2$, the induction hypothesis yields that w differs from v in $j-1$ positions of S. Since the images of x, y, z, w induce a 4-cycle in Q_l, also x, y, z, w induce a 4-cycle in H. The only 4-cycle in Q_k that contains all of y, z, w adds the vertex that differs from v in the $j-2$ positions of S where w differs, plus the two positions where y and z differ from w. This completes the proof that x has the desired property.

b) The k-dimensional cube Q_k has exactly $2^k k!$ automorphisms. (Part (a) is unnecessary.) Form automorphisms of Q_k by choosing a subset of the k coordinates in which to complement 0 and 1 and, independently, a permutation of the k coordinates. There are $2^k k!$ such automorphisms.

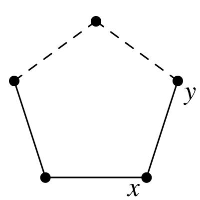
We prove that every automorphism has this form. Let $\mathbf{0}$ be the all-0 vertex. Let f be the inverse of an automorphism, and let v be the vertex mapped to $\mathbf{0}$ by f. The neighbors of v must be mapped to the neighbors of $\mathbf{0}$. If these choices completely determine f, then f complements the coordinates where v is nonzero, and the correspondence between the neighbors of $\mathbf{0}$ and the neighbors of v determines the permutation of the coordinates that expresses f as one of the maps listed above.

Suppose that x differs from v in coordinates $r_1, \ldots, r_j$. Let $u_1, \ldots, u_j$ be the neighbors of v differing from v in these coordinates. We prove that $f(x)$ is the k-tuple of weight j having 1 in the coordinates where $f(u_1), \ldots, f(u_j)$ have 1. We use induction on j.

For $j \le 1$, the claim follows by the definition of $u_1, \ldots, u_j$. For $j \ge 2$, let y and z be two neighbors of x that differ from v in $j-1$ coordinates. Let w be the common neighbor of y and z that differs from v in $j-2$ coordinates. By the induction hypothesis, $f(y)$ and $f(z)$ have weight $j-1$ (in the appropriate positions), and $f(w)$ has weight $j-1$. Since $f(x)$ must be the other common neighbor of $f(y)$ and $f(z)$, it has weight j, with 1s in the desired positions.

1.3.30. *The Petersen graph has twelve 5-cycles.* Let G be the Petersen graph. We show first that each edge of G appears in exactly four 5-cycles. For each edge $e = xy$ in G, there are two other edges incident to x and two others incident to y. Since G has no 3-cycles, we can thus extend xy at both ends to form a 4-vertex path in four ways. Since G has no 4-cycle, the endpoints of each such path are nonadjacent. By Proposition 1.1.38, there is exactly one vertex to add to such a path to complete a 5-cycle. Thus e is in exactly four 5-cycles.

When we sum this count over the 15 edges of G, we have counted 60 5-cycles. However, each 5-cycle has been counted five times—once for each of its edges. Thus the total number of 5-cycles in G is $60/5 = 12$.



1.3.31. *Combinatorial proofs with graphs.*

a) For $0 \le k \le n$, $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$. Consider the complete graph K_n, which has $\binom{n}{2}$ edges. If we partition the vertices of K_n into a k-set and an $(n-k)$-set, then we can count the edges as those within one block of the partition and those choosing a vertex from each. Hence the total number of edges is $\binom{k}{2} + \binom{n-k}{2} + k(n-k)$.

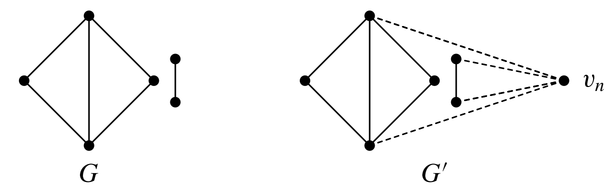
b) If $\sum n_i = n$, *then* $\sum \binom{n_i}{2} \le \binom{n}{2}$. Again consider the edges of K_n, and partition the vertices into sets with n_i being the size of the ith set. The left side of the inequality counts the edges in K_n having both ends in the same S_i, which is at most all of $E(K_n)$.

1.3.32. *For $n \ge 1$, there are $2^{\binom{n-1}{2}}$ simple even graphs with a fixed vertex set of size n.* Let A be the set of simple even graphs with vertex set $v_1, \ldots, v_n$. Since $2^{\binom{n-1}{2}}$ is the size of the set B of simple graphs with vertex set $v_1, \ldots, v_{n-1}$, we establish a bijection from A to B.

Given a graph in A, we obtain a graph in B by deleting v_n. To show that each graph in B arises exactly once, consider a graph $G \in B$. We form a new graph G' by adding a vertex v_n and making it adjacent to each vertex with odd degree in G, as illustrated below.

The vertices with odd degree in G have even degree in G'. Also, v_n itself has even degree because the number of vertices of odd degree in G is even. Thus $G' \in A$. Furthermore, G is the graph obtained from G' by deleting v_n, and every simple even graph in which deleting v_n yields G must have v_n adjacent to the same vertices as in G'.

Since there is a bijection from A to B, the two sets have the same size.



1.3.33. *Triangle-free graphs in which every two nonadjacent vertices have exactly two common neighbors.*

$n(G) = 1 + \binom{k+1}{2}$, *where k is the degree of a vertex x in G.* For every pair of neighbors of x, there is exactly one nonneighbor of x that they have as a common neighbor. Conversely, every nonneighbor of x has exactly one pair of neighbors of x in its neighborhood, because these are its common neighbors with x. This establishes a bijective correspondence between the pairs in $N(x)$ and the nonneighbors of x. Counting x, $N(x)$, and $\overline{N}(x)$, we have $n(G) = 1 + k + \binom{k}{2} = 1 + \binom{k+1}{2}$. Since this argument holds for every $x \in V(G)$, we conclude that G is k-regular.

Comment: Such graphs exist only for isolated values of k. Unique graphs exist for $k = 1, 2, 5$. Viewing the vertices as x, $N(x) = [k]$, and $\overline{N}(x) = \binom{[k]}{2}$, we have i adjacent to the pair $\{j, k\}$ if and only if $i \in \{j, k\}$. The lack of triangles guarantees that only disjoint pairs in $\binom{[k]}{2}$ can be adjacent, but each pair in $\binom{[k]}{2}$ must have exactly $k-2$ neighbors in $\binom{[k]}{2}$. For $k=5$, this implies that $\overline{N}(x)$ induces the 3-regular disjointness graph of $\binom{[5]}{2}$, which is the Petersen graph. Since the Petersen graph has girth 5 and diameter 2, each intersecting pair has exactly one common neighbor in $\overline{N}(x)$ in addition to its one common neighbor in $N(x)$, so this graph has the desired properties.

Numerical conditions eliminate $k \equiv 3 \pmod 4$, because G would be regular of odd degree with an odd number of vertices. There are stronger necessary conditions. After $k=5$, the next possibility is $k=10$, then 26, 37, 82, etc. A realization for $k=10$ is known to exist, but in general the set of realizable values is not known.

1.3.34. *If G is a kite-free simple n-vertex graph such that every pair of nonadjacent vertices has exactly two common neighbors, then G is regular.* Since nonadjacent vertices have common neighbors, G is connected. Hence it suffices to prove that adjacent vertices x and y have the same degree. To prove this, we establish a bijection from A to B, where $A = N(x) - N(y)$ and $B = N(y) - N(x)$.

Consider $u \in A$. Since $u \nleftrightarrow y$, there exists $v \in N(u) \cap N(y)$ with $v \neq x$. Since G is kite-free, $v \nleftrightarrow x$, so $v \in B$. Since x and v have common neighbors y and u, the vertex v cannot be generated in this way from another vertex of A. Hence we have defined an injection from A to B. Interchanging the roles of y and x yields an injection from B to A. Since these sets are finite, the injections are bijections, and $d(x) = d(y)$.

1.3.35. *If every induced k-vertex subgraph of a simple n-vertex graph G has the same number of edges, where $1 < k < n-1$, then G is a complete graph or an empty graph.*

a) If $l \ge k$ and G' is a graph on l vertices in which every induced k-vertex subgraph has m edges, then $e(G') = m\binom{l}{k}/\binom{l-2}{k-2}$. Counting the edges in all the k-vertex subgraphs of G' yields $m\binom{l}{k}$, but each edge appears in $\binom{l-2}{k-2}$ of these subgraphs, once for each k-set of vertices containing it. (Both sides of $\binom{l-2}{k-2}e(G') = m\binom{l}{k}$ count the ways to pick an edge of G' and a k-set of vertices in G' containing that edge. On the right, we pick the set first; on the left, we pick the edge first.)

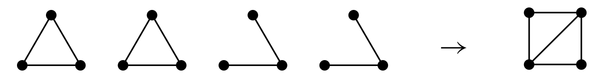
b) Under the stated conditions, $G = K_n$ or $G = \overline{K}_n$. Given vertices u and v, let A and B be the sets of edges incident to u and v, respectively. The set of edges with endpoints u and v is $A \cap B$. We compute

$$|A \cap B| = e(G) - \left|\overline{A \cap B}\right| = e(G) - \left|\overline{A} \cup \overline{B}\right| = e(G) - \left|\overline{A}\right| - \left|\overline{B}\right| + \left|\overline{A} \cap \overline{B}\right|.$$

In this formula, $\overline{A}$ and $\overline{B}$ are the edge sets of induced subgraphs of order $n-1$, and $\overline{A} \cap \overline{B}$ is the edge set of an induced subgraph of order $n-2$. By part (a), the sizes of these sets do not depend on the choice of u and v.

1.3.36. *The unique reconstruction of the graph with vertex-deleted subgraphs below is the kite.*

Proof 1. A vertex added to the first triangle may be joined to 0,1,2, or 3 of its vertices. We eliminate 0 and 1 because no vertex-deleted subgraph has an isolated vertex. We eliminate 3 because every vertex-deleted subgraph of K_4 is a triangle. Joining it to 2 yields the kite.



Proof 2. The graph G must have four vertices, and by Proposition 1.3.11 it has five edges. The only such simple graph is the kite.

1.3.37. *Retrieving a regular graph.* Suppose that H is a graph formed by deleting a vertex from a regular graph G. We have H, so we know $n(G) = n(H) + 1$, but we don't know the vertex degrees in G. If G is d-regular, then G has $dn(G)/2$ edges, and H has $dn(G)/2 - d$ edges. Thus $d = 2e(H)/(n(G)-2)$. Having determined d, we add one vertex w to H and add $d - d_H(v)$ edges from w to v for each $v \in V(H)$.

1.3.38. *A graph with at least 3 vertices is connected if and only if at least two of the subgraphs obtained by deleting one vertex are connected.* The endpoints of a maximal path are not cut-vertices. If G is connected, then the subgraphs obtained by deleted such vertices are connected, and there are at least of these.

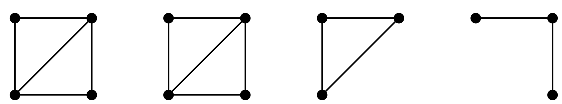
Conversely, suppose that at least two vertex-deleted subgraphs are connected. If $G - v$ is connected, then G is connected unless v is an isolated vertex. If v is an isolated vertex, then all the other subgraphs obtained by deleting one vertex are disconnected. Hence v cannot be isolated, and G is connected.

1.3.39. *Disconnected graphs are reconstructible.* First we show that G is connected if and only if it has at least two connected vertex-deleted subgraphs. Necessity holds, because the endpoints of a maximal path cannot be cut-vertices. If G is disconnected, then $G - v$ is disconnected unless v is an isolated vertex (degree 0) in G and $G - v$ is connected. This happens for at most one vertex in G.

After determining that G is disconnected, we obtain which disconnected graph it is from its vertex-deleted subgraphs. We aim to identify a connected graph M that is a component of G and a vds in the deck that arises by deleting a specified vertex u of M. Replacing $M - u$ by M in that subgraph will reconstruct G.

Among all components of all graphs in the deck, let M be one with maximum order. Since every component H of a potential reconstruction G appears as a component of some $G-v$, M cannot belong to any larger component of G. Hence M is a component of G. Let L be a fixed connected subgraph of M obtained by deleting a leaf u of some spanning tree of M. Then L is a component of $G-u$. We want to reconstruct G by substituting M for L in $G-u$; we must identify $G-u$. There may be several isomorphic copies of $G-u$.

As in the disconnected graph G shown above, M may appear as a component of every vds $G-v$. However, since M cannot be created by a vertex deletion, a vds with the fewest copies of M must arise by deleting a vertex of M. Among these, we seek a subgraph with the most copies of L as components, because in addition to occurrences of L as a component of G, we obtain an additional copy if and only if the deleted vertex of M can play the role of u. This identifies $G-u$, and we obtain G by replacing one of its components isomorphic to L with a component isomorphic to M.



1.3.40. *Largest graphs of specified types.*

a) Largest n-vertex simple graph with an independent set of size a.

Proof 1. Since there are no edges within the independent set, such a graph has at most $\binom{n}{2}-\binom{a}{2}$ edges, which equals $\binom{n-a}{2}+(n-a)a$. This bound is achieved by the graph consisting of a copy H of K_{n-a}, an independent set S of size a, and edges joining each vertex of H to each vertex of S.

Proof 2. Each vertex of an independent set of size a has degree at most $n-a$. Each other vertex has degree at most $n-1$. Thus $\sum d(v)\le a(n-a)+(n-a)(n-1)$. By the Degree-Sum Formula, $e(G)\le(n-a)(n-1+a)/2$. This formula equals those above and is achieved by the same graph, since this graph achieves the bound for each vertex degree.

b) The maximum size of an n-vertex simple graph with k components is $\binom{n-k+1}{2}$. The graph consisting of K_{n-k+1} plus $k-1$ isolated vertices has k components and $\binom{n-k+1}{2}$ edges. We prove that other n-vertex graphs with k components don't have maximum size. Let G be such a graph.

If G has a component that is not complete, then adding edges to make it complete does not change the number of components. Hence we may assume that every component is complete.

If G has components with r and s vertices, where $r\ge s>1$, then we move one vertex from the s-clique to the r-clique. This deletes $s-1$ edges and creates r edges, all incident to the moved vertex. The other edges remain the same, so we gain $r-s+1$ edges, which is positive.

Thus the number of edges is maximized only when every component is a complete graph and only one component has more than one vertex.

c) The maximum number of edges in a disconnected simple n-vertex graph is $\binom{n-1}{2}$, with equality only for K_1+K_{n-1}.

Proof 1 (using part (b)). The maximum over graphs with k components is $\binom{n-k+1}{2}$, which decreases as k increases. For disconnected graphs, $k\ge 2$. We maximize the number of edges when $k=2$, obtaining $\binom{n-1}{2}$.

Proof 2 (direct argument). Given a disconnected simple graph G, let S be the vertex set of one component of G, and let $t=|S|$. Since no edges join S and $\overline{S}$, $e(G)\le\binom{n}{2}-t(n-t)$. This bound is weakest when $t(n-t)$ is smallest, which for $1\le t\le n-1$ happens when $t\in\{1,n-1\}$. Thus always $e(G)\le\binom{n}{2}-1(n-1)=\binom{n-1}{2}$, and equality holds when $G=K_1+K_{n-1}$.

Proof 3 (induction on n). When $n=2$, the only simple graph with $e(G)>\binom{1}{2}=1$ is K_2, which is connected. For $n>2$, suppose $e(G)>\binom{n-1}{2}$. If $\Delta(G)=n-1$, then G is connected. Otherwise, we may select v with $d(v)\le n-2$. Then $e(G-v)>\binom{n-1}{2}-n+2=\binom{n-2}{2}$. By the induction hypothesis, $G-v$ is connected. Since $e(G)>\binom{n-1}{2}$ and G is simple, we have $d(v)>0$, so there is an edge from v to $G-v$, and G is also connected.

Proof 4 (complementation). If G is disconnected, then $\overline{G}$ is connected, so $e(\overline{G})\ge n-1$ and $e(G)\le\binom{n}{2}-(n-1)=\binom{n-1}{2}$. In fact, $\overline{G}$ must contain a spanning complete bipartite subgraph, which is as small as $n-1$ edges only when $\overline{G}=K_{1,n-1}$ and $G=K_1+K_{n-1}$.

1.3.41. *Every n-vertex simple graph with maximum degree $\lceil n/2\rceil$ and minimum degree $\lfloor n/2\rfloor-1$ is connected.* Let x be a vertex of maximum degree. It suffices to show that every vertex not adjacent to x has a common neighbor with x. Choose $y\notin N(x)$. We have $|N(x)|=\lceil n/2\rceil$ and $|N(y)|\ge\lfloor n/2\rfloor-1$. Since $y\nleftrightarrow x$, we have $N(x),N(y)\subseteq V(G)-\{x,y\}$. Thus

$$|N(x)\cap N(y)|=|N(x)|+|N(y)|-|N(x)\cup N(y)|\ge\lceil n/2\rceil+\lfloor n/2\rfloor-1-(n-2)=1.$$

1.3.42. *Strongly independent sets.* If S is an independent set with no common neighbors in a graph G, then the vertices of S have pairwise-disjoint closed neighborhoods of size at least $\delta(G)+1$. Thus there are at most $\lfloor n(G)/(\delta(G)+1)\rfloor$ of them. Equality is achievable for the 3-dimensional cube using $S=\{000,111\}$.

Equality is not achievable when $G=Q_4$, since with 16 vertices and minimum degree 4 it requires three parwise-disjoint closed neighborhoods of size 5. If $v\in S$, then no vertex differing from v in at most two places is in S. Also, at most one vertex differing from v in at least three places is in

S, since such vertices differ from each other in at most two places. Thus only two disjoint closed neighborhoods can be found in Q_4.

1.3.43. *Every simple graph has a vertex whose neighbors have average degree as large as the overall average degree.* Let $t(w)$ be the average degree of the neighbors of w. In the sum $\sum_{w\in V(G)} t(w) = \sum_{w\in V(G)}\sum_{y\in N(w)} d(y)/d(w)$, we have the terms $d(u)/d(v)$ and $d(v)/d(u)$ for each edge uv. Since $x/y + y/x \ge 2$ whenever x, y are positive real numbers (this is equivalent to $(x-y)^2 \ge 0$), each such contribution is at least 2. Hence $\sum t(w) \ge \sum_{uv\in E(G)} \frac{d(u)}{d(v)} + \frac{d(v)}{d(u)} \ge 2e(G)$. Hence the average of the neighborhood average degrees is at least the average degree, and the pigeonhole principle yields the desired vertex.

It is possible that every average neighborhood degree exceeds the average degree. Let G be the graph with $2n$ vertices formed by adding a matching between a complete graph and an independent set. Since G has $\binom{n}{2} + n$ edges and $2n$ vertices, G has average degree $(n+1)/2$. For each vertex of the n-clique, the neighborhood average degree is $n - 1 + 1/n$. For each leaf, the neighborhood average degree is n.

1.3.44. *Subgraphs with large minimum degree.* Let G be a loopless graph with average degree a.

a) If $x \in V(G)$, then $G' = G - x$ has average degree at least a if and only if $d(x) \le a/2$. Let a' be the average degree of G', and let n be the order of G. Deleting x reduces the degree sum by $2d(x)$, so $(n-1)a' = na - 2d(x)$. Hence $(n-1)(a'-a) = a - 2d(x)$. For $n > 1$, this implies that $a' \ge a$ if and only if $d(x) \le a/2$.

Alternative presentation. The average degree of G is $2e(G)/n(G)$. Since G' has $e(G) - d(x)$ edges, the average degree is at least a if and only if $\frac{2[e(G)-d(x)]}{n(G)-1} \ge a$. Since $e(G) = n(G)a/2$, we can rewrite this as $n(G)a - 2d(x) = 2e(G) - 2d(x) \ge an(G) - a$. By canceling $n(G)a$, we find that the original inequality is equivalent to $d(x) \le a/2$.

b) If $a > 0$, then G has a subgraph with minimum degree greater than $a/2$. Iteratively delete vertices with degree at most half the current average degree, until no such vertex exists. By part (a), the average degree never decreases. Since G is finite, the procedure must terminate. It ends only by finding a subgraph where every vertex has degree greater than $a/2$.

c) The result of part (b) is best possible. To prove that no fraction of a larger than $\frac{1}{2}a$ can be guaranteed, let G_n be an n-vertex tree. We have $a(G_n) = 2(n-1)/n = 2 - 2/n$, but subgraphs of G_n have minimum degree at most 1. Given $\beta > \frac{1}{2}$, we can choose n large enough so that $1 \le \beta a(G_n)$.

1.3.45. *Bipartite subgraphs of the Petersen graph.*

a) Every edge of the Petersen graph is in four 5-cycles. In every 5-cycle through an edge e, the edge e is the middle edge of a 4-vertex path. Such

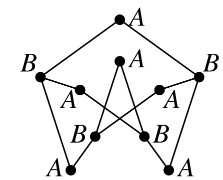
a path can be obtained in four ways, since each edge extends two ways at each endpoint. The neighbors at each endpoint of e are distinct and nonadjacent, since the girth is 5.

Since the endpoints of each such P_4 are nonadjacent, they have exactly one common neighbor. Thus each P_4 yields one 5-cycle, and each 5-cycle through e arises from such a P_4, so there are exactly four 5-cycles containing each edge.

b) The Petersen graph has twelve 5-cycles. Since there are 15 edges, summing the number of 5-cycles through each edge yields 60. Since each 5-cycle is counted five times in this total, the number of 5-cycles is 12.

c) The largest bipartite subgraph has twelve edges.

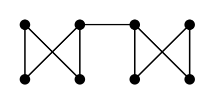
Proof 1 (breaking odd cycles). Each edge is in four 5-cycles, so we must delete at least 12/4 edges to break all 5-cycles. Hence we must delete at least three edges to have a bipartite subgraph. The illustration shows that deleting three is enough; the Petersen graph has a bipartite subgraph with 12 edges (see also the cover of the text).



Proof 2 (study of bipartite subgraphs). The Petersen graph G has an independent set of size 4, consisting of the vertices $\{ab, ac, ad, ae\}$ in the structural description. The 12 edges from these four vertices go to the other six vertices, so this is a bipartite subgraph with 12 edges.

Let X and Y be the partite sets of a bipartite subgraph H. If $|X| \le 4$, then $e(H) \le 12$, with equality only when X is an independent 4-set in G. Hence we need only consider the case $|X| = |Y| = 5$. To obtain $e(G) > 10$, some vertex $x \in X$ must have three neighbors in Y. The two nonneighbors of x in Y have common neighbors with x, and these must lie in $N(x)$, which is contained in Y. Hence $e(G[Y]) \ge 2$. Interchanging X and Y in the argument shows that also $e(G[X]) \ge 2$. Hence $e(H) \le 11$.

1.3.46. *When the algorithm of Theorem 1.4.2 is applied to a bipartite graph, it need not find the bipartite subgraph with the most edges.* For the bipartite graph below, the algorithm may reach the partition between the upper vertices and lower vertices.



This bipartite subgraph with eight edges has more than half of the edges at each vertex, and no further changes are made. However, the bipartite subgraph with the most edges is the full graph.

1.3.47. *Every nontrivial loopless graph G has a bipartite subgraph containing more than half its edges.* We use induction on $n(G)$. If $n(G) = 2$, then G consists of copies of a single edge and is bipartite. For $n(G) > 2$, choose $v \in V(G)$ that is not incident to all of $E(G)$ (at most two vertices can be incident to all of $E(G)$). Thus $e(G - v) > 0$. By the induction hypothesis, $G - v$ has a bipartite subgraph H containing more than $e(G)/2$ edges.

Let X, Y be a bipartition of H. If X contains at least half of $N_G(v)$, then add v to Y; otherwise add v to X. The augmented partition captures a bipartite subgraph of G having more than half of $E(G - v)$ and at least half of the remaining edges, so it has more than half of $E(G)$.

Comment. The statement can also be proved without induction. By Theorem 1.3.19, G has a bipartite subgraph H with at least $e(G)/2$ edges. By the proof of Theorem 1.3.19, equality holds only if $d_H(v) = d_G(v)/2$ for every $v \in V(G)$. Given an edge uv, each of u and v has exactly half its neighbors in its own partite set. Switching both to the opposite set will capture those edges while retaining the edge uv, so the new bipartite subgraph has more edges.

1.3.48. *No fraction of the edges larger than* $1/2$ *can be guaranteed for the largest bipartite subgraph.* If G_n is the complete graph K_{2n}, then $e(G_n) = \binom{2n}{2} = n(2n - 1)$, and the largest bipartite subgraph is $K_{n,n}$, which has n^2 edges. Hence $\lim_{n\to\infty} f(G_n)/e(G_n) = \lim_{n\to\infty} \frac{n^2}{2n^2-n} = \frac{1}{2}$. For large enough n, the fraction of the edges in the largest bipartite subgraph is arbitrarily close to $1/2$. (In fact, in every graph the largest bipartite subgraph has **more** than half the edges.)

1.3.49. *Every loopless graph G has a spanning k-partite subgraph H such that $e(H) \ge (1 - 1/k)e(G)$.*

Proof 1 (local change). Begin with an arbitrary partition of $V(G)$ into k parts $V_1, \ldots, V_k$, and consider the k-partite subgraph H containing all edges of G consisting of two vertices from distinct parts. Given a partition of $V(G)$, let $V(x)$ denote the part containing x. If in G some vertex x has more neighbors in V_j than in some other part, then shifting x to the other part increases the number of edges captured by the k-partite subgraph.

Since G has finitely many edges, this shifting process must terminate. It terminates when for each $x \in V(G)$ the number $|N(x) \cap V_i|$ is minimized by $V_i = V(x)$. Then $d_G(x) = \sum_i |N_G(x) \cap V_i| \ge k|N_G(x) \cap V(x)|$. We conclude that $|N_G(x) \cap V(x)| \le (1/k)d_G(x)$, and hence $d_H(x) \ge (1 - 1/k)d_G(x)$ for all $x \in V(G)$. By the degree-sum formula, $e(H) \ge (1 - 1/k)e(G)$.

Proof 2 (induction on n). We prove that when G is nontrivial, some such H has more than $(1 - 1/k)e(G)$ edges. This is true when $n = 2$. We procede by induction for $n > 2$. Choose a vertex $v \in V(G)$. By the induction hypothesis, $G - v$ has a spanning k-partite subgraph with more than $(1 - 1/k)e(G - v)$ edges. This subgraph partitions $V(G - v)$ into k partite sets. One of these sets contains at most $1/k$ neighbors of v. Add v to that set to obtain the desired k-partite subgraph H. Now $e(H) > (1 - 1/k)e(G - v) + (1 - 1/k)d_G(v) = (1 - 1/k)e(G)$.

1.3.50. For $n \ge 3$, the minimum number of edges in a connected n-vertex graph in which every edge belongs to a triangle is $\lceil 3(n - 1)/2 \rceil$. To achieve the minimum, we need only consider simple graphs. Say that connected graphs with each edge in a triangle are *good* graphs. For $n = 3$, the only such graph is K_3, with three edges.

When n is odd, a construction with the claimed size consists of $(n-1)/2$ triangles sharing a common vertex. When n is even, add one vertex to the construction for $n - 1$ and make it adjacent to both endpoints of one edge.

For the lower bound, let G be a smallest n-vertex good graph. Since G has fewer than $3n/2$ edges (by the construction), G has a vertex v of degree 2. Let x and y be its neighbors. Since each edge belongs to a triangle, $x \leftrightarrow y$. If $n > 3$, then we form G' by deleting v and, if xy have no other neighbor, contracting xy. Every edge of G' belongs to a triangle that contained it in G. The change reduces the number of vertices by 1 or 2 and reduces the number of edges by at least $3/2$ times the reduction in the number of vertices. By the induction hypothesis, $e(G') \ge \lceil 3(n(G') - 1)/2 \rceil$, and hence the desired bound holds for G.

1.3.51. Let G be a simple n-vertex graph.

a) $e(G) = \frac{\sum_{v\in V(G)} e(G-v)}{n-2}$. If we count up all the edges in all the subgraphs obtained by deleting one vertex, then each edge of G is counted exactly $n-2$ times, because it shows up in the $n - 2$ subgraphs obtained by deleting a vertex other than its endpoints.

b) If $n \ge 4$ and G has more than $n^2/4$ edges, then G has a vertex whose deletion leaves a graph with more than $(n - 1)^2/4$ edges. Since G has more than $n^2/4$ edges and $e(G)$ is an integer, we have $e(G) \ge (n^2 + 4)/4$ when n is even and $e(G) \ge (n^2 + 3)/4$ when n is odd (since $(2k + 1)^2 = 4k^2 + 4k + 1$, every square of an odd number is one more than a multiple of 4). Thus always we have $e(G) \ge (n^2 + 3)/4$.

By part (a), we have $\sum_{v\in V(G)} \frac{e(G-v)}{n-2} \ge (n^2 + 3)/4$. In the sum we have n terms. Since the largest number in a set is at least the average, there is a vertex v such that $\frac{e(G-v)}{n-2} \ge \frac{1}{n}\frac{n^2+3}{4}$. We rewrite this as

$$e(G - v) \ge \frac{(n^2 + 3)(n - 2)}{4n} = \frac{n^3 - 2n^2 + 3n - 6}{4n} = \frac{n^2 - 2n + 1}{4} + \frac{2n - 6}{4n}$$

When $n \geq 4$, the last term is positive, and we obtain the strict inequality $e(G - v) > (n-1)^2/4$.

c) Inductive proof that G contains a triangle if $e(G) > n^2/4$. We use induction on n. When $n \leq 3$, they only simple graph with more than $n^2/4$ edges is when $n = 3$ and $G = K_3$, which indeed contains a triangle. For the induction step, consider $n \geq 4$, and let G be an n-vertex simple graph with more than $n^2/4$ vertices. By part (b), G has a subgraph $G - v$ with $n-1$ vertices and more than $(n-1)^2/4$ edges. By the induction hypothesis, $G - v$ therefore contains a triangle. This triangle appears also in G.

1.3.52. *$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is the only n-vertex triangle-free graph of maximum size.* As in the proof of Mantel's result, let x be a vertex of maximum degree. Since $N(x)$ is an independent set, x and its non-neighbors capture all the edges, and we have $e(G) \leq (n - \Delta(G))\Delta(G)$. If equality holds, then summing the degrees in $V(G) - N(x)$ counts each edge exactly once. This requires that $V(G) - N(x)$ also is an independent set, and hence G is bipartite. If G is bipartite and has $(n-\Delta(G))\Delta(G)$ edges, then $G = K_{(n-\Delta(G)),\Delta(G)}$. Hence $e(G)$ is maximized only by $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.

1.3.53. *The bridge club with 14 members (no game can be played if two of the four people table have previously been partners): If each member has played with four others and then six additional games have been played, then the arrival of a new member allows a game to be played.* We show that the new player yields a set of four people among which no two have been partners. This is true if and only if the previous games must leave three people (in the original 14) among which no two have been partners.

The graph of pairs who have NOT been partners initially is K_{14}. For each game played, two edges are lost from this graph. At the breakpoint in the session, each vertex has lost four incident edges, so 28 edges have been deleted. In the remaining six games, 12 more edges are deleted. Hence 40 edges have been deleted. Since $e(K_{14}) = 91$, there remain 51 edges for pairs that have not yet been partners.

By Mantel's Theorem (Theorem 1.3.23), the maximum number of edges in a simple 14-vertex graph with no triangle is $\lfloor 14^2/4 \rfloor$. Since $51 > 49$, the graph of remaining edges has a triangle. Thus, when the 15th person arrives, there will be four people of whom none have partnered each other.

1.3.54. *The minimum number of triangles $t(G)$ in an n-vertex graph G and its complement.*

a) $t(G) = \binom{n}{3} - (n-2)e + \sum_{v\in V(G)} \binom{d(v)}{2}$. Let $d_1, \ldots, d_n$ denote the vertex degrees. We prove that the right side of the formula assigns weight 1 to the vertex triples that induce a triangle in G or $\overline{G}$ and weight 0 to all other triples. Among these terms, $\binom{n}{3}$ counts all triples, $(n-2)e$ counts those determined by an edge of G and a vertex off that edge, and $\sum \binom{d_i}{2}$ counts 1 for each pair of incident edges. In the table below, we group these contributions by how many edges the corresponding triple induces in G.

$t(G)$	in G	$\binom{n}{3}$	$-(n-2)e$	$\sum\binom{d_i}{2}$
1	3 edges	1	-3	3
0	2 edges	1	-2	1
0	1 edge	1	-1	0
1	0 edges	1	-0	0

b) $t(G) \geq n(n-1)(n-5)/24$. Begin with the formula for $k_3(G) + k_3(\overline{G})$ from part (a). Using the convexity of quadratic functions, we get a lower bound for the sum on the right by replacing the vertex degrees by the average degree $2e/n$. The bound is $\binom{n}{3} - (n-2)e + n\binom{2e/n}{2}$, which reduces to $\binom{n}{3} - 2e(\binom{n}{2} - e)/n$. As a function of e, this is minimized when $e = \frac{1}{2}\binom{n}{2}$. This substitution and algebraic simplification produce $t(G) \geq n(n-1)(n-5)/24$.

Comment. The proof of part (b) uses two minimizations. These imply that equality can hold only for a regular graph ($d_i = 2e/n$ for all i) with $e = \frac{1}{2}\binom{n}{2}$. There is such a regular graph if and only if n is odd and $(n-1)/2$ is even. Thus we need $n = 4k+1$ and G is $2k$-regular.

1.3.55. *Maximum size with no induced P_4. a) If G is a simple connected graph and $\overline{G}$ is disconnected, then $e(G) \leq \Delta(G)^2$, with equality only for $K_{\Delta(G),\Delta(G)}$.* Since $\overline{G}$ is disconnected, $\Delta(G) \geq n(G)/2$, with equality only if $G = K_{\Delta(G),\Delta(G)}$. Thus $e(G) = \sum d_i/2 \leq n(G)\Delta(G)/2 \leq \Delta(G)^2$. As observed, equality when $\overline{G}$ is disconnected requires $G = K_{\Delta(G),\Delta(G)}$.

b) If G is a simple connected graph with maximum degree D and no induced subgraph isomorphic to P_4, then $e(G) \leq D^2$. It suffices by part (a) to prove that $\overline{G}$ is disconnected when G is connected and P_4-free. We use induction on $n(G)$ for $n(G) \geq 2$; it is immediate when $n(G) = 2$. For the induction step, let v be a non-cut-vertex of G. The graph $G' = G - v$ is also P_4-free, so its complement is disconnected, by the induction hypothesis. Thus $V(G) - v$ has a vertex partition X, Y such that all of X is adjacent to all of Y in G. Since G is connected, v has a neighbor $z \in X \cup Y$; we may assume be symmetry that $z \in Y$. If $\overline{G}$ is connected, then $\overline{G}$ has a v, z-path. Let y be the vertex before z on this path; note that $y \in Y$. Also $\overline{G}$ connected requires $x \in X$ such that $vx \in E(\overline{G})$. Now $\{v, z, x, y\}$ induces P_4 in G.

1.3.56. *Inductive proof that for $\sum d_i$ even there is a multigraph with vertex degrees $d_1, \ldots, d_n$.*

Proof 1 (induction on $\sum d_i$). If $\sum d_i = 0$, then all d_i are 0, and the n-vertex graph with no edges has degree list d. For the induction step, suppose $\sum d_i > 0$. If only one d_i is nonzero, then it must be even, and the graph consisting of $n-1$ isolated vertices plus $d_i/2$ loops at one vertex has degree list d (multigraphs allow loops).

Otherwise, d has at least two nonzero entries, d_i and d_j. Replacing these with $d_i - 1$ and $d_j - 1$ yeilds a list d' with smaller even sum. By the induction hypothesis, some graph G' with degree list d'. Form G by adding an edge with endpoints u and v to G', where $d_{G'}(u) = d_i - 1$ and $d_{G'}(v) = d_j - 1$. Although u and v may already be adjacent in G', the resulting multigraph G has degree list d.

Proof 2 (induction on n). For $n = 1$, put $d_1/2$ loops at v_1. If d_n is even, put $d_n/2$ loops at v_n and apply the induction hypothesis. Otherwise, put an edge from v_n to some other vertex corresponding to positive d_i (which exists since $\sum d_i$ is even) and proceed as before.

1.3.57. *An n-tuple of nonnegative integers with largest entry k is graphic if the sum is even, $k < n$, and every entry is k or $k-1$.* Let $A(n)$ be the set of n-tuples satisfying these conditions. Let $B(n)$ be the set of graphic n-tuples. We prove by induction on n that n-tuples in $A(n)$ are also in $B(n)$. When $n = 1$, the only list in $A(n)$ is (0), and it is graphic.

For the induction step, let d be an n-tuple in $A(n)$, and let k be its largest element. Form d' from d by deleting a copy of k and subtracting 1 from k largest remaining elements. The operation is doable because $k < n$. To apply the induction hypothesis, we need to prove that $d' \in A(n-1)$. Since we delete an instance of k and subtract one from k other values, we reduce the sum by $2k$ to obtain d' from d, so d' does have even sum.

Let q be the number of copies of k in d. If $q > k+1$, then d' has ks and $(k-1)$s. If $q = k+1$, then d' has only $(k-1)$s. If $q < k+1$, then d' has $(k-1)$s and $(k-2)$s. Also, if $k = n-1$, then the first possibility cannot occur. Thus d' has length $n-1$, its largest value is less than $n-1$, and its largest and smallest values differ by at most 1. Thus $d' \in A(n-1)$, and we can apply the induction hypothesis to d'.

The induction hypothesis $(d' \in A(n-1)) \Rightarrow (d' \in B(n-1))$ tells us that d' is graphic. Now the Havel-Hakimi Theorem implies that d is graphic. (Actually, we use only the easy part of the HH Theorem, adding a vertex joined to vertices with desired degrees.)

1.3.58. *If d is a nonincreasing list of nonnegative integers, and d' is obtained by deleting d_k and subtracting 1 from the k largest other elements, then d is graphic if and only if d' is graphic.* The proof is like that of the Havel–Hakimi Theorem. Sufficiency is immediate. For necessity, let w be a vertex of degree d_k in a simple graph with degree sequence d. Alter G by 2-switches to obtain a graph in which w has the d_k highest-degree other vertices as neighbors. The argument to find a 2-switch increasing the number of desired neighbors of w is as in the proof of the Havel–Hakimi Theorem.

1.3.59. *The list $d = (d_1, \ldots, d_{2k})$ with $d_{2i} = d_{2i-1} = i$ for $1 \le i \le k$ is graphic.* This is the degree list for the bipartite graph with vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ defined by $x_r \leftrightarrow y_s$ if and only if $r+s > k$. Since the neighborhood of x_r is $\{y_k, y_{k-1}, \ldots, y_{k-r+1}\}$, the degree of x_r is r. Thus the graph has two vertices of each degree from 1 to k.

1.3.60. *Necessary and sufficient conditions for a list d to be graphic when d consists of k copies of a and $n-k$ copies of b, with $a \ge b \ge 0$.* Since the degree sum must be even, the quantity $ka + (n-k)b$ must be even. In addition, the inequality $ka \le k(k-1) + (n-k)\min\{k, b\}$ must hold, since each vertex with degree b has at most $\min\{k, b\}$ incident edges whose other endpoint has degree a. We construct graphs with the desired degree sequence when these conditions hold. Note that the inequality implies $a \le n-1$.

Case 1: *$b \ge k$ and $a \ge n-k$.* Begin with $K_{k,n-k}$, having partite sets X of size k and Y of size $n-k$. If $k(a-n+k)$ and $(n-k)(b-k)$ are even, then add an $(a-n+k)$-regular graph on X and a $(b-k)$-regular graph on Y. To show that this is possible, note first that $0 \le a-n+k \le k-1$ and $0 \le b-k \le a-k \le n-k-1$. Also, when pq is even, a q-regular graph on p vertices in a circle can be constructed by making each vertex adjacent to the $\lfloor q/2 \rfloor$ nearest vertices in each direction and also to the opposite vertex if q is odd (since then p is even).

Note that $k(a-n+k)$ and $(n-k)(b-k)$ have the same parity, since their difference $ak-(n-k)b$ differs from the given even number $ka+(n-k)b$ by an even amount. If they are both odd, then we delete one edge from $K_{k,n-k}$, and now one vertex in the subgraph on X should have degree $a-n+k+1$ and one in the subgraph on Y should have degree $b-k+1$. When pq is odd, such a graph on vertices $v_0, \ldots, v_{p-1}$ in a circle (q-regular except for one vertex of degree $q+1$) can be constructed by making each vertex adjacent to the $(q-1)/2$ nearest vertices in each direction and then adding the edges $\{v_i v_{i+(p-1)/2}$: $0 \le i \le (p-1)/2$. Note that all vertices are incident to one of the added edges, except that $v_{(p-1)/2}$ is incident to two of them.

Case 2: $k-1 \le a < n-k$. Begin by placing a complete graph on a set S of k vertices. These vertices now have degree $k-1$ and will become the vertices of degree a, which is okay since $a \ge b$. Put a set T of $n-k$ additional vertices in a circle. For each vertex in S, add $a-k+1$ consecutive neighbors in T, starting the next set immediately after the previous set ends. Since $a \le n-1$, each vertex in S is assigned $a-k+1$ distinct neighbors in T. Since $k(a-k+1) \le (n-k)b$ and the edges are distributed nearly equally to vertices of T, there is room to add these edges.

For the subgraph induced by T, we need a graph with $n-k$ vertices and $[(n-k)b - k(a-k+1)]/2$ edges and degrees differing by at most 1. The desired number of edges is integral, since $ka + (n-k)b$ is even, and it is nonnegative, since $k(a-k+1) \le (n-k)b$. The largest degree needed is $\lceil (n-k)b - k(a-k+1) \rceil\, n-k$. This is at most b, which is less than $n-k$ since $b \le a < n-k$. The desired graph now exists by Exercise 1.3.57.

Case 3: $b < k$ *and* $a \ge n-k$. Put the set S of size k in a circle. For each vertex in the set T of size $n-k$, assign b consecutive neighbors in S; these are distinct since $b < k$. Since $a \ge n-k$, no vertex of S receives too many edges. On S we put an almost-regular graph with k vertices and $[ak - b(n-k)]/2$ edges. Again, this number of edges is integral, and in the case specified it is nonnegative. Existence of such a graph requires $a - b(n-k)/k \le k-1$, which is equivalent to the given inequality $k(a-k+1) \le (n-k)b$. Now again Exercise 1.3.57 provides the needed graph.

Case 4: $b < k$ *and* $a < \min\{k-1, n-k\}$. Since $a < n-k$, also $b < n-k$. Therefore, we can use the idea of Case 1 without the complete bipartite graph. Again take disjoint vertex sets X of size k and Y of size $n-k$. If ka and $(n-k)b$ are even, then we use an a-regular graph on X and a b-regular graph on Y. As observed before, these exist.

Note that ka and $(n-k)b$ have the same parity, since their sum is given to be even. If they are both odd, then we put $\min\{k, n-k\}$ disjoint edges with endpoints in both X and Y. We now complete the graph with a regular graph of even degree on one of these sets and an almost-regular graph guaranteed by Exercise 1.3.57 on the other.

1.3.61. *If G is a self-complementary n-vertex graph and n is odd, then G has a vertex of degree $(n-1)/2$.* Let $d_1, \dots, d_n$ be the degree list of G in nonincreasing order. The degree list of $\overline{G}$ in nonincreasing order is $n-1-d_n, \dots, n-1-d_1$. Since $G \cong \overline{G}$, the lists are the same. Since n is odd, the central elements in the list yield $d_{(n+1)/2} = n-1-d_{(n+1)/2}$, so $d_{(n+1)/2} = (n-1)/2$.

1.3.62. *When n is congruent to 0 or 1 modulo 4, there is an n-vertex simple graph G with $\frac{1}{2}\binom{n}{2}$ edges such that $\Delta(G) - \delta(G) \le 1$.* This is satisfied by the construction given in the answer to Exercise 1.1.31.

More generally, let G be any $2k$-regular simple graph with $4k+1$ vertices, where $n = 4k+1$. Such a graph can be constructed by placing $4k+1$ vertices around a circle and joining each vertex to the k closest vertices in each direction. By the Degree-Sum Formula, $e(G) = (4k+1)(2k)/2 = \frac{1}{2}\binom{n}{2}$.

For the case where $n = 4k$, delete one vertex from the graph constructed above to form G'. Now $e(G') = e(G) - 2k = (4k-1)(2k)/2 = \frac{1}{2}\binom{n}{2}$.

1.3.63. *The non-negative integers $d_1 \ge \cdots \ge d_n$ are the vertex degrees of a loopless graph if and only if $\sum d_i$ is even and $d_1 \le d_2 + \cdots + d_n$. Necessity.* If such a graph exists, then $\sum d_i$ counts two endpoints of each edge and must be even. Also, every edge incident to the vertex of largest degree has its other end counted among the degrees of the other vertices, so the inequality holds.

Sufficiency. Specify vertices $v_1, \dots, v_n$ and construct a graph so that $d(v_i) = d_i$. Induction on n has problems: It is not enough to make d_n edges join v_1 and v_n degrees and apply the induction hypothesis to $(d_1 - d_n), d_2, \dots, d_{n-1}$. Although $d_1 - d_n \le d_2 + \cdots + d_{n-1}$ holds, $d_1 - d_n$ may not be the largest of these numbers.

Proof 1 (induction on $\sum d_i$). The basis step is $\sum d_i = 0$, realized by an independent set. Suppose that $\sum d_i > 0$; we consider two cases. If $d_1 = \sum_{i=2}^n d_i$, then the desired graph consists of d_1 edges from v_1 to $v_2, \dots, v_n$. If $d_1 < \sum_{i=2}^n d_i$, then the difference is at least 2, because the total degree sum is even. Also, at least two of the values after d_1 are nonzero, since d_1 is the largest. Thus we can subtract one from each of the last two nonzero values to obtain a list d' to which we can apply the induction hypothesis (it has even sum, and the largest value is at most the sum of the others. To the resulting G', we add one edge joining the two vertices whose degrees are the reduced values. (This can also be viewed as induction on $(\sum_{i=2}^n d_i) - d_1$.)

Proof 2 (induction on $\sum d_i$). Basis as above. Consider $\sum d_i > 0$. If $d_1 > d_2$, then we can subtract 1 from d_1 and from d_2 to obtain d' with smaller sum. Still $d_1 - 1$ is a largest value in d' and is bounded by the sum of the other values. If $d_1 = d_2$, then we subtract 1 from each of the two smallest values to form d'. If these are d_1 and d_2, then d' has the desired properties, and otherwise $\sum_{i=2}^n d_i$ exceeds d_1 by at least 2, and again d' has the desired properties. In each case, we can apply the induction hypothesis to d' and complete the proof as in Proof 1.

Proof 3 (local change). Every nonnegative integer sequence with even sum is realizable when loops and multiple edges are allowed. Given such a realization with a loop, we change it to reduce the number of loops without changing vertex degrees. Eliminating them all produces the desired realization. If we have loops at distinct vertices u and v, then we replace two loops with two copies of the edge uv. If we have loops only at v and have an edge xy between two vertices other than v, then we replace one loop and one copy of xy by edges vx and vy. Such an edge xy must exist because the sum of the degrees of the other vertices is as large as the degree of v.

1.3.64. *A simple graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ is connected if $d_j \ge j$ for all j such that $j \le n-1-d_n$.* Let $V(G) = \{v_1, \dots, v_n\}$, with $d(v_i) = d_i$, and let H be the component of G containing v_n; note that H has at least $1 + d_n$ vertices. If G is not connected, then G has another component H'. Let j be the number of vertices in H'. Since H has at least $1 + d_n$ vertices, we have $j \le n-1-d_n$. By the hypothesis, $d_j \ge j$. Since H' has j vertices, its maximum degree is at least d_j. Since $d_j \ge j$, there are at least $j+1$ vertices in H', which contradicts the definition of j. Hence G is in fact connected.

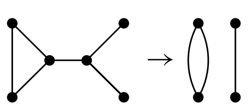
1.3.65. *If $D = \{a_i\}$ is a set of distinct positive integers, with $0 < a_1 < \cdots < a_k$, then there is a simple graph on $a_k + 1$ vertices whose set of vertex degrees (repetition allowed) is D.*

Proof 1 (inductive construction). We use induction on k. For $k = 1$, use K_{a_1+1}. For $k = 2$, use the join $K_{a_1} \vee \overline{K}_{a_2-a_1+1}$. That is, G consists of a clique Q with a_1 vertices, an independent set S with $a_2 - a_1 + 1$ vertices, and all edges from Q to S. The vertices of S have degree a_1, and those of Q have degree a_2.

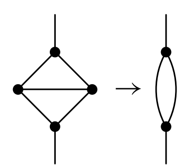
For $k \geq 2$, take a clique Q with a_1 vertices and an independent set S with $a_k - a_{k-1}$ vertices. Each vertex of S has neighborhood Q, and each vertex of Q is adjacent to all other vertices. Other vertices have a_1 neighbors in Q and none in S, so the degree set of $G - Q - S$ should be $\{a_2 - a_1, \ldots, a_{k-1} - a_1\}$. By the induction hypthesis, there is a simple graph H with $a_{k-1} - a_1 + 1$ vertices having this degree set (the degree set is smaller by 2). Using H for $G - Q - S$ completes G as desired.

Proof 2 (induction and complementation). Again use induction on k, using K_{a_1+1} when $k = 1$. For $k > 1$ and $0 < a_1 \cdots < a_k$, the complement of the desired graph with $a_1 + 1$ vertices has degree set $\{a_k - a_1, \ldots, a_k - a_{k-1}, 0\}$. By the induction hypothesis, there is a graph of order $a_k - a_1 + 1$ with degree set $\{a_k - a_1, \cdots, a_k - a_{k-1}\}$. Add a_1 isolated vertices and take the complement to obtain the desired graph G.

1.3.66. *Construction of cubic graphs not obtainable by expansion alone.* A simple cubic graph G that cannot be obtained from a smaller cubic graph by the expansion operation is the same as a cubic graph on which no erasure can be performed, since any erasure yielding a smaller H from G could be inverted by an expansion to obtain G from H. An edge cannot be erased by this operation if and only if one of the subsequent contractions produces a multiple edge. This happens if the other edges incident to the edge being erased belong to a triangle, or in one other case, as indicated below.

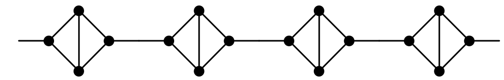


and



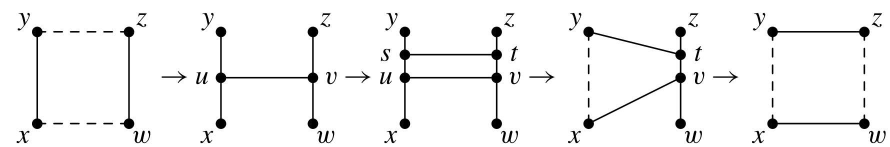
Finally, we need only provide a simple cubic graph with $4k$ vertices where every edge is non-erasable in one of these two ways. To do this place copies of $G_1, \ldots, G_k$ of $K_4 - e$ (the unique 4-vertex graph with 5 edges) around in a ring, and for each consecutive pair G_i, G_{i+1} add an edge joining

a pair of vertices with degree two in the subgraphs, as indicated below, where the wraparound edge has been cut.



1.3.67. *Construction of 3-regular simple graphs*

a) A 2-switch can be performed by performing a sequence of expansions and erasures. We achieve a 2-switch using two expansions and then two erasures as shown below. If the 2-switch deletes xy and zw and introduces xw and yz, then the first expansion places new vertices u and v on xy and zw, the second introduces s and t on the resulting edges ux and vz, the first erasure deletes su and its vertices, and the second erasure deletes tv and its vertices. The resulting vertices are the same as in the original graph, the erasures were legal because they created only edges that were not present originally, and we have deleted xy and zw and introduced xw and yz.



b) Every 3-regular simple graph can be obtained from K_4 by a sequence of expansions and erasures. Erasure is allowed only if no multiple edges result. Suppose H is the desired 3-regular graph. Every 3-regular graph has an even number of vertices, at least four. Any expansion of a 3-regular graph is a 3-regular graph with two more vertices. Hence successive expansions from K_4 produce a 3-regular graph G with $n(H)$ vertices. Since G and H have the same vertex degrees, there is a sequence of 2-switches from G to H. Since every 2-switch can be produced by a sequence of expansions and erasures, we can construct a sequence of expansions and erasures from K_4 to H by going through G.

1.3.68. *If G and H are X, Y-bigraphs, then $d_G(v) = d_H(v)$ for all $v \in X \cup Y$ if and only if there is a sequence of 2-switches that transforms G into H without ever changing the bipartition.* The condition is sufficient, since 2-switches do not change vertex degrees. For necessity, assume that $d_G(v) = d_H(v)$ for all v. We build a sequence of 2-switches transforming G to H.

Proof 1 (induction on $|X|$). If $|X| = 1$, then already $G = H$, so we may assume that $|X| > 1$. Choose $x \in X$ and let $k = d(x)$. Let S be a selection of k vertices of highest degree in Y. If $N(x) \neq S$, choose $y \in S$ and $y' \in Y - S$ so that $x \not\leftrightarrow y$ and $x \leftrightarrow y'$. Since $d(y) \geq d(y')$, there exists $x' \in X$ so that $y \leftrightarrow x'$ and $y' \not\leftrightarrow x'$. Switching $xy', x'y$ for $xy, x'y'$ increases

$|N(x) \cap S|$ with the same bipartition. Iterating this reaches $N(x) = S$; let G' be the resulting graph.

Doing the same in H yields graphs G' from G and H' from H such that $N_{G'}(x) = N_{H'}(x)$. Deleting x and applying the induction hypothesis to the graphs $G^* = G' - x$ and $H^* = H' - x$ completes the construction of the desired sequence of 2-switches.

Proof 2 (induction on number of discrepancies). Let F be the X, Y-bigraph whose edges are those belonging to exactly one of G and H. Let $d = e(F)$. Since G and H have identical vertex degrees, each vertex of F has the same number of incident edges from $E(G) - E(H)$ and $E(H) - E(G)$. When $d > 0$, F therefore has a cycle alternating between $E(G)$ and $E(H)$ (when we enter a vertex on an edge of one type, we can exit on the other type, we can't continue forever, and all cycles have even length).

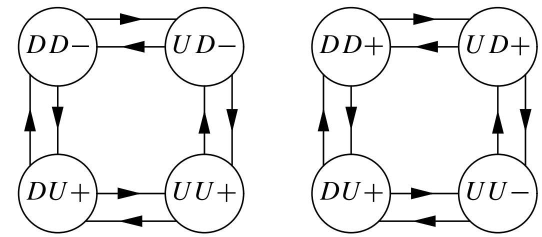
Let C be a shortest alternating cycle in F, with first $xy \in E(G) - E(H)$ and then $yx' \in E(H) - E(G)$ and $x'y' \in E(G) - E(H)$. We consider a 2-switch involving $\{x, y, x', y'\}$. If $y'x \in E(H) - E(G)$, then the 2-switch in G reduces d by 4. If $y'x \in E(G) - E(H)$, then we would have a shorter cycle in F. If $y'x \notin E(G) \cup E(H)$, then we perform the 2-switch in G; if $y'x \in E(G) \cup E(H)$, then we perform the 2-switch in H. Each of these last two cases yields a new pair of graphs with d reduced by 2, and the induction hypothesis applies to this pair to provide the rest of the exchanges.

1.4. DIRECTED GRAPHS

1.4.1. *Digraphs in the real world.* Many digraphs based on temporal order have no cycles. For example, given a set of football games, we can put an edge from game x to game y if game x ends before game y begins. The relation "is a parent of" also works.

Asymmetric digraphs without cycles often arise from tournaments. Each team plays every other team, and there is an edge for each game from the winner to the loser. The result can be without cycles, but usually cycles exist. Another example is the relation "has sent a letter to".

1.4.2. *If the first switch becomes disconnected from the wiring in the lightswitch system of Application 1.4.4, then the digraph for the resulting system is that below.*



1.4.3. *Every u, v-walk in a digraph contains a u, v-path.* The shortest u, v-walk contained in a u, v-walk W is a u, v-path, since the shortest walk has no vertex repetition.

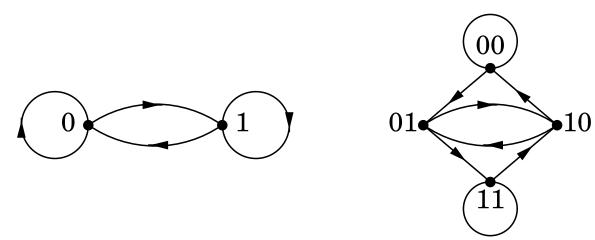
1.4.4. *Every closed walk of odd length in a digraph contains the edges of an odd cycle.* The proof follows that of the corresponding statement for graphs in Lemma 1.2.15, given that the definitions of walk and cycle require the head of each edge to be the tail of the next edge.

We use induction on the length l of a closed odd walk W. Basis step: $l = 1$. A closed walk of length 1 traverses a cycle of length 1.

Induction step: $l > 1$. Assume the claim for closed odd walks shorter than W. If W has no repeated vertex (other than first = last), then W itself forms a cycle of odd length. If vertex v is repeated in W, then we view W as starting at v and break W into two v, v-walks. Since W has odd length, one of these is odd and the other is even. The odd one is shorter than W. By the induction hypothesis, it contains an odd cycle, and this cycle appears in order in W.

1.4.5. *A finite directed graph contains a (directed) cycle if every vertex is the tail of at least one edge (has positive outdegree).* (The same conclusion holds if every vertex is the head of at least one edge.) Let G be such a graph, let P be a maximal (directed) path in G, and let x be the final vertex of P. Since x has at least one edge going out, there is an edge xy. Since P cannot be extended, y must belong to P. Now xy completes a cycle with the y, x-subpath of P.

1.4.6. *The De Bruijn graphs D_2 and D_3.*



1.4.7. *In an orientation of a simple graph with 10 vertices, the vertices can have distinct outdegrees.* Take the orientation of the complete graph with vertices $0, \dots, 9$ by orienting the edge ij from i to j if $i > j$. In this digraph, the outdegree of vertex i is i.

1.4.8. *There is an n-vertex tournament with $d^+(v) = d^-(v)$ for every vertex v if and only if n is odd.* If n is even, then $d^+(v) + d^-(v) = n - 1$ is odd, so the summands can't be equal integers. For odd n, we construct such a tournament.

Proof 1 (explicit construction). Place the n vertices equally spaced around a circle, and direct the edges from v to the $(n-1)/2$ vertices that follow v in the clockwise direction. After doing this for each vertex, the $(n-1)/2$ nearest vertices in the counterclockwise direction from v have edges directed to v, and each edge has been oriented.

Proof 2 (inductive construction). When $n = 1$, the 1-vertex tournament satisfies the degree condition. For $k > 1$, suppose that T is a tournament with $2k - 1$ vertices that satisfies the condition. Partition $V(T)$ into sets A and B with $|A| = k$ and $|B| = k - 1$. Add two vertices x and y. Add all edges from x to A, from A to y, from y to B, and from B to x. Each vertex in $V(T)$ now has one predecessor and one successor in $\{x, y\}$. We have $d^+(x) = k$, $d^-(x) = k - 1$, $d^+(y) = k - 1$, $d^-(y) = k$. Complete the construction of T' by adding the edge yx. Now T' is a tournament with $2k + 1$ vertices that satisfies the degree condition.

Proof 3 (Eulerian graphs). When n is odd, K_n is a connected even graph and hence is Eulerian. Orienting edges of K_n in the forward direction while following an Eulerian circuit yields the desired tournament.

1.4.9. *For each n, there is an n-vertex digraph in which the vertices have distinct indegrees and distinct outdegrees.* Using vertices $v_1, \dots, v_n$, let the edges be $\{v_i v_j : 1 \le i < j \le n\}$. Now $d^-(v_i) = i - 1$ and $d^+(v_i) = n - i$. Thus the indegrees are distinct, and the outdegrees are distinct.

1.4.10. *A digraph is strongly connected if and only if for each partition of the vertex set into nonempty sets S and T, there is an edge from S to T.* Given that D is strong, choose $x \in S$ and $y \in T$. Since D has an x,y-path, the path must leave S and enter T and do so along some edge.

Conversely, if there is such an edge for every partition, let S be the set of all vertices reachable from vertex x. If $S \ne V(D)$, then the hypothesis yield an edge leaving S, which adds a vertex to S. Since x was arbitrary, each vertex is reachable from every other, and D is strongly connected.

1.4.11. *In every digraph, some strong component has no entering edges, and some strong component has no exiting edges.*

Proof 1 (using cycles). Given a digraph D, form a digraph D^* with one vertex for each strong component of D. Let the strong components of D be $X_1, \dots, X_k$, with corresponding vertices $x_1, \dots, x_k$ in D^*. Put an edge from x_i to x_j in D^* if in D there is an edge from some vertex of X_i to some vertex of X_j. The problem is to show that D^* has a vertex with indegree 0 and a vertex with outdegree 0.

If such vertices do not exist, then D^* has a cycle (by Lemma 1.4.23). If D^* has a cycle, then the union of the strong components of D corresponding to the vertices of the cycle is a strongly connected subgraph of D containing all those components. This is a contradiction, because they were maximal strong subgraphs.

Proof 2 (extremality). For a vertex v in D, let $R(v)$ be the set of vertices reachable from v. Let u be a vertex minimizing $|R(u)|$. If $v \in R(u)$, then $R(v) \subseteq R(u)$, so $R(v) = R(u)$. Since $u \in R(u)$, also u is reachable from v. Thus $R(u)$ induces a strong subdigraph. By the definition of $R(u)$, no edges leave it, so it is a strong component. Applying the same argument to the reverse digraph yields a strong component with no entering edge.

1.4.12. *In a digraph the connection relation is an equivalence relation, and its equivalence classes are the vertex sets of the strong components.* We are defining x to be connected to y if the digraph has both an x,y-path and a y,x-path. The reflexive property holds using paths of length 0. The symmetric property holds by the definition.

For transitivity, consider an x,y-path P_1 and a y,z-path P_2. Let w be the first vertex of P_1 that belongs to P_2. Following P_1 from x to w and P_2 from w to z yields an x,z-path, by the choice of w. Applying this to obtain paths in both directions shows that the connection relation is transitive.

Since a strong component is a strongly connected subdigraph, its pairs of vertices satisfy the connection relation. Hence every strong component is contained in an equivalence class of the connection relation. In order to show that every equivalence class is contained in a strong component, we show that when x is connected to y, there is an x,y-path using only vertices of the equivalence class.

Let P be an x,y-path, and let Q be a y,x-path. The concatenation of Q with P is a closed walk in the digraph; let S be its vertex set. By following the walk, we find a u,v-walk for all $u, v \in S$. Such a walk contains a u,v-path. The same argument yields a v,u-path in the walk. Hence all pairs of vertices on it satisfy the connection relation, and we have found an x,y-path (and y,x-path) witin the equivalence class. Hence the subdigraph induced by the equivalence class is strongly connected.

1.4.13. *Strong components.*

a) Two maximal strongly connected subgraphs of a directed graph share no vertices. If strong components D_1, D_2 of D share a vertex v, then for all

$x \in V(D_1)$ and $y \in V(D_2)$, the union of an x, v-path in D_1 and a v, y-path in D_2 contains an x, y-path in D. Similarly, D has a y, x-path. Thus $D_1 \cup D_2$ is strongly connected.

b) The digraph D^* obtained by contracting the strong components of a digraph D is acyclic (D^* has a vertex v_i for each strong component D_i, with $v_i \to v_j$ if and only if $i \neq j$ and D has an edge from D_i to D_j). If D^* has a cycle with vertices $d_0, \ldots, d_{l-1}$, then D has strong components $D_0, \ldots, D_{l-1}$ such that D has an edge $u_i v_{i+1}$ from D_i to D_{i+1}, for each i (modulo l). If $x \in D_i$ and $y \in D_j$, this means that D contains an x, y-walk consisting of the concatenation of paths with successive endpoints $x, u_i, v_{i+1}, u_{i+1}, v_{i+2}, \ldots, u_{j-1}, v_j, y$. This walk contains an x, y-path. Since x, y were chosen arbitrarily from $D_0 \cup \cdots \cup D_{l-1}$, we conclude that $D_0 \cup \cdots \cup D_{l-1}$ is strongly connected, which contradicts $D_0, \ldots, D_{l-1}$ being maximal strongly connected subgraphs.

1.4.14. *If G is an n-vertex digraph with no cycles, then the vertices of G can be ordered as $v_1, \ldots, v_n$ so that if $v_i v_j \in E(G)$, then $i < j$.* If G has no cycles, then some vertex v has outdegree 0. Put v last in the ordering. Now $G - v$ also has no cycles, and we proceed iteratively. When we choose v_j, it has no successors among $v_1, \ldots, v_{j-1}$, so the desired condition on the edges holds.

1.4.15. *In the simple digraph with vertex set $\{(i, j) \in \mathbb{Z}^2 \colon 0 \le i \le m$ and $0 \le n\}$ and an edge from (i, j) to (i', j') if and only if (i', j') is obtained from (i, j) by adding 1 to one coordinate, there are $\binom{m+n}{n}$ paths from $(0, 0)$ to (m, n).* Traversing each edge adds one to each coordinate, so every such path has $m + n$ edges. We can record such a path as a 0, 1-list, recording 0 when we follow an edge that increases the first coordinate, 1 when we follow an edge that increases the second coordinate. Each list with m 0s and n 1s records a unique path. Since there are $\binom{m+n}{n}$ ways to form such a list by choosing positions for the 1s, the bijection implies that the number of paths is $\binom{m+n}{n}$.

1.4.16. *Fermat's Little Theorem.* Let $\mathbb{Z}_n$ denote the set of congruence classes of integers modulo a PRIME NUMBER n (the first printing of the second edition omitted this!). Multiplication by a positive integer a that is not a multiple of n defines a permutation of $\mathbb{Z}_n$, since $ai \equiv aj \pmod n$ yields $a(j - i) \equiv 0 \pmod n$, which requires n to divide $j - i$ when a and n are relatively prime. The functional digraph consists of pairwise disjoint cycles.

a) If G is the functional digraph with vertex set $\mathbb{Z}_n$ for the permutation defined by multiplication by a, then all cycles in G (except the loop on n) have length $l - 1$, where l is the least natural number such that $a^l \equiv a \pmod n$. This is the length of the cycle containing the element 1. Traversing a cycle of length k (not the cycle consisting of n) yields $xa^k \equiv x \pmod n$, or $x(a^k - 1) \equiv 0 \pmod n$, for some x not divisible by n. Since n is prime, this requires $a^k \equiv 1 \pmod n$, and hence $k \ge l - 1$. On the other hand $xa^{l-1} = x$, and hence $k \le l - 1$.

b) $a^{n-1} \equiv 1 \pmod n$. Since all nontrivial cycles have the same length, $l - 1$ divides $n - 1$. Let $m = (n - 1)/(l - 1)$. Now $a^{n-1} = a^{(l-1)m} = (a^{l-1})^m \equiv 1^m \equiv 1 \pmod n$.

1.4.17. *A (directed) odd cycle is a digraph with no kernel.* Let S be a kernel in an odd cycle C. Every vertex must be in S or have a successor in S. Since S is an independent set, exactly one of these two conditions holds at each vertex. Hence we must alternate between vertices in S and vertices not in S as we follow the C. We cannot alternate two conditions as we follow an odd cycle, so there is no kernel.

A digraph having an odd cycle as an induced subgraph and having a kernel. To an odd cycle, add one new vertex as a successor of each vertex on the cycle. The new vertex forms a kernel by itself.

1.4.18. *An acyclic digraph D has a unique kernel.*

Proof 1 (parity of cycles). By Theorem 1.4.16, a digraph with no odd cycles has at least one kernel. We show that a digraph with no even cycles has at most one kernel, by proving the contrapositive. If K and L are distinct kernels (each induces no edges), then every vertex of $K - L$ has a successor in $L - K$, and every vertex of $L - K$ has a successor in $K - L$.

Proof 2 (induction on $n(D)$). In a digraph with one vertex and no cycle, the vertex is a kernel. When $n(D) > 1$, the absence of cycles guarantees a vertex with outdegree 0 (Lemma 1.4.23). Such a vertex lies in every kernel, since it has no successor. Let $S' = \{v \in V(D) \colon d^+(v) = 0\}$. Note that S' induces no edges. Let D' be the subdigraph obtained from D by deleting S' and all vertices having successors in S'. The digraph D' has no cycles; by the induction hypothesis, D' has a unique kernel S''.

Let $S = S' \cup S''$. Since there are no edges from $V(D')$ to S', the set S is a kernel in D. Furthermore, S is the only kernel. We have argued that all of S' is present in every kernel, and independence of the kernel implies that no other vertex outside $V(D')$ is present. The lack of edges from $V(D')$ to S' implies that the remainder of the kernel must be a kernel in D', and there is only one such set.

1.4.19. *A digraph is Eulerian if and only if $d^+(v) = d^-(v)$ for every vertex v and the underlying graph has at most one nontrivial component.*

Necessity. Each passage through a vertex by a circuit uses an entering edge and an exiting edge; this applies also to the “last” and “first” edges of the circuit. Also, two edges can be in the same trail only when they lie in the same component of the underlying graph.

Sufficiency. We use induction on the number of edges, m. Basis step: When $m = 0$, a closed trail consisting of one vertex contains all the edges.

Induction step: Consider $m > 0$. With equal indegree and outdegree, each vertex in the nontrivial component of the underlying graph of our digraph G has outdegree at least 1 in G. By Lemma 1.2.25, G has a cycle C. Let G' be the digraph obtained from G by deleting $E(C)$.

Since C has 1 entering and 1 departing edge at each vertex, G' also has equal indegree and outdegree at each vertex. Each component of the underlying graph H' of G' is the underlying graph of some subgraph of G'. Since G' has fewer than m edges, the induction hypothesis yields an Eulerian circuit of each such subgraph of G'.

To form an Eulerian circuit of G, we traverse C, but when a component of H' is entered for the first time we detour along an Eulerian circuit of the corresponding subgraph of G', ending where the detour began. When we complete the traversal of C, we have an Eulerian circuit of G.

1.4.20. *A digraph is Eulerian if and only if indegree equals outdegree at every vertex and the underlying graph has at most one nontrivial component.* The conditions are necessary, since each passage through a vertex uses one entering edge and one departing edge.

For sufficiency, suppose that G is a digraph satisfying the conditions. We prove first that every non-extendible trail in G is closed. Let T be a non-extendible trail starting at u. Each time T passes through a vertex v other than u, it uses one entering edge and one departing edge. Thus upon each arrival at v, T has used one more edge entering v than departing v. Since $d^+(v) = d^-(v)$, there remains an edge on which T can continue. Hence a non-extendible trail can only end at v and must be closed.

We now show that a trail of maximal length in G must be an Eulerian circuit. Let T be a trail of maximum length; T must also be non-extendible, and hence T is closed. Suppose that T omits some edge e of G. Since the underlying graph of G has only one nontrivial component, it has a shortest path from e to the vertex set of T. Hence some edge e' not in T is incident to some vertex v of T. It may enter or leave v.

Since T is closed, there is a trail T' that starts and ends at v and uses the same edges as T. We now extend T' along e' (forward or backward depending on whether e leaves or enters v) to obtain a longer trail than T. This contradicts the choice of T, and hence T traverses all edges of G.

1.4.21. *A digraph has an Eulerian trail if and only if the underlying graph has only one nontrivial component and $d^-(v) = d^+(v)$ for all vertices or for all but two vertices, in which case in-degree and out-degree differ by one for the other two vertices.* Sufficiency: since the total number of heads equals the total number of tails, the vertices out of balance consist of x with an extra head and y with an extra tail. Add the directed edge xy and apply the characterization above for Eulerian digraphs.

1.4.22. *If D is a digraph with $d^-(v) = d^+(v)$ for every vertex v, except that $d^+(x) - d^-(x) = k = d^-(y) - d^+(y)$, then D contains k pairwise edge-disjoint x,y-paths.* Form a digraph D' by adding k edges from y to x. Since indegree equals outdegree for every vertex of D', the "component" of D' containing x and y is Eulerian. Deleting the added edges from an Eulerian circuit cuts it at k places; the resulting k directed trails are x,y-trails in the digraph D. As proved in Chapter 1, the edge set of every x,y-trail contains an x,y-path; the proof in Chapter 1 applies to both graphs and digraphs.

1.4.23. *Every graph G has an orientation such that $|d^+(v) - d^-(v)| \le 1$ for all v.*

Proof 1 (Eulerian circuits). Add edges to pair up vertices of odd degree (if any exist). Each component of this supergraph G' is Eulerian. Orient G' by following an Eulerian circuit in each component, orienting each edge forward as the circuit is traversed. The circuit leaves each vertex the same number of times as it enters, so the resulting orientation has equal indegree and outdegree at each vertex.

Deleting the edges of $E(G') - E(G)$ now yields the desired orientation of G, because at most one edge was added at each vertex to pair the vertices of odd degree. Deleting at most one incident edge at v produces difference at most one between $d^+(v)$ and $d^-(v)$.

Proof 2 (induction on $e(G)$). If $e(G) = 0$, then the claim holds. For $e(G) > 0$, if G has a cycle H, then orient H consistently, with no imbalance anywhere. If G has no cycle, then find a maximal path H and orient it consistently. This creates imbalance of 1 at the endpoints and 0 elsewhere. The endpoints have degree 1, so no further imbalance occurs there. In both cases, delete $E(H)$ and apply the induction hypothesis to complete the orientation.

1.4.24. *Not every graph has an orientation such that for every vertex subset, the numbers of edges entering and leaving differ by at most one.* Let G be a graph with at least four vertices such that every vertex degree is odd. Let D be an orientation of G. In D, no vertex of G has the same number of vertices entering and leaving. Let $S = \{v \in V : d^+(v) > d^-(v)\}$. Since each edge within S contributes the same amount to $\sum_{v\in S} d^+(v)$ and $\sum_{v\in S} d^-(v)$, there are $\sum_{v\in S} d^+(v) - \sum_{v\in S} d^-(v)$ more edges leaving S than entering. The difference is at least $|S|$. Similarly, for $\overline{S}$ the absolute difference is at least $|\overline{S}|$, so always some set has difference at least $n(G)/2$.

1.4.25. *Orientations and P_3-decomposition. a) Every connected graph has an orientation having at most one vertex with odd outdegree.*

Proof 1 (local change). Given an orientation of G with vertices x and y having odd outdegree, find an x,y-path P in the underlying graph and flip the orientation of every edge on P. This does not change the parity of the outdegree for any internal vertex of P, but it changes the parity of the outdegree for the endpoints, which previously had odd outdegree. Hence this operation reduces the number of vertices of odd outdegree by 2. We can apply this operation whenever at least two vertices have odd outdegree, so we can reduce the number of vertices with odd outdegree to 0 or 1.

Proof 2 (application of Eulerian circuits). Suppose that G has $2k$ vertices of odd degree. Add edges that pair these vertices to form an Eulerian supergraph G'. Follow an Eulerian circuit of G', starting from u along $uv \in E(G)$, producing an orientation of G as follows. Orient uv out from u; now u has odd outdegree and all other vertices have even outdegree. Subsequently, when the circuit traverses an edge $xy \in E(G)$, orient it so that x has even outdegree among the edges oriented so far. At each stage, the only vertex that can have odd outdegree among edges of G is the current vertex. The orientation chosen for the edges not in $E(G)$ is unimportant.

b) A simple connected graph with an even number of edges can be decomposed into paths with two edges. Since the sum of the outdegrees is the number of edges, the parity of the number of vertices with odd outdegree is the same as the parity of the number of edges. Hence part (a) implies that a connected graph with an even number of edges has an orientation in which every vertex has even outdegree. At each vertex, pair up exiting edges arbitrarily. Since G is simple, this decomposes G into copies of P_3.

1.4.26. *De Bruijn cycle for binary words of length 4, avoiding 0101 and 1010.* Make a vertex for each of the 8 sequences of length 3 from the alphabet $S = \{0, 1\}$. Put an edge from sequence a to sequence b, with label $\alpha \in S$, if b is obtained from a by dropping the first letter of a and appending α to the end. Traveling this edge from a corresponds to having α in sequence after a. We want our digraph to have 14 edges corresponding to the desired 14 words, and we want an Eulerian circuit through them to generate the cyclic arrangement of labels. The difference between this digraph and the De Bruijn digraph in Application 1.4.25 is omitting the two edges joining 010 and 101. The resulting digraph still has indegree = outdegree at every vertex, so it is Eulerian. One arrangement of labels generated by an Eulerian circuit is 00001001101111.

1.4.27. *De Bruijn cycle for any alphabet and length.* When A is an alphabet of size k, there exists a cyclic arrangement of k^l characters chosen from A such that the k^l strings of length l in the sequence are all distinct.

Idea: The indegree and outdegree is k at each vertex of the digraph constructed in the matter analogous to that for $k = 2$. Thus the digraph is Eulerian, and recording the edge labels along an Eulerian circuit yields the desired sequence. Below we repeat the details.

Define a digraph $D_{k,l}$ whose vertices are the $(l-1)$-tuples with elements in A. Place an edge from a to b if the last $n-2$ entries of a agree with the first $n-2$ entries of b. Label the edge with the last entry of b. For each vertex a, there are k ways to append a element of A to lengthen its name, and hence there are k edges leaving each vertex.

Similarly, there are k choices for a character deleted from the front of a predecessor's name to obtain name b, so each vertex has indegree k. Also, we can reach $b = (b_1, \ldots, b_{n-1})$ from any vertex by successively following the edges labeled $b_1, \ldots, b_{n-1}$. Since $D_{k,l}$ is strongly connected and has indegree equal to outdegree at every vertex, the characterization of Eulerian digraphs implies that $D_{k,l}$ is Eulerian.

Let C be an Eulerian circuit of $D_{k,l}$. When we are at the vertex with name $a = (a_1, \ldots, a_{n-1})$ while traversing C, the most recent edge had label a_{n-1}, because the label on an edge entering a vertex agrees with the last digit of the sequence at the vertex. Since we delete the front and shift the rest to obtain the rest of the label at the head, the successive earlier labels (looking backward) must have been $a_{n-2}, \ldots, a_1$ in order. If C next traverses an edge with label a_n, then the subsequence consisting of the n most recent edge labels at that time is $a_1, \ldots, a_n$.

Since the k^{l-1} vertex labels are distinct, and the edges leaving each vertex have distinct labels, and we traverse each edge from each vertex exactly once along C, the k^l strings of length l in the circular arrangement given by the edge labels along C are distinct.

1.4.28. *De Bruijn cycle for length 4 without the constant words.* Make a vertex for each of the m^3 sequences of length 3 from the alphabet S. Put an edge from sequence a to sequence b, with label $\alpha \in S$, if b is obtained from a by dropping the first letter and appending α to the end. Since there are m ways to append a letter, the out-degree of each vertex is m. For each sequence, there are m possible letters that could have been deleted to reach it, so the in-degree of each vertex is m.

Deleting the loops at the m constant vertices (aaa, bbb, etc.) reduces the indegree and outdegree at those vertices by 1, so the resulting digraph has equal indegree and outdegree at every vertex. Also the underlying graph is connected, since vertex abc can be reach from any other vertex by following the edge labeled a, then b, then c.

Thus an Eulerian circuit exists. Recording the edge labels while following an Eulerian circuit yields the desired arrangement. The 4-digit strings obtained are those formed by the 3-digit name of a vertex plus the label on an exiting edge. These $m^4 - m$ strings are distinct and avoid the constant words, since the loops were deleted from the digraph.

Alternative proof. If we know (from Exercise 1.4.27, for example) that there exists a De Bruijn cycle including the constant words, then we can simply delete one letter from each string of four consecutive identical letters, without using graph theory.

1.4.29. *A strong orientation of a graph that has an odd cycle also has an odd (directed) cycle.* Suppose that D is a strong orientation of a graph G that has an odd cycle $v_1, \dots, v_{2k+1}$. Since D is strongly connected, for each i there is a v_i, v_{i+1}-path in D. If for some i every such path has even length, then the edge between v_i and v_{i+1} points from v_{i+1} to v_i, since the other orientation would be a v_i, v_{i+1}-path of length 1 (odd). In this case, we have an odd cycle through v_i and v_{i+1}. Otherwise, we have a path of odd length from each v_i to v_{i+1}. Combining these gives a closed trail of odd length. In a digraph as well as in a graph (by the same proof), a closed odd trail contains the edges of an odd cycle.

1.4.30. *The maximum length of a shortest spanning closed walk in a strongly-connected n-vertex digraph is $\lfloor (n+1)^2/4 \rfloor$ if $n \ge 3$.* For the lower bound, let G consist of a u, v-path P of $n - l$ vertices, plus l vertices with edges from v and to u. When leaving a vertex not on P, P must be reached and traversed before the next vertex off P. Hence G requires $l(n - l + 1)$ steps to walk through every vertex, maximized by setting $l = \lfloor (n+1)/2 \rfloor$. The length of the walk is then $\lfloor (n+1)^2/4 \rfloor$.

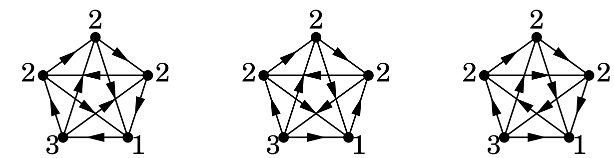
For any strongly-connected n-vertex digraph G, we obtain a spanning closed walk of length at most $\lfloor (n+1)^2/4 \rfloor$. Let m be the maximum length of a path in G; from each vertex to every other, there is a path of length at most m. Begin with a path P of length m; this visits $m + 1$ vertices. Next use paths to reach each of the remaining vertices in turn, followed by a path returning to the beginning of P. In this closed walk, $1 + (n - m - 1) + 1$ paths have been followed, each of length at most m. The total length is at most $m(n + 1 - m)$, which is bounded by $\lfloor (n+1)^2/4 \rfloor$.

1.4.31. *The smallest nonisomorphic pair of tournaments with the same score sequences have five vertices.*

At least five vertices are needed. The score sequence (outdegrees) of an n-vertex tournament can have only one 0 or $n - 1$. Nonisomorphic tournaments with such a vertex must continue to be nonisomorphic when that vertex is deleted. Hence a smallest nonisomorphic pair has no vertex with score 0 or $n - 1$. The only such score sequences with fewer than 5 vertices are 111 and 2211. The first is realized only by the 3-cycle. For 2211, name the low-degree vertices as v_1 and v_2 such that $v_1 \leftarrow v_2$, and name the high-degree vertices as v_3 and v_4 such that $v_3 \leftarrow v_4$. The only way to complete a tournament with this score sequence is now $N^+(v_1) = \{v_4\}$, $N^+(v_2) = \{v_1\}$, $N^+(v_3) = \{v_1, v_2\}$, and $N^+(v_4) = \{v_2, v_3\}$.

Five vertices suffice, by construction. On five vertices, the sequences to

consider are 33211, 32221, and 22222. There is only one isomorphic class with score sequence 22222, but there are more for the other two sequences. In fact, there are 3 nonisomorphic tournaments with score sequence 32221. They may be characterized as follows: (1) the bottom player beats the top player, and the three middle players induce a cyclic subtournament; (2) the top player beats the bottom player, and the three middle players induce a cyclic subtournament; (3) the top player beats the bottom player, and the three middle players induce a transitive subtournament.



Five vertices suffice, by counting. Each score sequence sums to 10 and has maximum outdegree at most 4; also there is at most one 4 and at most one 0. The possibilities are thus 43210, 43111, 42220, 42211, 33310, 33220, 33211, 32221, 22222. There are 2^{10} tournaments on five vertices; we show that they cannot fit into nine isomorphism classes. The isomorphism class consisting of a 5-cycle plus edges from each vertex to the vertex two later along the cycle occurs 4! times; once for each cyclic ordering of the vertices. Each of the other isomorphism classes occurs at most 5! times. Hence the nine isomorphism classes contain at most $24 + 8 \cdot 120$ of the 2^{10} tournaments. Since $1024 > 984$, there must be at least 10 isomorphism classes among the nine score sequences.

1.4.32. *Characterization of bigraphic sequences.* With $p = p_1, \dots, p_m$ and $q = q_1, \dots, q_n$, the pair (p, q) is **bigraphic** if there is a simple bipartite graph in which $p_1, \dots, p_m$ are the degrees for one partite set and $q_1, \dots, q_n$ are the degrees for the other.

If p has positive sum, then (p, q) is bigraphic if and only if (p', q') is bigraphic, where (p', q') is obtained from (p, q) by deleting the largest element Δ from p and subtracting 1 from each of the Δ largest elements of q. We follow the method of Theorem 1.3.31. Sufficiency of the condition follows by adding one vertex to a realization of the smaller pair.

For necessity, choose indices in a realization G so that $p_1 \ge \cdots \ge p_m$, $q_1 \ge \cdots \ge q_n$, $d(x_i) = p_i$, and $d(y_j) = q_j$. We produce a realization in which x_1 is adjacent to $y_1, \dots, y_{p_1}$. If $y_j \nleftrightarrow x_1$ for some $j \le p_1$, then $y_k \leftrightarrow x_1$ for some $k > p_1$. Since $q_j \ge q_k$, there exists x_i with $i > 1$ such that $x_i \in N(y_j) - N(y_k)$. We perform the 2-switch to replace $\{x_1 y_k, x_i y_j\}$ with $\{x_1 y_j, x_i y_k\}$. This reduces the number of missing neighbors, so we can obtain the desired realization. (Comment: the statement also holds when $m = 1$.)

1.4.33. *Bipartite 2-switch and 0,1-matrices with fixed row and column sums.* With a simple X,Y-bigraph G, we associate a 0,1-matrix $B(G)$ with rows indexed by X and columns indexed by Y. The matrix has a 1 in position i,j if and only if $x_i \leftrightarrow y_j$. Applying a 2-switch to G that exchanges $xy, x'y'$ for $xy', x'y$ (preserving the bipartition) affects $B(G)$ by interchanging the 0's and 1's in the 2 by 2 permutation submatrix induced by rows x, x' and columns y, y'. Hence there is a sequence of 2-switches transforming G to H without changing the bipartition if and only if there is a sequence of switches on 2 by 2 permutation submatrices that transforms $B(G)$ to $B(H)$.

Furthermore, G and H have the same bipartition and same vertex degrees if and only if $B(G)$ and $B(H)$ have the same row sums and the same column sums. Therefore, in the language of bipartite graphs the statement about matrices becomes "all bipartite graphs with the same bipartition and vertex degrees can be reached from each other using 2-switches preserving the bipartition." We prove either statement by induction. We use the phrasing of bipartite graphs.

Proof 1 (induction on m). If $m=1$, then already $G=H$. For $m>1$, let G be an X,Y-bigraph. Let x be a vertex of maximum degree in X, with $d(x)=k$. Let S be a set of k vertices of highest degree in Y. Using bipartition-preserving 2-switches, we transform G so that $N(x)=S$. If $N(x)\neq S$, we choose $y\in S$ and $y'\in Y-S$ so that $x \not\leftrightarrow y$ and $x\leftrightarrow y'$. Since $d(y)\ge d(y')$, we have $x'\in X$ so that $y\leftrightarrow x'$ and $y'\not\leftrightarrow x'$. Switching $xy', x'y$ for $xy, x'y'$ increases $|N(x)\cap S|$. Iterating this reaches $N(x)=S$. We can do the same thing in H to reach graphs G' from G and H' from H such that $N_{G'}(x)=N_{H'}(x)$. Now we can delete x and apply the induction hypothesis to the graphs $G^*=G'-x$ and $H^*=H'-x$ to complete the construction of the desired sequence of 2-switches.

Proof 2 (induction on number of discrepancies). Let F be the bipartite graph with the same bipartition as G and H consisting of edges belonging to exactly one of G and H. Let $d=e(F)$. Orient F by directing each edge of $G-E(H)$ from X to Y and each edge of $H-e(G)$ from Y to X. Since G,H have identical vertex degrees, in-degree equals outdegree at each vertex of F. If $d>0$, this implies that F contains a cycle. There is a 2-switch in G that introduces two edges of $E(G)-E(H)$ and reduces d by 4 if and only if F has a 4-cycle. Otherwise, Let C be a shortest cycle in F, and let x,y,x',y' be consecutive vertices on C. We have $xy\in E(G)-E(H)$, $x'y\in E(H)-E(G)$, and $x'y'\in E(G)-E(H)$. We also have $xy'\notin E(G)$, else we could replace these three edges of C by xy' to obtain a shorter cycle in F. We can now perform the 2-switch in G that replaces $xy, x'y'$ with $xy', x'y$. This reduces d by at least 2.

1.4.34. *If G and H are two tournaments on a vertex set V, then $d_G^+(v) =$*

$d_H^+(v)$ for all $v\in V$ if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3. Reversal of a 3-cycle changes no outdegree, so the condition is sufficient.

For necessity, let F be the subgraph of G consisting of edges oriented the opposite way in H. Since $d_G^+(v)=d_H^+(v)$ and $d_G^-(v)=d_H^-(v)$ for all v, every vertex has the same indegree and outdegree in F. Let x be a vertex of maximum degree in F, and let $S=N_F^+(x)$ and $T=N_F^-(x)$.

An edge from S to T in G completes a 3-cycle with x whose reversal in G reduces the number of pairs on which G and H disagree. An edge from T to S in H completes a 3-cycle with x whose reversal in H reduces the number of disagreements. If neither of these possibilities occurs, then G orients every edge of $S\times T$ from T to S, and H orients every such edge from S to T. Also F has edges from T to x. This gives every vertex of T higher outdegree than x in F, contradicting the choice of x.

1.4.35. *$p_1\le\cdots\le p_n$ is the sequence of outdegrees of a tournament if and only if $\sum_{i=1}^k p_i \ge \binom{k}{2}$ and $\sum_{i=1}^n p_i = \binom{n}{2}$.* *Necessity*. A tournament has $\binom{n}{2}$ edges in total, and any k vertices have out-degree-sum at least $\binom{k}{2}$ within the subtournament they induce.

Sufficiency. Given a sequence p satisfying the conditions, let $q_k = \sum_{i=1}^k p_k$ and $e_k = q_k - \binom{k}{2}$. We prove sufficiency by induction on $\sum e_k$. The only sequence p with $\sum e_k = 0$ is $0,1,\ldots,n-1$; this is realized by the *transitive* tournament T_n having $v_k\to v_j$ if and only if $k>j$. If $\sum e_k>0$, let r be the least k with $e_k>0$, and let s be the least index above r with $e_k=0$, which exists since $e_n=0$. We have $q_{s-1} > \binom{s-1}{2}$, $q_s=\binom{s}{2}$, and $q_{s+1}\ge\binom{s+1}{2}$. This yields $p_{s+1}\ge s$ and $p_s<s-1$, or $p_{s+1}-p_s\ge 2$. Similarly, if $r=1$ we have $p_1\ge 1$, and if $r>1$ we have $p_r-p_{r-1}\ge 2$.

Hence we can subtract one from p_r and add one to p_s to obtain a new sequence p' that is non-decreasing, satisfies the conditions, and reduces $\sum e_k$ by $s-r$. By the induction hypothesis, there is a tournament with score sequence p'. If $v_s\to v_r$ in this tournament, we can reverse this edge to obtain the score sequence p. If not, then the fact that $p'_s\ge p'_r$ implies there is another vertex u such that $v_s\to u$ and $u\to v_r$; obtain the desired tournament by reversing these two edges.

1.4.36. Let T be a tournament having no vertex with indegree 0.

a) If x is a king in T, then T has another king in $N^-(x)$. The subdigraph induced by the vertices of $N^-(x)$ is also a tournament; call it T'. Since every tournament has a king, T' has a king. Let y be a king in T'. Since x is a successor of y and every vertex of $N^+(x)$ is a successor of x, every vertex of $V(T)-V(T')$ is reachable from y by a path in T of length at most T. Hence y is also a king in the original tournament T.

b) T has at least three kings. Since T is a tournament, it has some king, x. By part (a), T has another king y in $N^-(x)$. By part (a) again, T has another king z in $N^-(y)$. Since $y \to x$, we have $x \notin N^-(y)$, and hence $z \neq x$. Thus x, y, z are three distinct kings in T.

c) For $n \geq 3$, an n-vertex tournament T with no source and only three kings. Let $S = \{x, y, z\}$ be a set of three vertices in $V(T)$. Let the subtournament on S be a 3-cycle. For all edges joining S and $V(T) - S$, let the endpoint in S be the tail. Place any tournament on $V(T) - S$. Now x, y, z are kings, but no vertex outside S is a king, because no edge enters S.

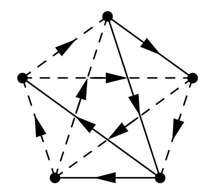
1.4.37. *Algorithm to find a king in a tournament T: Select $x \in V(T)$. If x has indegree 0, call it a king and stop. Otherwise, delete $\{x\} \cup N^+(x)$ from T to form T', and call the output from T' a king in T.* We prove the claims by induction on the number of vertices. The algorithm terminates, because it either stops by selecting a source (indegree 0) or moves to a smaller tournament. By the induction hypothesis, it terminates on the smaller tournament. Thus in each case it terminates and declares a king.

We prove by induction on the number of vertices that the vertex declared a king is a king. When there is only one vertex, it is a king. Suppose that $n(T) > 1$. If the initial vertex x is declared a king immediately, then it has outdegree $n - 1$ and is a king. Otherwise, the algorithm deletes x and its successors and runs on the tournament T' induced by the set of predecessors (in-neighbors) of x.

By the induction hypothesis, the vertex z that the algorithm selects as king in T' is a king in T', reaching each vertex of T' in at most two steps. It suffices to show that z is also a king in the full tournament. Since T' contains only predecessors of x, $z \to x$. Also, z reaches all successors of x in two steps through x. Thus z also reaches all discarded vertices in at most two steps and is a king in T.

1.4.38. *Tournaments with all players kings. a) If n is odd, then there is an tournament with n vertices such that every player is a king.*

Proof 1 (explicit construction). Place the players around a circle. Let each player defeat the $(n-1)/2$ players closest to it in the clockwise direction, and lose to the $(n-1)/2$ players closest to it in the counterclockwise direction. Since every pair of players is separated by fewer players around one side of the circle than the other, this gives a well-defined orientation to each edge. All players have exactly $(n-1)/2$ wins. Thus every outdegree is the maximum outdegree, and we have proved that every vertex of maximum outdegree in a tournament is a king. It is also easy to construct explicit paths. Each player beats the next $(n-1)/2$ players. The remaining $(n-1)/2$ players all lose to the last of these first $(n-1)/2$ players. The construction is illustrated below for five players.



Proof 2 (induction on n). For $n = 3$, every vertex in the 3-cycle is a king. For $n \geq 3$, given a tournament on vertex set S of size n in which every vertex is a king, we add two new vertices x, y. We orient $S \to x \to y \to S$. Every vertex of S reaches x in one step and y in two; x reaches y in one step and each vertex of S in two. Every vertex is a king. (The resulting tournaments are not regular.) Note: Since there is no such tournament when $n = 4$, one must also give an explicit construction for $n = 6$ to include in the basis. The next proof avoids this necessity.

Proof 3 (induction on n). For $n = 3$, we have the cyclic tournament. For $n = 5$, we have the cyclically symmetric tournament in which each vertex beats the two vertices that follow it on the circle. For $n > 5$, let T be an $(n-1)$-vertex tournament in which every vertex is a king, as guaranteed by the induction hypothesis. Add a new vertex x.

If n is odd, then partition $V(T)$ into pairs. For each pair, let a and b be the tail and head of the edge joining them, and add the edges xa and bx.

If n is even, then among any four vertices of $V(T)$ we can find a triple $\{u, v, w\}$ that induces a non-cyclic tournament. Pick one such triple, and partition the remaining vertices of $V(T)$ into pairs. Treat the edges joining x to these pairs as in the other case. Letting u be the vertex of the special triple with edges to the two other vertices, add edges xu, vx, and wx.

b) There is no tournament with four players in which every player is a king. Suppose G is such a tournament. A player with no wins cannot be a king. If some vertex has no losses, then no **other** vertex can be a king. Hence every player of G has 1 or 2 wins. Since the total wins must equal the total losses, there must be two players with 1 win and two players with 2 wins. Suppose x, y are the players with 1 win; by symmetry, suppose x beats y. Since x has no other win and y has exactly one win, the fourth player is not reached in two steps from x, and x is not a king.

1.4.39. *Every loopless digraph D has a vertex subset S such that $D[S]$ has no edges but every vertex is reachable from S by a path of length at most 2.*

Proof 1 (induction). The claim holds when $n(D) = 1$ and when there is a vertex with edges to all others. Otherwise, consider an arbitrary vertex x, and let $D' = D - x - N^+(x)$. Let S' be the subset of $V(D')$ guaranteed by the induction hypothesis. Observe that $S' \cap N^+(x) = \varnothing$. If $yx \in E(D)$ for some $y \in S'$, then $x \cup N^+(x)$ is reachable from y within two steps, and S' is the desired set S. Otherwise, the set $S = S' \cup \{x\}$ works.

Proof 2 (construction). Index the vertices as $v_1, \ldots, v_n$. Process the list in increasing order; when a vertex v_i is reached that has not been deleted, delete all successors of v_i with higher indices. Next process the list in decreasing order; when a vertex v_i is reached that has not been deleted (in either pass), delete all successors of v_i with lower indices.

The set S of vertices that are not deleted in either pass is independent. Every vertex deleted in the second pass has a predecessor in S. Every vertex deleted in the first pass can be reached from S directly or from a vertex deleted in the second pass, giving it a path of length at most two from S. Hence S has the desired properties.

Proof 3 (kernels). By looking at the reverse digraph, it suffices to show that every loopless digraph D has an independent set S that can be reached by a path of length at most 2 from each vertex outside S. Given a vertex ordering $v_1, \ldots, v_n$, decompose D into two acyclic spanning subgraphs G and H consisting of the edges that are forward and backwards in the ordering, respectively. All subgraphs of G and H are acyclic, and hence by Theorem 1.4.16 they have kernels. Let S be a kernel of the subgraph of G induced by a kernel T of H. Every vertex not in T has a successor in T, and every vertex in $T - S$ has a successor in S, so every vertex not in S has a path of length at most 2 to S. (Comment: The set S produced in this way is the same set produced in the reverse digraph by Proof 2. This proof is attributed to S. Thomasse on p. 163 of J. A. Bondy, Short proofs of classical theorems, *J. Graph Theory* 44 (2003), 159–165.)

1.4.40. *The largest unipathic subgraphs of the transitive tournament have* $\lfloor n^2/4 \rfloor$ *edges.* If a subgraph of T_n contains all three edges of any 3-vertex induced subtournament, then it contains two paths from the least-indexed of these vertices to the highest. Hence a unipathic subgraph must have as its underlying graph a triangle-free subgraph of K_n. By Mantel's Theorem, the maximum number of edges in such a subgraph is $\lfloor n^2/4 \rfloor$, achieved only by the complete equibipartite graph.

This leaves the problem of finding unipathic orientations of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ in T_n. Suppose G is such a subgraph, with partite sets X, Y. If there are four vertices, say $i < j < k < l$, that alternate from the two partite sets of G or have i, l in one set and j, k in the other, then the oriented bipartite subgraph induced by X, Y as partite sets has two i, l-paths. Hence when $n \ge 4$ all the vertices of X must precede all the vertices of Y, or vice versa. To obtain $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, we will have all edges ij such that $i \le \lfloor n/2 \rfloor$ and $j > \lfloor n/2 \rfloor$, or all edges such that $i \le \lceil n/2 \rceil$ and $j > \lceil n/2 \rceil$. Hence for $n \ge 4$ there are two extremal subgraphs when n is odd and only one when n is even. (There is only one when $n = 1$, and there are three when $n = 3$.)

1.4.41. *Given any listing of the vertices of a tournament, every sequence of switchings of consecutive vertices that induce a reverse edge leads to a list with no reverse edges in at most* $\binom{n}{2}$ *steps.* Under this algorithm, each switch changes the order of only one pair. Furthermore, the order of two elements in the list can change only when they are consecutive and induce a reverse edge. Hence each pair is interchanged at most once, and the algorithm terminates after at most $\binom{n}{2}$ steps with a spanning path.

1.4.42. *Every ordering of the vertices of a tournament that minimizes the sum of lengths of the feedback edges puts the vertices in nonincreasing order of outdegree.* For the ordering $v_1, \ldots, v_n$, the sum is the sum of $j - i$ over edges $v_j v_i$ such that $j > i$. Consider the interchange of v_i and v_{i+1}. If some vertex is a successor of both or predecessor of both, then the contribution to the sum from the edges involving it remains unchanged. If $x \in N^+(v_i) - N^+(v_{i+1})$, then the switch increases the contribution from these edges by 1. If $x \in N^+(v_{i+1}) - N^+(v_i)$, then the switch decreases the contribution from these edges by 1. If $v_i \to v_{i+1}$, then the switch increases the cost by 1, otherwise it decreases. Hence the net change in the sum of the lengths of feedback edges is $d^+(v_i) - d^+(v_{i+1})$.

This implies that if the ordering has any vertex followed by a vertex with larger outdegree, then the sum can be decreased. Hence minimizing the sum puts the vertices in nonincreasing order of outdegree. Furthermore, permuting the vertices of a given outdegree among themselves does not change the sum of the lengths of feedback edges, so every ordering in nonincreasing order of outdegree minimizes the sum.

2.TREES AND DISTANCE

2.1. BASIC PROPERTIES

2.1.1. *Trees with at most 6 vertices having specified maximum degree or diameter.* For maximum degree k, we start with the star $K_{1,k}$ and append leaves to obtain the desired number of vertices without creating a vertex of larger degree. For diameter k, we start with the path P_{k+1} and append leaves to obtain the desired number of vertices without creating a longer path. Below we list all the resulting isomorphism classes.

For $k = 0$, the only tree is K_1, and for $k = 1$, the only tree is K_2 (diameter or maximum degree k). For larger k, we list the trees in the tables. Let $T_{i,j}$ denote the tree with $i + j$ vertices obtained by starting with one edge and appending $i - 1$ leaves to one endpoint and $j - 1$ leaves at the other endpoint (note that $T_{1,k} = K_{1,k}$ for $k \geq 1$). Let Q be the 6-vertex tree with diameter 4 obtained by growing a leaf from a neighbor of a leaf in P_5. Let n denote the number of vertices.

maximum degree k

k / n	2	3	4	5
3	P_3			
4	P_4	$K_{1,3}$		
5	P_5	$T_{2,3}$	$K_{1,4}$	
6	P_6	$T_{3,3}, Q$	$T_{2,4}$	$K_{1,5}$

diameter k

k / n	2	3	4	5
3	P_3			
4	$K_{1,3}$	P_4		
5	$K_{1,4}$	$T_{2,3}$	P_5	
6	$K_{1,5}$	$T_{2,4}, T_{3,3}$	Q	P_6

2.1.2. *Characterization of trees.*

a) A graph is tree if and only if it is connected and every edge is a cut-edge. An edge e is a cut-edge if and only if e belongs to no cycle, so there are no cycles if and only if every edge is a cut-edge. (To review, edge $e = uv$ is a cut edge if and only if $G - e$ has no u, v-path, which is true if and only if G has no cycle containing e.)

b) A graph is a tree if and only if for all $x, y \in V(G)$, adding a copy of xy as an edge creates exactly one cycle. The number of cycles in $G + uv$ containing the new (copy of) edge uv equals the number of u, v-paths in G, and a graph is a tree if and only if for each pair u, v there is exactly one u, v-path. Note that the specified condition must also hold for addition of extra copies of edges already present; this excludes cliques.

2.1.3. *A graph is a tree if and only if it is loopless and has exactly one spanning tree.* If G is a tree, then G is loopless, since G is acyclic. Also, G is a spanning tree of G. If G contains another spanning tree, then G contains another edge not in G, which is impossible.

Let G be loopless and have exactly one spanning tree T. If G has a edge e not in T, then $T + e$ contains exactly one cycle, because T is a tree. Let f be another edge in this cycle. Then $T + e - f$ contains no cycle. Also $T+e-f$ is connected, because deleting an edge of a cycle cannot disconnect a graph. Hence $T + e - f$ is a tree different from T. Since G contains no such tree, G cannot contain an edge not in T, and G is the tree T.

2.1.4. *Every graph with fewer edges than vertices has a component that is a tree—TRUE.* Since the number of vertices or edges in a graph is the sum of the number in each component, a graph with fewer edges than vertices must have a component with fewer edges than vertices. By the properties of trees, such a component must be a tree.

2.1.5. *A maximal acyclic subgraph of a graph G consists of a spanning tree from each component of G.* We show that if H is a component of G and F is a maximal forest in G, then $F \cap H$ is a spanning tree of H. We may assume that F contains all vertices of G; if not, throw the missing ones in as isolated points to enlarge the forest. Note that $F \cap H$ contains no cycles, since F contains no cycles and $F \cap H$ is a subgraph of F.

We need only show that $F \cap H$ is a connected subgraph of H. If not, then it has more than one component. Since F is spanning and H is connected, H contains an edge between two of these components. Add this edge to F and $F \cap H$. It cannot create a cycle, since F previously did not contain a path between its endpoints. We have made F into a larger forest (more edges), which contradicts the assumption that it was maximal. (Note: the subgraph consisting of all vertices and no edges of G is a spanning subgraph of G; spanning means only that all the vertices appear, and says nothing about connectedness.

2.1.6. *Every tree with average degree a has $2/(2 - a)$ vertices.* Let the tree have n vertices and m edges. The average degree is the degree sum divided by n, the degree sum is twice m, and m is $n - 1$. Thus $a = \sum d_i/n = 2(n - 1)/n$. Solving for n yields $n = 2/(2 - a)$.

2.1.7. *Every n-vertex graph with m edges has at least $m - n + 1$ cycles.* Let k be the number of components in such a graph G. Choosing a spanning tree from each component uses $n-k$ edges. Each of the remaining $m-n+k$ edges completes a cycle with edges in this spanning forest. Each such cycle has one edge not in the forest, so these cycles are distinct. Since $k \ge 1$, we have found at least $m-n+1$ cycles.

2.1.8. *Characterization of simple graphs that are forests.*

a) A simple graph is a forest if and only if every induced subgraph has a vertex of degree at most 1. If G is a forest and H is an induced subgraph of G, then H is also a forest, since cycles cannot be created by deleting edges. Every component of H is a tree, which is an isolated vertex or has a leaf (a vertex of degree 1). If G is not a forest, then G contains a cycle. A shortest cycle in G has no chord, since that would yield a shorter cycle, and hence a shortest cycle is an induced subgraph. This induced subgraph is 2-regular and has no vertex of degree at most 1.

b) A simple graph is a forest if and only if every connected subgraph is an induced subgraph. If G has a connected subgraph H that is not an induced subgraph, then G has an edge xy not in H with endpoints in $V(H)$. Since H contains an x,y-path, $H+xy$ contains a cycle, and G is not a forest. Conversely, if G is not a forest, then G has a cycle C, and every subgraph of G obtained by deleting one edge from C is connected but not induced.

c) The number of components is the number of vertices minus the number of edges. In a forest, each component is a tree and has one less edge than vertex. Hence a forest with n vertices and k components has $n-k$ edges.

Conversely, every component with n_i vertices has at least $n_i - 1$ edges, since it is connected. Hence the number of edges in an n-vertex is n minus the number of components only if every component with n_i vertices has $n_i - 1$ edges. Hence every component is a tree, and the graph is a forest.

2.1.9. *For $2 \le k \le n-1$, the n-vertex graph formed by adding one vertex adjacent to every vertex of P_{n-1} has a spanning tree with diameter k.* Let $v_1, \dots, v_{n-1}$ be the vertices of the path in order, and let x be the vertex adjacent to all of them. The spanning tree consisting of the path $v_1, \dots, v_{k-1}$ and the edges $xv_{k-1}, \dots, xv_{n-1}$ has diameter k.

2.1.10. *If u and v are vertices in a connected n-vertex simple graph, and $d(u,v) > 2$, then $d(u) + d(v) \le n+1-d(u,v)$.* Since $d(u,v) > 2$, we have $N(u) \cap N(v) = \varnothing$, and hence $d(u)+d(v) = |N(u) \cup N(v)|$. Let $k = d(u,v)$. Between u and v on a shortest u,v-path are vertices $x_1, \dots, x_{k-1}$. Since this is a shortest u,v-path, vertices u, v and $x_2, \dots, x_{k-2}$ are forbidden from the neighborhoods of both u and v. Hence $|N(u) \cup N(v)| \le n+1-k$.

The inequality fails when $d(u,v) \le 2$, because in this case u and v can have many common neighbors. When $d(u,v) = 2$, the sum $d(u)+d(v)$ can be as high as $2n-4$.

2.1.11. *If x and y are adjacent vertices in a graph G, then always $|d_G(x,z) - d_G(y,z)| \le 1$.* A z,y-path can be extended (or trimmed) to reach x, and hence $d(z,x) \le d(z,y)+1$. Similarly, $d(z,y) \le d(z,x)+1$. Together, these yield $|d(z,x) - d(z,y)| \le 1$.

2.1.12. *Diameter and radius of $K_{m,n}$.* Every vertex has eccentricity 2 in $K_{m,n}$ if $m,n \ge 2$, which yields radius and diameter 2. For $K_{1,n}$, the radius is 1 and diameter is 2 if $n > 1$. The radius and diameter of $K_{1,1}$ are 1. The radius and diameter of $K_{0,n}$ are infinite if $n > 1$, and both are 0 for $K_{0,1}$.

2.1.13. *Every graph with diameter d has an independent set of size at least $\lceil (1+d)/2 \rceil$.* Let x, y be vertices with $d(x,y) = d$. Vertices that are nonconsecutive on a shortest x,y-path P are nonadjacent. Taking x and every second vertex along P produces an independent set of size $\lceil (1+d)/2 \rceil$.

2.1.14. *Starting a shortest path in the hypercube.* The distance between vertices in a hypercube is the number of positions in which their names differ. From u, a shortest u,v-path starts along any edge to a neighbor whose name differ from u in a coordinate where v also differs from u.

2.1.15. *The complement of a simple graph with diameter at least 4 has diameter at most 2.* The contrapositive of the statement is that if $\overline{G}$ has diameter at least 3, then G has diameter at most 3. Since $G = \overline{\overline{G}}$, this statement has been proved in the text.

2.1.16. *The "square" of a connected graph G has diameter $\lceil \mathrm{diam}\,(G)/2 \rceil$.* The square is the simple graph G' with $x \leftrightarrow y$ in G' if and only if $d_G(x,y) \le 2$. We prove the stronger result that $d_{G'}(x,y) = \lceil d_G(x,y)/2 \rceil$ for every $x, y \in V(G)$. Given an x,y-path P of length k, we can skip the odd vertices along P to obtain an x,y-path of length $\lceil k/2 \rceil$ in G'.

On the other hand, every x,y-path of length l in G' arises from a path of length at most $2l$ in G. Hence the shortest x,y-path in G' comes from the shortest x,y-path in G by the method described, and $d_{G'}(x,y) = \lceil d_G(x,y)/2 \rceil$. Hence

$$\mathrm{diam}\,(G') = \min_{x,y} d_{G'}(x,y) = \min_{x,y} \left\lceil \frac{d_G(x,y)}{2} \right\rceil = \left\lceil \min_{x,y} \frac{d_G(x,y)}{2} \right\rceil = \left\lceil \frac{\mathrm{diam}\,(G)}{2} \right\rceil.$$

2.1.17. *If an n-vertex graph G has $n-1$ edges and no cycles, then it is connected.* Let k be the number of components of G. If $k > 1$, then we adding an edge with endpoints in two components creates no cycles and reduces the number of components by 1. Doing this $k-1$ times creates a graph with $(n-1)+(k-1)$ edges that is connected and has no cycles. Such a graph is a tree and has $n-1$ edges. Therefore, $k = 1$, and the original graph G was connected.

2.1.18. *If G is a tree, then G has at least $\Delta(G)$ leaves.* Let $k = \Delta(G)$. Given $n > k \geq 2$, we cannot guarantee more leaves, as shown by growing a path of length $n - k - 1$ from a leaf of $K_{1,k}$.

Proof 1a (maximal paths). Deleting a vertex x of degree k produces a forest of k subtrees, and x has one neighbor w_i in the ith subtree G_i. Let P_i be a maximal path starting at x along the edge xw_i. The other end of P_i must be a leaf of G and must belong to G_i, so these k leaves are distinct.

Proof 1b (leaves in subtrees). Deleting a vertex x of degree k produces a forest of k subtrees. Each subtree is a single vertex, in which case the vertex is a leaf of G, or it has at least two leaves, of which at least one is not a neighbor of x. In either case we obtain a leaf of the original tree in each subtree.

Proof 2 (counting two ways). Count the degree sum by edges and by vertices. By edges, it is $2n - 2$. Let k be the maximum degree and l the number of leaves. The remaining vertices must have degree at least two each, so the degree sum when counted by vertices is at least $k + 2(n - l - 1) + l$. The inequality $2n - 2 \geq k + 2(n - l - 1) + 1$ simplifies to $l \geq k$. (Note: Similarly, degree $2(n - 1) - k$ remains for the vertices other than a vertex of maximum degree. Since all degrees are 1 or at least 2, there must be at least k vertices of degree 1.)

Proof 3: Induction on the number of vertices. For $n \leq 3$, this follows by inspecting the unique tree on n vertices. For $n > 3$, delete a leaf u. If $\Delta(T - u) = \Delta(T)$, then by the induction hypothesis $T - u$ has at least k leaves. Replacing u adds a leaf while losing at most one leaf from $T - u$. Otherwise $\Delta(T - u) = \Delta(T) - 1$, which happens only if the neighbor of u is the only vertex of maximum degree in T. Now the induction hypothesis yields at least $k - 1$ leaves in $T - u$. Replacing u adds another, since the vertex of maximum degree in T cannot be a leaf in $T - u$ (this is the reason for putting $n = 3$ in the basis step).

2.1.19. *If n_i denotes the number of vertices of degree i in a tree T, then $\sum in_i$ depends only on the number of vertices in T.* Since each vertex of degree i contributes i to the sum, the sum is the degree-sum, which equals twice the number of edges: $2n(T) - 2$.

2.1.20. *Hydrocarbon formulas C_kH_l.* The global method is the simplest one. With cycles forbidden, there are $k + l - 1$ "bonds" - i.e., edges. Twice this must equal the degree sum. Hence $2(k + l - 1) = 4k + l$, or $l = 2k + 2$.

Alternatively, (sigh), proof by induction. Basis step ($k = 1$): The formula holds for the only example. Induction step ($k > 1$): In the graph of the molecule, each H has degree 1. Deleting these vertices destroys no cycles, so the subgraph induced by the C-vertices is also a tree. Pick a leaf x in this tree. In the molecule it neighbors one C and three Hs. Replacing x and these three Hs by a single H yields a molecule with one less C that also satisfies the conditions. Applying the induction hypothesis yields $l = [2(k - 1) + 2] - 1 + 3 = 2k + 2$.

2.1.21. *If a simple n-vertex graph G has a decomposition into k spanning trees, and $\Delta(G) = \delta(G) + 1$, then $2k < n$, and G has $n - 2k$ vertices of degree $2k$ and $2k$ vertices of degree $2k - 1$.* Since every spanning tree of G has $n - 1$ edges, we have $e(G) = k(n - 1)$. Since $e(G) \leq n(n - 1)/2$ edges, this yields $k \leq n/2$. Equality requires $G = K_n$, but $\Delta(K_n) = \delta(K_n)$. Thus $2k < n$.

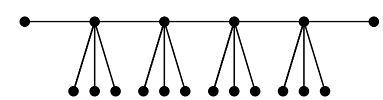
To determine the degree sequence, let l be the number of vertices of degree $\delta(G)$. By the degree-sum formula, $n\Delta(G) - l = 2kn - 2k$. Both sides are between two multiples of n. Since $0 < 2k < n$ and $0 < l < n$, the higher multiple of n is $n\Delta(G) = 2kn$, so $\Delta(G) = 2k$. It then also follows that $l = 2k$. Hence there are $n - 2k$ vertices of degree $2k$ and $2k$ vertices of degree $2k - 1$.

2.1.22. *A tree with degree list $k, k - 1, \ldots, 2, 1, 1, \ldots, 1$ has $2 + \binom{k}{2}$ vertices.* Since the tree has n vertices and $k - 1$ non-leaves, it has $n - k + 1$ leaves. Since $\sum_{i=1}^{k} i = k(k+1)/2$, the degrees of the vertices sum to $k(k+1)/2 + n - k$. The degree-sum is twice the number of edges, and the number of edges is $n - 1$. Thus $k(k + 1)/2 + n - k = 2n - 2$. Solving for n yields $n = 2 + \binom{k}{2}$.

2.1.23. *For a tree T with vertex degrees in $\{1, k\}$, the possible values of $n(T)$ are the positive integers that are 2 more than a multiple of $k - 1$.*

Proof 1 (degree-sum formula). Let m be the number of vertices of degree k. By the degree-sum formula, $mk + (n(T) - m) = 2n(T) - 2$, since T has $n(T) - 1$ edges. The equation simplifies to $n(T) = m(k - 1) + 2$. Since m is a nonnegative integer, $n(T)$ must be two more than a multiple of $k - 1$.

Whenever $n = m(k - 1) + 2$, there is such a tree (not unique for $m \geq 4$). Such a tree is constructed by adjoining $k - 2$ leaves to each internal vertex of a path of length $m + 1$, as illustrated below for $m = 4$ and $k = 5$.



Proof 2 (induction on m, the number of vertices of degree k). We proof that if T has m vertices of degree k, then $n(T) = m(k - 1) + 2$ If $m = 0$, then the tree must have two vertices.

For the induction step, suppose that $m > 0$. For a tree T with m vertices of degree k and the rest of degree 1, let T' be the tree obtained by deleting all the leaves. The tree T' is a tree whose vertices all have degree k in T. Let x be a leaf of T'. In T, x is adjacent to one non-leaf and to $k - 1$ leaves. Deleting the leaf neighbors of x leaves a tree T'' with $m - 1$ vertices of degree k and the rest of degree 1. By the induction hypothesis,

$n(T'') = (m-1)(k-1)+2$. Since we deleted $k-1$ vertices from T to obtain T'', we obtain $n(T) = m(k-1)+2$. This completes the induction step.

To prove inductively that all such values arise as the number of vertices in such a tree, we start with K_2 and iteratively expand a leaf into a vertex of degree k to add $k-1$ vertices.

2.1.24. *Every nontrivial tree has at least two maximal independent sets, with equality only for stars.* A nontrivial tree has an edge. Each vertex of an edge can be augmented to a maximal independent set, and these must be different, since each contains only one vertex of the edge. A star has exactly two maximal independent sets; the set containing the center cannot be enlarged, and the only maximal independent set not containing the center contains all the other vertices. If a tree is not a star, then it contains a path a, b, c, d. No two of the three independent sets $\{a, c\}$, $\{b, d\}$, $\{a, d\}$ can appear in a single independent set, so maximal independent sets containing these three must be distinct.

2.1.25. *Among trees with n vertices, the star has the most independent sets (and is the only tree with this many).*

Proof 1 (induction on n). For $n = 1$, there is only one tree, the star. For $n > 1$, consider a tree T. Let x be a leaf, and let y be its neighbor. The independent sets in T consist of the independent sets in $T-x$ and all sets formed by adding x to an independent set in $T-x-y$. By the induction hypothesis, the first type is maximized (only) when $T-x$ is a star. The second type contributes at most 2^{n-2} sets, and this is achieved only when $T-x-y$ has no edges, which requires that $T-x$ is a star with center at y. Thus both contributions are maximized when (and only when) T is a star with center y.

Proof 2 (counting). If an n-vertex tree T is not a star, then it contains a copy H of P_4. Of the 16 vertex subsets of $V(H)$, half are independent and half are not. If S is an independent set in T, then $S \cap V(H)$ is also independent. When we group the subsets of $V(T)$ by their intersection with $V(T) - V(H)$, we thus find that at most half the sets in each group are independent. Summing over all groups, we find that at most half of all subsets of $V(T)$, or 2^{n-1}, are independent. However, the star $K_{1,n-1}$ has $2^{n-1}+1$ independent sets.

2.1.26. *For $n \ge 3$, if G is an n-vertex graph such that every graph obtained by deleting one vertex of G is a tree, then $G = C_n$.* Let G_i be the graph obtained by deleting vertex v_i. Since G_i has $n-1$ vertices and is a tree, $e(G_i) = n-2$. Thus $\sum_{i=1}^{n} e(G_i) = n(n-2)$. Since each edge has two endpoints, each edge of G appears in $n-2$ of these graphs and thus is counted $n-2$ times in the sum. Thus $e(G) = n$.

Since G has n vertices and n edges, G must contain a cycle. Since G_i has no cycle, every cycle in G must contain v_i. Since this is true for all i, every cycle in G must contain every vertex. Thus G has a spanning cycle, and since G has n edges it has no additional edges, so $G = C_n$.

2.1.27. *If $n \ge 2$ and $d_1, \ldots, d_n$ are positive integers, then there exists a tree with these as its vertex degrees if and only if $d_n = 1$ and $\sum d_i = 2(n-1)$.* (Some graphs with such degree lists are not trees.) *Necessity:* Every n-vertex tree is connected and has $n-1$ edges, so every vertex has degree at least 1 (when $n \ge 2$) and the total degree sum is $2(n-1)$. *Sufficiency:* We give several proofs.

Proof 1 (induction on n). Basis step ($n = 2$): The only such list is $(1, 1)$, which is the degree list of the only tree on two vertices. Induction step ($n > 2$): Consider $d_1, \ldots, d_n$ satisfying the conditions. Since $\sum d_i > n$, some element exceeds 1. Since $\sum d_i < 2n$, some element is at most 1. Let d' be the list obtain by subtracting 1 from the largest element of d and deleting an element that equals 1. The total is now $2(n-2)$, and all elements are positive, so by the induction hypothesis there is a tree on $n-1$ vertices with d' as its vertex degrees. Adding a new vertex and an edge from it to the vertex whose degree is the value that was reduced by 1 yields a tree with the desired vertex degrees.

Proof 2 (explicit construction). Let k be the number of 1s in the list d. Since the total degree is $2n-2$ and all elements are positive, $k \ge 2$. Create a path $x, u_1, \ldots, u_{n-k}, y$. For $1 \le i \le n-k$, attach $d_i - 2$ vertices of degree 1 to u_i. The resulting graph is a tree (not the only one with this degree list), and it gives the proper degree to u_i. We need only check that we have the desired number of leaves. Counting x and y and indexing the list so that $d_1, \ldots, d_n \ge$, we compute the number of leaves as

$$2 + \sum_{i=1}^{n-k} (d_i - 2) = 2 - 2(n-k) + \sum_{i=1}^{n} d_i - \sum_{i=n-k+1}^{n} d_i = 2 - 2(n-k) + 2(n-1) - k = k.$$

Proof 3 (extremality). Because $\sum d_i = 2(n-1)$, which is even, there is a graph with n vertices and $n-1$ edges that realizes d. Among such graphs, let G (having k components) be one with the fewest components. If $k = 1$, then G is a connected graph with $n-1$ edges and is the desired tree.

If $k > 1$ and G is a forest, then G has $n-k$ edges. Therefore, G has a cycle. Let H be a component of G having a cycle, and let uv be an edge of the cycle. Let H' be another component of G. Because each d_i is positive, H' has an edge, xy. Replace the edges uv and xy by ux and vy (either uv or xy could be a loop.) Because uv was in a cycle, the subgraph induced by $V(H)$ is still connected. The deletion of vy might disconnect H', but each piece is now connected to $V(H)$, so the new graph G' realizes d with fewer components than G, contradicting the choice of G.

2.1.28. *The nonnegative integers $d_1 \ge \cdots \ge d_n$ are the degree sequence of some connected graph if and only if $\sum d_i$ is even, $d_n \ge 1$, and $\sum d_i \ge 2n-2$.* This claim does not hold for simple graphs because the conditions $\sum d_i$ even, $d_n \ge 1$, and $\sum d_i \ge 2n-2$ do not prevent $d_1 \ge n$, which is impossible for a simple graph. Hence we allow loops and multiple edges. Necessity follows because every graph has even degree sum and every connected graph has a spanning tree with $n-1$ edges. For sufficiency, we give several proofs.

Proof 1 (extremality). Since $\sum d_i$ is even, there is a graph with degrees $d_1, \ldots, d_n$. Consider a realization G with the fewest components; since $\sum d_i \ge 2n-2$, G has at least $n-1$ edges. If G has more than one component, then some component as many edges as vertices and thus has a cycle. A 2-switch involving an edge on this cycle and an edge in another component reduces the number of components without changing the degrees. The choice of G thus implies that G has only one component.

Proof 2 (induction on n). For $n = 1$, we use loops. For $n = 2$, if $d_1 = d_2$, then we use d_1 parallel edges. Otherwise, we have $n > 2$ or $d_1 > d_2$. Form a new list $d_1', \ldots, d_{n-1}'$ by deleting d_n and subtracting d_n units from other values. If $n \ge 3$ and $d_n = 1$, we subtract 1 from d_1, noting that $\sum d_i \ge 2n-2$ implies $d_1 > 1$. If $n \ge 3$ and $d_n > 1$, we make the subtractions from any two of the other numbers. In each case, the resulting sequence has even sum and all entries at least 1.

Letting $D = \sum d_i$, we have $\sum d_i' = D - 2d_n$. If $d_n = 1$, then $D - 2d_n \ge 2n-2-2 = 2(n-1)-2$. If $d_n > 1$, then $D \ge nd_n$, and so $D-2d_n \ge (n-2)d_n \ge 2n-4 = 2(n-1)-2$. Hence the new values satisfy the condition stated for a set of $n-1$ values. By the induction hypothesis, there is a connected graph G' with vertex degrees $d_1', \ldots, d_{n-1}'$.

To obtain the desired graph G, add a vertex v_n with $d_i - d_i'$ edges to the vertex with degree d_i, for $1 \le i \le n-1$. This graph G is connected, because a path from v_n to any other vertex v can be construct by starting from v_n to a neighbor and continuing with a path to v in G'.

Proof 3 (induction on $\sum d_i$ and prior result). If $\sum d_i = 2n-2$, then Exercise 2.1.27 applies. Otherwise, $\sum d_1 \ge 2n$. If $n = 1$, then we use loops. If $n > 1$, then we can delete 2 from d_1 or delete 1 from d_1 and d_2 without introducing a 0. After applying the induction hypothesis, adding one loop at v_1 or one edge from v_1 to v_2 restores the desired degrees.

2.1.29. *Every tree has a leaf in its larger partite set (in both if they have equal size).* Let X and Y be the partite sets of a tree T, with $|X| \ge |Y|$. If there is no leaf in X, then $e(T) \ge 2|X| = |X| + |X| \ge |X| + |Y| = n(T)$. This contradicts $e(T) < n(T)$.

2.1.30. *If T is a tree in which the neighbor of every leaf has degree at least 3, then some pair of leaves have a common neighbor.*

Proof 1 (extremality). Let P a longest path in T, with endpoint v adjacent to u. Since v is a leaf and u has only one other neighbor on P, u must have a neighbor w off P. If w has a neighbor $z \ne u$, then replacing (u, v) by (u, w, z) yields a longer path. Hence w is a leaf, and v, w are two leaves with a common neighbor.

Proof 2 (contradiction). Suppose all leaves of T have different neighbors. Deleting all leaves (and their incident edges) reduces the degree of each neighbor by 1. Since the neighbors all had degree at least 3, every vertex now has degree at least 2, which is impossible in an acyclic graph.

Proof 3 (counting argument). Suppose all k leaves of T have different neighbors. The $n-2k$ vertices other than leaves and their neighbors have degree at least 2, so the total degree is at least $k + 3k + 2(n-2k) = 2n$, contradicting $\sum d(v) = 2e(T) = 2n-2$.

Proof 4 (induction on $n(T)$). For $n = 4$, the only such tree is $K_{1,3}$, which satisfies the claim. For $n > 4$, let v be a leaf of T, and let w be its neighbor. If w has no other leaf as neighbor, but has degree at least 3, then $T - v$ is a smaller tree satisfying the hypotheses. By the induction hypothesis, $T - v$ has a pair of leaves with a common neighbor, and these form such a pair in T.

2.1.31. *A simple connected graph G with exactly two non-cut-vertices is a path.* **Proof 1** (properties of trees). Every connected graph has a spanning tree. Every leaf of a spanning tree is not a cut-vertex, since deleting it leaves a tree on the remaining vertices. Hence every spanning tree of G has only two leaves and is a path. Consider a spanning path with vertices $v_1, \ldots, v_n$ in order. If G has an edge v_iv_j with $i < j-1$, then adding v_iv_j to the path creates a cycle, and deleting $v_{j-1}v_j$ from the cycle yields another spanning tree with three leaves. Hence G has no edge off the path.

Proof 2 (properties of paths and distance). Let x and y be the non-cut-vertices, and let P be a shortest x, y-path. If $V(P) \ne V(G)$, then let w be a vertex with maximum distance from $V(P)$. By the choice of w, every vertex of $V(G) - V(P) - \{w\}$ is as close to $V(P)$ as w and hence reaches $V(P)$ by a path that does not use w. Hence w is a non-cut-vertex. Thus $V(P) = V(G)$. Now there is no other edge, because P was a shortest x, y-path.

2.1.32. *Characterization of cut-edges and loops.*

An edge of a connected graph is a cut-edge if and only if it belongs to every spanning tree. If G has a spanning tree T omitting e, then e belongs to a cycle in $T + e$ and hence is not a cut-edge in G. If e is not a cut-edge in G, then $G - e$ is connected and contains a spanning tree T that is also a spanning tree of G; thus some spanning tree omits e.

An edge of a connected graph is a loop if and only if it belongs to no spanning tree. If e is a loop, then e is a cycle and belongs to no spanning tree. If e is not a loop, and T is a spanning tree not containing e, then $T+e$ contains exactly one cycle, which contains another edge f. Now $T+e-f$ is a spanning tree containing e, since it has no cycle, and since deleting an edge from a cycle of the connected graph $T+e$ cannot disconnect it.

2.1.33. *A connected graph with n vertices has exactly one cycle if and only if it has exactly n edges.* Let G be a connected graph with n vertices. If G has exactly one cycle, then deleting an edge of the cycle produces a connected graph with no cycle. Such a graph is a tree and therefore has $n-1$ edges, which means that G has n edges.

For the converse, suppose that G has exactly n edges. Since G is connected, G has a spanning tree, which has $n-1$ edges. Thus G is obtained by adding one edge to a tree, which creates a graph with exactly one cycle.

Alternatively, we can use induction. If G has exactly n edges, then the degree sum is $2n$, and the average degree is 2. When $n=1$, the graph must be a loop, which is a cycle. When $n>2$, if G is 2-regular, then G is a cycle, since G is connected. If G is not 2-regular, then it has a vertex v of degree 1. Let $G'=G-v$. The graph G' is connected and has $n-1$ vertices and $n-1$ edges. By the induction hypothesis, G' has exactly one cycle. Since a vertex of degree 1 belongs to no cycle, G also has exactly one cycle.

2.1.34. *A simple n-vertex graph G with $n>k$ and $e(G)>n(G)(k-1)-\binom{k}{2}$ contains a copy of each tree with k edges.* We use induction on n. For the basis step, let G be a graph with $k+1$ vertices. The minimum allowed number of edges is $(k+1)(k-1)-\binom{k}{2}+1$, which simplifies to $\binom{k}{2}$. Hence $G=K_{k+1}$, and $T\subseteq G$.

For the induction step, consider $n>k+1$. If every vertex has degree at least k, then containment of T follows from Proposition 2.1.8. Otherwise, deleting a vertex of minimum degree (at most $k-1$) yields a subgraph G' on $n-1$ vertices with more than $(n-1)(k-1)-\binom{k}{2}$ edges. By the induction hypothesis, G' contains T, and hence $T\subseteq G$.

2.1.35. *The vertices of a tree T all have odd degree if and only if for all $e\in E(T)$, both components of $T-e$ have odd order.*

Necessity. If all vertices have odd degree, then deleting e creates two of even degree. By the Degree-sum Formula, each component of $T-e$ has an even number of odd-degree vertices. Together with the vertex incident to e, which has even degree in $T-e$, each component of $T-e$ has odd order.

Sufficiency.

Proof 1 (parity). Given that both components of $T-e$ have odd order, $n(T)$ is even. Now consider $v\in V(T)$. Deleting an edge incident to v yields a component containing v and a component not containing v, each of odd order. Together, the components not containing v when we delete the various edges incident to v are $d(v)$ pairwise disjoint subgraphs that together

contain all of $V(T)-\{v\}$. Under the given hypothesis, they all have odd order. Together with v, they produce an even total, $n(T)$. Hence the number of these subgraphs is odd, which means that the number of edges in T incident to v is odd.

Proof 2 (contradiction). Suppose that such a tree T_0 has a vertex v_1 of even degree. Let e_1 be the last edge on a path from a leaf to x. Let T_1 be the component of T_0-e_1 containing v_1. By hypothesis, T_1 has odd order, and v_1 is a vertex of odd degree in T_1. Since the number of odd-degree vertices in T_1 must be even, there is a vertex v_2 of T_1 (different from v_1) having even degree (in both T_1 and T).

Repeating the argument, given v_i of even degree in T_{i-1}, let e_i be the last edge on the v_{i-1},v_i-path in T_{i-1}, and let T_i be the component of $T_{i-1}-e_i$ containing v_i. Also T_i is the component of T_0-e_i that contains v_i, so T_i has odd order. Since v_i has odd degree in T_i, there must be another vertex v_{i+1} with even degree in T_i.

In this way we generate an infinite sequence $v_1,v_2,\ldots$ of distinct vertices in T_0. This contradicts the finiteness of the vertex set, so the assumption that T_0 has a vertex of even degree cannot hold.

2.1.36. *Every tree T of even order has exactly one subgraph in which every vertex has odd degree.*

Proof 1 (Induction). For $n(T)=2$, the only such subgraph is T itself. Suppose $n(T)>2$. Observe that every pendant edge must appear in the subgraph to give the leaves odd degree. Let x be an endpoint of a longest path P, with neighbor u. If u has another leaf neighbor y, add ux and uy to the unique such subgraph found in $T-\{x,y\}$. Otherwise, $d(u)=2$, since P is a longest path. In this case, add the isolated edge ux to the unique such subgraph found in $T-\{u,x\}$.

Proof 2 (Explicit construction). Every edge deletion breaks T into two components. Since the total number of vertices is even, the two components of $T-e$ both have odd order or both have even order. We claim that the desired subgraph G consists of all edges whose deletion leaves two components of odd order.

First, every vertex has odd degree in this subgraph. Consider deleting the edges incident to a vertex u. Since the total number of vertices in T is even, the number of resulting components other than u itself that have odd order must be odd. Hence u has odd order in G.

Furthermore, G is the only such subgraph. If e is a cut-edge of G, then in $G-e$ the two pieces must each have even degree sum. Given that G is a subgraph of T with odd degree at each vertex, parity of the degree sum forces G to e if $T-e$ has components of odd order and omit e if $T-e$ has components of even order.

Comment: Uniqueness also follows easily from symmetric difference. Given two such subgraphs G_1, G_2 , the degree of each vertex in the symmetric difference is even, since its degree is odd in each G_i. This yields a cycle in $G_1 \cup G_2 \subseteq T$, which is impossible.

2.1.37. *If T and T' are two spanning trees of a connected graph G, and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that both $T - e + e'$ and $T' - e' + e$ are spanning trees of G.* Deleting e from T leaves a graph having two components; let U, U' be their vertex sets. Let the endpoints of e be $u \in U$ and $u' \in U'$. Being a tree, T' contains a unique u, u'-path. This path must have an edge from U to U'; choose such an edge to be e', and then $T - e + e'$ is a spanning tree. Since e is the only edge of T between U and U', we have $e' \in E(T') - E(T)$. Furthermore, since e' is on the u, u'-path in T', e' is on the unique cycle formed by adding e to T', and thus $T' - e' + e$ is a spanning tree. Hence e' has all the desired properties.

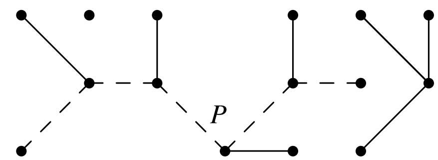
2.1.38. *If T and T' are two trees on the same vertex set such that $d_T(v) = d'_T(v)$ for each vertex v, then T' can be obtained from T' using 2-switches (Definition 1.3.32) with every intermediate graph being a tree.* Using induction on the number n of vertices, it suffices to show when $n \ge 4$ that we can apply (at most) one 2-switch to T to make a given leaf x be adjacent to its neighbor w in T'. We can then delete x from both trees and apply the induction hypothesis. Since the degrees specify the tree when n is at most 3, this argument also shows that at most $n - 3$ 2-switches are needed.

Let y be the neighbor of x in T. Note that w is not a leaf in T, since $d_{T'}(w) = d_T(w)$ and $xw \in E(T)$ and $n \ge 4$. Hence we can choose a vertex z in T that is a neighbor of w not on the x, w-path in T. Cutting xy and wz creates three components: x alone, one containing z, and one containing y, w. Adding the edges zy and xw to complete the 2-switch gives x its desired neighbor and reconnects the graph to form a new tree.

2.1.39. *If G is a nontrivial tree with $2k$ vertices of odd degree, then G decomposes into k paths.*

Proof 1 (induction and stronger result). We prove the claim for every forest G, using induction on k. Basis step ($k = 0$): If $k = 0$, then G has no leaf and hence no edge.

Induction step ($k > 0$): Suppose that each forest with $2k - 2$ vertices of odd degree has a decomposition into $k - 1$ paths. Since $k > 0$, some component of G is a tree with at least two vertices. This component has at least two leaves; let P be a path connecting two leaves. Deleting $E(P)$ changes the parity of the vertex degree only for the endpoints of P; it makes them even. Hence $G - E(P)$ is a forest with $2k - 2$ vertices of odd degree. By the induction hypothesis, $G - E(P)$ is the union of $k - 1$ pairwise edge-disjoint paths; together with P, these paths partition $E(G)$.



Proof 2 (extremality). Since there are $2k$ vertices of odd degree, at least k paths are needed. If two endpoints of paths occur at the same vertex of the tree, then those paths can be combined to reduce the number of paths. Hence a decomposition using the fewest paths has at most one endpoint at each vertex. Under this condition, endpoints occur only at vertices of odd degree. There are $2k$ of these. Hence there are at most $2k$ endpoints of paths and at most k paths.

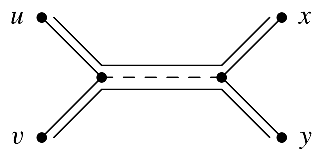
Proof 3 (applying previous result). A nontrivial tree has leaves, so $k > 0$. By Theorem 1.2.33, G decomposes into k trails. Since G has no cycles, all these trails are paths.

2.1.40. *If G is a tree with k leaves, then G is the union of $\lceil k/2 \rceil$ pairwise intersecting paths.* We prove that we can express G in this way using paths that end at leaves. First consider any way of pairing the leaves as ends of $\lceil k/2 \rceil$ paths (one leaf used twice when k is odd). Suppose that two of the paths are disjoint; let these be a u, v-path P and an x, y-path Q. Let R be the path connecting P and Q in G. Replace P and Q by the u, x-path and the v, y-path in G. These paths contain the same edges as P and Q, plus they cover R twice (and intersect). Hence the total length of the new set of paths is larger than before.

Continue this process; whenever two of the paths are disjoint, make a switch between them that increases the total length of the paths. This process cannot continue forever, since the total length of the paths is bounded by the number of paths ($\lceil k/2 \rceil$) times the maximum path length (at most $n - 1$). The process terminates only when the set of paths is pairwise intersecting. (We have not proved that some vertex belongs to all the paths.)

Finally, we show that a pairwise intersecting set of paths containing all the leaves must have union G. If any edge e of G is missing, then $G - e$ has two components H, H', each of which contains a leaf of G. Since e belongs to none of the paths, the paths using leaves in H do not intersect the paths using leaves in H'. This cannot happen, because the paths are pairwise intersecting.

(*Comment*: We can phrase the proof using extremality. The pairing with maximum total length has the desired properties; otherwise, we make a switch as above to increase the total length.)



2.1.41. *For $n \geq 4$, a simple n-vertex graph with at least $2n-3$ edges must have two cycles of equal length.* For such a graph, some component must have size at least twice its order minus 3. Hence we may assume that G is connected. A spanning tree T has $n-1$ edges and diameter at most $n-1$. Each remaining edge completes a cycle with edges of T. The lengths of these cycles belong to $\{3,\ldots,n\}$.

Since there are at least $n-2$ remaining edges, there are two cycles of the same length unless there are exactly $n-2$ remaining cycles and they create cycles of distinct lengths with the edge of T. This forces T to be a path. Now, after adding the edge e between the endpoints of T that produces a cycle of length n, the other remaining edges each produce two additional shorter cycles when added. These $2n-6$ additional cycles fall into the $n-3$ lengths $\{3,\ldots,n-1\}$. Since $2n-6 > n-3$ when $n \geq 4$, the pigeonhole principle yields two cycles of equal length.

2.1.42. *Extendible vertices.* In a nontrivial Eulerian graph G, a vertex is *extendible* if every trail beginning at v extends to an Eulerian circuit.

a) v is extendible if and only if $G - v$ is a forest.

Necessity. We prove the contrapositive. If $G - v$ is not a forest, then $G - v$ has a cycle C. In $G - E(C)$, every vertex has even degree, so the component of $G - E(C)$ containing v has an Eulerian circuit. This circuit starts and ends at v and exhausts all edges of G incident to v, so it cannot be extended to reach C and complete an Eulerian circuit of G.

Sufficiency. If $G-v$ is a forest, then every cycle of G contains v. Given a trail T starting at v, extend it arbitarily at the end until it can be extended no farther. Because every vertex has even degree, the process can end only at v. The resulting closed trail T' must use every edge incident to v, else it could extend farther. Since T' is closed, every vertex in $G - E(T')$ has even degree. If $G - E(T')$ has any edges, then minimum degree at least two in a component of $G - E(T')$ yields a cycle in $G - E(T')$; this cycle avoids v, since T' exhausted the edges incident to v. Since we have assumed that $G - v$ has no cycles, we conclude that $G - E(T')$ has no edges, so T' is an Eulerian circuit that extends T. (Sufficiency can also be proved by contrapositive.)

b) If v is extendible, then $d(v) = \Delta(G)$. An Eulerian graph decomposes into cycles. If this uses m cycles, then each vertex has degree at most

$2m$. By part (a) each cycle contains v, and thus $d(v) \geq 2m$. Hence v has maximum degree.

Alternatively, since each cycle contains v, an Eulerian circuit must visit v between any two visits to another vertex u. Hence $d(v) \geq d(u)$.

c) For $n(G) > 2$, all vertices are extendible if and only if G is a cycle. If G is a cycle, then every trail from a vertex extends to become the complete cycle. Conversely, suppose that all vertices are extendible. By part (a), every vertex lies on every cycle. Let C be a cycle in G; it must contain all vertices. If G has any additional edge e, then following the shorter part of C between the endpoints of e completes a cycle with e that does not contain all the vertices. Hence there cannot be an additional edge and $G = C$.

d) If G is not a cycle, then G has at most two extendible vertices. From part (c), we may assume that G is Eulerian but not a cycle. If v is extendible, then $G - v$ is a forest. This forest cannot be a path, since then G is a cycle or has a vertex of odd degree. Since $G - v$ is a forest and not a path, $G - v$ has *more* than $\Delta(G - v)$ leaves unless $G - v$ is a tree with exactly one vertex of degree greater than two. If $G-v$ has more than $\Delta(G-v)$ leaves, all in $N(v)$, then no vertex of $G - v$ has degree as large as v in G, and by part (b) no other vertex is extendible. In the latter case, the one other vertex of degree $d(v)$ may also be extendible, but all vertices except those two have degree 2.

2.1.43. *Given a vertex u in a connected graph G, there is a spanning tree of G that is the union of shortest paths from u to the other vertices.*

Proof 1 (induction on $n(G)$). When $n(G) = 1$, the vertex u is the entire tree. For $n(G) > 1$, let v be a vertex at maximum distance from u. Apply the induction hypothesis to $G - v$ to obtain a tree T in $G - v$. Shortest paths in G from u to vertices other than v do not use v, since v is farthest from u. Therefore, T consists of shortest paths in G from u to the vertices other than v. A shortest u,v-path in G arrives at v from some vertex of T. Adding the final edge of that path to T completes the desired tree in G.

Proof 2 (explicit construction). For each vertex other than u, choose an incident edge that starts a shortest path to u. No cycle is created, since as we follow any path of chosen edges, the distance from u strictly decreases. Also $n(G) - 1$ edges are chosen, and an acyclic subgraph with $n(G) - 1$ edges is a spanning tree. Since distance from u decreases with each step, the v,u-path in the chosen tree is a shortest v,u-path.

Comment: The claim can also be proved using BFS to grow the tree. Proof 1 is a short inductive proof that the BFS algorithm works. Proof 2 is an explicit description of the edge set produced by Proof 1.

2.1.44. *If a simple graph with diameter 2 has a cut-vertex, then its complement has an isolated vertex—TRUE.* Let v be a cut-vertex of a simple graph G with diameter 2. In order to have distance at most 2 to each vertex in the other component(s) of $G - v$, a vertex of $G - v$ must be adjacent to v. Hence v has degree $n(G) - 1$ in G and is isolated in $\overline{G}$.

2.1.45. *If a graph G has spanning trees with diameters 2 and l, then G has spanning trees with all diameters between 2 and l.*

Proof 1 (local change). The only trees with diameter 2 are stars, so G has a vertex v adjacent to all others. Given a spanning tree T with leaf u, replacing the edge incident to u with uv yields another spanning tree T'. For every destroyed path, a path shorter by 1 remains. For every created path, a path shorter by 1 was already present. Hence diam T' differs from diam T by at most 1. Continuing this procedure reaches a spanning tree of diameter 2 without skipping any values along the way, so all the desired values are obtained.

Proof 2 (explicit construction). Since G has a tree with diameter 2, it has a vertex v adjacent to all others. Every path in G that does not contain v extends to v and to an additional vertex if it does not already contain all vertices. Hence for $k < l$ there is a path P of length k in G that contains v as an internal vertex. Adding edges from v to all vertices not in P completes a spanning tree of diameter k.

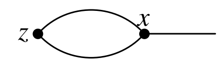
2.1.46. *For $n \geq 2$, the number of isomorphism classes of n-vertex trees with diameter at most 3 is $\lfloor n/2 \rfloor$.* If $n \leq 3$, there is only one tree, and its diameter is $n - 1$. If $n \geq 4$, every tree has diameter at least 2. There is one having diameter 2, the star. Every tree with diameter 3 has two centers, x, y, and every non-central vertex is adjacent to exactly one of x, y, so $d(x) + d(y) = n$. By symmetry, we may assume $d(x) \leq d(y)$. The unlabeled tree is now completely specified by $d(x)$, which can take any value from 2 through $\lfloor n/2 \rfloor$. Together with the star, the number of trees is $\lfloor n/2 \rfloor$.

2.1.47. *Diameter and radius.*

a) The distance function $d(u, v)$ satisfies the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$. A u, v-path of length $d(u, v)$ and a v, w-path of length $d(v, w)$ together form a u, w-walk of length $l = d(u, v) + d(v, w)$. Every u, w-walk contains a u, w-path among its edges, so there is a u, w-path of length at most l. Hence the shortest u, w-path has length at most l.

b) $d \leq 2r$, where d is the diameter of G and r is the radius of G. Let u, v be two vertices such that $d(u, v) = d$. Let w be a vertex in the center of G; it has eccentricity r. Thus $d(u, w) \leq r$ and $d(w, v) \leq r$. By part (a), $d = d(u, v) \leq d(u, w) + d(w, v) \leq 2r$.

c) Given integers r, d with $0 < r \leq d \leq 2r$, there is a simple graph with radius r and diameter d. Let $G = C_{2r} \cup H$, where $H \cong P_{d-r+1}$ and the cycle shares with H exactly one vertex x that is an endpoint of H. The distance from the other end of H to the vertex z opposite x on the cycle is d, and this is the maximum distance between vertices. Every vertex of H has distance at least r from z, and every vertex of the cycle has distance r from the vertex opposite it on the cycle. Hence the radius is at least r. The eccentricity of x equals r, so the radius equals r, and x is in the center.



2.1.48. *For $n \geq 4$, the minimum number of edges in an n-vertex graph with diameter 2 and maximum degree $n - 2$ is $2n - 4$.* The graph $K_{2,n-2}$ shows that $2n - 4$ edges are enough. We show that at least $2n - 4$ are needed. Let G be an n-vertex graph with diameter 2 and maximum degree $n - 2$. Let x be a vertex of degree $n - 2$, and let y be the vertex not adjacent to x.

Proof 1. Every path from y through x to another vertex has length at least 3, so diameter 2 requires paths from y to all of $V(G) - \{x, y\}$ in $G - x$. Hence $G - x$ is connected and therefore has at least $n - 2$ edges. With the $n - 2$ edges incident to x, this yields at least $2n - 4$ edges in G.

Proof 2. Let $A = N(y)$. Each vertex of $N(x) - A$ must have an edge to a vertex of A in order to reach y in two steps. These are distinct and distinct from the edges incident to y, so we have at least $|A| + |N(x) - A|$ edges in addition to those incident to x. The total is again at least $2n - 4$.

(Comment: The answer remains the same whenever $(2n - 2)/3 \leq \Delta(G) \leq n - 5$ but is $2n - 5$ when $n - 4 \leq \Delta(G) \leq n - 3$.)

2.1.49. *If G is a simple graph with* rad $G \geq 3$, *then* rad $\overline{G} \leq 2$. The radius is the minimum eccentricity. For $x \in V(G)$, there is a vertex y such that $d_G(x, y) \geq 3$. Let w be the third vertex from x along a shortest x, y-path (possibly $w = y$). For $v \in V(G) - \{x\}$, if $xv \notin E(\overline{G})$, then $xv \in E(G)$. Now $vw \notin E(G)$, since otherwise there is a shorter x, y-path. Thus x, w, v is an x, v-path of length 2 in $\overline{G}$. Hence for all $v \in V(G) - \{x\}$, there is an x, v-path of length at most 2 in $\overline{G}$, and we have $\varepsilon_{\overline{G}}(x) \leq 2$ and rad $(\overline{G}) \leq 2$.

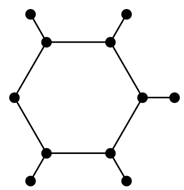
2.1.50. *Radius and eccentricity.*

a) The eccentricities of adjacent vertices differ by at most 1. Suppose that $x \leftrightarrow y$. For each vertex z, $d(x, z)$ and $d(y, z)$ differ by at most 1 (Exercise 2.1.11). Hence

$$\varepsilon(y) = \max_z d(y, z) \leq \max_z (d(x, z) + 1) = (\max_z d(x, z)) + 1 = \varepsilon(x) + 1.$$

Similarly, $\varepsilon(x) \leq \varepsilon(y) + 1$. The statement can be made more general: $|\varepsilon(x) - \varepsilon(y)| \leq d(x, y)$ for all $x, y \in V(G)$.

b) In a graph with radius r, the maximum possible distance from a vertex of eccentricity $r + 1$ to the center of G is r. The distance is at most r, since every vertex is within distance at most r of every vertex in the center, by the definitions of center and radius. The graph consisting of a cycle of length $2r$ plus a pendant edge at all but one vertex of the cycle achieves equality. All vertices of the cycle have eccentricity $r+1$ except the vertex opposite the one with no leaf neighbor, which is the unique vertex with eccentricity r. The leaves have eccentricity $r+2$, except for the one adjacent to the center.



2.1.51. *If x and y are distinct neighbors of a vertex v in a tree G, then* $2\varepsilon(v) \le \varepsilon(x) + \varepsilon(y)$. Let w be a vertex at distance $\varepsilon(v)$ from v. The vertex w cannot be both in the component of $G - xv$ containing x and in the component of $G - yv$ containing y, since this would create a cycle. Hence we may assume that w is in the component of $G - xv$ containing v. Hence $\varepsilon(x) \ge d(x, w) = \varepsilon(v) + 1$. Also $\varepsilon(y) \ge d(y, w) \ge d(v, w) - 1 = \varepsilon(v) - 1$. Summing these inequalities yields $\varepsilon(x) + \varepsilon(y) \ge \varepsilon(v) + \varepsilon(v)$.

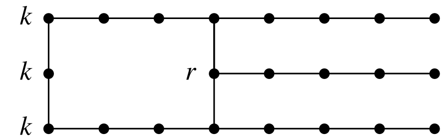
The smallest graph where this inequality can fail is the kite $K_4 - e$. Let v be a vertex of degree 2; it has eccentricity 2. Its neighbors x and y has degree 3 and hence eccentricity 1.

2.1.52. *Eccentricity of vertices outside the center.*

a) If G is a tree, then every vertex x outside the center of G has a neighbor with eccentricity $\varepsilon(x) - 1$. Let y be a vertex in the center, and let w be a vertex with distance at least $\varepsilon(x) - 1$ from x. Let v be the vertex where the unique x, w- and y, w-paths meet; note that v is on the x, y-path in G. Since $d(y, w) \le \varepsilon(y) \le \varepsilon(x) - 1 \le d(x, w)$, we have $d(y, v) \le d(x, v)$. This implies that $v \ne x$. Hence x has a neighbor z on the x, v-path in G.

This argument holds for every such w, and the x, v-path in G is always part of the x, y-path in G. Hence the same neighbor of x is always chosen as z. We have proved that $d(z, w) = d(x, w) - 1$ whenever $d(x, w) \ge \varepsilon(x) - 1$. On the other hand, since z is a neighbor of x, we have $d(z, w) \le d(x, w) + 1 \le \varepsilon(x) - 1$ for every vertex w with $d(x, w) < \varepsilon(x) - 1$. Hence $\varepsilon(z) = \varepsilon(x) - 1$.

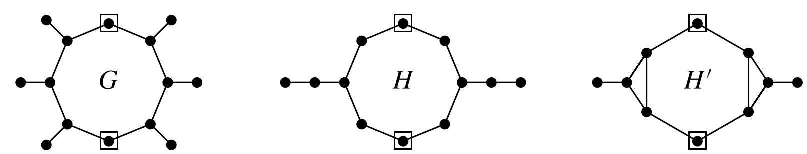
b) For all r and k with $2 \le r \le k < 2r$, there is a graph with radius r in which some vertex and its neighbors all have eccentricity k. Let G consist of a $2r$-cycle C and paths of length $k - r$ appended to three consecutive vertices on C. Below is an example with $r = 5$ and $k = 9$. The desired vertex is the one opposite the middle vertex of degree 3; vertices are labeled with their eccentricities.



2.1.53. *The center of a graph can be disconnected and can have components arbitrarily far apart.* We construct graphs center consists of two (marked) vertices separated by distance k. There are various natural constructions.

The graph G consists of a cycle of length $2k$ plus a pendant edge at all but two opposite vertices. These two are the center; other vertices of the cycle have eccentricity $k+1$, and the leaves have eccentricity $k+2$.

For even k, the graph H below consists of a cycle of length $2k$ plus pendant paths of length $k/2$ at two opposite vertices. For odd k, the graph H' consists of a cycle of length $2k$ plus paths of length $\lfloor k/2 \rfloor$ attached at one end to two opposite *pairs* of consecutive vertices.



2.1.54. *Centers in trees.*

a) A tree has exactly one center or has two adjacent centers.

Proof 1 (direct properties of trees). We prove that in a tree T any two centers are adjacent; since T has no triangles, this means it has at most two centers. Suppose u and v are distinct nonadjacent centers, with eccentricity k. There is a unique path R between them containing a vertex $x \notin \{u, v\}$. Given $z \in V(T)$, let P, Q be the unique u, z-path and unique v, z-path, respectively. At least one of P, Q contains x else $P \cup Q$ is a u, v-walk and contains a (u, v)-path other than R. If P passes through x, we have $d(x, z) < d(u, z)$; if Q, we have $d(x, z) < d(v, z)$. Hence $d(x, z) < \max\{d(u, z), d(v, z)\} \le k$. Since z is arbitrary, we conclude that x has smaller eccentricity than u and v. The contradiction implies $u \leftrightarrow v$.

Proof 2 (construction of the center). Let $P = x_1, \ldots, x_2$ be a longest path in T, so that $D = \operatorname{diam} T = d(x_1, x_2)$. Let $r = \lceil D/2 \rceil$. Let $\{u_1, u_2\}$ be the middle of P, with $u_1 = u_2$ if D is even. Label u_1, u_2 along P so that $d(x_i, u_i) = r$. Note that $d(v, u_i) \le r$ for all $v \in T$, else the (v, u_i)-path can be combined with the (u_i, x_i)-path or the (u_i, x_{3-i})-path to form a path longer than P. To show that no vertex outside $\{u_1, u_2\}$ can be a center, it suffices to show that every other vertex v has distance greater than r from x_1 or x_2.

The unique path from v to either x_1 or x_2 meets P at some point w (which may equal v). If w is in the u_1, x_2-portion of P, then $d(v, x_1) > r$. If w is in the u_2, x_1-portion of P, then $d(v, x_2) > r$.

b) A tree has exactly one center if and only if its diameter is twice its radius. Proof 3 above observes that the center or pair of centers is the middle of a longest path. The diameter of a tree is the length of its longest path. The radius is the eccentricity of any center. If the diameter is even, then there is one center, and its eccentricity is half the length of the longest path. If the diameter is odd, say $2k - 1$, then there are two centers, and the eccentricity of each is k, which exceeds $(2k - 1)/2$.

c) Every automorphism of a tree with an odd number of vertices maps at least one vertex to itself. The maximum distance from a vertex must be preserved under any automorphism, so any automorphism of any graph maps the center into itself. A central tree has only one vertex in the center, so it is fixed by any automorphism. A bicentral tree has two such vertices; they are fixed or exchange. If they exchange, then the two subtrees obtained by deleting the edge between the centers are exchanged by the automorphism. However, if the total number of vertices is odd, then the parity of the number of vertices in the two branches is different, so no automorphism can exchange the centers.

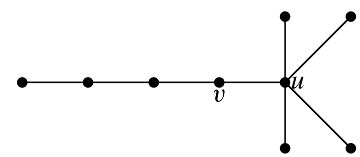
2.1.55. Given $x \in V(G)$, let $s(x) = \sum_{v \in V(G)} d(x, v)$. The *barycenter* of G is the subgraph induced by the set of vertices minimizing $s(x)$.

a) The barycenter of a tree is a single vertex or an edge. Let uv be an edge in a tree G, and let $T(u)$ and $T(v)$ be the components of $G - uv$ containing u and v, respectively. Note that $d(u, x) - d(v, x) = 1$ if $x \in V(T(v))$ and $d(u, x) - d(v, x) = -1$ if $x \in V(T(u))$. Summing the difference over $x \in V(G)$ yields $s(u) - s(v) = n(T(v)) - n(T(u))$.

As a result, $s(u_i) - s(u_{i+1})$ strictly decreases along any path $u_1, u_2, \ldots$; each step leaves more vertices behind. Considering two consecutive steps on a path x, y, z yields $s(x) - s(y) < s(y) - s(z)$, or $2s(y) < s(x) + s(z)$ whenever $x, z \in N(y)$. Thus the minimum of s cannot be achieved at two nonadjacent vertices, because it would be smaller at a vertex between them.

b) The maximum distance between the center and the barycenter in a tree of diameter d is $\lfloor d/2 \rfloor - 1$. By part (a), s is not minimized at a leaf when $n \geq 2$. Since every vertex is distance at most $\lfloor d/2 \rfloor$ from the center, we obtain an upper bound of $\lfloor d/2 \rfloor - 1$.

Part (a) implies that to achieve the bound of $\lfloor d/2 \rfloor - 1$ we need a tree having adjacent vertices u, v such that u is the neighbor of a leaf with eccentricity d, and the number of leaves adjacent to u is at least as large as $n(T(v))$. Since uv lies along a path of length d, we have at least $d - 1$ vertices in $T(v)$. Thus we need at least d vertices in $T(u)$ and at least $2d - 1$ vertices altogether. We obtain the smallest tree achieving the bound by merging an endpoint of P_d with the center of the star $K_{1,d-1}$. In the resulting tree, the barycenter u is the vertex of degree $d - 1$, and the distance between it and the center is $\lfloor d/2 \rfloor - 1$.



2.1.56. *Every tree T has a vertex v such that for all $e \in E(T)$, the component of $T - e$ containing v has at least $\lceil n(T)/2 \rceil$ vertices.*

Proof 1 (orientations). For each edge $xy \in E(T)$, we orient it from x to y if in $T - xy$ the component containing y contains at least $\lceil n(T)/2 \rceil$ vertices (there might be an edge which could be oriented either way). Denote the resulting digraph by $D(T)$.

If $D(T)$ has a vertex x with outdegree at least 2, then $T - x$ has two disjoint subtrees each having at least $\lceil n(T)/2 \rceil$ vertices, which is impossible. Now, since T does not contain a cycle, $D(T)$ does not contain a directed cycle. Hence $D(T)$ has a vertex v with outdegree 0. Since $D(T)$ has no vertex with outdegree at least two, every path in T with endpoint v is an oriented path to v in $D(T)$. Thus every edge xy points towards v, meaning that v is in a component of $T - xy$ with at least $\lceil n(T)/2 \rceil$ vertices.

The only flexibility in the choice of v is that an edge whose deletion leaves two components of equal order can be oriented either way, which yields two adjacent choices for v.

Proof 2 (algorithm). Instead of the existence proof using digraphs, one can march to the desired vertex. For each $v \in V(T)$, let $f(v)$ denote the minimum over $e \in E(T)$ of the order of the component of $T - e$ containing v. Note that $f(v)$ is achieved at some edge e incident to v.

Select a vertex v. If $f(v) < \lceil n(T)/2 \rceil$, then consider an edge e incident to v such that the order of the component of $T - e$ containing v is $f(v)$. Let u be the other endpoint of e. The component of $T - e$ containing u has more than half the vertices. For any other edge e' incident to u, the component of $T - e'$ containing u is strictly larger than the component of $T - e$ containing v. Hence $f(u) > f(v)$.

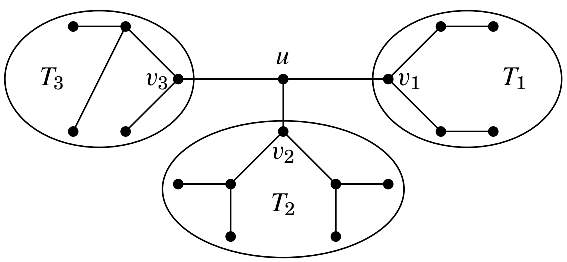
If $f(u) < \lceil n(T)/2 \rceil$, then we repeat the argument. Since f cannot increase indefinitely, we reach a vertex w with $f(v) \geq \lceil n(T)/2 \rceil$.

Uniqueness is as before; if two nonadjacent vertices have this property, then deleting edges on the path joining them yields a contradiction.

2.1.57. *a) If $n_1, \ldots, n_k$ are positive integers with sum $n-1$, then $\sum_{i=1}^{k} \binom{n_i}{2} \le \binom{n-1}{2}$.* The graph having pairwise disjoint cliques of sizes $n_1, \ldots, n_k$ has $\sum_{i=1}^{k} \binom{n_i}{2}$ edges and is a subgraph of K_{n-1}.

b) $\sum_{v \in V(T)} d(u, v) \le \binom{n}{2}$ when u is a vertex of a tree T. We use induction on n; the result holds trivially for $n = 2$. Consider $n > 2$. The graph $T - u$ is a forest with components $T_1, \ldots, T_k$, where $k \ge 1$. Because T is connected, u has a neighbor in each T_i; because T has no cycles, u has exactly one neighbor v_i in each T_i. If $v \in V(T_i)$, then the unique u, v-path in T passes through v_i, and we have $d_T(u, v) = 1 + d_{T_i}(v_i, v)$. Letting $n_i = n(T_i)$, we obtain $\sum_{v \in V(T_i)} d_T(u, v) = n_i + \sum_{v \in V(T_i)} d_{T_i}(v_i, v)$.

By the induction hypothesis, $\sum_{v \in V(T_i)} d_{T_i}(v_i, v) \le \binom{n_i}{2}$. If we sum the formula for distances from u over all the components of $T - u$, we obtain $\sum_{v \in V(T)} d_T(u, v) \le (n-1) + \sum_i \binom{n_i}{2}$. Now observe that $\sum \binom{n_i}{2} \le \binom{m}{2}$ whenever $\sum n_i = m$, because the right side counts the edges in K_m and the left side counts the edges in a subgraph of K_m (a disjoint union of cliques). Hence we have $\sum_{v \in V(T)} d_T(u, v) \le (n-1) + \binom{n-1}{2} = \binom{n}{2}$.



2.1.58. *If S and T are trees with leaf sets $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$, respectively, then $d_S(x_i, x_j) = d_T(y_i, y_j)$ for all $1 \le i \le j \le k$ implies that S and T are isomorphic.* It suffices to show that the numbers $d_S(x_i, x_j)$ determine S uniquely. That is, if S is a tree, then no other tree has the same leaf distances.

Proof 1 (induction on k). If $k = 2$, then S is a path of length $d(x_1, x_2)$. If $k > 2$, then a tree S with leaf distance set D has a shortest path P from x_k to a junction w. Since P has no internal vertices on paths joining other leaves, deleting $V(P) - \{w\}$ leaves a subtree with leaf set $\{x_1, \ldots, x_{k-1}\}$ realizing the distances not involving x_k. By the induction hypothesis, this distance set is uniquely realizable; call that tree S'. It remains only to show that the vertex w in $V(S')$ and $d_S(x_k, w)$ are uniquely determined.

Let $t = d_S(x_k, w)$. The vertex w must belong to the path Q joining some leaves x_i and x_j in S'. The paths from x_i and x_j to x_k in S together use the edges of Q, and each uses the path P from w to x_k. Thus $t = (d_S(x_i, x_k) + d_S(x_j, x_k) - d_S(x_i, x_j))/2$.

For arbitrary x_i and x_j, this formula gives the distance in S from x_k to the junction with the x_i, x_j-path. If w is not on the x_i, x_j-path, then the value of the formula exceeds t, since w is the closest vertex of S' to x_k. Hence $t = \min_{i,j<k}(d_S(x_i, x_k) + d_S(x_j, x_k) - d_S(x_i, x_j))/2$. For any i, j that achieves the minimum, $d_{S'}(x_i, w) = d_S(x_i, x_k) - t$, which identifies the vertex w in S'.

Thus there is only one w where the path can be attached and only one length of path that can be put there to form a tree realizing D.

Proof 2 (induction on $n(S)$). When $n(S) = 2$, there is no other tree with adjacent leaves. For $n(S) > 2$, let x_k be a leaf of maximum eccentricity; the eccentricity of a leaf is the maximum among its distances to other leaves.

If some leaf x_j has distance 2 from x_k, then they have a common neighbor. Deleting x_k yields a smaller tree S' with $k-1$ leaves, since the neighbor of x_k is not a leaf in S. The deletion does not change the distances among other leaves. By the induction hypothesis, there is only one way to assemble S' from the distance information, and to form S we must add x_k adjacent to the neighbor of x_j.

If no leaf has distance 2 from x_k, then the neighbor of x_k in S must have degree 2, because having two non-leaf neighbors would contradict the choice of x_k as a leaf of maximum eccentricity. Now $S - x_k$ has the same number of leaves but fewer vertices. The leaf x_k is replaced by x_k', and the distances from the kth leaf to other leaves are all reduced by 1. By the induction hypothesis, there is only one way to assemble $S - x_k$ from the distance information, and to form S we must add x_k adjacent to x_k'.

2.1.59. *If G is a tree with n vertices, k leaves, and maximum degree k, then $2\lceil (n-1)/k \rceil \le \operatorname{diam} G \le n - k + 1$, and the bounds are achievable, except that the lower bound is $2\lceil (n-1)/k \rceil - 1$ when $n \equiv 2 \pmod{k}$.* Let x be a vertex of degree k. Consider k maximal paths that start at x; these end at distinct leaves. If G has any other edge, it creates a cycle or leads to an additional leaf. Hence G is the union of k edge-disjoint paths with a common endpoint. The diameter of G is the sum of the lengths of two longest such paths.

Upper bound: Since the paths other than the two longest absorb at least $k-2$ edges, at most $n-k+1$ edges remain for the two longest paths; this is achieved by giving one path length $n-k$ and the others length 1.

Lower bound: If the longest and shortest of the k paths differ in length by more than 1, then shortening the longest while lengthening the shortest does not increase the sum of the two longest lengths. Hence the diameter is minimized by the tree G in which the lengths of any pair of the k paths differ by at most 1, meaning they all equal $\lfloor (n-1)/k \rfloor$ or $\lceil (n-1)/k \rceil$. There must be two of length $\lceil (n-1)/k \rceil$ unless $n \equiv 2 \pmod{k}$.

2.1.60. *If G has diameter d and maximum degree k, then $n(G) \le 1 + [(k-1)^d - 1]k/(k-2)$.* A single vertex x has at most k neighbors. Each of these has at most k other incident edges, and hence there are at most $k(k-1)$ vertices at distance 2 from x. Assuming that new vertices always get generated, the tree of paths from x has at most $k(k-1)^{i-1}$ vertices at distance i from x. Hence $n(G) \le 1 + \sum_{i=1}^{d} k(k-1)^{i-1} = 1 + k\frac{(k-1)^d - 1}{k-1-1}$. (Comment: C_5 and the Petersen graph are among the very few that achieve equality.)

2.1.61. *Every (k,g)-cage has diameter at most g.* (A (k,g)-*cage* is a graph with smallest order among k-regular graphs with girth at least g; Exercise 1.3.16 establishes the existence of such graphs).

Let G be a (k,g)-cage having two vertices x and y such that $d_G(x,y) > g$. We modify G to obtain a k-regular graph with girth at least g that has fewer vertices. This contradicts the choice of G, so there is no such pair of vertices in a cage G.

The modification is to delete x and y and add a matching from $N(x)$ to $N(y)$. Since $d(x,y) > g \ge 3$, the resulting smaller graph G' is simple. Since we have "replaced" edges to deleted vertices, G' is k-regular. It suffices to show that cycles in G' have length at least g. We need only consider cycles using at least one new edge.

Since $d_G(x,y) > g$, every path from $N(x)$ to $N(y)$ has length at least $g-1$. Also every path whose endpoints are within $N(x)$ has length at least $g-2$; otherwise, G has a short cycle through x. Every cycle through a new edge uses one new edge and a path from $N(x)$ to $N(y)$ or at least two new edges and at least two paths of length at least $g-2$. Hence every new cycle has length at least g.

2.1.62. *Connectedness and diameter of the 2-switch graph on spanning trees of G.* Let G be a connected graph with n vertices. The graph G' has one vertex for each spanning tree of G, with vertices adjacent in G' when the corresponding trees have exactly $n(G)-2$ common edges.

a) G' is connected.

Proof 1 (construction of path). For distinct spanning trees T and T' in G, choose $e \in E(T) - E(T')$. By Proposition 2.1.6, there exists $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G. Let $T_1 = T - e + e'$. The trees T and T_1 are adjacent in G'. The trees T_1 and T' share more edges than T and T' share. Repeating the argument produces a T,T'-path in G' via vertices $T, T_1, T_2, \ldots, T_k, T'$.

Formally, this uses induction on the number m of edges in $E(T) - E(T')$. When $m = 0$, there is a T,T'-path of length 0. When $m > 0$, we generate T_1 as above and apply the induction hypothesis to the pair T_1, T'.

Proof 2 (induction on $e(G)$). If $e(G) = n-1$, then G is a tree, and $G' = K_1$. For the induction step, consider $e(G) > n-1$. A connected n-vertex graph with at least n edges has a cycle C. Choose $e \in E(C)$. The graph $G-e$ is connected, and by the induction hypothesis $(G-e)'$ is connected. Every spanning tree of $G-e$ is a spanning tree of G, so $(G-e)'$ is the induced subgraph of $T(G)$ whose vertices are the spanning trees of G that omit e.

Since $(G-e)'$ is connected, it suffices to show that every spanning tree of G containing e is adjacent in G' to a spanning tree not containing e. If T contains e and T' does not, then there exists $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G omitting e. Thus $T - e + e'$ is the desired tree in $G-e$ adjacent to T in G'.

b) The diameter of G' is at most $n-1$, with equality when G has two spanning trees that share no edges. It suffices to show that $d_{G'}(T,T') = |E(T) - E(T')|$. Each edge on a path from T to T' in G' discards at most one edge of T, so the distance is at least $|E(T) - E(T')|$. Since for each $e \in E(T) - E(T')$ there exists $e' \in E(T') - E(T)$ such that $T - e + e' \in V(G')$, the path built in Proof 1 of part (a) has precisely this length.

Since trees in n-vertex graphs have at most $n-1$ edges, always $|E(T) - E(T')| \le n-1$, so $\operatorname{diam} G' \le n-1$ when G has n vertices. When G has two edge-disjoint spanning trees, the diameter of G' equals $n-1$.

2.1.63. *Every n-vertex graph with $n+1$ edges has a cycle of length at most $\lfloor (2n+2)/3 \rfloor$.* The bound is best possible, as seen by the example of three paths with common endpoints that have total length $n+1$ and nearly-equal lengths. Note that $\lfloor (2n+2)/3 \rfloor = \lceil 2n/3 \rceil$.

Proof 1. Since an n-vertex forest with k components has only $n-k$ edges, an n-vertex graph with $n+1$ edges has at least two cycles. Let C be a shortest cycle. Suppose that $e(C) > \lceil 2n/3 \rceil$. If $G - E(C)$ contains a path connecting two vertices of C, then it forms a cycle with the shorter path on C connecting these two vertices. The length of this cycle is at most

$$\tfrac{1}{2}e(C) + (e(G) - e(C) = e(G) - \tfrac{1}{2}e(C) < n + 1 - n/3 = (2n+3)/3.$$

If the length of this cycle is less than $(2n+3)/3$, then it is at most $(2n+2)/3$, and since it is an integer it is at most $\lfloor (2n+2)/3 \rfloor$.

If there is no such path, then no cycle shares an edge with C. Hence the additional cycle is restricted to a set of fewer than $n + 1 - \lceil 2n/3 \rceil$ edges, and again its length is less than $(2n+3)/3$.

Proof 2. We may assume that the graph is connected, since otherwise we apply the same argument to some component in which the number of edges exceeds the number of vertices by at least two. Consider a spanning tree T, using $n-1$ of the edges. Each of the two remaining edges forms a cycle when added to T. If these cycles share no edges, then the shortest has length at most $(n+1)/2$.

Hence we may assume that the two resulting cycles have at least one common edge; let x, y be the endpoints of their common path in T. Deleting the x, y-path in T from the union of the two cycles yields a third cycle. (The uniqueness of cycles formed when an edge is added to a tree implies that this edge set is in fact a single cycle.) Thus we have three cycles, and each edge in the union of the three cycles appears in exactly two of them. Thus the shortest of the three lengths is at most $2(n+1)/3$.

2.1.64. *If G is a connected graph that is not a tree, then G has a cycle of length at most* $2\operatorname{diam} G + 1$*, and this is best possible.* We use extremality for the upper bound; let C be a shortest cycle in G. If its length exceeds $2\operatorname{diam} G + 1$, then there are vertices x, y on C that have no path of length at most $\operatorname{diam} G$ connecting them along C. Following a shortest x, y-path P from its first edge off C until its return to C completes a shorter cycle. This holds because P has length at most k, and we use a portion of P in place of a path along C that has length more than k. We have proved that every shortest cycle in G has length at most $2\operatorname{diam} G + 1$.

The odd cycle C_{2k+1} shows that the bound is best possible. It is connected, is not a tree, and has diameter k. Its only cycle has length $2k+1$, so we cannot guarantee girth less than $2k+1$.

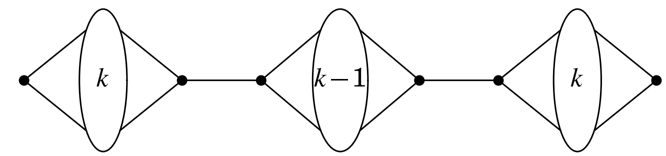
2.1.65. *If G is a connected simple graph of order n and minimum degree k, with $n-3 \ge k \ge 2$, then* $\operatorname{diam} G \le 3(n-2)/(k+1) - 1$*, with equality when $n-2$ is a multiple of $k+1$.* To interpret the desired inequality on $\operatorname{diam} G$, we let $d = \operatorname{diam} G$ and solve for n. Thus it suffices to prove that $n \ge (1 + \lfloor d/3 \rfloor)(k+1) + j$, where j is the remainder of d upon division by 3. Note that the inequality $n-3 \ge k$ is equivalent to $3(n-2)/(k+1) - 1 \ge 2$. Under this constraint, the result is immediate when $d \le 2$, so we may assume that $d \ge 3$.

Let $\langle v_0, \ldots, v_d \rangle$ be a path joining vertices at distance d. For a vertex x, let $N[x] = N(x) \cup \{x\}$. Let $S_i = N[v_{3i}]$ for $0 \le i < \lfloor d/3 \rfloor$, and let $S_{\lfloor d/3 \rfloor} = N[v_d]$. Since $d \ge 3$, there are $1 + \lfloor d/3 \rfloor$ such sets, pairwise disjoint (since we have a shortest v_0, v_d-path), and each has at least $k+1$ vertices. Furthermore, v_{d-2} does not appear in any of these sets if $j = 1$, and both v_{d-2} and v_{d-3} do not appear if $j = 2$. Hence n is as large as claimed.

To obtain an upper bound on d in terms of n, we write $\lfloor d/3 \rfloor$ as $(d-j)/3$. Solving for d in terms of n, we find in each case that $d \le 3(n-2)/(k+1) - 1 - j[1 - 3/(k+1)]$. Since $k \ge 2$, the bound $d \le 3(n-2)/(k+1) - 1$ is valid for every congruence class of d modulo 3.

When $n-2$ is a multiple of $k+1$, the bound is sharp. If $n-2 = k+1$, then deleting two edges incident to one vertex of K_n yields a graph with the desired diameter and minimum degree (also $\overline{C}_n$ suffices). For larger multiples, let $m = (n-2)/(k+1)$; note that $m \ge 2$. Begin with cliques $Q_1, \ldots, Q_m$ such that Q_1 and Q_m have order $k+2$ and the others have order $k+1$. For $1 \le i \le m$, choose $x_i, y_i \in Q_i$, and delete the edge $x_i y_i$.

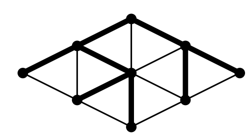
For $1 \le i \le m-1$, add the edge $y_i x_{i+1}$. The resulting graph has minimum degree k and diameter $3m-1$. The figure below illustrates the construction when $m = 3$; the ith ellipse represents $Q_m - \{x_i, y_i\}$. (There also exist regular graphs attaining the bound.)



2.1.66. *If $F_1, \ldots, F_m$ are forests whose union is G, then $m \ge \max_{H \subseteq G} \left\lceil \frac{e(H)}{n(H)-1} \right\rceil$.* From a subgraph H, each forest uses at most $n(H)-1$ edges. Thus at least $e(H)/(n(H)-1)$ forests are needed just to cover the edges of H, and the choice of H that gives the largest value of this is a lower bound on m.

2.1.67. *If a graph G has k pairwise edge-disjoint spanning trees in G, then for any partition of $V(G)$ into r parts, there are at least $k(r-1)$ edges of G whose endpoints are in different parts.* Deleting the edges of a spanning tree T that have endpoints in different parts leaves a forest with at least r components and hence at most $n(G) - r$ edges. Since T has $n(G) - 1$ edges, T must have at least $r-1$ edges between the parts. The argument holds separately for each spanning tree, yielding $k(r-1)$ distinct edges.

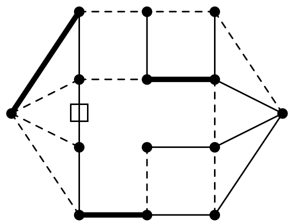
2.1.68. *A decomposition into two isomorphic spanning trees.* One tree turns into the other in the decomposition below upon rotation by 180 degrees.



2.1.69. *An instance of playing Bridg-it.* Indexing the 9 vertical edges as $g_{i,j}$ and the 16 horizontal/slanted edges as $h_{i,j}$, where i is the "row" index and j is the "column" index, we are given these moves:

Player 1:	$h_{1,1}$	$h_{2,3}$	$h_{4,2}$
Player 2:	$g_{2,2}$	$h_{3,2}$	$g_{2,1}$

After the third move of Player 1, the situation is as shown below. The bold edges are those seized by Player 1 and belong to both spanning trees. The two moves by Player 2 have cut the two edges that are missing.



The third move by Player 2 cuts the marked vertical edge. This cuts off three vertices from the rest of the solid tree. Player 1 must respond by choosing a dotted edge that can reconnect it. The choices are $h_{1,2}$, $h_{2,1}$, $h_{2,2}$, $h_{3,1}$, and $h_{4,1}$.

2.1.70. *Bridg-it cannot end in a tie.* That is, when no further moves can be made, one player must have a path connecting his/her goals.

Consider the graph for Player 1 formed in Theorem 2.1.17. At the end of the game, Player 1 has bridges on some of these edges, retaining them as a subgraph H, and the other edges have been cut by Player 2's bridges. Let C be the component of H containing the left goal for Player 1. The edges incident to $V(C)$ that have been cut correspond to a walk built by Player 2 that connects the goals for Player 2. This holds because successive edges around the outside of C are incident to the same "square" in the graph for Player 1, which corresponds to a vertex for Player 2. This can be described more precisely using the language of duality in planar graphs (Chapter 6).

2.1.71. *Player 2 has a winning strategy in Reverse Bridg-it.* A player building a path joining friendly ends is the *loser*, and it is forbidden to stall by building a bridge joining posts on the same end.

We use the same graph as in Theorem 2.1.17, keeping the auxiliary edge so that we start with two edge-disjoint spanning trees T and T'. An edge e that Player 1 can use belongs to only one of the trees, say T. The play by Player 1 will add e to T'. Since $e \in E(T) - E(T')$, Proposition 2.1.7 guarantees an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree. Player 2 makes a bridge to delete the edge e', and the strategy continues with the modified T' sharing the edge e with T. If the only edge of $E(T') - E(T)$ available to break the cycle in $T' + e$ is the auxiliary edge, then Player 1 has already built a path joining the goals and lost the game. The game continues always with two spanning trees available for Player 1, and it can only end with Player 1 completing the required path.

2.1.72. *If $G_1, \ldots, G_k$ are pairwise intersecting subtrees of a tree G, then G has a vertex in all of $G_1, \ldots, G_k$.* (A special case is the "Helly property" of the real line: pairwise intersecting intervals have a common point.)

Lemma: For vertices u, v, w in a tree G, the u,v-path P, the v,w-path Q, and the u,w-path R in G have a common vertex. Let z be the last vertex shared by P and R. They share all vertices up to z, since distinct paths cannot have the same endpoints. Therefore, the z,v-portion of P and the z,w-portion of R together form a v,w-path. Since G has only one v,w-path, this is Q. Hence z belongs to P, Q, and R.

Main result.

Proof 1 (induction on k). For $k = 2$, the hypothesis is the conclusion. For larger k, apply the inductive hypothesis to both $\{G_1, \ldots, G_{k-1}\}$ and $\{G_2, \ldots, G_k\}$. This yields a vertex u in all of $\{G_1, \ldots, G_{k-1}\}$ and a vertex v in all of $\{G_2, \ldots, G_k\}$. Because G is a tree, it has a unique u,v-path. This path belongs to all of $G_2, \ldots, G_{k-1}$. Let w be a vertex in $G_1 \cap G_k$. By the Lemma, the paths in G joining pairs in $\{u, v, w\}$ have a common vertex. Since the u,v-path is in $G_2, \ldots, G_{k-1}$, the w,u-path is in G_1, and the w,v-path is in G_k, the common vertex of these paths is in $G_1, \ldots, G_k$.

Proof 2 (induction on k). For $k = 3$, we let u, v, w be vertices of $G_1 \cap G_2$, $G_2 \cap G_3$, and $G_3 \cap G_1$, respectively. By the Lemma, the three paths joining these vertices have a common vertex, and this vertex belongs to all three subtrees. For $k > 3$, define the $k - 1$ subtrees $G_1 \cap G_k, \ldots, G_{k-1} \cap G_k$. By the case $k = 3$, these subtrees are pairwise intersecting. There are $k - 1$ of them, so by the induction hypothesis they have a common vertex. This vertex belongs to all of the original k trees.

2.1.73. *A simple graph G is a forest if and only if pairwise intersecting paths in G always have a common vertex.*

Sufficiency. We prove by contradiction that G is acyclic. If G has a cycle, then choosing any three vertices on the cycle cuts it into three paths that pairwise intersect at their endpoints. However, the three paths do not all have a common vertex. Hence G can have no cycle and is a tree.

Necessity. Let G be a forest. Pairwise intersecting paths lie in a single component of G, so we may assume that G is a tree. We use induction on the number of paths. By definition, two intersecting paths have a common vertex. For $k > 2$, let $P_1, \ldots, P_k$ be pairwise intersecting paths. Also $P_1, \ldots, P_{k-1}$ are pairwise intersecting, as are $P_2, \ldots, P_k$; each consists of $k - 1$ paths. The induction hypothesis guarantees a vertex u belonging to all of $P_1, \ldots, P_{k-1}$ and a vertex v belonging to all of $P_2, \ldots, P_k$. Since each of $P_2, \ldots, P_{k-1}$ contains both u and v and G has exactly one u,v-path Q, this path Q belongs to all of $P_2, \ldots, P_{k-1}$.

By hypothesis, P_1 and P_k also have a common vertex z. The unique z,u-path R lies in P_1, and the unique z,v-path S lies in P_k. Starting from z, let w be the last common vertex of R and S. It suffices to show that $w \in V(Q)$. Otherwise, consider the portion of R from w until it first reaches Q, the portion of S from w until it first reaches Q, and the portion of Q between these two points. Together, these form a closed trail and contain a cycle, but this cannot exist in the tree G. The contradiction implies that w belongs to Q and is the desired vertex.

2.1.74. *Every simple n-vertex graph G with $n-2$ edges is a subgraph of its complement.* (We need $e(G) < n-1$, since $K_{1,n-1} \not\subseteq \overline{K_{1,n-1}}$.)

We use induction on n. We will delete two vertices in the induction step, so we must include $n=2$ and $n=3$ in the basis. When $n=2$, we have $G = \overline{K}_2 \subseteq K_2 = \overline{G}$. When $n=3$, we have $G = K_2 + K_1 \subseteq P_3 = \overline{G}$.

For $n > 3$, let G be an n-vertex graph with $n-2$ edges. Suppose first that G has an isolated vertex x. Since $e(G) = n-2$, the Degree-Sum Formula yields a vertex y of degree at least 2. Let $G' = G - \{x, y\}$; this is a graph with $n-2$ vertices and at most $n-4$ edges. By the induction hypotheses, every graph with $n-2$ vertices and $n-4$ edges appears in its complement, so the same holds for smaller graphs (since they are contained in graphs with $n-4$ edges). A copy of G' contained in $\overline{G} - \{x, y\}$ extends to a copy of G in $\overline{G}$ by letting x represent y and letting y represent x.

Hence we may assume that G has no isolated vertices. Every non-tree component of G has at least as many edges as vertices, and trees have one less. Hence at least two components of G are trees. We may therefore choose vertices x and y of degree 1 with distinct neighbors. Let $N(x) = \{x'\}$ and $N(y) = \{y'\}$ with $x' \neq y'$. Let $G' = G - \{x, y\}$; this graph has $n-2$ vertices and $n-4$ edges. By the induction hypothesis, $G' \subseteq \overline{G'} = \overline{G} - x - y$. Let H be a copy of G' in $\overline{G} - x - y$. If x' or y' represents itself in H, then we let x and y switch identities to add their incident edges. Otherwise, we let x and y represent themselves to add their incident edges.

2.1.75. *Every non-star tree is (isomorphic to) a subgraph of its complement.*

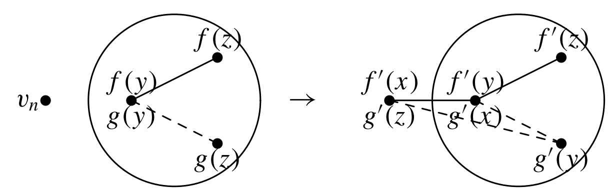
Proof 1 (loaded induction on n). We prove the stronger statement that, given an n-vertex tree T other than $K_{1,n-1}$, the graph K_n with vertex set $\{v_1, \ldots, v_n\}$ contains two edge-disjoint copies of T in which the two copies of each non-leaf vertex of T appear at distinct vertices. The only non-star tree with at most 4 vertices is the path P_4, which is self-complementary via a map that moves each vertex.

Now consider $n > 4$. We show first that T has a leaf x such that $T - x$ is not a star. If T is a path, let x be either leaf. Otherwise, T has at least three leaves; let P be a longest path in T, and let x be a leaf other than the endpoints of P. In either case, $T - x$ has a path of length at least 3.

Let $T' = T - x$, and let y be the neighbor of x in T. If y is not a leaf in T', then the induction hypothesis yields embeddings of T' in K_{n-1} in which y occurs at distinct vertices. We can extend both embeddings to K_n by placing x at v_n in each and adding the distinct edges to the images of y.

In this case the non-leaves of T are the same as the non-leaves of T', and the loaded claim holds for T.

If y is a leaf in T, we use the same argument unless $f(y) = g(y)$, where f, g are the mappings from $V(T')$ to $V(K_{n-1})$ for the two embeddings of T' guaranteed by the induction hypothesis. In this case, let z be the other neighbor of y; we have z as a non-leaf of T', and hence $f(z) \neq g(z)$. We cannot have both $g(z) = f(w)$ for some $w \in N(z)$ and $f(z) = g(u)$ for some $u \in N(z)$, because then the edge between $f(z)$ and $g(z)$ is used in both embeddings of T'. By symmetry, we may assume $f(z) \neq g(w)$ for all $w \in N(z)$. For T, we define $f', g' : V(T) \to V(K_n)$ for the edge-disjoint embeddings of T as follows: If $w \notin \{x, y, z\}$, let $f'(w) = f(w)$ and $g'(w) = g(w)$. For the other vertices, let $f'(z) = f(z)$, $f'(y) = f(y)$, $f'(x) = v_n$, $g'(z) = v_n$, $g'(y) = g(z)$, $g'(x) = g(y)$, as illustrated below. By construction the non-leaves of T have pairs of distinct images. The edges not involving x, y, z are mapped as before and hence become edge-disjoint subgraphs of $K_n - \{v_n, f(y), f(z), g(z)\}$. The path x, y, z is explicitly given edge-disjoint images under f', g'. This leaves only the edges involving z. Those under f are the same as under f'. The shift of z from $g(z)$ to $g'(z) = v_n$ does not produce a common edge because $f'(z) = f(z)$ is not the image under g of any neighbor of z.



Proof 2. (induction on $n(T)$ by deleting two leaves—proof due to Fred Galvin). To cover the basis step, we prove first that the claim is true when T has a path P of length at least 3 that includes a endpoint of every edge (see "caterpillars" in Section 2.2). First we embed P in its complement so that every vertex moves. If $n(P)$ is even, say $n(P) = 2k$, then we apply the vertex permutation $\binom{1,2,\ldots,k,k+1,\ldots,2k}{2,4,\ldots,2k,1,\ldots,2k-1}$. When $n(P) = 2k-1$, we use $\binom{1,2,\ldots,k,k+1,\ldots,2k-1}{2k-1,2k-3,\ldots,1,2k,\ldots,2}$. Now, since every vertex on P has moved, we can place the remaining leaves at their original positions and add incident edges from $\overline{T}$ to make them adjacent to their desired neighbors.

All non-star trees with at most six vertices have such a path P. For the induction step, consider a tree T with $n(T) > 6$. Let u and v be endpoints of a longest path in T, so $d(u, v) = \operatorname{diam} T$, and let $T' = T - u - v$. Let x and y be the neighbors of u and v, respectively. If T is not a star and T' is a star, then T is embeddable in its complement using the construction above.

If T' is not a star, then by the induction hypothesis T' embeds in $\overline{T'}$. If the embedding puts x or y at itself, then adding the edges xv and yu yields a copy of T in $\overline{T}$. Otherwise, make u adjacent to the image of x and v adjacent to the image of y to complete the copy of T in $\overline{T}$.

2.1.76. *If $A_1, \ldots, A_n$ are distinct subsets of $[n]$, then there exists $x \in [n]$ such that $A_1 \cup \{x\}, \ldots, A_n \cup \{x\}$ are distinct.* We need to find an element x such that no pair of sets differ by x. Consider the graph G with $V(G) = \{A_1, \ldots, A_n\}$ and $A_i \leftrightarrow A_j$ if only if A_i and A_j differ by the addition or deletion of a single element. Color (label) an edge A_iA_j by the element in which the endpoints differ. Any color that appears in a cycle of G must appear an even number of times in that cycle, because as we traverse the cycle we return to the original set. Hence a subgraph F formed by selecting one edge having each edge-label that appears in G will contain no cycles and must be a forest. Since a forest has at most $n-1$ edges, there must be an element that does not appear on any edge and can serve as x.

2.2. SPANNING TREES & ENUMERATION

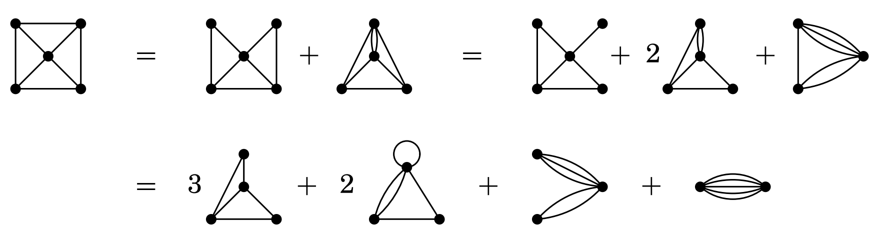
2.2.1. *Description of trees by Prüfer codes.* We use the fact that the degree of a vertex in the tree is one more than the number of times it appears in the corresponding code.

a) The trees with constant Prüfer codes are the stars. The $n-1$ labels that don't appear in the code have degree 1 in the tree; the label that appears $n-2$ times has degree $n-1$.

b) The trees whose codes contain two values are the double-stars. Since $n-2$ labels don't appear in the code, there are $n-2$ leaves in the tree.

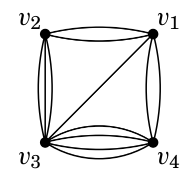
c) The trees whose codes have no repeated entries are the paths. Since $n-2$ labels appear once and two are missing, $n-2$ vertices have degree 2, and two are leaves. All trees with this degree sequence are paths.

2.2.2. *The graph $K_1 \vee C_4$ has 45 spanning trees.* For each graph G in the computation below, we mean $\tau(G)$.



$$= \quad 3\cdot 8 + 2\cdot 5 + 3\cdot 2 + 5 \quad = \quad 45$$

2.2.3. *Application of the Matrix Tree Theorem.* The matrix $Q = D - A$ for this graph appears on the right below. All rows and columns sum to 0. If we delete any row and column and take the determinant, the result is 106, which is the number of spanning trees. Alternatively, we could apply the recurrence. The number of trees not containing the diagonal edge is $2\cdot 3\cdot 4 + 3\cdot 4\cdot 2 + 4\cdot 2\cdot 2 + 2\cdot 2\cdot 3$, which is 76. The number of trees containing the diagonal edge is $5\cdot 6$, which is 30.

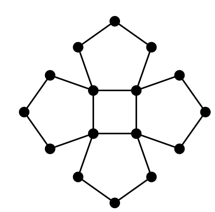


$$\begin{pmatrix} 5 & -2 & -3 & 0 \\ -2 & 5 & -1 & -2 \\ -3 & -1 & 8 & -1 \\ 0 & -2 & -4 & 6 \end{pmatrix}$$

2.2.4. *If a graph G with m edges has a graceful labeling, then K_{2m+1} decomposes into copies of G.* As in the proof of Theorem 2.2.16, view the vertices modulo $2m+1$. Let $a_1, \ldots, a_n$ be the vertex labels on in a graceful labeling of G. By definition, $0 \le a_j \le m$ for each j. For $0 \le i \le 2m$, the ith copy of G uses vertices $i + a_1, \ldots, i + a_n$. Each copy uses one edge from each difference class, and the successive copies use distinct edges from a class, so each edge of K_{2m+1} appears in exactly one of these copies of G.

2.2.5. *The graph below has 2000 spanning trees.* The graph has 16 vertices and 20 edges; we must delete five edges to form a spanning tree. The 5-cycles are pairwise edge-disjoint; we group the deleted edges by the 5-cycles. Each 5-cycle must lose an edge; one 5-cycle will lose two. To avoid disconnecting the graph, one edge lost from the 5-cycle that loses two must be on the 4-cycle, and thus the 4-cycle is also broken.

Every subgraph satisfying these rules is connected with 15 edges, since every vertex has a path to the central 4-cycle, and there is a path from one vertex to the next on the 4-cycle via the 5-cycles that lose just one edge). Hence these are the spanning trees. We can pick the 5-cycle that loses two edges in 4 ways, pick its second lost edge in 4 ways, and pick the edge lost from each remaining 5-cycle in five ways, yielding a total of $4\cdot 4\cdot 5\cdot 5\cdot 5$ spanning trees. The product is 2000.



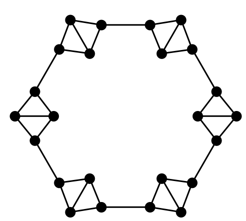
2.2.6. *The 3-regular graph that is a ring of m kites (shown below for $m = 6$) has $2m8^m$ spanning trees.* Call the edges joining kites the "link edges". Deleting two link edges disconnects the graph, so each spanning tree omits at most one link edge.

If a spanning tree uses $m - 1$ link edges, then it also contains a spanning tree from each kite. By Example 2.2.6, each kite has eight spanning trees. (Each such spanning tree has three edges; each choice of three edges works except the two forming triangles, and $8 = \binom{5}{2} - 2$.)

To form a spanning tree of this type, we pick one of the m link edges to delete and pick a spanning tree from each kite in 8^k ways. Thus there are $m8^{k-1}$ spanning trees of this sort.

The other possibility is to use all m link edges. Now we must have exactly one kite where the vertices of degree 2 in the kite are not connected by a path within the kite. Since we avoid cycles and spanning trees but must connect the two 3-valent vertices of the kite out to the rest of the graph, we retain exactly two edge from the kite that is cut. Each way of choosing two edges to retain works exept the two that form a path between the 2-valent vertex through one 3-valent vertex: $8 = \binom{5}{2} - 2$.

Since we pick one kite to cut in m ways, pick one of 8 ways to cut it, and pick one of 8 spanning trees in each other kite, there are $m8^m$ spanning trees of this type, for $2m8^m$ spannning trees altogether.



2.2.7. *$K_n - e$ has $(n-2)n^{n-3}$ spanning trees.*

Proof 1 (symmetry and Cayley's Formula—*easiest!*). By Cayley's Formula, there are n^{n-2} spanning trees in K_n. Since each has $n - 1$ edges, there are $(n-1)n^{n-2}$ pairs (e, T) such that T is a spanning tree in K_n and $e \in E(T)$. When we group these pairs according to the $\binom{n}{2}$ edges in K_n, we divide by $\binom{n}{2}$ to obtain $2n^{n-3}$ as the number of trees containing any given edge, since by symmetry each edge of K_n appears in the same number of spanning trees.

To count the spanning trees in $K_n - e$, we subtract from the total number of spanning trees in K_n the number that contain the particular edge e. Subtracting $t = 2n^{n-3}$ from n^{n-2} leaves $(n-2)n^{n-3}$ spanning trees in K_n that do not contain e.

Proof 2 (Prüfer correspondence). Given vertex set $[n]$, we count the trees not containing the edge between $n - 1$ and n. In the algorithm to generating the Prüfer code of a tree with vertex set $[n]$, we never delete vertex n. Also, we do not delete vertex $n-1$ unless $n-1$ and n are the only leaves, in which case the remaining tree at that stage is a path (because it is a tree with only two leaves).

If the tree contains the edge $(n-1, n)$, then $(n-1, n)$ will be the final edge, and the label last written down is $n - 1$ or n. If not, then the path between $n - 1$ and n has at least two edges, and we will peel off vertices from one end until only the edge containing n remains. The label n is never recorded during this process, and neither is $n - 1$. Thus a Prüfer code corresponds to a tree not containing $(n-1, n)$ if and only if the last term of the list is not $n - 1$ or n, and there are $(n-2)n^{n-3}$ such lists.

Proof 3 (Matrix Tree Theorem). For $K_n - e$, the matrix $D - A$ has diagonal $n-1, \dots, n-1, n-2, n-2$, with positions $n-1, n$ and $n, n-1$ equal to 0 and all else -1. Delete the last row and column and take the determinant to obtain the number of spanning trees. To compute the determinant, apply row and column operations as follows: 1) add the $n - 2$ other columns to the first so the first column becomes $1, \dots, 1, 0$. 2) subtract the first row from all but the last, so the first row is $1, -1, \dots, -1$, the last is $0, -1, \dots, -1, n-2$, and the others are 0 except for n on the diagonal. The interior rows can then be used to reduce this to a diagonal matrix with entries $1, n, \dots, n, n-2$, whose determinant is $(n-2)n^{n-3}$.

2.2.8. *With vertex set $[n]$, there are $\binom{n}{2}(2^{n-2} - 2)$ trees with $n - 2$ leaves and $n!/2$ trees with 2 leaves.* Every tree with two leaves is a path (paths along distinct edges incident to a vertex of degree k leads to k distinct leaves, so having only two leaves in a tree implies maximum degree 2). Every tree with $n - 2$ leaves has exactly two non-leaves. Each leaf is adjacent to one of these two vertices, with at least one leaf neighbor for each of the two vertices. These trees are the "double-stars".

To count paths directly, the vertices of a path in order form a permutation of the vertex set. Following the path from the other end produces another permutation. On the other hand, every permutation arises in this way. Hence there are two permutations for every path, and the number of paths is $n!/2$.

To count double-stars directly, we pick the two central vertices in one of $\binom{n}{2}$ ways and then pick the set of leaves adjacent to the lower of the two central vertices. This set is a subset of the $n - 2$ remaining vertex labels, and it can be any subset other than the full set and the empty set. The number of ways to do this is the same no matter how the central vertices is chosen, so the number of double-stars is $\binom{n}{2}(2^{n-2} - 2)$.

To solve this using the Prüfer correspondence, we count Prüfer codes for paths and for double-stars. In the Prüfer code corresponding to a tree, the labels of the leaves are the labels that do not appear.

For paths (two leaves), the other $n-2$ labels must each appear in the Prüfer code, so they must appear once each. Having chosen the leaf labels in $\binom{n}{2}$ ways, there are $(n-2)!$ ways to form a Prüfer code in which all the other labels appear. The product is $n!/2$.

For double-stars ($n-2$ leaves), exactly two labels appear in the Prüfer code. We can choose these two labels in $\binom{n}{2}$ ways. To form a Prüfer code (and thus a tree) with these two labels as non-leaves, we choose an arbitrary nonempty proper subset of the positions $1, \ldots, n-2$ for the appearances of the first label. There are $2^{n-2}-2$ ways to do this step. Hence there are $\binom{n}{2}(2^{n-2}-2)$ ways to form the Prüfer code.

2.2.9. *There are $(n!/k!)S(n-2, n-k)$ trees on a fixed vertex set of size n that have exactly k leaves.* Consider the Prüfer sequences of trees. The leaves of a tree are the labels that do not appear in the sequence. We can choose the labels of the leaves in $\binom{n}{k}$ ways. Given a fixed set of leaves, we must count the sequences of length $n-2$ in which the remaining $n-k$ labels all appear. Each label occupies some set of positions in the sequence. We partition the set of positions into $n-k$ nonempty parts, and then we can assign these parts to the labels in $(n-k)!$ ways to complete the sequence. The number of ways to perform the partition, by definition, is $S(n-2, n-k)$. Since these operations are independent, the total number of legal Prüfer sequences is $\binom{n}{k}(n-k)!S(n-2, n-k)$.

2.2.10. *$K_{2,m}$ has $m2^{m-1}$ spanning trees.* Let X, Y be the partite sets, with $|X| = 2$. Each spanning tree has one vertex of Y as a common neighbor of the vertices in X; it can be chosen in m ways. The remaining vertices are leaves; for each, we choose its neighbor in X in one of two ways. Every spanning tree is formed this way, so there are $m2^{m-1}$ trees.

Alternatively, note that $K_{2,m}$ is obtained from the two-vertex multigraph H with m edges by replacing each edge with a path of 2 edges. Since H itself has m spanning trees, Exercise 2.2.12 allows the spanning trees of $K_{2,m}$ to be counted by multiplying m by a factor of $2^{e(H)-n(H)+1} = 2^{m-1}$.

$K_{2,m}$ has $\lfloor (m+1)/2 \rfloor$ isomorphism classes of spanning trees. The vertices in X have one common neighbor, and the isomorphism class is determined by splitting the remaining $m-1$ vertices between them as leaves. We attach k leaves to one neighbor and $m-1-k$ to the other, where $0 \le k \le \lfloor (m-1)/2 \rfloor$. Hence there are $\lfloor (m+1)/2 \rfloor$ isomorphism classes.

2.2.11. $\tau(K_{3,m}) = m^2 3^{m-1}$. Let X, Y be the partite sets, with $|X| = 3$. A spanning tree must have a single vertex in Y adjacent to all of X or two vertices in Y forming P_5 with X. In each case, the remaining vertices of Y are distributed as leaf neighbors arbitrarily to the three vertices of X; each has a choice among the three vertices of X for its neighbor. Hence there are $m3^{m-1}$ spanning trees of the first type and $[3m2(m-1)/2]3^{m-2}$ trees of the second type. and then the remaining vertices in the other

2.2.12. *The effect of graph transformations on the number τ of spanning trees.* Let G be a graph with n vertices and m edges.

a) If H is obtained from G by replacing every edge with k parallel edges, then $\tau(H) = k^{n-1}\tau(G)$.

Proof 1 (direct combinatorial argument). Each spanning tree T of G yields k^{n-1} distinct spanning trees of H by choosing any one of the k copies of each edge in T. This implies $\tau(H) \ge k^{n-1}\tau(G)$. Also, every tree arises in this way. A tree T in H uses at most one edge between each pair of vertices. Since T is connected and acyclic, the edges in G whose copies are used in T form a spanning tree of G that generates T. Hence $\tau(H) \le k^{n-1}\tau(G)$.

Proof 2 (induction on m using the recurrence for τ). If $m = 0$, then $\tau(G) = \tau(H) = 0$, unless $n = 1$, in which case $1 = k^0 \cdot 1$. If $m > 0$, choose $e \in E(G)$. Let H' be the graph obtained from H by contracting all k copies of e. Let H'' be the graph obtained from H by deleting all k copies of e. The spanning trees of H can be grouped by whether they use a copy of e (they cannot use more than one copy). There are $k \times \tau(H')$ of these trees that use a copy of e and $\tau(H'')$ that do not. We can apply the induction hypothesis to H' and H'', since each arises from a graph with fewer than m edges by having k copies of each edge: H' from $G \cdot e$ and H'' from $G - e$. Thus

$$\begin{aligned}\tau(H) = k \times \tau(H') + \tau(H'') &= k \cdot k^{n-2}\tau(G \cdot e) + k^{n-1}\tau(G - e) \\ &= k^{n-1}[\tau(G \cdot e) + \tau(G - e)] = k^{n-1}\tau(G).\end{aligned}$$

Proof 3 (matrix tree theorem). Let Q, Q' be the matrices obtained from G, G', from which we delete one row and column before taking the determinant. By construction, $Q' = kQ$. When we take the determinant of a submatrix of order $n-1$, we thus obtain $\tau(G') = k^{n-1}\tau(G)$.

b) If H is obtained from G by replacing each $e \in E(G)$ with a path $P(e)$ of k edges, then $\tau(H) = k^{m-n+1}\tau(G)$.

Proof 1 (combinatorial argument). A spanning tree T of G yields k^{m-n+1} spanning trees of H as follows. If $e \in E(T)$, include all of $P(e)$. If $e \notin E(T)$, use all but one edge of $P(e)$. Choosing one of the k edges of $P(e)$ to omit for each $e \in E(G) - E(T)$ yields k^{m-n+1} distinct trees (connected and acyclic) in H. Again we must show that all spanning trees have been generated. A tree T' in H omits at most one edge from each path $P(e)$, else some vertex in $P(e)$ would be separated from the remainder of H. Let T be the spanning subgraph of G with $E(T) = \{e \in E(G)\colon P(e) \subseteq T'\}$. If T'

is connected and has no cycles, then the same is true of T, and T' is one of the trees generated from T as described above.

Proof 2 (induction on m). The basis step $m = 0$ is as in (a). For $m > 0$, select an edge $e \in E(G)$. The spanning trees of H use k or $k-1$ edges of $P(e)$. These two types are counted by $\tau(H')$ and $\tau(H'')$, where H' is the graph obtained from H by contracting all edges in $P(e)$, and H'' is the graph obtained from H by deleting $P(e)$ (except for its end-vertices). Since these graphs arise from $G \cdot e$ and $G - e$ (each with $m-1$ edges) by replacing each edge with a path of length k, applying the induction hypothesis yields

$$\begin{aligned}\tau(H) = \tau(H') + k \cdot \tau(H'') &= k^{(m-1)-(n-1)+1}\tau(G \cdot e) + k[k^{(m-1)-n+1}\tau(G-e)] \\ &= k^{m-n+1}[\tau(G \cdot e) + \tau(G-e)] = k^{m-n+1}\tau(G).\end{aligned}$$

2.2.13. *Spanning trees in $K_{n,n}$.* For each spanning tree T of $K_{n,n}$, a list $f(T)$ of pairs of integers (written vertically) is formed as follows: Let u, v be the least-indexed leaves of the remaining subtree that occur in X and Y. Add the pair $\binom{a}{b}$ to the sequence, where a is the index of the neighbor of u and b is the index of the neighbor of v. Delete $\{u, v\}$ and iterate until $n-2$ pairs are generated and one edge remains.

a) Every spanning tree of $K_{n,n}$ has a leaf in each partite set, and hence f is well-defined. If each vertex of one partite set has degree at least 2, then at least $2n$ edges are incident to this partite set, which are too many to have in a spanning tree of a graph with $2n$ vertices.

b) f is a bijection from the set of spanning trees of $K_{n,n}$ to the set of $n-1$-element lists of pairs of elements from $[n]$, and hence $K_{n,n}$ has n^{2n-2} spanning trees. We use an analogue of Prüfer codes. Consider $K_{n,n}$ with partite sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. For each spanning tree T, we form a sequence $f(T)$ of $n-1$ pairs of integers chosen from $[n]$ by recording at each step the ordered pair of subscripts of the neighbors of the least-indexed leaves of T remaining in X and Y, and then deleting these leaves. What remains is a spanning tree in a smaller balanced biclique, so by part (a) the process is well-defined.

Since there n^{2n-2} such lists, it suffices to show that f establishes a bijection from the set of spanning trees of $K_{n,n}$ to the set of lists.

From a list L of $n-1$ pairs of integers chosen from $[n]$, we generate a tree $g(L)$ with vertex set $X \cup Y$. We begin with $X \cup Y$, no edges, and each vertex unmarked. At the ith step, when the ith ordered pair is $\binom{a(i)}{b(i)}$, let u be the least index of an unmarked vertex in Y that does not appear in first coordinates of L at or after position i, and let v be the least index of an unmarked vertex in X that does not appear in second coordinates of L at or after position i. We add the edges $x_{a(i)}y_u$ and $y_{b(i)}x_v$, and then we mark x_v and y_u to eliminate them from further consideration. After $n-1$ pairs, we add one edge joining the two remaining unmarked vertices.

After the ith step, we have $2n - 2i$ components, each containing one unmarked vertex. This follows by induction on i; it holds when $i = 0$. Since indices cannot be marked until after they no longer appear in the list, the two edges created in the ith step join pairs of unmarked vertices. By the induction hypothesis, these come from four different components, and the two added edges combine these into two, each keeping one unmarked vertex. Thus adding the last edge completes the construction of a tree.

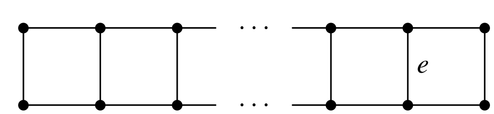
In computing $f(T)$, a label no longer appears in the sequence after it is deleted as a leaf. Hence the vertices marked at the ith step in computing $g(L)$ are precisely the leaves deleted at the ith step in computing $f(g(L))$, which also records $\binom{a(i)}{b(i)}$. Thus $L = f(g(L))$. Similarly, the leaves deleting at the ith step in computing $f(T)$ are the vertices marked at the ith step in computing $g(f(T))$, which yields $T = g(f(T))$. Hence each maps inverts the other, and both are bijections.

2.2.14. *The number of trees with vertices $1, \ldots, r+s$ that have partite sets of sizes r and s is $\binom{r+s}{s}s^{r-1}r^{s-1}$ if $r \neq s$.* It suffices to count the Prüfer codes for such trees. The factor $\binom{r+s}{r}$ counts the assignments of labels to the two partite sets (half that amount if $r = s$). When deleting a vertex in computing the Prüfer code, we record a vertex of the other partite set. Since an edge remains at the end of the construction, the final code has $s-1$ entries from the r-set and $r-1$ entries from the s-set.

It suffices to show that the sublists formed from each partite set determine the full list, because there are $s^{r-1}r^{s-1}$ such pairs of sublists. In reconstructing the code and tree from the pair of lists, the next leaf to be "finished" by receiving its last edge is the least label that is unfinished and doesn't appear in the remainder of the list. The remainder of the list is the remainder of the two sublists. We know which set contains the next leaf to be finished. Its neighbor comes from the other set. This tells us which sublist contributes the next element of the full list. Iterating this merges the two sublists into the full Prüfer code.

When $r = s$, the given formula counts the lists twice.

2.2.15. *For $n \geq 1$, the number of spanning trees in the graph G_n with $2n$ vertices and $3n-2$ edges pictured below satisfies the recurrence $t_n = 4t_{n-1} - t_{n-2}$ for $n \geq 3$, with $t_1 = 1$ and $t_2 = 4$.*

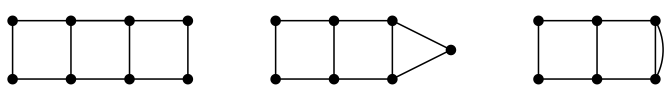


(*Comment:* The solution to the recurrence is $t_n = \frac{1}{2\sqrt{3}}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]$.) Using the recurrence, this follows by induction on n.) We derive the recurrence. Let $t_n = \tau(G_n)$.

Proof 1 (direct argument for recurrence). Each spanning tree in G_n uses two or three of the three rightmost edges. Those with two of the rightmost edges are obtained by adding any two of those edges to any spanning tree of G_{n-1}. Thus there are $3t_{n-1}$ such trees. To prove the recurrence $t_n = 4t_{n-1} - t_{n-2}$, it suffices to show that there are $t_{n-1} - t_{n-2}$ spanning trees that contain the three rightmost edges.

Such trees cannot contain the second-to-last vertical edge e. Therefore, deleting the three rightmost edges and adding e yields a spanning tree of G_{n-1}. Furthermore, each spanning tree of G_{n-1} using e arises exactly once in this way, because we can invert this operation. Hence the number of spanning trees of G_n containing the three rightmost edges equals the number of spanning trees of G_{n-1} containing e. The number of spanning trees of G_{n-1} that don't contain e is t_{n-2}, so the number of spanning trees of G_{n-1} that do contain e is $t_{n-1} - t_{n-2}$.

Proof 2 (deletion/contraction recurrence). Applying the recurrence introduces graphs of other types. Let H_n be the graph obtained by contracting the rightmost edge of G_n, and let F_{n-1} be the graph obtained by contracting one of the rightmost edges of H_n. Below we show G_4, H_4, and F_3.



By using $\tau(G) = \tau(G-e) + \tau(G \cdot e)$ on a rightmost edge e and observing that a pendant edge appears in all spanning trees while a loop appears in none, we obtain

$$\begin{aligned}\tau(G_n) &= \tau(G_{n-1}) + \tau(H_n)\\ \tau(H_n) &= \tau(G_{n-1}) + \tau(F_{n-1})\\ \tau(F_n) &= \tau(G_n) + \tau(H_{n-1})\end{aligned}$$

Substituting in for $\tau(H_n)$ and then for $\tau(F_{n-1})$ and then for $\tau(H_{n-1})$ yields the desired recurrence:

$$\begin{aligned}\tau(G_n) &= \tau(G_{n-1}) + \tau(G_{n-1}) + \tau(F_{n-1}) = 2\tau(G_{n-1}) + \tau(G_{n-1}) + \tau(H_{n-2})\\ &= 3\tau(G_{n-1}) + \tau(G_{n-1}) - \tau(G_{n-2}) = 4\tau(G_{n-1}) - \tau(G_{n-2}).\end{aligned}$$

2.2.16. *Spanning trees in $K_1 \vee P_n$.* The number a_n of spanning trees satisfies $a_n = a_{n-1} + 1 + \sum_{i=1}^{n-1} a_i$ for $n > 1$, with $a_1 = 1$. Let $x_1, \ldots, x_n$ be the vertices of the path in order, and let z be the vertex off the path. There are a_{n-1} spanning trees not using the edge zx_n; they combine the edge $x_{n-1}x_n$

with a spanning tree of $K_1 \vee P_{n-1}$. Among trees containing zx_n, let i be the highest index such that all of the path $x_{i+1}, \ldots, x_n$ appears in the tree. For each i, there are a_i such trees, since the specified edges are combined with a spanning tree of $K_1 \vee P_i$. The term 1 corresponds to $i = 0$; here the entire tree is $P_n \cup zx_n$. This exhausts all possible spanning trees.

2.2.17. *Cayley's formula from the Matrix Tree Theorem.* The number of labeled n-vertex trees is the number of spanning trees in K_n. Using the Matrix Tree Theorem, we compute this by subtracting the adjacency matrix from the diagonal matrix of degrees, deleting one row and column, and taking the determinant. All degrees are $n-1$, so the initial matrix is $n-1$ on the diagonal and -1 elsewhere. Delete the last row and column. We compute the determinant of the resulting matrix.

Proof 1 (row operations). Add every row to the first row does not change the determinant but makes every entry in the first row 1. Now add the first row to every other row. The determinant remains unchanged, but every row below the first is now 0 everywhere except on the diagonal, where the value is n. The matrix is now upper triangular, so the determinant is the product of the diagonal entries, which are one 1 and $n-2$ copies of n. Hence the determinant is n^{n-2}, as desired.

Proof 2 (eigenvalues). The determinant of a matrix is the product of its eigenvalues. The eigenvalues of a matrix are shifted by λ when λI is added to the matrix. The matrix in question is $nI_{n-1} - J_{n-1}$, where I_{n-1} is the $n-1$-by-$n-1$ identity matrix and J_{n-1} is the $n-1$-by-$n-1$ matrix with every entry 1. The eigenvalues of $-J_{n-1}$ are $-(n-1)$ with multiplicity 1 and 0 with multiplicity $n-2$. Hence the eigenvalues of the desired matrix are 1 with multiplicity 1 and n with multiplicity $n-2$. Hence the determinant is n^{n-2}, as desired.

2.2.18. *Proof that $\tau(K_{r,s}) = s^{r-1}r^{s-1}$ using the Matrix Tree Theorem.* The adjacency matrix of $K_{r,s}$ is $\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{pmatrix}$, where $\mathbf{0}$ and $\mathbf{1}$ denote matrices of all 0s and all 1s, and both the row partition and the column partition consist of r in the first block and s in the second block. The diagonal matrix of degrees is $\begin{pmatrix}sI_r & 0\\ 0 & rI_s\end{pmatrix}$, where I_n is the identity matrix of order n. Hence we may delete the first row and column to obtain $Q^* = \begin{pmatrix}sI_{r-1} & -\mathbf{1}\\ -\mathbf{1} & rI_s\end{pmatrix}$.

We apply row and column operations that do not change the determinant. We subtract column $r-1$ (last of the first block) from the earlier columns and subtract column r (first of the second block) from the later columns. This yields the matrix on the left below, where the values outside the matrix indicate the number of rows or columns in the blocks. Now, we add to row $r-1$ the earlier rows and add to row r the later rows, yielding the matrix on the right below.

$$\begin{array}{c@{}c} & \begin{array}{cccc} r-2 & 1 & 1 & s-1 \end{array} \\ \begin{array}{c} r-2\\ 1\\ 1\\ s-1 \end{array} & \begin{pmatrix} sI_{r-2} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ -s\mathbf{1} & s & -1 & \mathbf{0} \\ \mathbf{0} & -1 & r & -r\mathbf{1} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & rI_{s-1} \end{pmatrix} \end{array} \qquad \begin{array}{c@{}c} & \begin{array}{cccc} r-2 & 1 & 1 & s-1 \end{array} \\ \begin{array}{c} r-2\\ 1\\ 1\\ s-1 \end{array} & \begin{pmatrix} sI_{r-2} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & s & -r+1 & \mathbf{0} \\ \mathbf{0} & -s & r & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & rI_{s-1} \end{pmatrix} \end{array}$$

Adding row $r-1$ to row r now makes row r all zero except for a single 1 in position r (on the diagonal). Adding row r to the first $r-2$ rows (and $r-1$ times row r to row $r-1$) now leaves the 1 in row r as the only nonzero entry in column r. Also, the s in column $r-1$ of row $r-1$ is now the only nonzero entry in row $r-1$. Hence we can add $1/s$ times row $r-1$ to each of the last $s-1$ rows to eliminate the other nonzero entries in column $r-1$.

The resulting matrix is diagonal, with diagonal entries consisting of $r-1$ copies of s, one copy of 1, and $s-1$ copies of r. Since adding a multiple of a row or column to another does not change the determinant, the determinant of our original matrix equals the determinant of this diagonal matrix. The determinant of a diagonal matrix is the product of its diagonal entries, so the determinant is $s^{r-1}r^{s-1}$.

2.2.19. *The number t_n of labeled trees on n vertices satisfies the recurrence $t_n = \sum_{k=1}^{n-1} k\binom{n-2}{k-1} t_k t_{n-k}$.* For an arbitrary labeled tree on n vertices, delete the edge incident to v_2 on the path from v_2 to v_1. This yields labeled trees on k and $n-k$ vertices for some k, where v_1 belongs to the tree on k vertices and v_2 to the tree on $n-k$ vertices. Each such pair arises from exactly $k\binom{n-2}{k-1}$ labeled trees on n vertices. To see this, reverse the process. First choose the $k-1$ other vertices to be in the subtree containing v_1. Next, choose a tree on k labeled vertices and a tree on $n-k$ labeled vertices (any such choice could arise by deleting the specified edge of a tree on n vertices). Finally, reconnect the tree by adding an edge from v_2 to any one of the k vertices in the tree containing v_1. This counts the trees such that the subtree containing v_1 has k vertices, and summing this over k yields t_n.

2.2.20. *A d-regular graph G has a decomposition into copies of $K_{1,d}$ if and only if G is bipartite.* If G has bipartition X, Y, then for each $x \in X$ we include the copy of $K_{1,d}$ obtained by taking all d edges incident to x. Since every edge has exactly one endpoint in X, and every vertex in X has degree d, this puts every edge of G into exactly one star in our list.

If G has a $K_{1,d}$-decomposition, then we let X be the set of centers of the copies of $K_{1,d}$ in the decomposition. Since G is d-regular, each copy of $K_{1,d}$ uses all edges incident to its center. Since the list is a decomposition, each edge is in exactly one such star, so X is an independent set. Since every edge belongs to some $K_{1,d}$ centered in X, there is no edge with both endpoints outside X. Thus the remaining vertices also form an independent set, and G has bipartition $X, \overline{X}$.

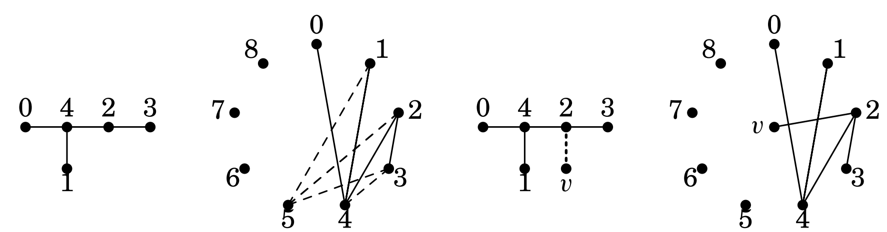
Alternative proof of sufficiency. If G is not bipartite, then G contains an odd cycle. When decomposing a d-regular graph into copies of $K_{1,d}$, each subgraph used consists of all d edges incident to a single vertex. Hence each vertex occurs only as a center or only as a leaf in these subgraphs. Also, every edge joins the center and the leaf in the star containing it. These statements require that centers and leaves alternate along a cycle, but this cannot be done in an odd cycle.

2.2.21. *Decomposition of $K_{2m-1,2m}$ into m spanning paths.* We add a vertex to the smaller partite set and decomposition $K_{2m,2m}$ into m spanning cycles. Deleting the added vertex from each cycle yields pairwise edge-disjoint spanning paths of $K_{2m-1,2m}$.

Let the partite sets of $K_{2m,2m}$ be $x_1, \ldots, x_{2m}$ and $y_1, \ldots, y_{2m}$. Let the kth cycle consist of the edges of the forms $x_i y_{i+2k-1}$ and $x_i y_{i+2k}$, where subscripts above $2m$ are reduced by $2m$. These sets are pairwise disjoint and form spanning cycles.

2.2.22. *If G is an n-vertex simple graph having a decomposition into k spanning trees, and $\Delta(G) = \delta(G) + 1$, then G has $n - 2k$ vertices of degree $2k$ and $2k$ vertices of degree $2k-1$.* Each spanning tree has $n-1$ edges, so $e(G) = k(n-1)$. Note that $k < n/2$, since G is simple and is not K_n (since it is not regular). If G has r vertices of minimum degree and $n-r$ of maximum degree, then the Degree-Sum Formula yields $2k(n-1) = n\Delta(G) - r$. Since $1 \le r \le n$, we conclude that $\Delta(G) = 2k = r$.

2.2.23. *If the Graceful Tree Conjecture holds and $e(T) = m$, then K_{2m} decomposes into $2m-1$ copies of T.* Let $T' = T - u$, where u is a leaf of T with neighbor v. Let w be a vertex of K_{2m}. Construct a cyclic T'-decomposition of $K_{2m} - w$ using a graceful labeling of T' as in the proof of Theorem 2.2.16. Each vertex serves as v in exactly one copy of T'. Extend each copy of T' to a copy of T by adding the edge to w from the vertex serving as v. This exhausts the edges to w and completes the T-decomposition of G.

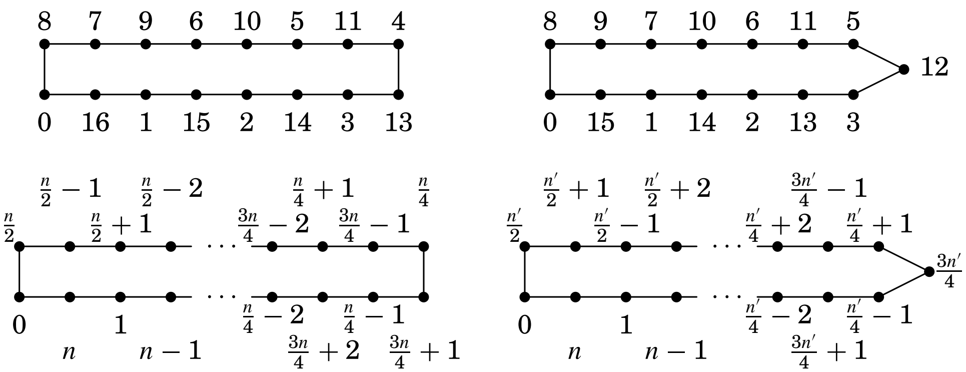


2.2.24. *Of the n^{n-2} trees with vertex set $\{0, \ldots, n-1\}$, how many are gracefully labeled by their vertex names?* This question was incorrectly posed. It should be *of the graphs with vertex set* $\{0,\dots,n-1\}$ *that have* $n-1$ *edges, how many are gracefully labeled by their vertex names?* Such a graph has k choices for the placement of the edge with difference $n-k$, since the lower endpoint can be any of $\{0,\dots,k-1\}$. Hence the number of graphs is $(n-1)!$.

2.2.25. *If a graph G is graceful and Eulerian, then $e(G)$ is congruent to 0 or 3 mod 4.* Let f be a graceful labeling. The parity of the sum of the labels on an edge is the same as the parity of their difference. Hence the sum $\sum_{v\in V(G)} d(v)f(v)$ has the same parity as the sum of the edge differences. The first sum is even, since G is Eulerian. The second has the same parity as the number of odd numbers in the range from 1 to $e(G)$. This is even if and only if $e(G)$ is congruent to 0 or 3 mod 4, which completes the proof.

2.2.26. *The cycle C_n is graceful if and only if 4 divides n or $n+1$.* The necessity of the condition is a special case of Exercise 2.2.25. For sufficiency, we provide a construction for each congruence class. We show an explicit construction ($n=16$ and $n=15$) and a general construction for each class. In the class where $n+1$ is divisible by 4, we let n' denote $n+1$. When n is divisible by 4, let $n'=n$.

The labeling uses a base edge joining 0 and $n'/2$, plus two paths. The bottom path, starting from 0, alternates labels from the top and bottom to give the large differences: $n, n-1$, and so on down to $n'/2+1$. The top path, starting from $n'/2$, uses labels working from the center to give the small differences: 1, 2, and so on up to $n'/2-1$. The label next to $n'/2$ is $n'/2-1$ when 4 divides n, otherwise $n'/2+1$. When chosen this way, the two paths reach the same label at their other ends to complete the cycle: $n/4$ in the even case, $3n'/4$ in the odd case. Checking this ensures that the intervals of labels used do not overlap. Note that the value $3n/4$ is not used in the even case, and $n'/4$ is not used in the odd case.

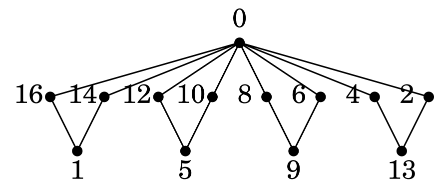


2.2.27. *The graph consisting of k copies of C_4 with one common vertex is graceful.* The construction is illustrated below for $k=4$. Let x be the

central vertex. Let the neighbors of x be $y_0,\dots,y_{2k-1}$, and let the remaining vertices be $z_0,\dots,z_{k-1}$, such that $N(z_i)=\{y_{2i}, y_{2i+1}\}$.

Define a labeling f by $f(x)=0$, $f(y_i)=4k-2i$, and $f(z_i)=4i+1$. The labels on $y_1,\dots,y_{2k}$ are distinct positive even numbers, and those on $z_1,\dots,z_k$ are distinct odd numbers, so f is injective, as desired. The differences on the edges from x are the desired distinct even numbers.

The differences on the remaining edges are odd and less than $2k$; it suffices to show that their values are distinct. Involving z_i, the differences are $4k-1-8i$ and $4k-3-8i$. Starting from z_0 through increasing i, these are $4k-1, 4k-3, 4k-9, 4k-11, \dots$. Starting from z_{k-1} through decreasing i, these are $-4k+5, -4k+7, -4k+13, -4k+15, \dots$. The absolute values are distinct, as needed.



2.2.28. Given positive integers $d_1,\dots,d_n$, there exists a caterpillar with vertex degrees $d_1,\dots,d_n$ if and only if $\sum d_i = 2n-2$. If there is such a caterpillar, it is a tree and has $n-1$ edges, and hence the vertex degrees sum to $n-2$. Hence the condition is necessary. There are various proofs of sufficiency, which construct a caterpillar with these degrees given only the list $d_1,\dots,d_n$ of positive numbers with sum $2n-2$.

Proof 1 (explicit construction). We may assume that $d_1\ge\cdots\ge d_k > 1 = d_{k+1}=\cdots=d_n$. Begin with a path of length $k+1$ with vertices $v_0,\dots,v_{k+1}$. Augment these vertices to their desired degrees by adding d_i-2 edges (and leaf neighbors) at v_i, for $1\le i\le k$. This creates a caterpillar with vertex degrees $d_1,\dots,d_k$ for the nonleaves. We must prove that it has $n-k$ leaves, which is the number of 1s in the list.

Including v_0 and v_{k+1}, the actual number of leaves in the caterpillar we constructed is $2+\sum_{i=1}^{k}(d_i-2)$. This equals $2-2k+(\sum_{i=1}^{n} d_i)-\sum_{i=k+1}^{n} d_i$. Since $\sum_{i=1}^{n} d_i = 2n-2$, the number of leaves is $(2-2k)+(2n-2)-(n-k)=n-k$, as desired. We have created an n-vertex caterpillar with vertex degrees $d_1,\dots,d_n$.

Proof 2 (induction on n). Basis step ($n=2$): the only list is 1, 1, and the one graph realizing this is a caterpillar. Induction step ($n>2$): n positive numbers summing to $2n-2$ must include at least two 1s; otherwise, the sum is at least $2n-1$. If the remaining numbers are all 2s, then P_n is a caterpillar with the desired degrees. Otherwise, some d_i exceeds 2; by symmetry, we may assume that this is d_1. Let d' be the list obtained by reducing d_1 by one and deleting one of the 1s. The list d' has $n-1$ entries, all positive, and it sums to $2n-4 = 2(n-1)-2$. By the induction hypothesis, there is a caterpillar G' with degree list d'.

Let x be a vertex of G' with degree d'_1. Since $d_1 > 2$, we have $d'_1 \geq 2$, and hence x is on the spinal path. Growing a leaf at x yields obtain a larger caterpillar G with degree list d. This completes the induction step.

2.2.29. *Every tree transforms to a caterpillar with the same degree list by operations that delete an edge and add another rejoining the two components.* Let P be a longest path in the current tree T. If P is incident to every edge, then T is a caterpillar. Otherwise a path P' of length at least two leaves P at some vertex x. Let uv be an edge of P', with u between x and v, and let y be a neighbor of x on P.

Cut xy and add yu. Now cut uv and add vx. Each operation has the specified type, and together they form a 2-switch preserving the vertex degrees. Also, the new tree has a path whose length is that of P plus $d_T(x,u)$.

Since the length of a path cannot exceed the number of vertices, this process terminates. It can only terminate when the longest path is incident to all edges and the tree is a caterpillar.

2.2.30. *A connected graph is a caterpillar if and only if it can be drawn on a channel without edge crossings.*

Necessity. If G is a caterpillar, let P be the spine of G. Draw P on a channel by alternating between the two sides of the channel. The remaining edges of G consist of a leaf and a vertex of P. If u, v, w are three consecutive vertices on P, then v has an "unobstructed view" of the other side of the channel between the edges vu and vw. Each leaf x adjacent to v can be placed in that portion of the other bank, and the edge vx can then be drawn straight across the channel without crossing another edge.

Sufficiency. Suppose that G is drawn on a channel. The endpoints of an edge e cannot both have neighbors in the same direction along the channel, since that would create a crossing. Hence G has no cycle, since a cycle would leave an edge and return to it via the same direction along the channel. We conclude that G is a tree.

If G contains the 7-vertex tree that is not a caterpillar, then let v be its central vertex. The three neighbors of v occur on the other side of the channel in some order; let u be the middle neighbor. The other edge incident to u must lie in one direction or the other from uv, contradicting the preceding paragraph. Hence G avoids the forbidden subtree and is a caterpillar.

(Alternatively, we can prove this directly by moving along the channel to extract the spine, observing that the remainder of the tree must be leaves attached to the spine.)

2.2.31. *Every caterpillar has an up/down labeling.* Constructive proof. Let $P = v_0, \ldots, v_k$ be a longest path in a caterpillar G with m edges; by the argument above P is the spine of G. We iteratively construct a graceful labeling f for G. Define two parameters l, u that denote the biggest low label and smallest high label used; after each stage the unused labels are $\{l+1, \ldots, u-1\}$. Let r denote the lowest edge difference achieved; after each stage $r, \ldots, m$ have been achieved.

Begin by setting $f(v_0) = 0$ and $f(v_1) = m$; hence $l = 0$, $u = m$, $r = m$. Before stage i, we will have $\{f(v_i), f(v_{i-1})\} = \{l, u\}$; this is true by construction before stage 1. Suppose this is true before stage i, along with the other claims made for l, u, d. Let $d = d_G(v_i)$. In stage i, label the $d-1$ remaining neighbors of v_i with the $d-1$ numbers nearest $f(v_{i-1})$ that have not been used, ending with v_{i+1}. Since we start with $|f(v_i) - f(v_{i-1})| = u - l = r$, the new differences are $r-1, \ldots, r-d+1$, which have not yet been achieved. To finish stage i, reset r to $r-d+1$; also, if $f(x_{i-1}) = l$ reset l to $l+d-1$, but if $f(x_{i-1}) = u$ reset u to $u-d+1$. Now stage i is complete, and the claims about l, u, r are satisfied as we are ready to start stage $i+1$: $\{f(v_{i+1}), f(v_i)\} = \{l, u\}$, $r = u-l$, and the edge differences so far are $r, \ldots, m$. After stage $k-1$, we have assigned distinct labels in $\{0, \ldots, m\}$ to all $m+1$ vertices, and the differences of labels of adjacent vertices are all distinct, so we have constructed a graceful labeling.

The 7-vertex tree that is not a caterpillar has no up/down labeling. In an up/down labeling of a connected bipartite graph, one partite set must have all labels above the threshold and the other have all labels below the threshold. Also, we can interchange the high side and the low side by subtracting all labels from $n-1$. Hence for this 7-vertex tree we may assume the labels on the vertices of degree 2 are the high labels 4,5,6. Since 0,6 must be adjacent, this leaves two cases: 0 on the center or 0 on the leaf next to 6. In the first case, putting 1 or 2 next to 6 gives a difference already present, but with 3 next to 6 we can no longer obtain a difference of 1 on any edge. In the second case, we can only obtain a difference of 5 by putting 1 on the center, but now putting 2 next to 5 gives two edges with difference 3, while putting 2 next to 4 and 3 next to 5 give two edges with difference 2. Hence there is no way to complete an up/down labeling.

2.2.32. *There are $2^{n-4} + 2^{\lfloor (n-4)/2 \rfloor}$ isomorphism classes of n-vertex caterpillars.* We describe caterpillars by binary lists. Each 1 represents an edge on the spine. Each 0 represents a pendant edge at the spine vertex between the edges corresponding to the nearest 1s on each side. Thus n-vertex caterpillars correspond to binary lists of length $n-1$ with both end bits being 1.

We can generate the lists for caterpillars from either end of the spine; reversing the list yields a caterpillar in the same isomorphism class. Hence we count the lists, add the symmetric lists, and divide by 2. There are 2^{n-3} lists of the specified type. To make a symmetric list, we specify $\lceil (n-3)/2 \rceil$ bits. Thus the result is $(2^{n-3} + 2^{\lceil (n-3)/2 \rceil})/2$.

2.2.33. *If T is an orientation of a tree such that the heads of the edges are all distinct, then T is a union of paths from the root (the one vertex that is not a head), and each each vertex is reached by one path from the root.* We use induction on n, the number of vertices. For $n = 1$, the tree with one vertex satisfies all the conditions. Consider $n > 1$. Since there are $n-1$ edges, some vertex is not a tail. This vertex v is not the root, since the root is the tail of all its incident edges. Since the heads are distinct, v is incident to only one edge and is its head. Hence $T - v$ is an orientation of a smaller tree where the heads of the edges are distinct. By the induction hypothesis, it is a tree of paths from the root (one to each vertex), and replacing the edge to v preserves this desired conclusion for the full tree.

2.2.34. *An explicit de Bruijn cycle of length 2^n is generated by starting with n 0's and subsequently appending a 1 when doing so does not repeat a previous string of length n (otherwise append a 0).* A de Bruijn cycle is formed by recording the successive edge labels along an Eulerian circuit in the de Bruijn digraph. The vertices of the de Bruijn digraph are the 2^{n-1} binary strings of length $n-1$. From each vertex two edges depart, labeled 0 and 1. The edge 0 leaving v goes to the vertex obtained by dropping the first bit of v and appending 0 at the end. The edge 1 leaving v goes to the vertex obtained by dropping the first bit of v and appending 1 at the end.

Let v_0 denote the all-zero vertex, and let e be the loop at v_0 labeled 0. The $2^{n-1} - 1$ edges labeled 0 other than e form a tree of paths in to v_0. (Since a path along these edges never reintroduces a 1, it cannot return to a vertex with a 1 after leaving it.) Starting at v_0 along edge e means starting with n 0's. Algorithm 2.4.7 now tells us to follow the edge labeled 1 at every subsequent step unless it has already been used; that is, unless appending a 1 to the current list creates a previous string of length n. Theorem 2.4.9 guarantees that the result is an Eulerian circuit.

2.2.35. *Tarry's Algorithm (The Robot in the Castle).* The rules of motion are: 1) After entering a corridor, traverse it and enter the room at the other end. 2) After entering a room whose doors are all unmarked, mark I on the door of entry. 3) When in a room having an unmarked door, mark O on some unmarked door and exit through it. 4) When in a room having all doors marked, find one not marked O (if one exists), and exit through it. 5) When in a room having all doors marked O, STOP.

When in a room other than the original room u, the number of entering edges that have been used exceeds the number of exiting edges. Thus an exiting door has not yet been marked O. This implies that the robot can only terminate in the original room u.

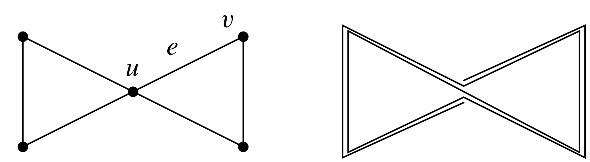
The edges marked I grow from u a tree of paths that can be followed back to u. The rules for motion establish an ordering of the edges leaving each room so that the edge labeled I (for a room other than u) is last.

In order to terminate in u or to leave a room v by the door marked I, every edge entering the room must have been used to enter it, including all edges marked I at the other end. Therefore, for every room actually entered, the robot follows all its incident corridors in both directions.

Thus it suffices to show that every room is reached. Let V be the set of all rooms, and let S be the set of rooms reached in a particular robot tour. If $S \neq V$, then since the castle is connected there is a corridor joining rooms $s \in S$ and $r \notin S$ (the shortest path between S and $\overline{S}$. Since every reached vertex has its incidence corridors followed in both directions, the corridor sr is followed, and r is also reached. The contradiction yields $S = V$.

Comment. Consider a digraph in which each corridor becomes a pair of oppositely-directed edges. Thus indegree equals outdegree at each vertex. The digraph is Eulerian, and the edges marked I form an intree to the initial vertex. The rules for the robot produce an Eulerian circuit by the method in Algorithm 2.4.7.

The portion of the original tour after the initial edge $e = uv$ is not a tour formed according to the rules for a tour in $G - e$, because in the original tour no door of u is ever marked I. If e is not a cut-edge, then tours that follow e, follow $G - e$ from v, and return along e do not include tours that do not start and end with e. There may be such tours, as illustrated below, so such a proof falls into the induction trap.



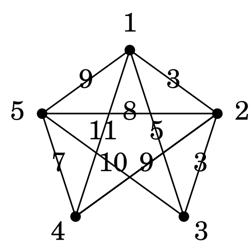
2.3. OPTIMIZATION AND TREES

2.3.1. *In an edge-weighting of K_n, the total weight on every cycle is even if and only if the total weight on every triangle is even.* Necessity is trivial, since triangles are cycles. For sufficiency, suppose that every triangle has even weight. We use induction on the length to prove that every cycle C has even weight. The basis step, length 3, is given by hypothesis. For the induction step, consider a cycle C and a chord e. The chord creates two shorter cycles C_1, C_2 with C. By the induction hypothesis, C_1 and C_2 have even weight. The weight of C is the sum of their weights minus twice the weight of e, so it is still even.

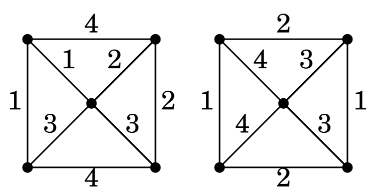
2.3.2. *If T is a minimum-weight spanning tree of a weighted graph G, then the u, v-path in T need not be a minimum-weight u, v-path in G.* If G is a cycle of length of length at least 3 with all edge weights 1, then the cheapest path between the endpoints of the edge omitted by T has cost 1, but the cheapest path between them in T costs $n(G) - 1$.

2.3.3. *Computation of minimum spanning tree.* The matrix on the left below corresponds to the weighted graph on the right. Using Kruskal's algorithm, we iteratively select the cheapest edge not creating a cycle. Starting with the two edges of weight 3, the edge of weight 5 is forbidden, but the edge of weight 7 is available. The edge of weight 8 completes the minimum spanning tree, total weight 21. Note that if the edge of weight 8 had weight 10, then either of the edges of weight 9 could be chosen to complete the tree; in this case there would be two spanning trees with the minimum value.

$$\begin{pmatrix} 0 & 3 & 5 & 11 & 9 \\ 3 & 0 & 3 & 9 & 8 \\ 5 & 3 & 0 & \infty & 10 \\ 11 & 9 & \infty & 0 & 7 \\ 9 & 8 & 10 & 7 & 0 \end{pmatrix}$$



2.3.4. *Weighted trees in $K_1 \vee C_4$.* On the left, the spanning tree is unique, using all edges of weights 1 and 2. On the right it can use either edge of weight 2 and either edge of weight 3 plus the edges of weight 1.



2.3.5. *Shortest paths in a digraph.* The direct i to j travel time is the entry $a_{i,j}$ in the first matrix below. The second matrix recordes the least i to j travel time for each pair i, j. These numbers were determined for each i by iteratively updating candidate distances from i and then selecting the closest of the unreached set (Dijkstra's Algorithm). To do this by hand,

make an extra copy of the matrix and use crossouts to update candidate distances in each row, using the original numbers when updating candidate distances. The answer can be presented with more information by drawing the tree of shortest paths that grows from each vertex.

$$\begin{pmatrix} 0 & 10 & 20 & \infty & 17 \\ 7 & 0 & 5 & 22 & 33 \\ 14 & 13 & 0 & 15 & 27 \\ 30 & \infty & 17 & 0 & 10 \\ \infty & 15 & 12 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 10 & 15 & 25 & 17 \\ 7 & 0 & 5 & 20 & 24 \\ 14 & 13 & 0 & 15 & 25 \\ 30 & 25 & 17 & 0 & 10 \\ 22 & 15 & 12 & 8 & 0 \end{pmatrix}$$

2.3.6. *In an integer weighting of the edges of K_n, the total weight is even on every cycle if and only if the subgraph consisting of the edges with odd weight is a spanning complete bipartite subgraph.*

Sufficiency. Every cycle contains an even number of edges from a spanning complete bipartite subgraph.

Necessity. Suppose that the total weight on every cycle is even. We claim that every component of the spanning subgraph consisting of edges with even weight is a complete graph. Otherwise, it has two vertices x, y at distance 2, which induce P_3 with their common neighbor z. Since xy has odd weight, x, y, z would form a cycle with odd total weight.

If the spanning subgraph of edges with even weight has at least three components, then selecting one vertex from each of three components yields a triangle with odd weight. Hence there are at most two components. This implies that the complement (the graph of edges with odd weight) is a spanning complete bipartite subgraph of G.

2.3.7. *A weighted graph with distinct edge weights has a unique minimum-weight spanning tree (MST).*

Proof 1 (properties of spanning trees). If G has two minimum-weight spanning trees, then let e be the lightest edge of the symmetric difference. Since the edge weights are distinct, this weight appears in only one of the two trees. Let T be this tree, and let T' be the other. Since $e \in E(T) - E(T')$, there exists $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree. By the choice of e, $w(e') > w(e)$. Now $w(T' + e - e') < w(T')$, contradicting the assumption that T' is an MST. Hence there cannot be two MSTs.

Proof 2 (induction on $k = e(G) - n(G) + 1$). If $k = 0$, then G is a tree and has only one spanning tree. If $k > 0$, then G is not a tree; let e be the heaviest edge of G that appears in a cycle, and let C be the cycle containing e. We claim that e appears in no MST of G. If T is a spanning tree containing e, then T omits some edge e' of C, and $T - e + e'$ is a cheaper spanning tree than T. Since e appears in no MST of G, every MST of G is an MST of $G - e$. By the induction hypothesis, there is only one such tree.

Proof 3 (Kruskal's Algorithm). In Kruskal's Algorithm, there is no choice if there are no ties between edge weights. Thus the algorithm can produce only one tree. We also need to show that Kruskal's Algorithm can produce every MST. The proof in the text can be modified to show this; if e is the first edge of the algorithm's tree that is not in an MST T', then we obtain an edge e' with the same weight as e such that $e' \in E(T') - E(T)$ and e' is available when e is chosen. The algorithm can choose e' instead. Continuing to modify the choices in this way turns T into T'.

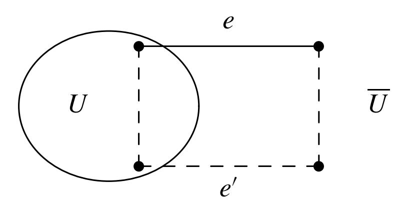
2.3.8. *No matter how ties are broken in choosing the next edge for Kruskal's Algorithm, the list of weights of a minimum spanning tree (in nondecreasing order) is unique.* We consider edges in nondecreasing order of cost. We prove that after considering all edges of a particular cost, the vertex sets of the components of the forest built so far is the same independent of the order of consideration of the edges of that cost. We prove this by induction on the number of different cost values that have been considered. At the start, none have been considered and the forest consists of isolated vertices.

Before considering the edges of cost x, the induction hypothesis tells us that the vertex sets of the components of the forest are fixed. Let H be a graph with a vertex for each such component, and put two vertices adjacent in H if G has an edge of cost x joining the corresponding two components. Suppose that H has k vertices and l components. Independent of the order in which the algorithm consider the edges of cost x, it must select some $k-l$ edges of cost x in G, and it cannot select more, since this would create a cycle among the chosen edges.

2.3.9. *Among the cheapest spanning trees containing a spanning forest F is one containing the cheapest edge joining components of F.* Let T be a cheapest spanning tree containing F. If $e \notin E(T)$, then $T+e$ contains exactly one cycle, since T has exactly one u,v-path. Since u,v belong to distinct components of F, the u,v-path in T contains another edge e' between distinct components of F. If e' costs more than e, then $T' = T - e' + e$ is a cheaper spanning tree containing F, which contradicts the choice of T. Hence e' costs the same as e, and T' contains e and is a cheapest spanning tree containing F. Applying this statement at every step of Kruskal's algorithm proves that Kruskal's algorithm finds a minimum weight spanning tree.

2.3.10. *Prim's algorithm produces a minimum-weight spanning tree.* Let v_1 be the initial vertex, let T be the tree produced, and let T^* be an optimal tree that agrees with T for the most steps. Let e be the first edge chosen for T that does not appear in T^*, and let U be the set of vertices in the subtree of T that has been grown before e is added. Adding e to T^* creates a cycle C; since e links U to $\overline{U}$, C must contain another edge e' from U to $\overline{U}$.

Since $T^* + e - e'$ is another spanning tree, the optimality of T^* yields $w(e') \le w(e)$. Since e' is incident to U, e' is available for consideration when e is chosen by the algorithm; since the algorithm chose e, we have $w(e) \le w(e')$. Hence $w(e) = w(e')$, and $T^* + e - e'$ is a spanning tree with the same weight as T^*. It is thus an optimal spanning tree that agrees with T longer than T^*, which contradicts the choice of T^*.



2.3.11. *Every minimum-weight spanning tree achieves the minimum of the maximum weight edge over all spanning trees.* Let T be a minimum-weight spanning tree, and let T^* be one that minimizes the maximum weight edge. If T does not, then T has an edge e whose weight is greater than the weight of every edge in T^*. If we delete e from T, Then we can find an edge $e^* \in E(T^*)$ that joins the components of $T - e$, since T^* is connected. Since $w(e) > w(e^*)$, the weight of $T - e + e'$ is less than the weight of T, which contradicts the minimality of T. Thus T has the desired property.

2.3.12. *The greedy algorithm cannot guarantee minimum weight spanning paths.* This fails even on four vertices with only three distinct vertex weights. If two incident edges have the minimum weight a, such as $a = 1$, the algorithm begins by choosing them. If the two edges completing a 4-cycle with them have maximum weight c, such as $c = 10$, then one of those must be chosen to complete a path of weight $2a + c$. However, if the other two edges have intermediate weight b, such as $b = 2$, there is a path of weight $2b + a$, which will be cheaper whenever $b < (a + c)/2$. For $n > 4$, the construction generalizes in many possible ways using three weights $a < b < c$. A path of length $n-2$ having weight a for each edge and weight c for the two edges completing the cycle yields a path of weight $(n-2)a + c$ by the greedy algorithm, but if all other weights equal b there is a path of weight $2b + (n-3)a$, which is cheaper whenever $b < (a + c)/2$.

2.3.13. *If T and T' are spanning trees in a weighted graph G, with T having minimum weight, then T' can be changed into T by steps that exchange one edge of T' for one edge of T so that the edge set is always a spanning tree and the total weight never increases.* It suffices to find one such step whenever T' is different from T; the sequence then exists by using induction on the number of edges in which the two trees differ.

Choose any $e' \in E(T') - E(T)$. Deleting e' from T' creates two components with vertex sets U, U'. The path in T between the endpoints of e' must have an edge e from U to U'; thus $T' - e' + e$ is a spanning tree. We want to show that $w(T' - e' + e) \le w(T')$.

Since e is an edge of the path in T between the endpoints of e', the edge e belongs to the unique cycle in T created by adding e' to T. Thus $T + e' - e$ is also a spanning tree. Because $T - e + e'$ is a spanning tree and T has minimum weight, $w(e) \le w(e')$. Thus $T' - e' + e$ moves from T' toward T without increasing the weight.

2.3.14. *When e is a heaviest edge on a cycle G in a connected weighted graph G, there is a minimum spanning tree not containing e.* Let T be a minimum spanning tree in G. If $e \in E(T)$, then $T - e$ has two components with vertex sets U and U'. The subgraph $C - e$ is a path with endpoints in U and U'; hence it contains an edge e' joining U and U'. Since $w(e') \le w(e)$ by hypothesis, $T - e + e'$ is a tree as cheap as T that avoids e.

Given a weighted graph, iteratively deleting a heaviest non-cut-edge produces a minimum spanning tree. A non-cut-edge is an edge on a cycle. A heaviest such edge is a heaviest edge on that cycle. We have shown that some minimum spanning tree avoids it, so deleting it does not change the minimum weight of a spanning tree. This remains true as we delete edges. When no cycles remain, we have a connected acyclic subgraph. It is the only remaining spanning tree and has the minimum weight among spanning trees of the original graph.

2.3.15. *If T is a minimum spanning tree of a connected weighted graph G, then T omits some heaviest edge from every cycle of G.*

Proof 1 (edge exchange). Suppose e is a heaviest edge on cycle C. If $e \in E(T)$, then $T - e$ is disconnected, but $C - e$ must contain an edge e' joining the two components of $T - e$. Since T has minimum weight, $T - e + e'$ has weight as large as T, so $w(e') \ge w(e)$. Since e has maximum weight on C, equality holds, and T does not contain all the heaviest edges from C.

Proof 2 (Kruskal's algorithm). List the edges in order of increasing weight, breaking ties by putting the edges of a given weight that belong to T before those that don't belong to T. The greedy algorithm (Kruskal's algorithm) applied to this ordering L yields a minimum spanning tree, and it is precisely T. Now let C be an arbitrary cycle in G, and let $e_1, \ldots, e_k$ be the edges of C in order of appearance in L; $e_k = uv$ is a heaviest edge of C. It suffices to show that e_k does not appear in T. For each earlier edge e_i of C, either e_i appears in T or e_i is rejected by the greedy algorithm because it completes a cycle. In either case, T contains a path between the endpoints of e_i. Hence when the algorithm considers e_k, it has already selected edges that form paths joining the endpoints of each other edge of C. Together, these paths form a u, v-walk, which contains a u, v-path. Hence adding e_k would complete a cycle, and the algorithm rejects e_k.

2.3.16. *Four people crossing a bridge.* Name the people 10, 5, 2, 1, respectively, according to the number of minutes they take to cross when walking alone. To get across before the flood, they can first send {1, 2} in time 2. Next 1 returns with the flashlight in time 1, and now {5, 10} cross in time 10. Finally, 2 carries the flashlight back, and {1, 2} cross together again. The time used is $2 + 1 + 10 + 2 + 2 = 17$. The key is to send 5 and 10 together to avoid a charge of 5.

To solve the problem with graph theory, make a vertex for each possible state. A state consists of a partition of the people into the two banks, along with the location of the flashlight. There is an edge from state A to state B if state A is obtained from state B by moving one or two people (and the flashlight) from the side of A that has the flashlight to the other side. The problem is to find a shortest path from the initial state $(10, 5, 2, 1, F|\varnothing)$ to the final state $(\varnothing|10, 5, 2, 1, F)$. Dijkstra's algorithm finds such a path.

There are many vertices and edges in the graph of states. The path corresponding to the solution in the first paragraph passes through the vertices $(10, 5|2, 1, F)$, $(10, 5, 1, F|2)$, $(1|10, 5, 2, F)$, $(2, 1, F|10, 5)$, $(10, 5, 2, 1)$.

2.3.17. *The BFS algorithm computes $d(u, z)$ for every $z \in V(G)$.* The algorithm declares vertices to have distance k when searching vertices declared to have distance $k - 1$. Since vertices are searched in the order in which they are found, all vertices declared to have distance less than $k - 1$ are searched before any vertices declared to have distance $k - 1$.

We use induction on $d(u, z)$. When $d(u, z) = 0$, we have $u = z$, and initial declaration is correct. When $d(u, z) > 0$, let W be the set of all neighbors of z along shortest z, u-paths. Since $d(u, W) = d(u, z) - 1$, the induction hypothesis implies that the algorithm computes $d(u, v)$ correctly for all $v \in W$. Also, the preceding paragraph ensures that z will not be found before any vertices of W are searched. Hence when a vertex of W is searched, z will be found and assigned the correct distance.

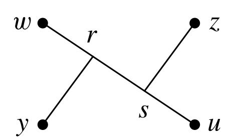
2.3.18. *Use of Breadth-First Search to compute the girth of a graph.* When running BFS, reaching a vertex that is already in the list of vertices already reached creates a second path from the root to that vertex. Following one path and back the other is a closed path in which the edges reaching the new vertex occur only once, so they lie on a cycle.

When the root is a vertex of a shortest cycle, the sum of the two lengths to the reached vertex is the length of that cycle. The sum can never be smaller. Thus we run BFS from each vertex as root until we find a vertex repeatedly, record the sum of the lengths of the two paths, and take the smallest value of this over all choices of the root.

2.3.19. *Computing diameter of trees.* From a arbitrary vertex w, we find a maximally distant vertex u (via BFS), and then we find a vertex v maximally distant from u (via BFS). We show that $d(y,z) \le d(u,v)$ for all $y, z \in V(T)$. Because v is at maximum distance from u, this holds if $u \in \{y,z\}$, so we may assume that $u \notin \{y,z\}$.

We use that each vertex pair in a tree is connected by a unique path. Let r be the vertex at which the w,y-path separates from the w,u-path. Let s be the vertex at which the w,z-path separates from the w,u-path. By symmetry, we may assume that r is between w and s. Since $d(w,u) \ge d(w,z)$, we have $d(s,u) \ge d(s,z)$. Now

$$d(y,z) = d(y,r)+d(r,s)+d(s,z) \le d(y,r)+d(r,s)+d(s,u) = d(y,u) \le d(v,u).$$



2.3.20. *Minimum diameter spanning tree.* An MDST is a spanning tree in which the maximum length of a path is as small as possible. Intuition suggests that running Dijkstra's algorithm from a vertex of minimum eccentricity (a center) will produce an MDST, but this may fail.

a) Construct a 5-vertex example of an unweighted graph (edge weights all equal 1) in which Dijkstra's algorithm can be run from some vertex of minimum eccentricity and produce a spanning tree that does not have minimum diameter. Answer: the chin of the bull.

(Note: when there are multiple candidates with the same distance from the root, or multiple ways to reach the new vertex with minimum distance, the choice in Dijkstra's algorithm can be made arbitrarily.)

b) Construct a 4-vertex example of a weighted graph such that Dijkstra's algorithm cannot produce an MDST when run from any vertex.

2.3.21. *Algorithm to test for bipartiteness.* In each component, run the BFS search algorithm from a given vertex x, recording for each newly found vertex a distance one more than the distance for the vertex from which it is found. By the properties of distance, searching from a vertex v to find a vertex w may discover $d(x,w) = d(x,v) - 1$ or $d(x,w) = d(x,v)$ or $d(x,w) = d(x,v) + 1$ (if w is not yet in the set found).

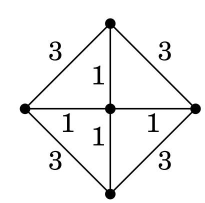
If the second case ever arises, then we have adjacent vertices at the same distance from x, and there is an odd cycle in the graph. Otherwise, at the end we form a bipartition that partions the vertices according to the parity of their distance from x.

2.3.22. *The Chinese Postman Problem in the k-dimensional cube Q_k, with every edge having weight 1.* If k is even, then no duplicate edges are needed, since Q_k is k-regular; total cost is $k2^{k-1}$. If k is odd, then a duplicated edge is needed at every vertex. It suffices to duplicate the matching across the last coordinate. Thus the total cost in this case is $(k+1)2^{k-1}$.

2.3.23. *The Lazy Postman.* The postman's trail must cover every edge and contribute even degree to each vertex except the start P and end H. In the example given, C,D,G,H have the wrong parity. Hence the duplicated edges must consist of two paths that pair these vertices (with least total distance), since this will change the degree parity only for the ends of the paths. If we pair them as DG and CH, then the shortest paths are DEIFG and CBEIH, totaling 18 extra (obviously not optimal since both use EI). If CG and DH, then the paths are (CBEIFG or CPAFG) and DEIH, totaling 18 in either case. If CD and GH, then the paths are CBED and GFIH, totaling 15. Hence the edges in the paths CBED and GFIH are traveled twice; all others are traveled once.

2.3.24. *Chinese Postman Problem.* Solving the Chinese Postman problem on a weighted graph with $2k$ vertices of odd degree requires duplicating the edges in a set of k trails that pair up the vertices of odd degree as endpoints. The only vertices of a trail that have odd degree in the trail are its endpoints. If some u,v-trail T in the optimal solution is not a path, then it contains a u,v-path P. In P, every vertex degree is even, except for the endpoints. Hence using P instead of T to join u and v does not change the parity on any vertex and yields smaller total weight.

Since no edge need be used thrice, the duplicated trails in an optimal solution are pairwise edge-disjoint. As in the example below, they need not be vertex-disjoint. With four vertices of odd degree, two paths are required, and the cheapest way is to send both through the central vertex.



2.3.25. *If G is an n-vertex rooted plane tree in which every vertex has 0 or k children, then $n = tk + 1$ for some integer t.*

Proof 1 (Induction). We use induction on the number of non-leaf vertices. When there are no such vertices, the root is the only vertex, and the formula works with $t = 0$. When the tree T is bigger, find a leaf at maximum distance from the root, and let x be its parent. By the choice of x, all children of x are leaves. Deleting the children of x yields a tree T' with one less non-leaf vertex and k fewer total vertices. By the induction hypothesis, $n(T') = tk + 1$ for some t, and thus $n(T) = (t + 1)k + 1$.

Proof 2 (Degree counting). If T has n vertices, then it has $n - 1$ edges, and the degree-sum is $2n - 2$. If $n > 1$, then the root has degree l, the other $t - 1$ non-leaf vertices each have degree $k + 1$, and the $n - t$ leaves each have degree 1. Thus $2n - 2 = k + (t - 1)(k + 1) + (n - t)$. This simplifies to $n = tk + 1$.

2.3.26. *A recurrence relation to count the binary trees with $n+1$ leaves.* Let a_n be the desired number of trees. When $n = 0$, the root is the only leaf. When $n > 0$, each tree has some number of leaves, k, in the subtree rooted at the left child of the root, where $1 \le k \le n$. We can root any binary tree with k leaves at the left child and any binary tree with $n-k+1$ leaves at the right child. Summing over k counts all the trees. Thus $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$ for $n > 0$, with $a_0 = 1$. (*Comment:* These are the *Catalan numbers*.)

2.3.27. *A recurrence relation for the number of rooted plane trees with n vertices.* Let a_n be the desired number of trees. When $n = 1$, there is one tree. When $n > 1$, the root has a child. The subtree rooted at the leftmost child has some number of vertices, k, where $1 \le k \le n - 1$. The remainder of the tree is a rooted subtree with the same root as the original tree; it has $n-k$ vertices. We can combine any tree of the first type with any tree of the second type. Summing over k counts all the trees. Thus $a_n = \sum_{1=k}^{n-1} a_k a_{n-k}$ for $n > 1$, with $a_1 = 1$. (*Comment:* This is the same sequence as in the previous problem, with index shifted by 1.)

2.3.28. *A code with minimum expected length for messages with relative frequencies 1,2,3,4,5,5,6,7,8,9.* Iteratively combining least-frequent items and reading paths from the resulting tree yields the codes below. Some variation in the codes is possible, but not in their lengths. The average length (weighted by frequency!) is 3.48.

frequency	1	2	3	4	5	5	6	7	8	9
code	00000	00001	0001	100	101	110	111	001	010	011
length	5	5	4	3	3	3	3	3	3	3

2.3.29. *Computation of an optimal code.* Successive combination of the cheapest pairs leads to a tree. For each letter, we list the frequency and the depth of the corresponding leaf, which is the length of the associated codeword. The assignment of codewords is not unique, but the set (with multiplicities) of depths for each frequency is. Given frequencies f_i, with associated lengths l_i and total frequency T, the expected length per character is $\sum f_i l_i / T$. For the given frequencies, this produces expected length

of $(7\cdot 4+6\cdot 19+5\cdot 21+4\cdot 26+3\cdot 30)/100 = 4.41$ bits per character, which is less than the 5 bits of ASCII.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	$\varnothing$
9	2	2	4	12	2	3	2	9	1	1	4	2	6	8	2	1	6	4	6	4	2	2	1	2	1	2
3	6	6	5	3	6	5	6	3	7	7	5	6	4	4	6	7	4	5	4	5	6	6	7	6	6	5

2.3.30. *Optimal code for powers of 1/2.*

a) the two smallest probabilities are equal. Let p_n, p_{n-1} be smallest and second smallest probabilities in the distribution. Each probability other than p_n is a multiple of p_{n-1}. If $p_n < p_{n-1}$, then the sum of all the probabilities is not a multiple of p_{n-1}. This contradicts $\sum_{i=1}^{n} p_i = 1$, since 1 is a multiple of p_{n-1}.

b) The expected message length of the optimal (Huffman) code for such a distribu tion is $-\sum p_i \lg p_i$. We use induction on n to prove that each item with probability $(1/2)^k$ is assign to a leaf at length k from the root; this yields the stated formula. For $n = 1$ and $p_1 = 1$, the one item has message length 0, as desired. For larger n, the Huffman tree is obtained by finding the optimal tree for the smaller set $q_1, \ldots, q_{n-1}$ (where $q_{n-1} = p_n + p_{n-1}$ and $q_i = p_i$ for $1 \le i \le n$) and extending the tree at the leaf for q_{n-1} to leaves one deeper for p_{n-1} and p_n. By part (a), $q_{n-1} = 2p_{n-1} = 2p_n$. By the induction hypothesis, the depth of the leaf for q_{n-1} is $-\lg q_{n-1}$, and for $p_1, \ldots, p_{n-2}$ it is as desired. The new leaves for p_{n-1}, p_n have depth $+1 - \lg q_{n-1} = -\lg p_{n-1} = -\lg p_n$, as desired.

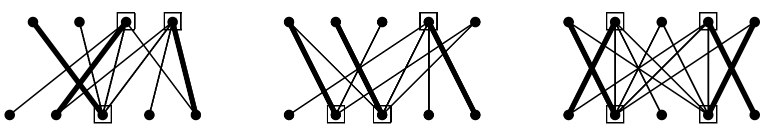
2.3.31. *For every probability distribution $\{p_i\}$ on n messages and every binary code for these messages, the expected length of a code word is at least $-\sum p_i \lg p_i$.* Proof by induction on n. For $n = 1 = p_1$, the entropy and the expected length for the optimal code both equal 0; there is no need to use any digits. For $n > 1$, let W be the words in an optimal code, with W_0, W_1 denoting the sets of code words starting with 0,1, respectively. If all words start with the same bit, then the code is not optimal, because the code obtained by deleting the first bit of each word has smaller expected length. Hence W_0, W_1 are codes for smaller sets of messages. Let q_0, q_1 be the sum of the probabilities for the messages in W_0, W_1. Normalizing the p_i's by q_0 or q_1 gives the probability distributions for the smaller codes. Because the words within W_0 or W_1 all start with the same bit, their expected length is at least 1 more than the optimal expected length for those distributions.

Applying the induction hypothesis to both W_0 and W_1, we find that the expected length for W is at least $q_0[1-\sum_{i\in W_0} \frac{p_i}{q_0} \lg \frac{p_i}{q_0}] + q_1[1-\sum_{i\in W_1} \frac{p_i}{q_1} \lg \frac{p_i}{q_1}]$ $= 1 - \sum_{i\in W_0} p_i(\lg p_i - \lg q_0) - \sum_{i\in W_1} p_i(\lg p_i - \lg q_1) = 1 + q_0 \lg q_0 + q_1 \lg q_1 - \sum p_i \lg p_i$. It suffices to prove that $1+q_0 \lg q_0 + q_1 \lg q_1 \ge 0$ when $q_0 + q_1 = 1$. Because $f(x) = x \lg x$ is convex for $0 < x < 1$ (since $f''(x) = 1/x > 0$), we have $1 + f(x) + f(1-x) \ge 1 + 2f(.5) = 0$.

3.MATCHINGS AND FACTORS

3.1. MATCHINGS AND COVERS

3.1.1. *Examples of maximum matchings.* In each graph below, we show a matching in bold and mark a vertex cover of the same size. We know that the matching has maximum size and the vertex cover has minimum size because the size of every matching in a graph is at most the size of every vertex cover in the graph.



3.1.2. *The minimum size of a maximal matching in C_n is $\lceil n/3 \rceil$.* Suppose that C_n has a matching of size k. The $n-2k$ unmatched vertices fall into k "buckets" between matched edges around the cycle. The matching is maximal if and only if none of the buckets contain two vertices. By the pigeonhole principle, there must be some such bucket if $n-2k \geq k+1$; hence a matching of size k cannot be maximal if $k \leq (n-1)/3$.

On the other hand, if $n-2k \leq k$ and $k \leq n/2$, then we can match two vertices and skip one until we have skipped $n-2k$, which avoids having two vertices in one bucket. Hence C_n has a maximal matching of size $k \leq n/2$ if and only if $3k \geq n$. We conclude that the minimum size of a maximal matching in C_n is $\lceil n/3 \rceil$.

3.1.3. *If $S \subseteq V(G)$ is saturated by some matching in G, then S is saturated by some maximum matching.* If M saturates S, then the characterization of maximum matchings implies that a maximum matching M^* can be obtained from M by a sequence of alternating path augmentations. Although edges may be lost in an augmentation, each augmentation continues to saturated all saturated vertices and enlarges the saturated set by 2. Thus S remains saturated in M^*

When $S \subseteq V(G)$ is saturated by some matching in G, it need not be true that S is saturated by every maximum matching. When G is an odd cycle, all maximum matchings omit distinct vertices.

3.1.4. Let G be a simple graph.

$\alpha(G) = 1$ *if and only if G is a complete graph.* The independence number is 1 if and only if no two vertices are nonadjacent.

$\alpha'(G) = 1$ *if and only if G consiste of isolated vertices plus a triangle or nontrivial star.* Deleting the vertices of one edge in such a graph leaves no edges remaining. Conversely, suppose that $\alpha'(G) = 1$, and ignore the isolated vertices. Let v be a vertex of maximum degree. If every edge is incident to v, then G is a star. Otherwise, an edge e not incident to v shares an endpoint with every edge incident to v. Since e has only two endpoints, $d(v) = 2$, and there is only one such edge e. Thus G is a triangle.

$\beta(G) = 1$ *if and only if G is a nontrivial star plus isolated vertices.* The vertex cover number is 1 if and only if one vertex is incident to all edges.

$\beta'(G) = 1$ *if and only if $G = K_2$.* Since every edge covers two vertices, $\beta'(G) = 1$ requires that $n(G) = 2$, and indeed $\beta'(K_2) = 1$.

3.1.5. $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$ *for every graph G.* Form an independent set S by iteratively selecting a remaining vertex for S and deleting that vertex and all its neighbors. Each step adds one vertex to S and deletes at most $\Delta(G)+1$ vertices from G. Hence we perform at least $n(G)/(\Delta(G)+1)$ steps and obtain an independent set at least that big.

3.1.6. *If T is a tree with n vertices and independence number k, then $\alpha'(T) = n-k$.* The vertices outside a maximum independent set form a vertex cover of size $n-k$. Since trees are bipartite, the König–Egerváry Theorem then applies to yield $\alpha'(T) = \beta(T) = n - \alpha(T) = n-k$.

3.1.7. *A graph G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H having no isolated vertices.* If G is bipartite, then every subgraph H of G is bipartite, and by König's Theorem the number of edges of H needed to cover $V(H)$ equals $\alpha(H)$ if H has no isolated vertices. If G is not bipartite, then G contains an odd cycle H, and this subgraph H has no isolated vertices and requires $\alpha(H)+1$ edges to cover its vertices.

3.1.8. *Every tree T has at most one perfect matching.*

Proof 1 (contradiction). Let M and M' be perfect matchings in a tree. Form the symmetric difference of the edge sets, $M \triangle M'$. Since the matchings are perfect, each vertex has degree 0 or 2 in the symmetric difference, so every component is an isolated vertex or a cycle. Since the tree has no cycle, every vertex must have degree 0 in the symmetric difference, which means that the two matchings are the same.

Proof 2 (induction). For the basis step, a tree with one vertex has no perfect matching; a tree with two vertices has one. For the induction step, consider an arbitrary tree T on $n > 2$ vertices, and consider a leaf v. In any perfect matching, v must be matched to its neighbor u. The remainder of any matching is a matching in $T-\{u,v\}$. Since each perfect matching in T must contain the edge uv, the number of perfect matchings in T equals the number of perfect matchings in $T-\{u,v\}$.

Each component of $T-\{u,v\}$ is a tree; by the induction hypothesis, each component has at most one perfect matching. The number of perfect matchings in a graph is the product of the number of perfect matchings in each component, so the original T has at most one perfect matching. (More generally, a forest has at most one perfect matching.)

3.1.9. *Every maximal matching in a graph G has size at least $\alpha'(G)/2$.*

Proof 1 (counting and contrapositive). Let M^* be a maximum matching, and let M be another matching. We show that if $|M| < \alpha'(G)/2$, then M is not a maximal matching. Since M saturates $2|M|$ vertices and $|M| < \alpha'(G)/2$, we conclude that M saturates fewer than $\alpha'(G)$ vertices. This means that M cannot saturate a vertex of every edge of M^*, and there is some edge of M^* that can be added to enlarge M.

Proof 2 (augmenting paths). Let M be a maximal matching, and let M^* be a maximum matching. Consider the symmetric difference $F = M\triangle M^*$. Since the number of edges from M and M^* in a component of F differ by at most one, the symmetric difference contains at least $|M^*|-|M|$ augmenting paths. Since M is maximal, each augmenting path must contain an edge of M (an M-augmenting path of length one is an edge that can be added to M). Thus $|M^*|-|M| \le |M|$, and we obtain $|M| \ge |M^*|/2 = \alpha'(G)/2$.

Proof 3 (vertex covers). When M is a maximal matching, then the vertices saturated by M form a vertex cover (if an edge had no vertex in this set, then it could be added to M). Since every vertex cover has size at least $\alpha'(G)$, we obtain $2|M| \ge \beta(G) \ge \alpha'(G)$.

3.1.10. *If M and N are matchings in a graph G and $|M| > |N|$, then there are matchings M' and N' in G such that $\left|M'\right| = |M|-1$, $\left|N'\right| = |N|+1$, $M'\cap N' = M\cap N$, and $M'\cup N' = M\cup N$.* Consider the subgraph F of G consisting of the edges in the symmetric difference $M\triangle N$; this consists of all edges belonging to exactly one of M and N. Since each of M and N is a matching, every vertex has at most one incident edge in M and at most one incident edge in N. Hence the degree of every vertex in F is at most 2.

The components of a graph with maximum degree 2 are paths and cycles. A path or cycle in F alternates edges between M and N. Since $|M| > |N|$, F has a component with more edges of M than of N. Such a component can only be a path P that starts and ends with an edge of M. Form M' from M by replacing $M\cap E(P)$ with $N\cap E(P)$; this reduces the size by one. Form N' from N by replacing $N\cap E(P)$ with $M\cap E(P)$; this increases the size by one. Since we have only switched edges belonging to exactly one of the sets, we have not changed the union or intersection.

3.1.11. *If C and C' are cycles in a graph G, then $C\triangle C'$ decomposes into cycles.* Since even graphs decompose into cycles (Proposition 1.2.27), it suffices to show that $C\triangle C'$ has even degree at each vertex. The set of edges in a cycle that are incident to v has even size (2 or 0). The symmetric difference of any two sets of even size has even size, since always $|A\triangle B| = |A|+|B|-2|A\cap B|$.

3.1.12. *If C and C' are cycles of length k in a graph with girth k, then $C\triangle C'$ is a single cycle if and only if $C\cap C'$ is a single nontrivial path.*

Sufficiency. If $C\cap C'$ is a single path P, then the other paths in C and C' between the endpoints of P share only their endpoints, and hence $C\triangle C'$ is their union and is a single cycle.

Necessity. We know that $C\cap C'$ must have an edge, since otherwise C and C' are edge-disjoint and $C\triangle C'$ has two cycles. Also we may assume that C and C' are distinct, since otherwise $C\triangle C'$ has no edges.

Suppose that P and P' are distinct maximal paths in $C\cap C'$. Now C is the union of four paths P, Q, P', Q', and C' is the union of four paths P, R, P', R'. Note that Q and Q' may share edges with R and R'. By symmetry, we may assume that P' is no longer than P and that Q and R share the same endpoint of P.

If Q and R also share the same endpoint of P', then $Q\cup R$ and $Q'\cup R'$ both form closed walks in which (by the maximality of P and P') some edge appears only once. If Q and R do not share the same endpoint of P', then $P'\cup Q\cup R$ and $P'\cup Q'\cup R'$ both form closed walks in which the edges of P' appear only once. In each case, the two closed walks each each contain a cycle, and the sum of their lengths is less than $2k$. This yields a cycle of length less than k in G, which is impossible.

Comment: The statement can fail for longer cycles. In the 3-dimensional cube Q_3, there are two 6-cycles through antipodal vertices, and their symmetric difference consists of two disjoint 4-cycles.

3.1.13. *In an X,Y-bigraph G, if $S\subseteq X$ is saturated by a matching M and $T\subseteq Y$ is saturated by a matching M', then there is a matching that saturates both S and T.* Let F be a subgraph of G with edge set $M\cup M'$. Since each vertex has at most one incident edge from each matching, F has maximum degree 2. Each component of F is an alternating path or an alternating cycle (alternating between M and M'). From a component that is an alternating cycle or an alternating path of odd length, we can choose the edges of M or of M' to saturate all the vertices of the component.

Let P be a component of F that is a path of even length. The edge at one end of P is in M; the edge at the other end is in M'. Also P starts and ends in the same partite set. If it starts and ends in X, then the ends cannot both be in S, because only one endpoint of P is saturated by M.

Choosing the edges of M in this component will thus saturate all vertices of S and T contained in $V(P)$. Similarly, choosing the edges of M' from any component of F that is a path of even length with endpoints in Y will saturate all the vertices of $S \cup T$ in that component.

3.1.14. *Matchings in the Petersen graph.*

Deleting any perfect matching leaves C_5+C_5. Deleting a perfect matching leaves a 2-regular spanning subgraph, which is a disjoint union of cycles. Since the Petersen graph has girth 5, the only possible coverings of the vertices by disjoint cycles are $C_5 + C_5$ and C_{10}.

If a 10-cycle exists, with vertices $[v_1, \ldots, v_{10}]$ in order, then the remaining matching consists of chords. Two consecutive vertices cannot neighbor their opposite vertices on the cycle, since that creates a 4-cycle. Similarly, the neighbors must be at least four steps away on the cycle. Hence we may assume by symmetry that $v_1 \leftrightarrow v_5$. Now making v_{10} adjacent to any of $\{v_6, v_5, v_4\}$ creates a cycle of length at most 4, so there is no way to insert the remaining edges.

a) The Petersen graph has twelve 5-cycles. Each edge extends to P_4 in four ways by picking an incident edge at each endpoint. Since the graph has diameter two and girth 5, every P_4 belongs to exactly one 5-cycle through an additional vertex. Since there are 15 edges, we have generated 60 5-cycles, but each 5-cycle is generated five times.

b) The Petersen graph has six perfect matchings. Since the Petersen graph has girth five, the five remaining edges incident to any 5-cycle form a perfect matching, and deleting them leaves a 5-cycle on the complementary vertices. Hence the 5-cycles group into pairs of 5-cycles with a matching between them. Since every matching leaves C_5+C_5, every matching arises in this way, and by part (b) there are six of them.

3.1.15. *Matchings in k-dimensional cubes.*

a) For $k \geq 2$, if M is a perfect matching of Q_k, then there are an even number of edges in M whose endpoints differ in coordinate i. Let V_0 and V_1 be the sets of vertices having 0 and 1 in coordinate i, respectively. Each has even size. Since the vertices of V_r not matched to V_{1-r} must be matched within V_r, the number of vertices matched by edges to V_{1-r} must be even.

b) Q_3 has nine perfect matchings. There are four edges in each such matching, with an even number distributed to each coordinate. The possible distributions are $(4, 0, 0)$ and $(2, 2, 0)$. There are three matchings of the first type. For the second type, we pick a direction to avoid crossing, pick one of the two matchings in one of the 4-cycles, and then the choice of the matching in the other 4-cycle is forced to avoid making all four edges change the same coordinate. Hence there are $3 \cdot 2 \cdot 1$ perfect matchings of the second type.

3.1.16. *When $k \geq 2$, the k-dimensional hypercube Q_k has at least $2^{(2^{k-2})}$ perfect matchings.*

Proof 1 (induction on k). Let m_k denote the number of perfect matchings. Note that $m_2 = 2$, which satisfies the inequality. When $k > 2$, we can choose matchings independently in each of two disjoint subcubes of dimension $k-1$. The number of such matchings is m_{k-1}^2. By the induction hypothesis, this is at least $(2^{2^{k-3}})^2$, which equals $2^{2^{k-2}}$.

Comment: Since we could choose the two disjoint subcubes in k ways, we can recursively form km_{k-1}^2 perfect matchings in this way, some of which are counted more than once.

Proof 2 (direct construction). Pick two coordinates. There are 2^{k-2} copies of Q_2 in which those two coordinates vary, and two choices of a perfect matching in each copy of Q_2. This yields $2^{2^{k-2}}$ perfect matchings. (Since we can choose the two coordinates in $\binom{k}{2}$ ways, we can generate $\binom{k}{2}2^{2^{k-2}}$ perfect matchings, but there is some repetition.)

3.1.17. *In every perfect matching in the hypercube Q_k, there are exactly $\binom{k-1}{i}$ edges that match vertices with weight i to vertices with weight $i+1$,* where the *weight* of a vertex is the number of 1s in its binary k-tuple name.

Proof 1 (induction on i). Since the vertex of weight 0 must match to a vertex of weight 1, the claim holds when $i = 0$. For the induction step, the induction hypothesis yields $\binom{k-1}{i-1}$ vertices of weight $i-1$ matched to vertices of weight i. The remaining vertices of weight i must match to vertices of weight $i+1$. Since $\binom{k}{i} - \binom{k-1}{i-1} = \binom{k-1}{i}$, the claim follows.

Proof 2 (canonical forms). Let M^* be the matching where every edge matches vertices with 0 and 1 in the last coordinate. The number of edges matching weight i to weight $i+1$ is the number of choices of i ones from the first $k-1$ positions, which is $\binom{k-1}{i}$.

It now suffices to prove that every perfect matching M has the same weight distribution as M^*. The symmetric difference of M and M^* is a union of even cycles alternating between M and M^*, plus isolated vertices saturated by the same edge in both matchings. It suffices to show that the weight distribution on each cycle is the same for both matchings.

The edges joining vertices of weights i and $i+1$ along a cycle C alternate appearing with increasing weight and with decreasing weight, since weight changes by 1 along each edge. For the same reason, the number of edges along C from a vertex to the next appearance of a vertex with the same weight is even. Since C alternates between M and M^*, this means that the edges joining vertices of weights i and $i+$ alternate between M and M^*. Hence there is the same number of each type, as desired.

3.1.18. *The game of choosing adjacent vertices, where the last move wins.* Suppose that G has a perfect matching M. Whenever the first player chooses a vertex, the second player takes its mate in M. This vertex is available, because after each move of the second player the set of vertices visited forms a set of full edges in M, and the first player cannot take two vertices at a time. Thus with this strategy, the second player can always make a move after any move of the first player and never loses.

If G has no perfect matching, then let M be a maximum matching in G. The first player starts by choosing a vertex not covered by M. Thereafter, whenever the second player chooses a vertex x, the first player chooses the mate of x in M. The vertex x must be covered by M, else x completes an M-augmenting path using all the vertices chosen thus far. Thus the first player always has a move available and does not lose.

3.1.19. *A family $A_1, \ldots, A_m$ of subsets of Y has a system of distinct representatives if and only if $\left|\bigcup_{j\in S} A_i\right| \ge |S|$ for every $S \subseteq [m]$.* Form an X, Y-bigraph G with $X = \{1, \ldots, m\}$ and $Y = \{y_1, \ldots, y_n\}$. Include the edge iy_j if and only if $y_j \in A_i$. A set of edges in G is a matching if and only if its endpoints in Y form a system of distinct representatives for the sets indexed by its endpoints in X. The family has a system of distinct representatives if and only if G has a matching that saturates X.

It thus suffices to show that the given condition is equivalent to Hall's condition for saturating X. If $S \subseteq X$, then $N_G(S) = \bigcup_{i\in S} A_i$, so $|N(S)| \ge |S|$ if and only if $\left|\bigcup_{i\in S} A_i\right| \ge |S|$.

3.1.20. *An extension of Hall's Theorem using stars with more than two vertices.* We form an X, Y-bigraph G with partite sets $X = x_1, \ldots, x_n$ for the trips and $Y = y_1, \ldots, y_m$ for the people, and edge set $\{x_i y_j\colon$ person j likes trip $i\}$. To fill each trip to its capacity c_i, we seek a subgraph whose components are stars, with degree c_i at x_i.

Form an X', Y-bigraph G' by making n_i copies of each vertex x_i. Now G has the desired stars if and only if G' has a matching that saturates X'. Thus the desired condition for G should become Hall's Condition for G'.

In G', the neighborhoods of the copies of a vertex of x are the same. Hence Hall's Condition will hold if and only if it holds whenever $S \subseteq X'$ consists of all copies of each vertex of X for which it includes any copies. That is, Hall's Condition reduces to requiring $|N(T)| \ge \sum_{x_i\in T} c_i$ for all $T \subseteq X$. This condition is necessary, since the trips in T need this many distinct people. It is sufficient, because it implies Hall's Condition for G'.

3.1.21. *If G is an X, Y-bigraph such that $|N(S)| > |S|$ whenever $\varnothing \ne S \subset X$, then every edge of G belongs to some matching that saturates X.* Let xy be an edge of G, with $x \in X$ and $y \in Y$, and let $G' = G - x - y$. Each set $S \subseteq X - \{x\}$ loses at most one neighbor when y is deleted. Combining this with the hypothesis yields $|N_{G'}(S)| \ge |N_G(S)| - 1 \ge |S|$. Thus G' satisfies Hall's Condition and has a matching that saturates $X - \{x\}$. With the edge xy, this completes a matching in G that contains xy and saturates X.

3.1.22. *A bipartite graph G has a perfect matching if and only if $|N(S)| \ge |S|$ for all $S \subseteq V(G)$.* This conclusion does not hold for non-bipartite graphs. In an odd cycle, we obtain neighbors for a set of vertices by taking the vertices immediately following them on the cycle. Thus $|N(S)| \ge |S|$ for all $S \subseteq V$, but the graph has no perfect matching. Complete graphs of odd order also form counterexamples. For bipartite graphs, we give two proofs.

Proof 1 (graph transformation). Let G' be a bipartite graph consisting of two disjoint copies of G, where each partite set in G' consists of one copy of X and one copy of Y; call these X' and Y'. Then G' has a perfect matching if and only if G has a perfect matching. Since $|X'| = |Y'|$, G' has a perfect matching if and only if it has a matching that completely saturates X'.

By Hall's Theorem, G' has a matching saturating X' if and only if $|N(S')| \ge |S'|$ for all $S' \subseteq X'$. Given $S' \subseteq X'$, let $T_1 = S' \cap X$ and $T_2 = S' - T_1$. Let $S \subseteq V(G)$ be the set of vertices consisting of T_1 in X plus the vertices of Y having copies in T_2. This establishes a bijection between subsets S' of X' and subsets S of $V(G)$, with $|S'| = |S|$. Also $|N(S')| = |N(S)|$, by the construction of G'.

Hence Hall's condition is satisfied for G' if and only if the condition of this problem holds in G. In summary, we have shown

$$[G \text{ has a 1-factor}] \Leftrightarrow G' \text{ has a 1-factor} \Leftrightarrow$$
$$|N(S')| \ge |S'| \text{ for all } S' \subseteq X' \Leftrightarrow |N(S)| \ge |S| \text{ for all } S \subseteq V(G).$$

Proof 2 (by Hall's Theorem). *Necessity:* Let M be a perfect matching, and let S be a subset of $V(G)$. Vertices of S are matched to distinct vertices of $N(S)$ by M, so $|N(S)| \ge |S|$. *Sufficiency:* If $|N(S)| \ge |S|$ for all $S \subseteq V$, then $|N(A)| \ge |A|$ for all $A \subseteq X$. By Hall's Theorem, the graph thus has a matching M that saturates X. Thus $|Y| \ge |X|$, and the condition $|N(Y)| \ge |Y|$ yields $|X| \ge CY$. Thus $|Y| = |X|$, and M is a perfect matching.

3.1.23. *Alternative proof of Hall's Theorem.* Given an X, Y-bigraph G, we prove that Hall's Condition suffices for a matching that saturates X. Let $m = |X|$. For $m = 1$, the statement is immediate.

Induction step: $m > 1$. If $|N(S)| > |S|$ for every nonempty proper subset $S \subset X$, select any neighbor y of any vertex $x \in X$. Deleting y reduces the size of the neighborhood of each subset of $X - \{x\}$ by at most 1. Hence Hall's Condition holds in $G' = G - x - y$. By the induction hypothesis, G' has a matching that saturates $X - \{x\}$, which combines with xy to form a matching that saturates X.

Otherwise, $|N(S)| = |S|$ for some nonempty proper subset $S \subset X$. Let $G_1 = G[S \cup N(S)]$, and let $G_2 = G - V(G_1)$. Because the neighbors of vertices in S are confined to $N(S)$, Hall's Condition for G implies Hall's Condition for G_1. For G_2, consider $T \subseteq X - S$. Since $|N_G(T \cup S)| \geq |T \cup S|$, we obtain

$$N_{G_2}(T) = N_G(T \cup S) - N_G(S) \geq |T \cup S| - |S| = |T|.$$

Thus Hall's Condition holds for both G_1 and G_2. By the induction hypothesis, G_1 has a matching that saturates S, and G_2 has a matching that saturates $X - S$. Together, they form a matching that saturates X.

3.1.24. *A square matrix of nonnegative integers is a sum of k permutation matrices if and only if each row and column sums to k.* If A is the sum of k permutation matrices, then each matrix adds one to the sum in each row and column, and each row or column of A has sum k.

For the converse, let A be a square matrix with rows and columns summing to k. We use induction on k to express A as a sum of k permutation matrices. For $k = 1$, A is a permutation matrix.

For $k > 1$, form a bipartite graph G with vertices $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ so that the number of edges joining x_i and y_j is $a_{i,j}$. The graph G is bipartite and regular, so by the Marriage Theorem it has a perfect matching. Let $b_{i,j} = 1$ if $x_i y_j$ belongs to this matching and $b_{i,j} = 0$ otherwise; the resulting matrix B is a permutation matrix. Each row and column of B has exactly one 1. Thus $A' = B - A$ is a nonnegative integer matrix whose rows and columns sum to $k - 1$. Applying the induction hypothesis to A' yields $k - 1$ additional permutation matrices that with B sum to A.

3.1.25. *A nonnegative doubly stochastic matrix can be expressed as a convex combination of permutation matrices.* For simplicity, we allow multiples of doubly stochastic matrices and prove a superficially more general statement. We use induction on the number of nonzero entries to prove that if Q is a matrix of nonnegative entries in which every row and every column sums to t, then Q can be expressed as a linear combination of permutation matrices with nonnegative coefficients summing to t.

If Q has exactly n nonzero entries, then Q is t times a permutation matrix, because Q must have at least one nonzero entry in every row and column. If Q has more nonzero entries, begin by defining a bipartite graph G with $x_i \leftrightarrow y_j$ if and only if $Q_{i,j} > 0$. If G has a perfect matching, then the edges $x_i y_{\sigma(i)}$ of the matching correspond to a permutation σ with permutation matrix P.

Let ε be the minimum (positive) value in the positions of Q corresponding to the 1's in P. The matrix $Q' = Q - \varepsilon P$ is a nonnegative matrix with fewer nonzero entries than Q, and row and column sums $t - \varepsilon$. By the induction hypothesis, we can express Q' as a nonnegative combination

$\sum_{i=1}^{m} c_i P_i$, with $\sum_{i=1}^{m} c_i = t - \varepsilon$. Hence $Q = \sum c_i P_i + \varepsilon P$. With $c_{m+1} = \varepsilon$ and $P_{m+1} = P$, we have expressed Q in the desired form.

It remains to prove that G has a perfect matching; we show that it satisfies Hall's condition. If S is a subset of X corresponding to a particular set of rows in Q, we need only show that these rows have nonzero entries in at least $|S|$ columns altogether. This follows because the total nonzero amount in the rows S is $t\,|S|$. Since each column contains only a total of t, it is not possible to contain a total of $t\,|S|$ in fewer than $|S|$ columns.

Comment. When the entries of Q are rational, the result follows directly from the Marriage Theorem. Multiplying Q by the least common denominator d of its positive entries converts it to an integer matrix in which all rows and columns sum to d. The entry in position i, j now is the number of edges joining x_i and y_j in a d-regular bipartite graph (multiple edges allowed). The Marriage Theorem implies that the graph has a perfect matching. By induction on d, it can be decomposed into perfect matchings. These matchings correspond to permutation matrices. In the expression of Q as a convex combination of these matrices, we give weight a/d to a permutation matrix arising a times in the list of matchings.

3.1.26. *Achieving columns with all suits.* The cards in an n by m array have m values and n suits, with each value on one card in each suit.

a) It is always possible to find a set of m cards, one in each column, having the m different values. Form a X, Y-bigraph in which X represents the columns and Y represents the values, with r edges from $x \in X$ to $y \in Y$ if value y appears r times in column x. Since each column contains n cards and each value appears in n positions (once in each suit), the multigraph is n-regular. By the Marriage Corollary to Hall's Theorem, every nontrivial regular bipartite graph has a perfect matching. (This applies also when multiple edges are present, which can occur here.) A perfect matching selects m distinct values occurring in the m columns.

(Using Hall's Theorem directly, a set S of k columns contains nk cards. Since there are n cards of each value, S contains cards of at least k values. Hence the graph satisfies Hall's condition and has a perfect matching.)

b) By a sequence of exchanges of cards of the same value, the cards can be rearranged so that each column consists of n cards of distinct suits. Making each column consist of n cards of different suits is equivalent to spreading each suit across all columns. The full result follows by induction on n, with $n = 1$ as a trivial basis step.

For the induction step, when $n > 1$, use part (a) to find cards of distinct values representing the m columns. Then perform at most one exchange for each value to bring the values in a single suit to those positions. Positions within a column are unimportant, so we can treat the other suits as an instance of the problem with $n-1$ suits. We apply the induction hypothesis to fix up the remaining suits.

The problem can always be solved using at most $mn - \sum_{k\le n} \lceil m/k \rceil$ exchanges. The worst case requires at least $\lfloor m/n \rfloor \, n(n-1)/2$ exchanges.

3.1.27. *The second player can force a draw in a positional game if $a \ge 2b$, where a is the minimum size of a winning set and b is the maximum number of winning sets containing a particular position.* Let P be the set of positions, and let $W_1, \dots, W_m$ be the winning sets of positions. With $|P| = n$, let G be the bipartite graph on $n+2m$ vertices with partite sets $P = \{p_1, \dots, p_n\}$ and $W = \{w_1, \dots, w_m\} \cup \{w'_1, \dots, w'_m\}$ by creating two edges $p_i w_j$ and $p_i w'_j$ for each incidence $p_i \in W_j$ of a position in a winning set.

If G has a matching M that saturates W, then Player 2 can use M to force a draw. When the position taken by Player 1 on a given move is matched to one of $\{w_i, w'_i$ in M, Player 2 responds by taking the position matched to the other one of these two elements. Player 1 thus can never obtain all the positions of a winning set. (If Player 1 takes an unmatched position, Player 2 can respond by taking any available position.)

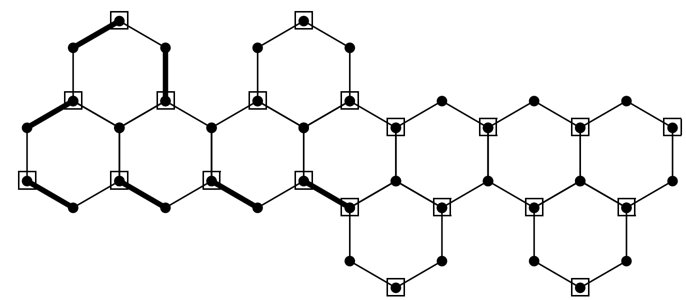
The existence of such a matching follows from $a \ge 2b$ by Hall's condition. If $S \subseteq W$, then S has representatives (w or w') of at least $|S|/2$ winning sets. Since each position appears in at most b winning sets, the number of positions in the union of these winning sets is at least $a(|S|/2)/b \ge |S|$. Thus $|N(S)| \ge |S|$ for every $S \subseteq W$.

3.1.28. *A graph with no perfect matching.*

Proof 1 (vertex cover). The graph has 42 vertices, so a perfect matching would have 21 edges. The marked vertices form a vertex cover of size 20. The edges of a matching must be covered by distinct vertices in a vertex cover, so there is no matching with more than 20 edges.

Proof 2 (Hall's condition). Using two labels X and Y, we obtain a bipartition of the graph. Partite set X consists of the marked vertices in the left half of the picture and the unmarked vertices in the right half. This is an independent set of size 21, and the remaining vertices also form an independent set Y of size 21.

Hall's Condition is a necessary condition for a perfect matching; we show that Hall's Condition does not hold. Let S be the subset of X consisting of the 11 unmarked vertices in the right half of the graph. The neighbors of vertices in S are the 10 marked vertices in the right half of the graph. Thus $|N(S)| < |S|$.



Proof 3 (other dual problems). In every graph $\alpha'(G) + \beta'(G) = n(G)$, so it suffices to show that at least 22 edges are needed to cover $V(G)$. Also $\beta'(G) \ge \alpha(G)$ always, since distinct edges are needed to cover the vertices of an independent set. Thus it suffices to show that G has an independent set of size at least 22. Such a set is given by the unmarked vertices above (the complement of a vertex cover).

Proof 4 (augmenting paths). Having found a matching M of size 20, one can prove that there is no perfect matching by following all possible M-alternating paths from one M-unsaturated vertex to show that none reaches the other unsaturated vertex. In this particular example, this method is not too difficult.

Proof 5 (symmetry and case analysis). The graph has two edges whose deletion leaves two isomorphic components of order 21. Since 21 is odd, a perfect matching must use exactly one of the two connecting edges. By symmetry, we may assume it is the one in bold above. This forces a neighboring vertex of degree 2 to be matched to its other neighbor, introducing other bold edge. Repeating this argument yields a path of bold edges forced into the matching. As soon as this leaves a vertex with no available neighbor, we have proved that a perfect matching cannot be completed.

3.1.29. *Every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$.* Each vertex of G covers at most $\Delta(G)$ edges. Since all edges must be covered in a vertex cover, this yields $\beta(G) \ge e(G)/\Delta(G)$. By the König–Egerváry Theorem, $\alpha'(G) = \beta(G)$ when G is bipartite. Thus $\alpha'(G) \ge e(G)/\Delta(G)$.

Every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k. Such a graph G is a simple bipartite graph with partite sets of size n. Thus $\Delta(G) \le n$, and we compute $\alpha'(G) \ge e(G)/\Delta(G) > (k-1)n/n = k-1$. Thus G has a matching of size k.

3.1.30. *The maximum number of edges in a simple bipartite graph that has no matching with k edges and no star with l edges is $(k-1)(l-1)$.* If G is a bipartite graph having no matching with k edges, then G has a vertex cover using at most $k-1$ vertices. If G is a simple graph having no star with l edges, then each vertex covers at most $l-1$ edges. Hence the vertex cover covers at most $(k-1)(l-1)$ edges, which must be all the edges of G. The bound is achieved by $(k-1)K_{1,l-1}$.

3.1.31. *Hall's Theorem from the König–Egerváry Theorem.* By the König–Egerváry Theorem, an X,Y-bigraph G fails to have a matching that saturates X if and only if G has a vertex cover of size less than $|X|$. Let Q be such a cover, with $R = Q \cap X$ and $T = Q \cap Y$. Because Q is a vertex cover, there is no edge from $X - R$ to $Y - T$, which means that $N(X-R) \subseteq T$. This yields

$$|N(X-R)| \le |T| = |Q| - |R| < |X| - |R| = |X - R|.$$

We have used the König–Egerváry Theorem to show that absence of a matching that saturates X yields a violation of Hall's Condition. Thus Hall's Condition is sufficient for such a matching.

Similarly, if $|N(S)| < |S|$ for some $S \subseteq X$, then $N(S) \cup X - S$ is a vertex cover of size less than $|X|$, and there is no matching of size $|X|$. Hence Hall's Condition also is necessary.

3.1.32. *If G is a bipartite graph with partite sets X, Y, then $\alpha'(G) = |X| - \max_{S\subseteq X}(|S| - |N(S)|)$.* Let $d = \max(|S| - |N(S)|)$. The case $S = \varnothing$ implies that $d \ge 0$. Choose $T \subseteq X$ such that $|T| - |N(T)| = d$. Because saturated vertices of T must have distinct neighbors in any matching and only $|T| - d$ neighbors are available, every matching leaves at least d vertices (of T) unsaturated. Thus $\alpha'(G) \le |X| - d$.

To prove that G has a matching as large as $|X| - d$, we form a new graph G' by adding d vertices to the partite set Y and making all of them adjacent to all of X. This adds d vertices to $N(S)$ for each $S \subseteq X$, which yields $|N_{G'}(S)| \ge |S|$ for all $S \subseteq X$. By Hall's Theorem, G' has a matching saturating all of X. When we delete the new vertices of G', we lose at most d edges of the matching. Hence what remains is a matching of size at least $|X| - d$ in G, as desired.

3.1.33. *König–Egerváry from Exercise 3.1.32.* Always $\alpha'(G) \le \beta(G)$, so it suffices to show that a bipartite graph G has a matching and a vertex cover of the same size. Consider an X,Y-bigraph G in which S is a subset of X with maximum deficiency. By part (a), $\alpha'(G) = |X| - |S| + |N(S)|$.

Let $R = (X - S) \cup (N(S))$. By the definition of $N(S)$, there are no edges joining S and $Y - N(S)$. Therefore, R is a vertex cover of G. The size of R is $|X| - |S| + |N(S)|$, which equals $\alpha'(G)$. Thus G has a matching and a vertex cover of the same size, as desired.

3.1.34. *When G is an X,Y-bigraph with no isolated vertices and the deficiency of a set S is $|S| - |N(S)|$, the graph G has a matching that saturates X if and only if each subset of Y has deficiency at most $|Y| - |X|$.* Using Hall's Theorem, it suffices to show that $|N(S)| \ge |S|$ for all $S \subseteq X$ if and only if $|T| - |N(T)| \le |Y| - |X|$ for all $T \subseteq Y$.

We rewrite the latter condition as $|X| - |N(T)| \le |Y| - |T|$. Since every vertex of $X - N(T)$ has no neighbor in T, we have $N(X - N(T)) \subseteq Y - T$. If Hall's Condition holds, then applying it with $S = X - N(T)$ yields $|X| - |N(T)| = |S| \le |N(S)| \le |Y| - |T|$, which is the desired condition.

Conversely, suppose that $|X| - |N(T)| \le |Y| - |T|$ for all $T \subseteq Y$. Given $S \subseteq X$, let $T = Y - N(S)$. Since T omits all neighbors of vertices in S, we have $S \subseteq X - N(T)$. Now $|S| \le |X| - |N(T)| \le |Y| - |T| = |N(S)|$. Hence Hall's Condition holds.

3.1.35. *A bipartite graph G in $K_{X,Y}$ fails to have $(k+1)K_2$ as an induced subgraph if and only if each $S \subseteq X$ has a subset of size at most k with neighborhood $N(S)$.* For any $S \subseteq X$, let T be a minimal subset of S with neighborhood $N(S)$. By the minimality of T, G has an induced matching of size $|T|$. Hence if G has no induced matching of size k, each $S \subseteq X$ has a subset of size at most k with neighborhood $N(S)$. Conversely, if $k + 1K_2$ does occur as an induced subgraph, then the set of its vertices in X have no subset of size at most k with the same neighborhood.

3.1.36. *If a bipartite graph G has a matching saturating a partite set X of size m, then at most $\binom{m}{2}$ edges of G belong to no matching of size m.* There are at least three distinguishable ways to get the bound. The most direct one, and the one that suggests the extremal graph proving that no smaller bound is possible, considers pairs of vertices.

Proof 1. First note that, after renumbering the vertices so the edges of the given matching are $\{x_i y_i\}$, every edge involving any other vertex of Y belongs to a perfect matching, so it suffices to restrict attention to the subgraph induced by X and $y_1, \ldots, y_m$. Consider the edges $x_i y_j$ and $x_j y_i$, for $j < i \le m$. If both edges are present, then neither belongs to no maximum matching, since they can be exchanged for $x_i y_i$ and $x_j y_j$ in the original matching. Taking one from each such pair bounds the number of unmatchable edges by $\binom{m}{2}$. On the other hand, taking all $x_i y_j$ with $j < i \le m$ yields a graph in which $\{x_i y_i\}$ is the unique maximum matching. If any edge $x_i y_i$ is deleted from this graph, then $|\mathrm{Adj}(\{x_1, \ldots, x_i\})| = i - 1$.

Proof 2. Induction on m. As before, reduce attention to the edges between X and $y_1, \ldots, y_m$. Among X, let x be the vertex of maximum degree in this subgraph. It has degree at most m, and its deletion leaves a graph satisfying the hypotheses for $m-1$. Allowing for the matched edge involving x, this gives a bound of $\binom{m-1}{2} + m - 1 = \binom{m}{2}$ on the unmatchable edges. Note that x coresponds to x_m in the example above.

Proof 3. Let S be an arbitrary subset of X. If $|N(S)| > |S|$ for every proper nonempty subset S of X, then every edge belongs to a perfect matching. To show this, delete the endpoints of an edge xy. This reduces the adjacency set of any S not containing x by at most 1, so the reduced graph has a perfect matching, and replacing xy yields a perfect matching of the original graph containing xy. So, assume there is an S with $|N(S)| = |S| = s$, and let $N(S) = T$. Then the subgraphs induced by $S \cup T$, $(X - S) \cup T$, and $(X - S) \cup (Y - T)$ partition the edges of G. The first and last have perfect matchings, and an edge there fails to appear in a perfect matching of G if and only if it appears in no perfect matching of the subgraph. No edge of the middle graph appears in a perfect matching of G. By induction and the fact that $|T| = |S|$, the bound on the number of edges that appear in no perfect matching of G is $\binom{s}{2} + s(m - s) + \binom{m-s}{2} = \binom{m}{2}$.

3.1.37. Let G be an X, Y-bigraph having a matching that saturates X.

a) If $S, T \subseteq X$ are sets such that $|N(S)| = |S|$ and $|N(T)| = |T|$, then $|N(S \cap T)| = |S \cap T|$.

Proof 1 (manipulation of sets). Since G has a matching that saturates X, $|N(S \cap T)| \ge |S \cap T|$, and $|N(S \cup T)| \ge |S \cup T|$. Also $N(S \cup T) = N(S) \cup N(T)$ and $N(S \cap T) \subseteq N(S) \cap N(T)$. Together, these statements yield

$$\begin{aligned}|S \cup T| + |S \cap T| &\le |N(S \cup T)| + |N(S \cap T)| \le |N(S) \cup N(T)| + |N(S) \cap N(T)| \\ &= |N(S)| + |N(T)| = |S| + |T|\end{aligned}$$

Since the two ends of this string of expressions are equal, the inequalities along the way hold with equality. In particular, $|N(S \cap T)| = |S \cap T|$ and $|N(S \cup T)| = |S \cup T|$.

Proof 2 (characterization of sets with no excess neighbors). If M is a matching that saturates X, then $|N(S)| = |S|$ if and only if every vertex of $N(S)$ is matched by M to a vertex of S. If $|N(S)| = |S|$ and $|N(T)| = |T|$, then every vertex of $N(S)$ is matched into S and every vertex of $N(T)$ is matched into T. Since $N(S \cap T) \subseteq N(S) \cap N(T)$, we conclude that every vertex of $N(S \cap T)$ is matched into $S \cap T$, and therefore $|N(S \cap T)| = |S \cap T|$.

b) There is a vertex $x \in X$ such that every edge incident to x belongs to some matching that saturates X. We use induction on $|X|$. If $|N(S)| > |S|$ for every nonempty proper subset of X, then Hall's Condition holds for the graph obtained by deleting the endpoints of any edge. Thus each edge can be combined with a matching saturating what remains of X in the graph obtained by deleting its endpoints, so every edge of G belongs to some matching saturating X.

In the remaining case, there is a nonempty proper subset $S \subseteq X$ such that $|N(S)| = |S|$. Let G_1 be the subgraph of G induced by $S \cup N(S)$, and let G_2 be the subgraph obtained by deleting $S \cup N(S)$. As in the proof of Hall's Theorem, the graph G_2 obtained by deleting $S \cup N(S)$ satisfies Hall's Condition (the proof of $\left|N_{G_2}(T)\right| \ge |T|$ follows from $|N_G(T \cup S)| \ge |T \cup S|$). Thus G_2 has a matching saturating $X - S$.

The subgraph G_1 also satisfies Hall's condition, since it retains all neighbors of each vertex of S. By the induction hypothesis, S has a vertex x such that every edge incident to x belongs to a matching in G_1 that saturates S. These matchings can be combined with a single matching in G_2 that saturates $X - S$ to obtain matchings in G that saturate X. Hence the vertex x serves as the desired vertex in G.

It appears that part (a) is not needed to solve part (b).

3.1.38. *Pairing up farms and hunting ranges.* Suppose the unit of area is the size of one range. Let G be the bipartite graph between hunting ranges and farms formed by placing an edge between a hunting range and a farm if the area of their intersection is at least ε, where $\varepsilon = 4/(n+1)^2$ if n is odd and $\varepsilon = 4/[n(n+2)]$ if n is even. We prove that this graph has a perfect matching, which yields the desired assignments.

Let H be the union of some set of k hunting ranges. Let $f_1 \ge \cdots \ge f_n$ be the areas of intersection with H of the farms, and let F be the set of k farms having largest intersection with H. If $f_k = \alpha$, then the area of H is bounded by $\sum_{i=1}^{k-1} g_i + \alpha(n + 1 - k) \le k - 1 + \alpha(n + 1 - k)$. It also equals k, so we have $\alpha \ge 1/(n + 1 - k)$. Since we have k farms meeting H with area at least $1/(n+1-k)$, we find for each farm in F a hunting range contained in H that intersects the farm with area at least $1/[k(n+1-k)] \ge \varepsilon$. Hence any set of k hunting ranges has at least k neighbors in G, which guarantees the matching.

Note that ε is the largest possible guaranteed minimum intersection. Let $k = \lceil n/2 \rceil$. With hunting ranges in equal strips, we can arrange that some set of $k - 1$ farms intersects each of the first k hunting ranges with area $1/k$, and the remaining farms intersect each of the first k hunting ranges with area ε, since $(k - 1)/k + (n + 1 - k)\varepsilon = 1$. Now one of the first k hunting ranges must be matched with area ε.

3.1.39. $\alpha(G) \le n(G) - e(G)/\Delta(G)$. Let S be an independent set of size $\alpha(G)$. Since $V(G) - S$ is a vertex cover, summing the vertex degrees in $V(G) - S$ provides an upper bound on $e(G)$. Thus $e(G) \le (n(G) - \alpha(G))\Delta(G)$, which is equivalent to the desired inequality.

If G is regular, then $\alpha(G) \le n(G)/2$. In the previous inequality, set $e(G) = n(G)\Delta(G)/2$.

3.1.40. *If G is a bipartite graph, then $\alpha(G) = n(G)/2$ if and only if G has a perfect matching.* Since $\alpha(G) = n(G) - \beta(G) = n(G) - \alpha'(G)$ by Lemma 3.1.21 and the König–Egerváry Theorem, we have $\alpha(G) = n(G)/2$ if and only if $\alpha'(G) = n(G)/2$.

3.1.41. (corrected statement) *If G is a nonbipartite n-vertex graph with exactly one cycle C, then $\alpha(G) \ge (n-1)/2$, with equality if and only if $G - V(C)$ has a perfect matching.* The cycle C must have odd length, say k. Let e be an edge of C, and let $G' = G - e$. The graph G' is bipartite, so $\alpha(G-e) \ge n/2$. An independent set S in $G-e$ is also independent in G unless it contains both endpoints of e. If $|S| > n/2$, then we can afford to drop one of these vertices. If $|S| = n/2$, then we can take the other partite set instead to avoid the endpoints of e. In each case, $\alpha(G) \ge (n-1)/2$.

If $G - V(C)$ has a perfect matching, then an independent set is limited to $(k-1)/2$ vertices of C and $(n-k)/2$ vertices outside C, so $\alpha(G) \le (n-1)/2$ and equality holds.

For the converse, observe that deleting $E(C)$ leaves a forest F in which each component has a vertex of G. Let H be a component of F, with x being its vertex on C, and let r be its order. If $H-x$ has no perfect matching, then $\alpha'(H-x) \le r/2-1$ (that is, it cannot equal $(r-1)/2$). Now $\beta(H-x) \le r/2-1$, by König-Egerváry, and $\alpha(H-x) \ge r/2$, since the complement of a vertex cover is an independent set. Since this independent set does not use x, we can combine it with an independent set of size at least $(n-r)/2$ in the bipartite graph $G - V(H)$ to obtain $\alpha(G) \ge n/2$. Since this holds for each component of F, $\alpha(G) = (n-1)/2$ requires a perfect matching in $G - V(C)$. (This direction can also be proved by induction on $n-k$.)

3.1.42. *The greedy algorithm produces an independent set of size at least $\sum_{v\in V(G)} \frac{1}{d(v)+1}$ in a graph G.* The algorithm iteratively selects a vertex of minimum degree in the remaining graph and deletes it and its neighbors. We prove the desired bound by induction on the number of vertices.

Basis step: $n=0$. When there are no vertices, there is no contribution to the independent set, and the empty sum is also 0.

Induction step $n>0$. Let x be a vertex of minimum degree, let $S = \{x\} \cup N(x)$, and let $G' = G - S$. The algorithm selects x and then seeks an independent set in G'. We apply the induction hypothesis to G' to obtain a lower bound on the contribution that the algorithm obtains from G'. Thus the size of the independent set found in G is at least $1 + \sum_{v\in V(G')} \frac{1}{d_{G'}(v)+1}$.

Note that $\{x\} \cup N(x)$ is a set of size $d_G(x)+1$, and the choice of x as a vertex of minimum degree implies that each vertex in this set contributes at most $d_G(x)+1$ to the desired sum. Thus

$$\begin{aligned} 1 + \sum_{v\in V(G')} \frac{1}{d_{G'}(v)+1} &= \sum_{v\in S} \frac{1}{d_G(x)+1} + \sum_{v\in V(G)-S} \frac{1}{d_{G-S}(v)+1} \\ &\ge \sum_{v\in S} \frac{1}{d_G(v)+1} + \sum_{v\in V(G)-S} \frac{1}{d_G(v)+1}\ . \end{aligned}$$

Thus the algorithm finds an independent set at least as large as desired.

3.1.43. *Consequences of Gallai's Theorem* (G has no isolated vertices).

a) A maximal matching M is a maximum matching if and only if it is contained in a minimum edge cover. If M is a maximal matching, then the smallest edge cover L containing M adds one edge to cover each M-unsaturated vertex, since no edge covers two M-unsaturated vertices. We have $|L| = |M| + (n - 2|M|) = n - |M|$. By Gallai's Theorem (Theorem 3.1.22), $|M| = \alpha'(G)$ if and only if $|L| = \beta'(G)$.

b) A minimal edge cover L is a minimum edge cover if and only if it contains a maximum matching. As observed in proving Theorem 3.1.22, every minimal edge cover consists of disjoint stars. The largest matching contained in a disjoint union of stars consists of one edge from each component. The size of this matching is $n - |L|$. Hence a minimal edge cover L has size $n - \alpha'(G)$ if and only if L contains a matching of size $\alpha'(G)$.

3.1.44. *If G is a simple graph in which the sum of the degrees of any k vertices is less than $n-k$, then every maximal independent set in G has more than k vertices.* Let S be an independent set. If $|S| \le k$, then the sum of the degrees of the vertices in S is less than $n-k$. This means that some vertex x outside S is not a neighbor of any vertex in S, and hence x can be added to form an independent set containing S. Thus maximal independent sets must have more than k vertices.

3.1.45. *If xy and xz are α-critical edges in G and $y \not\leftrightarrow z$, then G contains an induced odd cycle (through xy and xz).* Let Y, Z be maximum stable sets in $G - xy$ and $G - xz$, respectively. Since Y, Z are not independent in G, we have $x, y \in Y$ and $x, z \in Z$.

Proof 1. Let $H = G[Y \triangle Z]$. Since $x \in Y \cap Z$, H is a bipartite graph with bipartition $Y - Z, Z - Y$. If some component of H has partite sets of different sizes, then we can substitute the larger for the smaller in Y or Z to obtain a stable set in G of size exceeding $\alpha(G)$.

If y and z belong to different components H_y and H_z of H, then let S be the union of $V(H_y) \cap Z$, $V(H_z) \cap Y$, one partite set of each other component of H, and $Y \cap Z$. Since $x \in Y \cap Z$ and $y, z \notin S$, the set S is independent in G. Also, $|S| = |Y| = |Z| > \alpha(G)$. Hence y and z belong to the same component of H. A shortest y,z-path in H is a chordless path of odd length in G, and it completes a chordless odd cycle with zx and xy.

Proof 2. Let $H' = G[(Y \triangle Z)] \cup \{x\}$. Note that $|Y \triangle Z|$ is even, since $|Y| = |Z|$. Let $2k = |Y \triangle Z|$. If H' is bipartite, then it has an independent set of size at least $k+1$, which combines with $Y \cap Z - \{x\}$ to form an independent set of size $\alpha(G)+1$ in G. Hence H' has an odd cycle. Since H is bipartite, this cycle passes through x. Since $N_{H'}(x) = \{y, z\}$, the odd cycle contains the desired edges.

3.1.46. *A graph has domination number 1 if and only if some vertex neighbors all others.* This is immediate from the definition of dominating set.

3.1.47. *The smallest tree where the vertex cover number exceeds the domination number is P_6.* In a graph with no isolated vertices, every vertex cover is a dominating set, since every vertex is incident to an edge, and at least one endpoint of that edge is in the set. Hence $\gamma(G) \le \beta(G)$. We want a tree where the inequality is strict.

If $\gamma(G) = 1$, then a single vertex is adjacent to all others, and since G is a tree there are no other edges, so $\beta(G) = 1$. Hence we need $\gamma(G) \ge 2$ and $\beta(G) \ge 3$. A tree is bipartite, so $al'(G) = \beta(G) \ge 3$. A matching of size 3 requires at least 6 vertices. There are two isomorphism classes of 6-vertex trees with perfect matchings, and P_6 is the only one having a dominating set of size 2. (Smaller trees can also be excluded by case analysis instead of using the König–Egerváry Theorem.)

3.1.48. $\gamma(C_n) = \gamma(P_n) = \lceil n/3 \rceil$. With maximum degree 2, vertices can dominate only two vertices besides themselves. Therefore, $\lceil n/3 \rceil$ is a lower bound. Picking every third vertex starting with the second (and using the last when n is not divisible by 3) yields a dominating set of size $\lceil n/3 \rceil$.

3.1.49. *In a graph G without isolated vertices, the complement of a minimal dominating set is a dominating set, and hence $\gamma(G) \le n(G)/2$.* Let S be a minimal dominating set. Every vertex of $\overline{S}$ has a neighbor in S, and by minimality this fails when a vertex is omitted from S. For each $x \in S$, there is thus a vertex of $\overline{S}$ whose only neighbor in S is x. In particular, every $x \in S$ has a neighbor in $\overline{S}$, which means that $\overline{S}$ is a dominating set.

Since S and $\overline{S}$ are disjoint dominating sets, one of them has size at most $n(G)/2$.

3.1.50. *If G is a n-vertex graph without isolated vertices, then $\gamma(G) \le n - \beta'(G) \le n/2$.*

Proof 1. Since every edge covers at most two vertices, always $\beta'(G) \ge n/2$. As discussed in the proof of Theorem 3.1.22, the components of a minimum edge cover are stars, and the number of stars is $n - \beta'(G)$. Since the union of these stars is a spanning subgraph, choosing the centers of these stars yields a dominating set.

Proof 2. Since $\alpha'(G) = n - \beta'(G)$ by Theorem 3.1.22, it suffices to show that $\gamma(G) \le \alpha'(G)$. In a maximum matching M, the two endpoints of an edge in M cannot have distinct unsaturated neighbors. Also the unsaturated neighbors all have saturated neighbors. Therefore, picking from each edge of M the endpoint having unsaturated neighbor(s) (or either if neither has such a neighbor) yields a dominating set of size $\alpha'(G)$.

Construction of n-vertex graphs with domination number k, for $1 \le k \le n/2$. Form G from a matching of size k by selecting one vertex from each edge and adding edges to make these vertices pairwise adjacent.

3.1.51. *Domination in an n-vertex simple graph G with no isolated vertices.*

a) $\lceil n/(1+\Delta(G) \rceil \le \gamma(G) \le n - \Delta(G)$. Each vertex takes care of itself and at most $\Delta(G)$ others; thus $\gamma(G)(1+\Delta(G)) \ge n$. For the upper bound, note that the set consisting of all vertices *except* the neighbors of a vertex of maximum degree is a dominating set.

b) $(1 + \operatorname{diam} G)/3 \le \gamma(G) \le n - \lceil 2\operatorname{diam} G/3 \rceil$. Let P be a shortest u,v-path, where $d(u,v) = \operatorname{diam} G$. Taking the vertices at distances $1, 4, 7, \dots$ along P from u yields a set that dominates all the vertices of P (v is also needed if $\operatorname{diam} G$ is divisible by 3. Even if all the vertices off P are needed to augment this to a dominating set, we still have used at most $n - \lceil 2\operatorname{diam} G/3 \rceil$ vertices.

For the lower bound, the vertices at distances $0, 3, 6, \dots$ from u along P must be dominated by distinct vertices in a dominating set; a vertex dominating two of them would yield a shorter u,v-path. This yields the lower bound.

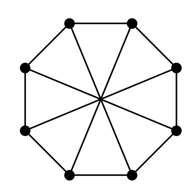
Both bounds hold with equality for the path P_n.

3.1.52. *If the diameter of G is at least 3, then $\gamma(\overline{G}) \le 2$.* Let u and v be two vertices such that $d_G(u,v) = 3$. The set $\{u,v\}$ is a dominating set in $\overline{G}$, because u and v have no common neighbors in G. For $x \in V(G) - \{u,v\}$, at least one of $\{u,v\}$ is nonadjacent to x in G and therefore adjacent to it in $\overline{G}$.

3.1.53. *Examples with specified domination number.*

A $5k$-vertex graph with domination number $2k$ and minimum degree 2. Begin with kC_5 and add edges to form a cycle using one vertex from each 5-cycle. Two vertices must be used from each original 5-cycle, and this suffices for a dominating set.

A 3-regular graph G with $\gamma(G) = 3n(G)/8$. In a 3-regular graph, each vertex dominates itself and three others. In the 8-vertex graph below, deletion of any vertex and its neighbors leaves P_4, which cannot be dominated by one additional vertex. Hence the domination number is 3.



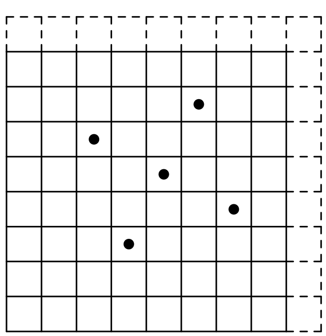
3.1.54. *The Petersen graph has domination number 3 and total domination number 4.* Each vertex dominates itself and three others, so at least three vertices are needed. Since the graph has diameter 2, the neighbors of a single vertex form a dominating set.

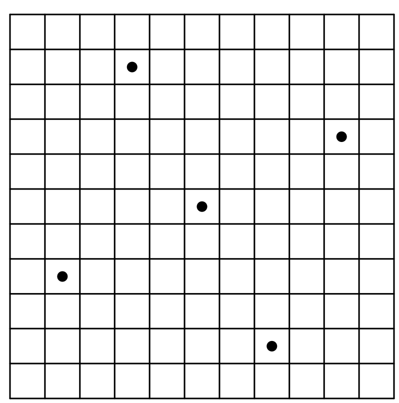
A total dominating set S must include a neighbor of every vertex in S. Hence S must contain two adjacent vertices. This pair leaves four undominated vertices. Adding a neighbor of the original pair dominates at most three of these, since the graph is 3-regular. Hence $|S| \geq 4$. One vertex and its neighbors form a total dominating set of size four.

3.1.55. *Dominating sets in the hypercube Q_4.* Since Q_4 is 4-regular, each vertex dominates itself and four others. Now $n(Q_4) = 16$ yields $\gamma(Q_4) \geq \lceil 16/5 \rceil = 4$. Since $\{0000, 0111, 1100, 1011\}$ is an independent dominating set and $\{0000, 0001, 1110, 1111\}$ is a total dominating set, the domination, independent domination, and total domination numbers all equal 4.

Adding two vertices to this total dominating set of size 4 completes a connected dominating set of size 6. We show there is no smaller connected dominating set. A connected 5-vertex subgraph contains two incident edges. Let S be the set of three vertices in two such edges. The set T of vertices undominated by S has size 6. Each neighbor of a vertex of S dominates at most two vertices in T. Each vertex of T dominates at most three vertices in T, except for one vertex that dominates itself and four others (For example, if $S = \{0000, 0001, 0010\}$, then the high-degree vertex of $Q_4[T]$ is 1111.) To dominate T with only two additional vertices, we must therefore use the high-degree vertex of T. However, its distance to S is 3, so it cannot be used to complete a connected set of size 5.

3.1.56. *Five pairwise non-attacking queens can control an 8-by-8 chessboard.* As shown below, they can also control a 9-by-9 chessboard. Five queens still suffice for an 11-by-11 chessboard, but this configuration does not exist on the 8-by-8 board.





3.1.57. *An n-vertex tree with domination number 2 in which the minimum size of an independent dominating set is $\lfloor n/2 \rfloor$.* Consider the tree of diameter 3 with two central vertices u and v in which one central vertex has $\lfloor (n-2)/2 \rfloor$ leaf neighbors and the other has $\lceil (n-2)/2 \rceil$ leaf neighbors. The set $\{u, v\}$ is a dominating set, but these cannot both appear in an independent dominating set. If u does not appear in a dominating set, then all its leaf neighbors must appear. We also must include at least one vertex from the set consisting of v and its leaf neighbors, since these are not dominated by the other leaves. Hence the independent dominating set must have at least $\lfloor (n-2)/2 \rfloor + 1$ vertices.

3.1.58. *Every $K_{1,r}$-free graph G has an independent dominating set of size at most $(r-2)\gamma(G) - (r-3)$.* Let S be a minimum dominating set in G. Let S' be a maximal independent subset of S. Let $T = V(G) - R$, where R is the set $N(S') \cup S'$ of vertices dominated by S'. Let T' be a maximal independent subset of T.

Since T' contains no neighbor of S', $S' \cup T'$ is independent. Since S' is a maximal independent subset of S, every vertex of $S - S'$ has a neighbor in S'. Similarly, T' dominates $T - T'$. Hence $S' \cup T'$ is a dominating set.

It remains to show that $\left|S' \cup T'\right| \leq (r-1)\gamma(G) - (r-3)$. Each vertex of $S - S'$ has at most $r-2$ neighbors in T', since it has a neighbor in S', and $S' \cup T'$ is independent, and G is $K_{1,r}$-free. Since S is dominating, each vertex of T' has at least one neighbor in $S - S'$. Hence $\left|T'\right| \leq (r-2)\left|S - S'\right|$, which yields $\left|S' \cup T'\right| \leq (r-2)|S| - (r-3)\left|S'\right|$. Since $|S| = \gamma(G)$ and $\left|S'\right| \geq 1$, we conclude that $\left|S' \cup T'\right| \leq (r-2)\gamma(G) - (r-3)$.

3.1.59. *In a graph G of order n, the minimum size of a connected dominating set is n minus the maximum number ℓ of leaves in a spanning tree.* For the upper bound, deleting the leaves in a spanning tree with ℓ leaves yields a connected dominating set of size $n - \ell$.

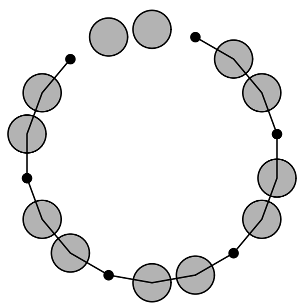
For the lower bound, we form a spanning tree T of G by taking a spanning tree in the subgraph induced by a connected dominating set S and adding each remaining vertex as a neighbor of one of these. Thus $|S|$ is n minus the number of leaves in T. Since this number of leaves is at most ℓ, we have $|S| \leq n = \ell$.

3.1.60. *A graph with minimum degree k and no connected dominating set of size less than $3n(G)/(k+1) - 2$.* Form G from a cyclic arrangement of $3m$ pairwise-disjoint cliques of sizes $\lceil k/2 \rceil, \lfloor k/2 \rfloor, 1, \lceil k/2 \rceil, \lfloor k/2 \rfloor, 1, \cdots$ in order by making each vertex adjacent to every vertex in the clique before it and the clique after it. Each vertex is adjacent to all but itself in its own clique and the two neighboring cliques, so G is k-regular.

A graph has a connected dominating set of size r if and only if it has a spanning tree with at most r non-leaf vertices (Exercise 3.1.59). Hence we show that spanning trees in G must have many non-leaves. Let S be the set of vertices in the cliques of size 1 in the construction.

In a subgraph of G having all pairs of nearest vertices in S joined by paths through the two intervening cliques, there is a cycle. If at least two such pairs are not joined by such paths, then the subgraph is disconnected. Therefore, every spanning tree of G contains a path P directly connecting all but one of the successive pairs, as shown below.

At least one endpoint of P must be a non-leaf. If one endpoint of P is a leaf, then the other endpoint of P and some vertex in the untouched clique next to it must be non-leaves. In either case, we have obtained at least $3m-2$ non-leaves. Since $m = n/(k+1)$, we have the desired bound.



3.2. ALGORITHMS AND APPLICATIONS

3.2.1. *A weighted graph with four vertices where the maximum weight matching is not a maximum size matching.* Let $G = P_4$, and give the middle edge greater weight than the sum of the other weights.

3.2.2. *Use of the Hungarian Algorithm to test for the existence of a perfect matching in a bipartite graph G.* Given that the partite sets of G have size n, form a weighted matching problem on $K_{n,n}$ in which the edges of G have weight 1 and the edges not in G have weight 0. There is a perfect matching in G if and only if the solution to the weighted matching problem is n.

3.2.3. *Multiplicity of stable matchings.* With men u, v and women a, b, there may be two stable matching. Suppose the preferences are $u : a > b$, $v : b > a$, $a : v > u$, $b : u > v$. If both men get their first choices, then they prefer no one to their assigned partner, so the matching is stable. The same argument applies when the women get their first choices. However, the matchings with men getting their first choices and women getting their first choices are different.

3.2.4. *Stable matchings under proposal algorithm.* Consider the preference orders listed below.

Men $\{u, v, w, x, y, z\}$	Women $\{a, b, c, d, e, f\}$
$u : a > b > d > c > f > e$	$a : z > x > y > u > v > w$
$v : a > b > c > f > e > d$	$b : y > z > w > x > v > u$
$w : c > b > d > a > f > e$	$c : v > x > w > y > u > z$
$x : c > a > d > b > e > f$	$d : w > y > u > x > z > v$
$y : c > d > a > b > f > e$	$e : u > v > x > w > y > z$
$z : d > e > f > c > b > a$	$f : u > w > x > v > z > y$

When men propose, the steps of the algorithm are as below. For each round, we list the proposals by u, v, w, x, y, z in order, followed by the resulting rejections. Round 1: a, a, c, c, c, d; $a \times v$, $c \times w$, $c \times y$. Round 2: a, b, b, c, d, d; $b \times v$, $d \times z$. Round 3: a, c, b, c, d, e; $c \times x$. Round 4: a, c, b, a, d, e; $a \times u$. Round 5: b, c, b, a, d, e; $b \times u$. Round 6: d, c, b, a, d, e; $d \times u$. Round 7: c, c, b, a, d, e; $c \times u$. Round 8: f, c, b, a, d, e; stable matching.

When women propose, the steps of the algorithm are as below. For each round, we list the proposals by a, b, c, d, e, f in order, followed by the resulting rejections. Round 1: z, y, v, w, u, u; $u \times e$. Round 2: z, y, v, w, v, u; $v \times e$. Round 3: z, y, v, w, x, u; stable matching.

Note that the pairs uf and vc occur in both results, and in all other cases the women are happier when the women propose and the men are happier when the men propose.

3.2.5. *Maximum weight transversal.* For each matrix below, we underscore a maximum weight transversal, and the labels on the rows and columns form a cover whose total cost equals the weight of the transversal.

For every position (i, j), the label on row i plus the label on column j is at least the entry in position (i, j) in the matrix. Hence the labeling is feasible for the dual problem. Equality between the sum of the labels in a feasible labeling and the sum of the entries of a transversal implies that the transversal is one of maximum weight and feasible labeling is one of minimum weight, because *every* feasible labeling has sum as large as the weight of *every* matching (since the positions in the matching must be covered disjointly by the labels).

$$\begin{array}{cc} & \begin{matrix} 0 & 0 & 1 & 0 & 2 \end{matrix} \\ \begin{matrix} 4 \\ 3 \\ 4 \\ 7 \\ 8 \end{matrix} & \begin{pmatrix} \underline{4} & 4 & 4 & 3 & 6 \\ 1 & 1 & \underline{4} & 3 & 4 \\ 1 & \underline{4} & 5 & 3 & 5 \\ 5 & 6 & 4 & 7 & \underline{9} \\ 5 & 3 & 6 & \underline{8} & 3 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 3 & 1 & 2 & 1 & 0 \end{matrix} \\ \begin{matrix} 7 \\ 6 \\ 6 \\ 5 \\ 5 \end{matrix} & \begin{pmatrix} 7 & 8 & \underline{9} & 8 & 7 \\ 8 & 7 & 6 & \underline{7} & 6 \\ 9 & 6 & 5 & 4 & \underline{6} \\ \underline{8} & 5 & 7 & 6 & 4 \\ 7 & \underline{6} & 5 & 5 & 5 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 0 & 0 & 1 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 7 \\ 3 \\ 7 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 2 & 3 & 4 & \underline{5} \\ 6 & \underline{7} & 8 & 7 & 2 \\ 1 & 3 & \underline{4} & 4 & 5 \\ 3 & 6 & 2 & \underline{8} & 7 \\ \underline{4} & 1 & 3 & 5 & 4 \end{pmatrix} \end{array}$$

Review of the "Hungarian Algorithm" for maximum weighted matching in the assignment problem. Find a feasible vertex labeling for the dual, i.e. weights $l(v)$ such that $l(x_i)+l(y_i) \ge w(ij)$. (This can be done by using the maximum in each row as the row label, with 0's for the columns.) Subtract out to find the "excess value" matrix $l(x_i)+l(y_j)-w(ij)$. Find a maximum matching and minimum cover in the equality subgraph (0's in the excess matrix). If this is a perfect matching, its value equals the dual value $\sum l(v)$ being minimized, hence is optimal. If not, let S be the set of rows not in the cover, T the set of columns in the cover, and ε the minimum excess value in the uncovered positions. Subtract ε from the row labels in S, add ε to the row labels in T, readjust the excess matrix, and iterate. (Note: any minimum cover can be used, and we know from the proof of the König–Egerváry Theorem that we can obtain a minimum cover by using $T \cup (X - S)$, where T and S are the subsets of Y and X reachable by alternating paths from the unsaturated vertices in the row-set X.)

If the matching was not complete, then $|S| > |T|$ and $\sum l(v)$ decreases, which guarantees the finiteness of the algorithm. The positions are of four types, corresponding to edges from S to T, $X-S$ to T, S to $Y-T$, and $X-S$ to $Y-T$. The change to the excess in the four cases is $0, +\varepsilon, -\varepsilon, 0$, respectively. Note that ε was defined to be the minimum excess corresponding to edges from S to $Y-T$, so every excess remains positive. For the first matrix above, the successive excess matrices computed in the algorithm could look like those below. These are not unique, because different matchings could be chosen in the equality subgraphs. The entries in the matching and the rows and columns in the cover ($X-S$ and T) are indicated with underscores.

$$
\begin{array}{c} \\ 6\\4\\5\\9\\\underline{8} \end{array}
\!\!\begin{array}{c} \begin{array}{ccccc} 0&0&\underline{0}&0&\underline{0} \end{array} \\ \begin{pmatrix} 2&2&2&3&0\\ 3&3&\underline{0}&1&0\\ 4&1&0&2&\underline{0}\\ 4&3&5&2&0\\ 3&5&2&\underline{0}&5 \end{pmatrix} \end{array}
\to
\begin{array}{c} \\ 5\\\underline{3}\\\underline{4}\\8\\\underline{8} \end{array}
\!\!\begin{array}{c} \begin{array}{ccccc} 0&0&1&0&\underline{1} \end{array} \\ \begin{pmatrix} 1&1&2&2&0\\ 2&2&\underline{0}&0&0\\ 3&\underline{0}&0&1&0\\ 3&2&5&1&\underline{0}\\ 3&5&3&\underline{0}&6 \end{pmatrix} \end{array}
\to
\begin{array}{c} \\ 4\\3\\4\\7\\8 \end{array}
\!\!\begin{array}{c} \begin{array}{ccccc} 0&0&1&0&2 \end{array} \\ \begin{pmatrix} \underline{0}&0&1&1&0\\ 2&2&\underline{0}&0&1\\ 3&\underline{0}&0&1&1\\ 2&1&4&0&\underline{0}\\ 3&5&3&\underline{0}&7 \end{pmatrix} \end{array}
$$

3.2.6. *Finding a transversal of minimum weight.* Let the rows correspond to vertices $x_1, \ldots, x_5$, the columns to vertices $y_1, \ldots, y_5$, and let the weight of edge $x_i y_j$ be the value in position ij. Optimality of the answer can be proved by exhibiting an optimal matching and exhibiting a feasible labeling for the dual problem that has the same total value.

Alternatively, finding a minimum transversal is the same as finding a minimum weight perfect matching in the corresponding graph, which corresponds to a maximum weight matching in the weighted graph obtained by subtracting all the weights from a fixed constant. In the example given,

we could subtract the weights from 13, and then the answer would be $5 \cdot 13$ minus the maximum weight of a transversal in the resulting matrix.

In the direct approach, the dual problem is to maximize $\sum l(v)$ subject to $l(x_i)+l(y_j) \le w(x_i y_j)$. Subtracting the labels from the weights yields the "reduced cost" matrix. At each iteration, we determine the equality subgraph and ε as before, but this time add ε to the labels of vertices in S (rows not in the cover) and subtract ε from the labels of vertices in T (columns in the cover). Since $|S| > |T|$, $\sum l(v)$ increases. Every matching has weight at least $\sum l(v)$. When G_l contains a complete matching, $\min \sum w$ and $\max \sum l$ are attained and equal.

In the matrix below, the underscored positions form a minimum-weight transversal; the weight is 30. In the dual problem, the indicated labeling has total value 30, and the labels $l(x_i)$ and $l(y_j)$ sum to at most the matrix entry $w_{i,j}$. Hence these solutions are optimal.

$$
\begin{array}{c} \\ 4\\2\\4\\5\\4 \end{array}
\!\!\begin{array}{c} \begin{array}{ccccc} 0&1&3&5&2 \end{array} \\ \begin{pmatrix} \underline{4}&5&8&10&11\\ 7&6&\underline{5}&7&4\\ 8&\underline{5}&12&9&6\\ 6&6&13&10&\underline{7}\\ 4&5&7&\underline{9}&8 \end{pmatrix} \end{array}
$$

3.2.7. *The Bus Driver Problem.* Bus drivers are paid overtime for the time by which their routes in a day exceed t. There are n bus drivers, n morning routes with durations $x_1, \ldots, x_n$, and n afternoon routes with durations $y_1, \ldots, y_n$. Assign to the edge $a_i b_j$ the weight $w_{i,j} = \max\{0, x_i + y_j - t\}$. The problem is then to find the perfect matching of minimum total weight. Index the morning runs so that $x_1 \ge \cdots \ge x_n$. Index the afternoon runs so that $y_1 \ge \cdots \ge y_n$. A feasible solution matches a_i to $b_{\sigma(i)}$ for some permutation σ of $[n]$. If there exists $i < j$ with $\sigma(i) > \sigma(j)$, then we have

$$\alpha = w_{i,\sigma(i)} + w_{j,\sigma(j)} = \max\{0, x_i + y_{\sigma(i)} - t\} + \max\{0, x_j + y_{\sigma(j)} - t\}$$

$$\beta = w_{i,\sigma(j)} + w_{j,\sigma(i)} = \max\{0, x_i + y_{\sigma(j)} - t\} + \max\{0, x_j + y_{\sigma(i)} - t\}$$

It suffices to prove that $\alpha \ge \beta$, because then there exists a minimizing permutation with no inversion. The nonzero terms in the maximizations have the same sum for each pair. Also,

$$x_i + y_{\sigma(i)} - t \ge x_i + y_{\sigma(j)} - t \ge x_j + y_{\sigma(j)} - t$$

$$x_i + y_{\sigma(i)} - t \ge x_j + y_{\sigma(i)} - t \ge x_j + y_{\sigma(j)} - t$$

If the central terms in the inequalities are both positive, then α is at least their sum, which equals β. If both are nonpositive, then $\alpha \geq 0 = \beta$. If the first is positive and the second nonpositive, then

$$\alpha = x_i + y_{\sigma(i)} - t \geq x_i + y_{\sigma(j)} - t = \beta.$$

If instead the second is positive, then

$$\alpha = x_i + y_{\sigma(i)} - t \geq x_j + y_{\sigma(i)} - t = \beta.$$

3.2.8. *When the weights in a matrix are the products of nonnegative numbers associated with the rows and columns, a maximum weight transversal is obtained by pairing the row having the kth largest row weight with the column having the kth largest column weight, for each k.* We show that all other pairings are nonoptimal. If the weights are not matched in order, then there exist indices i, j such that $a_i > a_j$ but the weight b matched with a_i is less than the weight b' matched with a_j. To show that switching these assignments increases the total weight, we compute

$$\begin{aligned} a_i b' + a_j b &= a_i b + a_i(b' - b) + a_j(b - b') + a_j b' \\ &= a_i b + a_j b' + (a_i - a_j)(b' - b) > a_i b + a_j b' \end{aligned}$$

When the weights in a matrix are the sums of nonnegative numbers associated with the rows and columns, every transversal has the same weight. Since a transversal uses one element in each row and each column, $w_{i,j} = a_i + b_j$ means that every transversal has total weight $\sum a_i + \sum b_j$.

3.2.9. *One-sided preferences.* There are k seminars and n students, each student to take one seminar. The ith seminar will have k_i students, where $\sum k_i = n$. Each student ranks the seminars; we seek a stable assignment where no two students can both improve by switching.

Form an X, Y-bigraph where X is the set of students and Y has k_i vertices for each vertex i. For each edge from student x_j to a vertex representing the ith seminar, let the weight be k minus the rank of the ith seminar in the preference of x_j.

A maximum weight matching in this weighting is a stable assignment, since if two students can both improve by trading assignments, then the result would be a matching of larger weight.

3.2.10. *Weighted preferences need not be stable.* Consider men $\{x_1, x_2, x_3\}$ and women $\{y_1, y_2, y_3\}$. Each assigns $3 - i$ points to the ith person in his or her preference list. Hence we indicate a preference order by a triple whose entries are $0, 1, 2$ in some order, with the position of the integer i being the index of the person of the opposite sex to whom this person assigns i points.

In the matrix below, the preference vectors of the men and women label the rows and coloumns, respectively. An entry in the matrix is the sum

of the points assigned to that potential edge by the two people. The underlined diagonal is the only matching that uses the maximum entry in each row and column, so it is the only maximum-weight matching. However, it is not a stable matching, because man x_1 and woman y_1 prefer each other to their assigned mates.

$$\begin{array}{c c} & \begin{array}{ccc} 120 & 120 & 210 \end{array} \\ \begin{array}{c} 120 \\ 021 \\ 120 \end{array} & \begin{pmatrix} 2 & 3 & 3 \\ 1 & 4 & 2 \\ 2 & 2 & 0 \end{pmatrix} \end{array}$$

The example can be extended for all larger numbers of men and women by adding pairs who are each other's first choice and are rated last by the six people in this example.

3.2.11. *In the result of the Gale-Shapley Proposal Algorithm with men proposing, every man receives a mate at least as high on his list as in any other stable matching.* We prove that under the G-S Algorithm with men proposing, no man is ever rejected by any woman who is matched to him in any stable matching. This yields the result, since each man's sequence of proposals proceeds downward from the top of his list, and he can only wind up with a woman less desirable than his most desirable match over all stable matchings if he is rejected by some women who matches him in some stable matching.

Consider the first time when some man x is rejected by a woman a to whom he is matched in some stable matching M. The rejection occurs because a has a proposal from a man y higher than x on her list. In M, man y is matched to some woman b. Since M is stable, y cannot prefer a to b. Thus b appears above a on the list for y. But now the decreasing property of proposals from men implies that y has proposed to b in the G-S Algorithm before proposing to a. If y is now proposing to a, then y was previously rejected by b. Since y is matched to b in M, this contradicts our hypothesis that a rejecting x was the first rejection involving a pair that occurs in some stable matching.

3.2.12. *The Stable Roommates Problem defined by the preference orderings below has no stable matching.* There are only three matchings to consider: $ab|cd$, $ac|bd$, and $ad|bc$. In each, two non-paired people prefer each other to their current roommates. The problematic pairs are $\{b, c\}$, $\{a, b\}$, $\{a, c\}$ in the three matchings, respectively.

$$\begin{array}{l} a : b > c > d \\ b : c > a > d \\ c : a > b > d \\ d : a > b > c \end{array}$$

3.2.13. In the stable roommates problem, suppose that each individual declares a top portion of the preference list as "acceptable". Define the *acceptability graph* to be the graph whose vertices are the people and whose edges are the pairs of people who rank each other as acceptable. Prove that all sets of rankings with acceptability graph G lead to a stable matching if and only if G is bipartite. (Abeledo–Isaak [1991]).

In the stable roommates problem with each individual declaring a top portion of the preference list as "acceptable", and the acceptability graph being the graph on the people whose edges are the mutually acceptable pairs, all sets of rankings with acceptability graph G allow stable matchings if and only if G is bipartite. If G is bipartite, then we view the two partite sets as the two groups in the classical stable matching problem (isolated vertices may be added to make the partite sets have equal size). The unacceptable choices for an individual x may be put in any order, since they are all (equally) unacceptable, so we can ensure that all choices for x that are in the same partite set appear at the bottom of the preference order for x. In the outcome of the Gale-Shapley Proposal Algorithm, there is no pair (x, a) from opposite partite sets such that a and x prefer each other to their assigned mates. Also no x prefers an individual in its own partite set to the person assigned to x, since all individuals in its own partite set are unacceptable. Hence the stable matching produced for the bipartite version is also stable in the original problem.

If G is not bipartite, then G has an odd cycle $[x_1, \ldots, x_k]$. Define a set of rankings such that x_i prefers x_{i+1} to x_{i-1} (indices modulo k), and x_i prefers x_{i-1} to all others. The preferences of people not on the cycle are irrelevant. Since the cycle has odd length, the people on the cycle cannot be paired up using edges of the cycle. Given a candidate matching M, we may assume by symmetry that x_1 is not matched to x_2 or to x_k in M. Now x_1 prefers x_2 to M-mate of x_1, and x_2 prefers x_1 to the M-mate of x_2 (which might be x_3). Hence the matching M is not stable. Thus there is no stable matching for these preferences, which means that this acceptability graph does not always permit a stable matching.

3.2.14. *In the Proposal Algorithm with men proposing, no man is every rejected by all the women.*

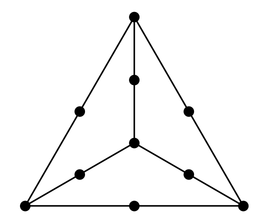
Proof 1. By Theorem 3.2.18, the Proposal Algorithm succeeds, so it ends with each men being accepted before being rejected by all women.

Proof 2. Once a woman has received a proposal, she thereafter receives a proposal on each round, since the key observation is her sequence of "maybe"s is nondecreasing in her list. If a round has j rejections and $n-j$ "maybe"s, then the $n-j$ unrejected men are distinct, since men propose to exactly one woman on each round.

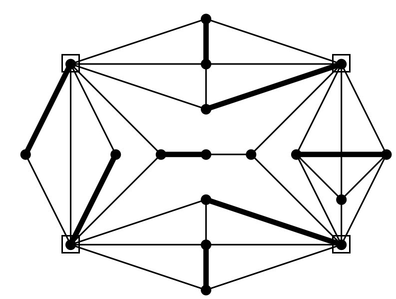
When a man has been rejected by k women, those k women have received proposals, and thereafter by the remarks above they always receive proposals from k distinct men. In particular, when a man has been rejected by $n-1$ women, on the next round they receive proposals from $n-1$ distinct men other than him, and he proposes to the remaining women, so the algorithm ends successfully on that step.

3.3. MATCHINGS IN GENERAL GRAPHS

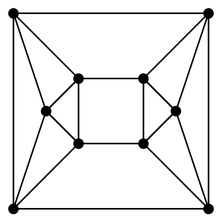
3.3.1. *The graph G below has no 1-factor.* Deleting the four vertices with degree 3 leaves six isolated vertices; thus $o(G-S) > |S|$ for this set S.



3.3.2. *The maximum size of a matching in the graph G below is 8.* A matching of size 8 is shown. Since $n(G) = 18$, it suffices to show that G has no perfect matching. For this we present a set S such that $o(G-S) > |S|$, violating Tutte's condition. Such a set S is marked. (Note: the smallest vertex cover has size 9, so duality using vertex cover is not adequate.)



3.3.3. *k-factors in the 4-regular graph below.* The full graph is a 4-factor, and the spanning subgraph with no edges is a 0-factor. There is a 2-factor consisting of the outer 4-cycle and the 6-cycle on the remaining vertices. Since these cycles have even length, taking alternating edges from both cycles yields a 1-factor. Deleting the edges of the 1-factor leaves a 3-factor.



3.3.4. *A k-regular bipartite graph is r-factorable if and only if r divides k.* The edges incident to a single vertex demonstrate necessity. For sufficiency, a k-regular bipartite graph has a perfect matching, and hence by induction on k is 1-factorable; take unions of the 1-factors in groups of r.

3.3.5. *Join of graphs G and H.* As long as G and H have at least one vertex each, $G \vee H$ is connected (it has $K_{n(G),n(H)}$ as a spanning subgraph).

In forming $G \vee H$, every vertex of G gains $n(H)$ neighbors in H, and every vertex in H gains $n(G)$ neighbors in G. Hence $\Delta(G \vee H) = \max\{\Delta(G) + n(H), \Delta(H) + n(G)\}$.

3.3.6. *A tree T has a perfect matching if and only if $o(T - v) = 1$ for every $v \in V(T)$.* *Necessity*. Let M be a perfect matching in T in which u is the vertex matched to v. Each component of $T - v$ not containing u must have a perfect matching and hence even order. The component containing u is matched by M except for u, so it has odd order.

Sufficiency. **Proof 1** (construction of matching). Suppose that $o(T - v) = 1$ for all $v \in V(T)$. Each vertex has a neighbor in one component of odd order. We claim that pairing each w to its neighbor in the odd component of $T - w$ yields a matching. It suffices to prove that if u is the neighbor of v in the unique odd component T_1 of $T - v$, then v is the neighbor of u in the unique odd component T_2 of $T - u$. Since $o(T - v) = 1$, the components of $T - v$ other than T_1 have even order. The subtree T_2 consists of these components and edges from these to v. Hence T_2 includes some even vertex sets and $\{v\}$, and T_2 thus has odd order.

Proof 2 (induction on $n(T)$). The claim is immediate for $n(T) = 2$. If $n(T) > 2$ and $o(T - v) = 1$ for all v, then the neighbor w of any leaf u has only one leaf neighbor. Let $T' = T - \{u, w\}$. The components of $T' - v$ are the same as the components of $T - v$, except that one of them in $T - v$ includes $\{u, w\}$ and the corresponding component of $T' - v$ omits them. Hence the parities are the same, and $o(T' - v) = 1$ for all $v \in V(T')$. By the induction hypothesis, T' has a perfect matching, and adding the edge uw to this completes a perfect matching in T.

(Comment: It is also possible to do the induction step by deleting an arbitrary vertex, but it is then a bit more involved to prove that every

component T' of the forest left by matching v to its neighbor in the odd component of $T - v$ satisfies the condition $o(T' - x) = 1$ for all x.

Proof 2a (induction and extremality). The basis again is $n(T) = 2$. For $n(T) > 2$, let P be a longest path. Let x be an endpoint of P, with neighbor y. Since $o(T - y) = 1$ and P is a longest path, $d_T(y) = 2$. Deleting x and y yields a tree T' such that $o(T' - v) = o(T - v) = 1$ for all $v \in V(T')$, since x and y lie in the same component of $T' - v$. Hence the induction hypothesis yields a perfect matching in T', which combines with xy to form a perfect matching in T.

Proof 3 (Tutte's Condition). By Tutte's Theorem, it suffices to prove for all $S \subseteq V(T)$ that $o(T - S) \le |S|$. We prove this by induction on $|S|$. Since $o(T - v) = 1$, we have $n(T)$ even, and hence $o(T - \varnothing) = 0$. When $|S| = 1$, the hypothesis $o(T - v) = 1$ yields the desired inequality for $S = \{v\}$.

For the induction step, suppose that $|S| > 1$. Let T' be the smallest subtree of T that contains all of S. Note that all leaves of T' are elements of S. Let v be a leaf of T', and let $S' = S - \{v\}$. By the induction hypothesis, $o(T - S') \le |S'| = |S| - 1$. It suffices to show that when we delete v from $T - S'$, the number of odd components increases by at most 1.

Let T'' be the component of $T - S'$ containing v. Deleting v from $T - S'$ replaces T'' with the components of $T'' - v$. We worry only if $T'' - v$ has at least two odd components. Since v is a leaf of T', all of S' lies in one component of $T - v$. Hence the components of $T'' - v$ are the same as the components of $T - v$ except for the one component of $T - v$ containing S'.

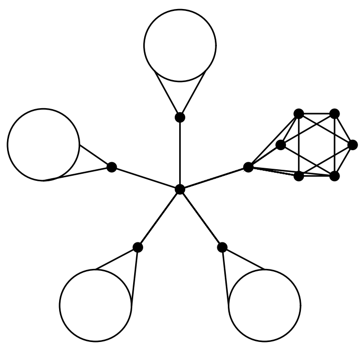
Since $o(T - v) = 1$, we can have two odd components in $T'' - v$ only if the one odd component of $T - v$ is a component of $T'' - v$ and the component of $T'' - v$ that is not a component of $T - v$ is also odd. Since the remaining components of $T'' - v$ are even, this means that T'' itself has odd order (it includes v and the two odd components of $T'' - v$). Therefore, the replacement of T'' with the components of $T'' - v$ increases the number of odd components only by one. We conclude that $o(T - S) \le o(T - S') + 1 \le |S'| + 1 = |S|$, which completes the induction step.

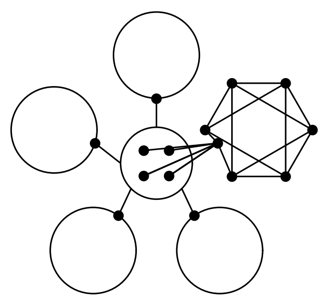
3.3.7. *There exist k-regular simple graphs with no perfect matching.* When k is even, K_{k+1} is a k-regular graph with no perfect matching, since it has an odd number of vertices. When k is odd, there are two usual types of constructions.

Construction 1. Begin with k disjoint copies of K_{k+1}. Delete $(k-1)/2$ disjoint edges from each copy, which drops the degree of $k - 1$ vertices in each copy to $k - 1$. Add a new vertex v_i to the ith copy, joining it to each of these vertices of degree $k - 1$. Add one final vertex x joined to $v_1, \dots, v_k$. The graph has been constructed to be k-regular. Deleting x leaves k components of order $k + 2$ (odd); hence the graph fails Tutte's condition and has no perfect matching.

A slight variation is to start with k copies of $K_{k-1,k}$, add a matching of size $(k-1)/2$ to the larger side in each copy, and join the leftover vertices from each larger side to a final vertex x.

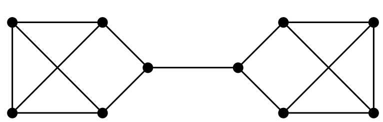
Construction 2. Begin with k disjoint copies of K_{k+1}. Subdivide one edge in each copy, which introduces k new vertices of degree 2. To raise their degree to k, add an independent set of $k-2$ additional vertices in the center joined to each of these k vertices. Deleting the $k-2$ vertices in the center violates Tutte's condition.





3.3.8. *No graph with a cut-vertex is 1-factorable.* Suppose v is a cut-vertex of G. If G is 1-factorable, then G has even order, and $G - v$ has a component H of odd order. For any 1-factor using an edge incident to v whose other endpoint is not in H, the vertices of H cannot all be matched. The contradiction implies there is no 1-factorization.

A 3-regular simple graph having a 1-factor and connectivity 1.



3.3.9. *Every graph G with no isolated vertices has a matching of size at least $n(G)/(1+\Delta(G))$.* We use induction on the number of edges. In the induction step, we will delete an edge whose endpoints have degree at least 2 (other edge deletions would isolate a vertex). This tells us what we need to cover in the basis step.

Basis step: every edge of G is incident to a vertex of degree 1. In such a graph, every component has at most one vertex of degree exceeding 1, and thus each component is a star. We form a matching using one edge from each component. Since the number of vertices in each component is 1

plus the degree of the central vertex, the number of components is at least $n(G)/(1+\Delta(G))$.

Induction step: G has an edge e whose endpoints have degree at least 2. Since $G' = G - e$ has no isolated vertex, we can apply the induction hypothesis to obtain $\alpha'(G) \ge \alpha'(G') \ge n(G')/(1+\Delta(G')) \ge n(G)/(1+\Delta(G))$.

3.3.10. *The maximum possible value of $\beta(G)$ in terms of $\alpha'(G)$ is $2\alpha'(G)$.* If G has a maximal matching of size k, then the $2k$ endpoints of these edges form a set of vertices covering the edges, because any uncovered edge could be added to the matching. Hence $\beta(G) \le 2\alpha'(G)$. A graph consisting of k disjoint triangles has $\alpha' = k$ and $\beta = 2k$, so the inequality is best possible. These values also hold for the graph K_{2k+1}, since we cannot omit two vertices from a vertex cover of K_{2k+1}. More generally, every disjoint union of cliques of odd order satisfies $\beta(G) = 2\alpha'(G)$.

3.3.11. *A graph G has a matching that saturates a set $T \subseteq V(G)$ if and only if for all $S \subseteq V(G)$, the number of odd components of $G - S$ contained in $G[T]$ is at most $|S|$.*

Necessity. Saturating T requires saturating each vertex in the odd components of $G[T]$, which uses a vertex of S for each such component.

Sufficiency. Form G' by adding a set U of $n(G)$ new vertices adjacent to each other and to every vertex of $G - T$. We claim that G' satisfies Tutte's Condition. Each $S' \subseteq V(G')$ that contains all of U has size at least $n(G)$. Since $G' - S'$ has at most $n(G)$ vertices, it has at most $|S'|$ odd components.

When $U \not\subseteq S'$, what remains of G' outside of T is a single component. Letting $S = V(G) \cap S'$, the number of odd components in $G' - S'$ is thus at most one more than the number of odd components of $G - S$ contained in T. This yields $o(G' - S') \le |S'| + 1$, which suffices since $n(G')$ is even.

Since G' satisfies Tutte's Condition, G' has a perfect matching. The edges used to saturate T all lie in G, since no edges were added from T to U. Hence these edges form a matching in G that saturates T.

3.3.12. *Extension of König–Egerváry to general graphs.* A *generalized cover* of G is a collection of vertex subsets $S_1, \ldots, S_k$ and T such that each S_i has odd size and every edge of G has one endpoint in T or both endpoints in some S_i. The *weight* of a generalized cover is $|T| + \sum(|S_i| - 1)/2$.

The minimum weight $\beta^*(G)$ of a generalized cover equals the maximum size $\alpha'(G)$ of a matching. Always $\alpha'(G) \le \beta^*(G)$, because a matching uses at most $(|S_i|-1)/2$ edges within S_i and at most $|T|$ edges incident to T, and there are no edges not of this type when $S_1, \ldots, S_k$ and T form a generalized cover. For equality, it suffices to exhibit a generalized cover with weight equal to $\alpha'(G)$.

The Berge–Tutte formula says that $2\alpha'(G) = \min_T\{n - d(T)\}$, where $d(T) = o(G-T) - |T|$ is the deficiency of a vertex set T. Let T be a *maximal* set among those having maximum deficiency. For this choice of T, there are no components of (positive) even order in $G-T$, since we could add to T a leaf of a spanning tree of such a component to obtain a larger set T' with the same deficiency. Let $S_1, \ldots, S_k$ be the vertex sets of the components of $G-T$. By construction, this is a generalized cover. Because $2\alpha'(G) = n - d(T)$, we have $k = |T| + d(T)$. Thus

$$\beta^*(G) = |T| + \sum_{i=1}^{k}(|S_i| - 1)/2 = (n + |T| - k)/2 = \alpha'(G).$$

3.3.13. *Proof of Tutte's Theorem from Hall's Theorem.* Given a graph G such that $o(G-S) \le |S|$ for all $S \subseteq V(G)$, we prove that G has a perfect matching. Let T be a maximal vertex subset such that $o(G-T) = |T|$.

a) Every component of $G-T$ is odd, and $T \ne \varnothing$. If $G-T$ has an even component C, then let v be a leaf of a spanning tree of C. Now $|T \cup \{v\}| = |T| + 1 = o(G-T) + 1 = o(G-(T\cup\{v\}))$, which contradicts the maximality of T. Thus $G-T$ has no even components.

Since $o(G-\varnothing) \le 0$, the graph G has no odd components. Since $G-T$ has no even components, we have $|T| > 0$, and $G-T$ is smaller than G.

b) If C is a component of $G-T$, then Tutte's Condition holds for every subgraph of C obtained by deleting one vertex. Since $C-x$ has even order, a violation requires $o(C-x-S) \ge |S| + 2$. Adding this inequality to $|T| = o(G-T)$ and $|\{x\}| = 1$ yields

$$|T \cup x \cup S| \le o(G-T) - 1 + o(C-x-S) = o(G-T-x-S),$$

which contradicts the maximality of T.

c) The bipartite graph H formed from G by contracting the components of $G-T$ (and deleting edges within T) satisfies Hall's Condition for a matching that saturates the partite set opposite T. There is an edge from a vertex $t \in T$ to a component C of $G-T$ if and only if $N_G(t)$ contains a vertex of C. For $A \subset \mathbf{C}$, let $B = N_H(A)$. The elements of A are odd components of $G-B$; hence $|A| \le o(G-B)$. Since Tutte's condition yields $o(G-B) \le |B|$, we have $|N_H(A)| \ge |A|$.

d) The final proof. By Hall's Theorem and part (c), H has a matching that saturates $\mathbf{C}$. This matching yields $o(G-T) = |T|$ pairwise disjoint edges from odd components of $G-T$ to T. By part (a), these are all the components of $G-T$ These edges saturates one vertex from each component of $G-T$. By part (c) and the induction hypothesis, the vertices remaining in each component of $G-T$ are saturated by a perfect matching of that subgraph. The union of the matchings created is a perfect matching of G.

3.3.14. *If G is a simple graph with $\delta(G) \ge k$ and $n(G) \ge 2k$, then $\alpha'(G) \ge k$.* Let $n = n(G)$. By the Berge–Tutte Formula, it suffices to show that the deficiency $o(G-S) - |S|$ is at most $n - 2k$ for every $S \subseteq V(G)$. We prove this by contradiction; suppose that $o(G-S) - |S| > n - 2k$.

Let $s = |S|$. We have $o(G-S) > n - 2k + s$. Thus there are more than $n - 2k + s$ vertices outside S. Together with S, we have $n > n - 2k + 2s$. Thus $s < k$. With $s < k$, a vertex outside S has fewer than k neighbors in S, and $\delta(G) \ge k$ implies that no odd components of $G-S$ are single vertices.

Indeed, every component of $G-S$ has at least $1 + k - s$ vertices. Thus we can improve our earlier inequality: $(1+k-s)(n-2k+s+1) + s \le n$. This simplifies to $(k-s)(n-2k+s-1) < 0$. Since $n \ge 2k$, both factors on the left are positive, which yields a contradiction.

3.3.15. *Every 3-regular graph G with at most two cut-edges has a 1-factor.* Since G has at most two cut-edges, at most two odd components of $G-S$ have one edge to S; the remainder have at least three edges to S (using the parity of degrees). With $\left|[S, \overline{S}]\right| = m$, this yields $3|S| \ge m \ge 3o(G-S) - 4$. Thus $|S| \ge o(G-S) - 4/3$. Since $n(G)$ is even, $|S|$ and $o(G-S)$ have the same parity, which means that $o(G-S)$ exceeds $|S|$ only if is greater by at least 2. This contradicts $o(G-S) \le |S| + 4/3$. Hence Tutte's condition holds, and Tutte's Theorem implies that G has a 1-factor.

3.3.16. *If G is k-regular and remains connected when any $k-2$ edges are deleted, then G has a 1-factor.* By Tutte's Theorem, it suffices to show that $o(G-S) \le |S|$ for every $S \subseteq V(G)$. This follows for $S = \varnothing$ from the assumption that $n(G)$ is even; hence we may assume that $S \ne \varnothing$. Let H be an odd component of $G-S$, and let m be the number of edges joining H to S. In the subgraph H, the sum of the degrees is $kn(H) - m$. Since this must be even and $n(H)$ is odd, k and m must have the same parity.

By the hypothesis, there are at least $k-1$ edges between H and S. The requirement of equal parity thus yields $m \ge k$. Summing over all odd components of $G-S$ yields at least $k \cdot o(G-S)$ edges between S and $V(G) - S$. Since the degree sum of the vertices in S is exactly $k|S|$, we obtain $k \cdot o(G-S) \le k|S|$, or $o(G-S) \le |S|$.

3.3.17. *Under the conditions of Exercise 3.3.16, each edge belongs to some 1-factor in G.* We want to show that $G' = G-x-y$ has a 1-factor. By Tutte's Theorem, since G has even order, it suffices to show that $o(G'-S') \le \left|S'\right| + 1$ for all $S' \subset V(G')$. Equivalently, $o(G-S) \le |S| - 1$ for all $S \subseteq V(G)$ that contain $\{x, y\}$.

Let l be the number of edges between S and an odd component H of $G-S$; the hypothesis yields $l \ge k-1$. The sum $kn(H) - m$ of the vertex degrees in H must be even, but $n(H)$ is odd, so k and m must have the same parity; we conclude that $l \ge k$. Summing over all odd components of $G-S$, we have $m \ge k \cdot o(G-S)$, where m is the number of edges between S and the rest of the graph. Since G is k-regular and $G[S]$ contains the edge xy, we have $m \le k|S| - 2$. Thus $o(G-S) \le |S| - 2/k$. Since $o(G-S)$ and $|S|$ are integers, we have the needed inequality.

3.3.18. *Construction of a k-regular graph with no 1-factor (when k is odd), such that deleting any $k-3$ edges leaves a connected graph.* We make the graph simple and connected under the deletion of any $k-3$ vertices, which is a stronger requirement.

The Tutte set S will have size $k-2$, leaving k components in $G-S$. Each component of $G-S$ consists of $K_{k-2,k-1}$ plus a cycle added through the vertices in the larger partite set. This gives those vertices degree k. Add a matching from the vertices in the smaller partite set to S. Now G is k-regular and has a Tutte set, so it has no 1-factor.

When any $k-3$ vertices are deleted to form G', what remains in each component H of $G-S$ is connected, due to the spanning biclique. Also some edge from H to S remains. If exactly one vertex of S remains, then G' is now connected. If more than one vertex of S remains, then any two are connected in G' by a path through some component of $G-S$.

3.3.19. *Every 3-regular simple graph with no cut-edge decomposes into copies of P_4 (the 4-vertex path).* By Petersen's Theorem, a 3-regular simple graph G with no cut-edge has a 1-factor M. Deleting the edges of M from G leaves a 2-factor H, which is a disjoint union of cycles. Choose an orientation D for the 2-factor so that each vertex has one edge in and one edge out (that is, make the components of the 2-factor into directed cycles).

We let each edge of M be the central edge in a copy of P_4. The other two edges in the copy of P_4 containing $uv \in M$ are the edges leaving u and v in D; let these be uw and vz. These three edges form P_4 if $w \neq z$. We have $w \neq z$ because each vertex has only one entering edge in D.

The central edges of these P_4's are precisely the $n(G)/2$ edges of M. Each edge of H appears in exactly one of the constructed P_4's, since every edge outside M follows exactly one vertex in D. Thus the copies of P_4 formed in this way are pairwise edge-disjoint and cover $E(G)$.

3.3.20. *A 3-regular simple graph G has a 1-factor if and only if it decomposes into copies of P_4.*

Necessity. Deleting the edges of a 1-factor M from G leaves a 2-factor H, which is a disjoint union of cycles. Choose an orientation D for the 2-factor by choosing a consistent orientation around each cycle.

Let each edge of M be the central edge in a copy of P_4. The other two edges in the copy of P_4 containing $uv \in M$ are the edges leaving u and v in D; let these be uw and vz. These three edges form P_4 when $w \neq z$, which holds since each vertex has only one entering edge in D.

The central edges of these P_4s are the $n(G)/2$ edges of M. Each edge of the 2-factor also appears in exactly one constructed P_4, since each such edge is the tail of exactly one vertex in D. Thus these copies of P_4 are pairwise edge-disjoint and cover $E(G)$.

Sufficiency. A P_4-decomposition of a 3-regular graph G has $n(G)/2$ subgraphs, since $e(G) = 3n(G)/2$ and $e(P_4) = 3$. No edge-disjoint copies of P_4 have a common internal vertex v, since that would give v degree at least 4. Hence the middle edges in the subgraphs of the decomposition form a matching of size $n(G)/2$ and hence a 1-factor.

3.3.21. *If G is a $2m$-regular graph, and T is a tree with m edges and diameter less than the girth of G, then G decomposes into copies of T.* We prove a stronger result. Consider an arbitrary labeling of $V(T)$ with $\{1, \dots, m+1\}$. We prove by induction on m that G has a T-decomposition such that each vertex of G appears in $m+1$ copies of T, once with each label. Call this a *labeled T-decomposition* of G. The trivial necessary degree conditions are satisfied because the sum of the vertex degrees in T is $2m$. There will be $n(G)$ copies of T, independent of the value of m.

For $m = 0$ (or $m = 1$), the claim is immediate. For $m \geq 1$, let H be a 2-factor of G, and let i be a leaf of T, with neighbor j. Note that the distance in T from j to any other vertex of T is at most $\mathrm{diam}\,(T) - 1$. The induction hypothesis guarantees a labeled $T-i$-decomposition of $G - E(H)$. For each vertex w in each cycle in H, we add to the copy of $T-i$ with j at w by adding the edge to the next vertex in the cycle, which will then receive label i. This vertex does not already appear in this copy of $T-i$, because the girth of G exceeds the diameter of T.

3.3.22. *Hall's Theorem follows from Tutte's Theorem.* Given an X,Y-bigraph G, let H be the graph obtained from G by adding one vertex to Y if $n(G)$ is odd and then adding edges to turn Y into a clique.

a) G has a matching of size $|X|$ if and only if H has a 1-factor. Each edge of a matching in G has one vertex of X and one vertex of Y. Since $H[Y]$ is a clique, we can pair the remaining vertices arbitrarily to obtain a 1-factor in H from a matching of size $|X|$ in G. Conversely, if H has a 1-factor, it must use $|X|$ edges to saturate X, since $H[X]$ is an independent set. These edges from the desired matching in G.

b) If G satisfies Hall's Condition ($|N(S)| \geq |S|$ for all $S \subseteq X$), then H satisfies Tutte's Condition ($o(H-T) \leq |T|$ for all $T \subseteq V(H)$). Since $H[Y \cap T]$ is a clique, the odd components obtained by deleting T are the vertices of X whose neighbors all lie in T and perhaps the one large remaining component. Let $S = \{x \in X \colon N(x) \subseteq T \cap Y\}$. Since G satisfies Hall's Condition, $|S| \leq |T \cap Y| \leq |T|$. Thus $o(H-T) \leq |T| + 1$. Since $n(H)$ is even, $o(H-T)$ and $|T|$ have the same parity, and thus $o(H-T) \leq |T|$. Thus H satisfies Tutte's Condition.

c) Tutte's Theorem implies Hall's Theorem. The necessity of Hall's condition is immediate (any subset of X must have as many neighbors as elements to be completely matched). For sufficiency as a consequence of

Tutte's Theorem, we form H from G as described above. Since G satisfies Hall's Condition, part (b) implies that H satisfies Tutte's Condition. Tutte's Theorem now implies that H has a 1-factor. Part (a) now implies that G has a matching saturating X.

3.3.23. Consider a connected claw-free graph G of even order.

a) In a spanning tree of G generated by Breadth-First Search, any two vertices with a common parent other than the root are adjacent. Let r be the root, and let u, v be children of s, with $s \neq r$. In a Breadth-First Search tree, the path from the root to any vertex is a shortest such path. We have $d(u,r) = d(v,r) = d(s,r)+1$. An edge from u or v to the parent t of s would establish a shorter path to the root of length $d(s,r)$. Hence there is no such edge. To avoid having s, t, u, v induce a claw, u and v must be adjacent.

b) G has a 1-factor. We use induction on $n(G)$. Basis step ($n(G) = 2$): G must be K_2.

Induction step ($n(G) > 2$): Let T be a Breadth-First Search tree from r. Let u be a vertex at maximum distance from r, and let s be the parent of v. If s has no other child, then $T' = T - \{u, s\}$ is connected. if s has another child v, then let $T' = T - \{u, v\}$.

In each case, T' is connected, so $G'[V(T')]$ is a smaller connected claw-free graph of even order. The induction hypothesis guarantees a perfect matching in G'. To this matching we add the edge between the two vertices we deleted to obtain G' (in the first case, the edge us exists because s is the parent of u; in the second case, the edge uv exists by part (a).)

3.3.24. *Maximum number of edges with no 1-factor. A maximal n-vertex graph with no 1-factor consists of m vertices of degree $n-1$, with the remaining $n-m$ vertices inducing a union of $m+2$ cliques of odd order.* Since adding an edge cannot reduce α' or increase it by more than one, we may assume that $\alpha'(G) = n/2 - 1$. Hence $\max_{S\subseteq V}\{o(G-S) - |S|\} = 2$; the maximum matching omits 2 vertices. Let S be a set achieving equality, so $o(G-S) = |S|+2$. Each component of $G-S$ must induce a clique, $G-S$ has no component of even order (else add edges from even to odd components), and vertices of S have degree $n-1$, all because adding the edges that would be missing if any of these failed would not reduce the deficiency of S. This completely describes the maximal graphs.

The maximum number of edges in a graph with minimum degree $k < n/6 - 2$ and no 1-factor is $\binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$. We assume n is even. Let G be a maximal n-vertex graph with no 1-factor. Let $f(k) = \binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$. We first observe that there is a maximal graph having no 1-factor that has $f(k)$ edges and minimum degree k; the graph is $K_k \vee ((k+1)K_1 + K_{n-2k-1})$. This is an example of the structure above with $m = k$. This construction is valid when $n - 2k - 1 > 0$, which requires only

$k \le n/2 - 1$. Nevertheless, $f(k)$ is not always the maximum size of a graph H with minimum degree k and no 1-factor. For $n \ge 8$, we can build counterexamples when $k < n/2 - 1$ and k is at least $n/6$ (approximately). The smallest counterexample occurs when $n = 8$ and $k = 2$. We have $f(2) = 16$ and $f(3) = 18$. We obtain a graph with minimum degree 2, no 1-factor, and 17 edges by deleting one edge from $K_3 \vee (5K_1)$.

Suppose H has a vertex of degree $k < n/6-2$ and has no perfect matching; we obtain an upper bound on $e(H)$. Augment H by adding edges to obtain a maximal supergraph G having no 1-factor; note that $\delta(G) = l \ge k$. By direct computation, $f(t+1) - f(t) = 3t - n + 4$. If $t \ge n/2$, then there is no graph with minimum degree at least t that has no 1-factor. Since $f(t)$ is a parabola centered at $t = (n-4)/3$ and $\delta(G) < n/2$, we have $f(k) \ge f(l)$ if $k < n/6 - 2$. Therefore, it suffices to prove that if $\delta(G) = k$ and G has the form described above, then $e(G) \le f(k)$.

If v does not have degree $n-1$, then v belongs to a clique of size $d(v) - m + 1$ in $G-S$, which is odd. If any two components of $G-S$ have sizes $p \ge q \ge k - m + 3$, then we gain $2p$ and lose $2q - 4$ edges by moving two vertices from the smaller to the larger clique in $G-S$, still maintaining the same minimum degree. Hence for fixed m, $e(G)$ is maximized by using cliques of size $k-m+1$ for all but one component of $G-S$. Now the degree sum in G is $m(n-1)+(k-m+1)(m+1)k+[n-(k-m+1)(m+1)][n-(k-m+1)(m+1)]$. If $m < k/2$, then replacing m by $k-m$ increases this, since only the first term changes. Hence we may assume $m \ge k/2$. If $k > m \ge k/2 - 1$, then we can increase m by moving two vertices from a small clique to S, since $2(m+1) \ge 2(k-m-1)$ guarantees that we can then take vertices from the other small components to make new components of size $k-m-1$. This increases the number of edges (computation omitted), so we may assume $m = k$. Now $e(G) = f(k)$.

3.3.25. *A graph G is factor-critical if and only if $n(G)$ is odd and $o(G-S) \le |S|$ for all nonempty $S \subseteq V(G)$. Necessity.* Factor-critical graphs are those where every subgraph obtained by deleting one vertex has a 1-factor. Thus factor-critical graphs have odd order. Given a nonempty subset S of $V(G)$, let v be a member of S, and let $G' = G - v$. Since G' has a 1-factor and $G - S = G' - S'$, we have $o(G-S) = o(G'-S') \le |S'| = |S| - 1$. Thus the desired inequality holds for S.

Sufficiency. Suppose that $n(G)$ is odd and $o(G-S) \le |S|$ for all nonempty $S \subseteq V(G)$. Given a vertex $v \in V(G)$, let $G' = G - v$. Consider $S' \subseteq V(G')$, and let $S = S' \cup \{v\}$. Since G has odd order, the quantities $|S|$ and $o(G-S)$ have different parity. The hypothesis thus yields $o(G-S) \le |S| - 1$. Since $G' - S' = G - S$, we have $o(G'-S') = o(G-S) \le |S| - 1 = |S'|$. By Tutte's Theorem, $G - v$ has a 1-factor.

3.3.26. *If M is a matching in a graph G, and u is an M-unsaturated vertex, and G has no M-augmenting path that starts at u, then u is unsaturated in some maximum matching in G.* We use induction on the difference between $|M|$ and $\alpha'(G)$. If the difference is 0, then already u is unsaturated in some maximum matching. If M is not a maximum matching, then there is an M-augmenting path P. Since u is not an endpoint of P, then $M \triangle E(P)$ is a larger matching M' that does not saturate u.

If no M'-augmenting path starts at u, then by the induction hypothesis u is unsaturated in some maximum matching. Suppose that an M'-augmenting path P' starts at u. Since no M-augmenting path starts at u, P' shares an edge with P. Form an M-alternating path by following P' from u until it first reaches a vertex of P; this uses no edges of $M' - M$. Then follow P to the end in whichever direction continues the M-alternating path. Since both endpoints of P are M-unsaturated, this completes an M-augmenting path starting at u, which contradicts the hypothesis.

3.3.27. *Proof of Tutte's 1-Factor Theorem from correctness of the Blosson Algorithm.*

a) If G has no perfect matching, and M is a maximum matching in G, and S and T are the sets generated when running the Blossom Algorithm from u, then $|T| < |S| \le o(G - T)$. Since M is a maximum matching, no M-augmenting path is found. At the start of the algorithm, $|T| = 0 < 1 = |S|$. As the algorithm proceeds, exploring a vertex $v \in S$ leads to consideration of a neighbor y of v that is not in T. If y is unsaturated in M, then an M-augmenting path is found; this case is forbidden by hypothesis. If y is saturated but not yet in S, then y is added to T and its mate is added to S. This augments both T and S by one element, maintaining $|S| = |T| + 1$.

Finally, if y is saturated and lies in S, then a blossom is established. Every matched edge in the blossom has one vertex of T and one vertex of S, and the blossom is shrunk into the single vertex of S on it that is not matched to another vertex along the blossom. Hence the "current" S and T shrink by the same number of vertices. Therefore, we always maintain $|S| = |T| + 1$.

Now consider the second inequality. When the algorithm ends, the current S is an independent set with no edges to vertices that have not been reached, because if an edge has been found between vertices of S, then a blossom would have been shrunk, and an edge to an unreached vertex would have been explored. Hence deleting T leaves at least $|S|$ odd components. (There may be more among the unreached vertices; the algorithm does not explore unsaturated edges from T.)

These isolated vertices in the final S correspond to odd components of $G - T$ in the original graph, because shrinking of a blossom loses an even number of vertices. Since all edges leaving the blossom are explored, the final T disconnects everything in the blossom from what is outside of it. Since also the blossom corresponds to an odd number of vertices in the original graph, deleting T from the original graph must leave an odd component among these vertices. One such set exists for each vertex in the final S (at the end of the algorithm).

b) Proof of Tutte's 1-Factor Theorem. If $o(G - T) \le |T|$ for every vertex subset T, then the algorithm cannot end by finding a set T such that $|T| < |S| \le o(G - T)$. Hence it can only end by finding an augmentation, and this continues until a 1-factor is found.

3.3.28. *a) Reduction of the f-factor problem to the f-solubility problem.* It suffices to prove that G has an f-factor if and only if the graph H obtained by replacing each edge by a path of length 3 is f'-soluble, where f' is the extension of f obtained by defining f' to equal 1 on all the new vertices.

Suppose that x, a, b, y is the path in H representing the edge xy in G. If G has an f-factor, then H is f'-soluble by giving weights $1, 0, 1$ or $0, 1, 0$ to the successive edges on the path depending on whether xy is or is not in the f-factor.

Conversely, if H is f'-soluble, then because every edge of H is incident to a vertex with $f' = 1$, every edge is used with weight 1 or 0, and the weights along the 3-edge path representing an edge xy must be $1, 0, 1$ or $0, 1, 0$. At a vertex x there must be exactly $f(x)$ paths of the first type, so we obtain an f-factor of G by using the edges corresponding to these paths.

b) Reduction of f-solubility to 1-factor. Let H be the graph formed from G by replacing each vertex $v \in V(G)$ with an independent set of $f(v)$ vertices. Now G is f-soluble if and only if H has a 1-factor; collapsing or expanding the vertices turns the solution of one problem into the solution of the other.

3.3.29. *Tutte's f-factor condition and graphic sequences.* For disjoint sets Q, T, let $e(Q, T)$ denote the number of edges from Q to T. For a function h defined on $V(G)$, let $h(S) = \sum_{v \in S} h(v)$ for $S \subseteq V(G)$.

For $f \colon V(G) \to \mathbb{N}_0$, Tutte proved that G has an f-factor if and only if

$$q(S, T) + f(T) - d_{G-S}(T) \le f(S)$$

for all choices of disjoint subsets $S, T \subseteq V(G)$, where $q(S, T)$ is the number of components Q of $G - S - T$ such that $e(Q, T) + f(V(Q))$ is odd. *a)* **The Parity Lemma**. *The quantity $\delta(S, T)$ has the same parity as $f(V)$ for disjoint sets $S, T \subseteq V(G)$, where $\delta(S, T) = f(S) - f(T) + d_{G-S}(T) - q(S, T)$.* We use the observation that the parity of the number of odd values in a set of integers equals the parity of the sum of the set.

We fix S and use induction on $|T|$. When $T = \varnothing$, we have $f(T) - d_{G-S}(T) = 0$. Also $m(Q, T) = 0$ for each component Q of $G - S - T$, so we can sum over the components to obtain $q(S, \varnothing) \equiv f(\overline{S}) \pmod 2$. Signs don't matter, so $\delta(S, \varnothing) \equiv f(\overline{S}) - f(S) \equiv f(V(G)) \pmod 2$.

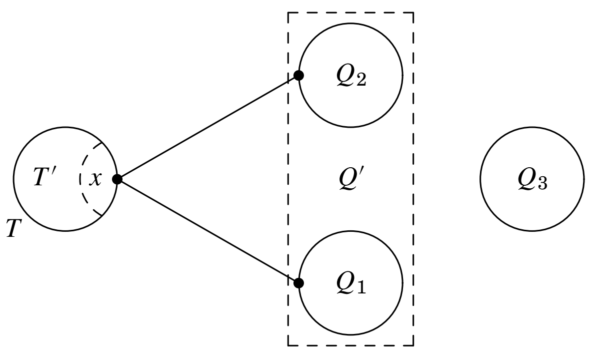
For $T \neq \varnothing$, we compare $\delta(S, T)$ and $\delta(S, T - x)$; it suffices to show that the difference is even. Let $T' = T - x$. In computing the difference, the contributions of $-f(S)$ cancel, as do the sums over T'. This leaves

$$\delta(S, T) - \delta(S, T') = q(S, T) - q(S, T') + f(x) - d_{G-S}(x).$$

The contributions to $q(S, T)$ and $q(S, T')$ from components of $G - S - T$ having no neighbors of x also cancel. The components having neighbors of x combine with x to form one large component in $G - S - T'$ with vertex set $Q' \cup \{x\}$. Our initial observation about parity yields

$$q(S, T) - q(S, T') \equiv f(Q') + m(Q', T) - [f(Q' \cup \{x\}) + m(Q' \cup \{x\}, T')] \pmod 2.$$

The edges from x to Q' count in $m(Q', T)$, the edges from x to T' count in $m(Q' \cup \{x\}, T')$, and the edges from Q' to T' count in both. Thus $m(Q', T) - m(Q' \cup \{x\}, T') \equiv d_{G-S}(x) \pmod 2$, and we have $q(S, T) - q(S, T') \equiv f(x) + d_{G-S}(x) \pmod 2$. This has the same parity as the rest of the difference, yielding $\delta(S, T) - \delta(S, T') \equiv 0 \pmod 2$.



b) Let $d_1, \ldots, d_n$ be nonnegative integers with $\sum d_i$ even and $d_1 \geq \cdots \geq d_n$. If $G = K_n$ and $f(v_i) = d_i$, then G has an f-factor if and only if $\sum_{i=1}^{k} d_i \leq (n - 1 - s)k + \sum_{i=n+1-s}^{n} d_i$ for all k, s with $k + s \leq n$. Such an f-factor exists if and only if $f(V) = \sum d_i$ is even and Tutte's condition holds. Since $d_{K_n - S}(v) = n - 1 - |S|$ for all $v \in T$, the f-factor condition requires that $f(T) \leq f(S) + (n - 1 - |S|)\,|T| - q(S, T)$ for any disjoint sets $S, T \subseteq V(G)$. With $|T| = k$ and $|S| = s$, the condition becomes $f(T) \leq f(S) + (n - 1 - s)k - q(S, T)$.

Necessity. Applied with the k vertices of largest degree in T and the s vertices of smallest degree in S and using $q(S, T) \geq 0$, the f-factor condition yields the desired inequality.

Sufficiency. It suffices to establish inequality shown above to be equivalent to the f-factor condition. Since $K_n - S - T$ is connected, always $q(S, T) \leq 1$. Since $\sum f(v_i)$ is even, the two sides of the inequality have the same parity, by the Parity Lemma. It therefore suffices to prove that $f(T) \leq f(S) + (n - 1 - s)k$ when S and T are disjoint. With $s = |S|$ and $k = |T|$, it suffices to prove the inquality when T consists of the k vertices of largest degree and S consists of the s vertices of smallest degree. It then becomes the given inequality for $d_1, \ldots, d_n$.

c) Nonnegative integers $d_1, \ldots, d_n$ with $d_1 \geq \cdots \geq d_n$ are the vertex degrees of a simple graph if and only if $\sum d_i$ is even and $\sum_{i=1}^{k} d_i \leq k(k - 1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for $1 \leq k \leq n$. Any realization can be viewed as an f-factor of K_n, where $f(v_i) = d_i$. Thus it suffice to show that this condition is equivalent to the condition in part (b).

For fixed $|S|, |T|$, this inequality is always satisfied if and only if it is satisfied when $T = \{x_i, \ldots, x_k\}$ and $S = \{x_{n+1-s}, \ldots, x_n\}$, in which case it becomes $\sum_{i=1}^{k} f_i \leq \sum_{i=n+1-s}^{n} f_i + (n - 1 - s)k = (n - 1)k + \sum_{i=n+1-s}^{n} (f_i - k)$. This is always satisfied if and only if it is satisfied when the right side attains its minimum over $0 \leq s \leq n - k$, which happens when $n + 1 - s = \min\{i\colon i > k \text{ and } f_i < k\}$. Since $(n - 1)k = k(k - 1) + (n - k)k$, the value is then $k(k - 1) + \sum_{i=k+1}^{n} \min\{k, f_i\}$.

4.CONNECTIVITY AND PATHS

4.1. CUTS AND CONNECTIVITY

4.1.1. *Statements about connectedness.*

a) Every graph with connectivity 4 is 2-connected—TRUE. If the minimum number of vertices whose deletion disconnects G is 4, then deletion of fewer than two vertices leaves G connected. Also, K_5 is 2-connected.

b) Every 3-connected graph has connectivity 3—FALSE. Every graph with connectivity greater than 3, such as $K_{4,4}$, is 3-connected.

c) Every k-connected graph is k-edge-connected—TRUE. Always $\kappa'(G) \geq \kappa(G)$, which means that every disconnecting set of edges in a k-connected graph has size at least k.

d) Every k-edge-connected graph is k-connected—FALSE. The graph consisting of two $k+1$-cliques sharing a single vertex ($K_1 \vee 2K_k$) is k-edge-connected but not k-connected.

4.1.2. *If e is a cut-edge of G, then e contains a cut-vertex of G unless e is a component of G.* If e is a component of G, then the vertices of e are not cut-vertices; deleting a vertex of degree 1 cannot disconnect a graph. Otherwise, e has an endpoint v with degree greater than one; we claim v is a cut-vertex. Let u be the other endpoint of e, and let w be another neighbor of v. Then G has a u,w-path through v, but every u,w path in G uses e, so G has no u,w-path in $G-v$.

4.1.3. *If a simple graph G is not a complete graph and is not k-connected, then G has a separating set of size $k-1$.*

Proof 1 (verifying definition). Since G' arises by adding edges to G, it is connected. If G' has a cut-vertex v, then v is also a cut-vertex in G, since $G-v$ is a spanning subgraph of $G'-v$. Since neighbors of v in G are adjacent in G', they cannot be in different components of $G'-v$. Hence $G'-v$ has only one component.

Proof 2 (weak duality). Let x,y be vertices of G. Since G is connected, it has an x,y-path $v_0,\ldots,v_k$. In G', both $v_0,v_2,v_4,\ldots,v_k$ and $v_0,v_1,v_3,\ldots,v_k$ are x,y-paths, and they are internally disjoint. Thus at least two vertices must be deleted to separate x and y.

4.1.4. *A graph G is k-connected if and only if $G \vee K_r$ is $k+r$-connected.* To separate $G \vee K_r$, one must delete all of the added vertices, since they are adjacent to all vertices. Since deleting them leaves G, a set is a separating set in $G \vee K_r$ if and only if it contains the r vertices outside G and the remainder is a separating set in G. Thus the minimum size of a separating set in $G \vee K_r$ is r more than the minimum size of a separating set in G.

4.1.5. *If G' is obtained from a connected graph G by adding edges joining pairs of vertices whose distance in G is 2, then G' is 2-connected.*

Proof 1 (definition of 2-connected). Since G' is obtained by adding edges to G, G' is also connected. If G' has a cut-vertex v, then v is also a cut-vertex in G, since $G-v$ is a spanning subgraph of $G'-v$. By construction, neighbors of v in G are adjacent in G', and hence they cannot be in different components of $G'-v$. Hence $G'-v$ has only one component.

Proof 2 (weak duality). Let x,y be vertices of G. Since G is connected, it has an x,y-path $v_0,\ldots,v_k$. In G', both $v_0,v_2,v_4,\ldots,v_k$ and $v_0,v_1,v_3,\ldots,v_k$ are x,y-paths, and they are internally disjoint. Thus at least two vertices must be deleted to separate x and y.

Proof 3 (induction on $n(G)$). When $n(G)=3$, $G'=K_3$, which is 2-connected. When $n(G)>3$, let v be a leaf of a spanning tree in G. Since $G-v$ is connected and $G'-v=(G-v)'$, the induction hypothesis implies that $G'-v$ is 2-connected. Since v has at least two neighbors in G', the Expansion Lemma implies that G' also is 2-connected.

4.1.6. *A connected graph with blocks $B_1,\ldots,B_k$ has $\left(\sum_{i=1}^k n(B_i)\right)-k+1$ vertices.* We use induction on k. Basis step: $k=1$. A graph that is a single block B_1 has $n(B_1)$ vertices.

Induction step: $k>1$. When G is not 2-connected, there is a block B that contains only one of the cut-vertices; let this vertex be v, and index the blocks so that $B_k=B$. Let $G'=G-(V(B)-\{v\})$. The graph G' is connected and has blocks $B_1,\ldots,B_{k-1}$. By the induction hypothesis, $n(G')=\left(\sum_{i=1}^{k-1} n(B_i)\right)-(k-1)+1$. Since we deleted $n(B_k)-1$ vertices from G to obtain G', the number of vertices in G is as desired.

4.1.7. *The number of spanning trees of a connected graph is the product of the numbers of spanning trees of each of its blocks.* We use induction on k. Basis step: $k=1$. In a graph that is a single block, the spanning trees of the graph are the spanning trees of the block.

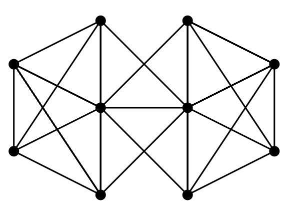
Induction step: $k>1$. Let v be a cut-vertex of G. The graph G is the union of two graphs G_1,G_2 that share only v. A subgraph is a spanning tree of G if and only if it is the union of a spanning tree in G_1 and a spanning tree in G_2. Since we can combine any spanning tree of G_1 with any spanning tree of G_2 to make a spanning tree of G, the number of spanning trees of G is the product of the number in G_1 and the number in G_2.

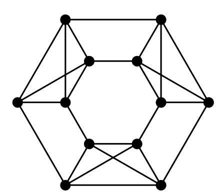
Also the blocks of G are the blocks of G_1 together with the blocks of G_2. Applying the induction hypothesis, we take the product of the numbers of spanning trees in the blocks of G_1 and multiply it by the product of the numbers of spanning trees in the blocks of G_2 to obtain the number of spanning trees of G.

4.1.8. *For the graph G on the left below, $\kappa(G) = 2$, $\kappa'(G) = 4$, and $\delta(G) = 4$. For the graph H on the right, $\kappa(H) = \kappa'(H) = \delta(H) = 4$.* The vertices all have degree 4, except that the vertices in the "center" of the drawing of G have degree 7.

In G, these two vertices form a separating set, and the graph has no cut-vertex, so $\kappa(G) = 2$. By Corollary 4.1.13, if there is an edge cut with fewer than four edges, it must have at least five vertices on each side. Proposition 4.1.12 states that $\left|[S,\overline{S}]\right| = [\sum_{v\in S} d(v)] - 2e(G[S])$. Since one of the sides has at least one of the vertices of degree 7, we may assume that $\sum_{v\in S} d(v) \ge 23$. To obtain $\left|[S,\overline{S}]\right| \le 3$, this requires $2e(G[S]) \ge 20$. Thus the subgraph induced by S must be K_5, but G does not contain K_5.

In H, it suffices to show that there is no separating set S of size 3, since $\kappa(H) \le \kappa'(H) \le \delta(H) = 4$. To show this, let x, y be two nonadjacent vertices of $H - S$. By a small case analysis, one shows that in each direction around the central portion of the graph, there are two x,y-paths sharing no internal vertices. Thus four vertices must be deleted to break all x,y-paths.

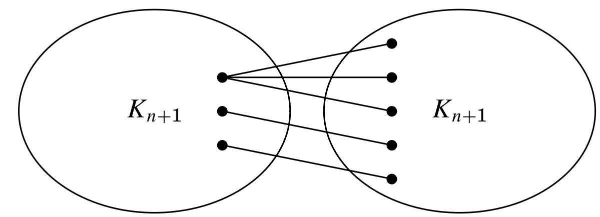




4.1.9. *Given nonnegative integers with $l \le m \le n$, there is a simple graph with $\kappa = l$, $\kappa' = m$, and $\delta = n$.* Begin with two disjoint copies of K_{n+1}. This yields $\delta = n$, and we will add a few more edges. Pick l vertices from the first clique and m vertices from the second. Add m edges between them in such a way that each of the special vertices belongs to at least one of the new edges. The construction is illustrated below with $m = 3$ and $l = 5$.

Deleting the m special edges disconnects the graph, as does deleting the l special vertices in the first copy of K_{n+1}. Since we are using $n+1$ vertices in each of the complete subgraphs, $l \le m \le n$ guarantees that the

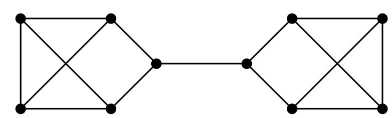
minimum degree remains n and that there really are two components remaining after the deletions. No smaller set disconnects the graph, because the connectivity of the complete subgraphs is n.



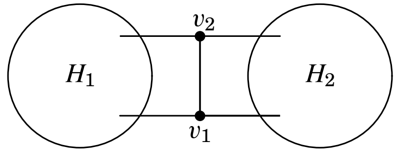
4.1.10. *The graph below is the smallest 3-regular simple graph with connectivity 1.* Since the graph below is 3-regular and has connectivity 1, it suffices to show that every 3-regular simple graph with connectivity 1 has at least 10 vertices.

Proof 1 (case analysis). Let v be a cut-vertex of a 3-regular simple graph G with connectivity 1. Each component H of $G - v$ has one or two neighbors of v. Since the neighbors of v have degree 3, H also has a vertex u not adjacent to v. Since $d(u) = 3$, $n(H) \ge 4$. Since G has at least two such components plus v, we have $n(G) \ge 4 + 4 + 1 = 9$. By the degree-sum formula, no 3-regular graph has order 9, so $n(G) \ge 10$.

Proof 2 (using edge-connectivity). Since $\kappa = \kappa'$ for 3-regular graphs, we seek the smallest 3-regular connected graph G having a cut-edge e. The graph $G - e$ has two components, each having one vertex of degree 2 and the rest of degree 3. Since it has a vertex of degree 3, such a component has at least four vertices. Since it has an even number of vertices of degree 3, each component has at least five vertices.



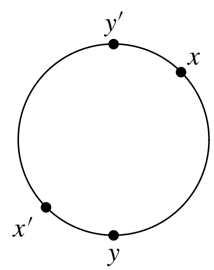
4.1.11. *$\kappa' = \kappa$ when $\Delta(G) \le 3$.* Let S be a minimum vertex cut ($|S| = \kappa(G)$). Since $\kappa(G) \le \kappa'(G)$ always, we need only provide an edge cut of size $|S|$. Let H_1 and H_2 be two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2. Since $\Delta(G) \le 3$, v cannot have two neighbors in H_1 and two in H_2. For each such v, delete the edge to a member of $\{H_1, H_2\}$ in which v has only one neighbor. These $\kappa(G)$ edges break all paths from H_1 to H_2 except in the case drawn below, where a path can come into S via v_1 and leave via v_2. Here we simply choose the edge to H_1 for each v_i.



4.1.12. *A k-regular k-connected graph when k is odd.* For $n > k = 2r + 1$ and $r \ge 1$, we show that the Harary graph $H_{k,n}$ is k-connected. The graph consists of n vertices $v_0, \ldots, v_{n-1}$ spaced equally around a circle, with each vertex adjacent to the r nearest vertices in each direction, plus the "special" edges $v_i v_{i+\lfloor n/2 \rfloor}$ for $0 \le i \le \lfloor (n-1)/2 \rfloor$. When n is odd, $v_{\lfloor n/2 \rfloor}$ has two incident special edges.

To prove that $\kappa(G) = k$, consider a separating set S. Since $G - S$ is disconnected, there are nonadjacent vertices x and y such that every x, y-path passes through S. Let $C(u, v)$ denote the vertices encountered when moving from u to v clockwise along the circle (omitting u and v). The cut S must contain r consecutive vertices from each of $C(x, y)$ and $C(y, x)$ in order to break every x, y-path (otherwise, one could start at x and always take a step in the direction of y). Hence $|S| \ge k$ unless S contains exactly r consecutive vertices in each of $C(x, y)$ and $C(y, x)$.

In this case, we claim that there remains an x, y-path using a special edge involving x or y. Let x' and y' be the neighbors of x and y along the special edges, using v_0 as the neighbor when one of these is $v_{\lfloor n/2 \rfloor}$. Label x and y so that $C(x, y)$ is smaller than $C(y, x)$ (diametrically opposite vertices require n even and are adjacent). Note that $\left|C(x', y')\right| \ge |C(x, y)| - 1$ (the two sets have different sizes when n is odd if $x = v_i$ and $y = v_j$ with $0 \le j < \lfloor n/2 \rfloor \le i \le n - 1$). Because $|C(x, y)| \ge r$, we have $\left|C(x', y')\right| \ge r - 1$. Therefore, when we delete r consecutive vertices from $C(y, x)$, all of $C(y, x') \cup \{x'\}$ or $\{y'\} \cup C(y', x)$ remains. Therefore at least one of the two x, y-paths with these sets as the internal vertices remains in $G - S$.



4.1.13. *Numerical argument for edge-connectivity of $K_{m,n}$.*

a) Size of $[S, \overline{S}]$. Let X and Y be the partite sets of $K_{m,n}$, with $|X| = m$ and $|Y| = n$. Consider $S \subseteq V(K_{m,n})$ such that $|S \cap X| = a$ and $|S \cap Y| = b$.

Now $[S, \overline{S}]$ consists of the edges from $S \cap X$ to $\overline{S} \cap Y$ and from $S \cap Y$ to $\overline{S} \cap X$. There are $a(n - b) + b(m - a)$ such edges.

b) $\kappa'(K_{m,n}) = \min\{m, n\}$. If $m + n = 1$, then the answer is 0, by convention, as desired. Otherwise, we assume that $m \le n$ and consider an edge cut $[S, \overline{S}]$. In the notation of part (a), we have $0 \le a \le m$ and $0 \le b \le n$ and $0 < a+b < m+n$. If $0 < a < m$, then $a(n-b)+b(m-a) \ge (n-b)+b = n \ge m$. If $a = 0$, then $b > 0$ and $a(n - b) + b(m - a) = bm \ge m$. If $a = m$, then $b < n$ and $a(n - b) + b(m - a) = m(n - b) \ge m$. Thus $a(n - b) + b(m - a) \ge m$ in all cases, with equality when $a = 0$ and $b = 1$.

c) $K_{3,3}$ has no edge cut with seven edges. Since $K_{3,3}$ has six vertices, every connected spanning subgraph of $K_{3,3}$ has at least five edges. Hence deleting any five or more edges of $K_{3,3}$ leaves a disconnected subgraph. No set of seven edges is an edge cut, because 7 does not occur in the set of values of $a(3 - b) + b(3 - a)$. Writing this in the form $3(a + b) - 2ab$, achieving 7 requires $a + b \ge 3$. Also, a, b must also have different parity, since $2ab$ even implies that $a + b$ must be odd to obtain 7. The remaining cases, $(1, 2)$, $(0, 3)$, $(2, 3)$, do not yield 7.

4.1.14. *If G is a connected graph and for every edge e there are cycles C_1 and C_2 such that $E(C_1) \cap E(C_2) = \{e\}$, then G is 3-edge-connected.* It suffices to show that no set of two edges disconnects G. Consider $e, e' \in E(G)$. Since G has two cycles through e', $G - e'$ is connected. Since G has two cycles that share only e, at least one of these cycles still exists in $G - e'$. Therefore, e lies on a cycle in $G - e'$ and is not a cut-edge of $G - e'$. We have proved that deleting both e' and e leaves a connected subgraph. The argument holds for each edge pair, so G is 3-edge-connected.

The Petersen graph satisfies this condition (hence is 3-edge-connected).

Proof 1 (symmetry and disjointness description). The underlying set [5] is in the disjointness definition of the Petersen graph can be permuted to turn each edge into any other. Hence it suffices to prove that the condition holds for one edge. In particular, the edge $(12, 34)$ is the only common edge in the two 5-cycles $(12, 34, 51, 23, 45)$ and $(12, 34, 52, 14, 35)$.

Proof 2 (properties of the graph). Alternatively, let x and y be the endpoints of an edge in the Petersen graph. Since the girth is 5, the neighbors u, v of x and w, z of y form an independent set of size 4. Let a be the unique common neighbor of u and w, and let b be the common neighbor of v and y; these are distinct since the girth is 5. Since a, b, x, y, u, v, w, z are eight distinct vertices, we have constructed cycles with vertices (u, x, y, w, a) and (v, x, y, z, b) that share only xy.

4.1.15. *The Petersen graph is 3-connected.* Since the Petersen graph G is 3-regular, it suffices by Theorem 4.1.11 to prove that G is 3-edge-connected. Let $[S, \overline{S}]$ be a minimum edge cut. If $\left|[S, \overline{S}]\right| < 3$, then

$[\sum_{v\in S} d(v)] - 2e(G[S]) \le 2$, by Proposition 4.1.12. We may compute this from either side of the cut, so we may assume that $|S| \le |\overline{S}|$.

Since G has no cycle of length less than 5, when $|S| < 5$ we have $e(G[S]) \le |S| - 1$. This yields $3|S| - 2(|S| - 1) \le 2$, which simplifies to $|S| \le 0$. This is impossible for nonempty S. For $|S| = 5$, we obtain $3|S| - 2|S| \le 2$, which again is false. Hence no edge cut has size less than 3.

4.1.16. *The Petersen graph has an edge cut of size m if and only if $3 \le m \le 12$.* Since the graph has 10 vertices, we consider edge cuts of the form $[S, \overline{S}]$ for $1 \le |S| \le 5$. Since $|[S, \overline{S}]| = \sum_{v\in S} d(v) - 2e(G[S]) = 3|S| - 2e(G[S])$, we consider the number of edges in $G[S]$. Since the girth is 5, all induced subgraphs with at most four vertices are forests.

The independent sets with up to four vertices yield cuts of sizes $3, 6, 9, 12$. Deleting two adjacent vertices and their neighbors leaves $2K_2$, so there induced subgraphs with two to four vertices that have one edge, yielding cuts of sizes $4, 7, 10$. Deleting the vertices of a P_3 and their neighbors leaves $2K_1$, so there are induced subgraphs with three to five vertices that have two edges, yielding cuts of sizes $5, 8, 11$.

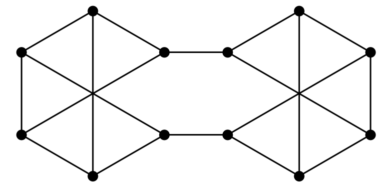
Let $e(S)$ denote $e(G[S])$. An edge cut of size less than 3 requires $3|S| - 2e(S) \le 2$, or $e(S) \ge (3/2)|S| - 1$. Since $e(S) \le |S| - 1$ when $|S| \le 4$, we combine the two inequalities to obtain $|S| \le 0$, which is impossible. (For $|S| = 5$, $e(S) \le |S|$ yields $|S| \le 2$, again a contradiction.)

Similarly, an edge cut of size more than 12 requires $2e(S) \le 3|S| - 13$. With $|S| \le 5$, this yields $2e(S) \le 2$, but there is no 5-vertex induced subgraph with only one edge.

4.1.17. *Deleting an edge cut of size 3 in the Petersen graph isolates a vertex.* Proposition 4.1.12 yields $|[S, \overline{S}]| = 3|S| - 2e(G[S])$. Thus $|[S, \overline{S}]| = 3$ requires S and $\overline{S}$ to have odd size. Let S be the smaller side of the cut. When $|S| = 5$, the induced subgraph has at most 5 edges, and the cut has size at least $3 \cdot 5 - 2 \cdot 5 = 5$. When $|S| = 3$, the induced subgraph has at most 2 edges, and the cut has size at least $3 \cdot 3 - 2 \cdot 2 = 5$. Hence $|S| = 1$ for a cut of size 3.

4.1.18. *Every triangle-free simple graph with minimum degree at least 3 and order at most 11 is 3-edge-connected.* Let $[S, \overline{S}]$ be an edge cut of size less than 3, with $|S| \le |\overline{S}|$. Let $k = |S|$. Since $\delta(G) \ge 3$ and $[S, \overline{S}] \le 2$, the Degree-Sum Formula yields $e(G[S]) \ge (3k - 2)/2$. Since $G[S]$ is triangle-free, Mantel's Theorem (Section 1.3) yields $e(G[S]) \le \lfloor k^2/4 \rfloor$. Hence $k^2/4 \ge (3k - 2)/2$. For positive integer k, this inequality is valid only when $k \ge 6$. Since the smaller side of the cut has at most five vertices, we obtain a contradiction, and there is no edge cut of size at most 2.

The bound of 11 is sharp. The 12-vertex 3-regular triangle-free graph below is not 3-edge-connected.



4.1.19. *a) If $\delta(G) \ge n - 2$ for a simple n-vertex graph G, then $\kappa(G) = \delta(G)$.* If $\delta = n - 1$, then $G = K_n$, which has connectivity $n - 1$. If $\delta = n - 2$, then when u and v are nonadjacent the other $n - 2$ vertices are all common neighbors of u and v. It is necessary to delete all common neighbors of some pair of vertices to separate the graph, so $\kappa \ge n - 2 = \delta$.

b) Construction of graphs with $\delta = n - 3$ and $\kappa < \delta$. For any $n \ge 4$, let $G = K_n - E(C_4)$; i.e., G is formed by deleting the edges of a 4-cycle from a clique. The subgraph induced by these four vertices is $2K_2$, so deleting the other $n - 4$ vertices of G disconnects the graph. However, G has 4 vertices of degree $n - 3$ and $n - 4$ of degree $n - 1$, so $\kappa(G) < \delta(G)$.

4.1.20. *Every simple n-vertex graph G with $\delta(G) \ge (n + k - 2)/2$ is k-connected, and this is best possible.* We do not consider $k = n$, because we have adopted the convention that no n-vertex graph is n-connected. To see that the result is best possible, consider $K_{k-1} \vee (K_{\lfloor (n-k+1)/2 \rfloor} + K_{\lceil (n-k+1)/2 \rceil})$. This graph has a separating set of size $k - 1$, and its minimum degree is $k - 1 + \lfloor (n - k + 1)/2 \rfloor - 1 = \lfloor (n + k - 3)/2 \rfloor$. There are several ways to prove that $\delta \ge (n + k - 2)/2$ ensures k-connectedness.

Proof 1 (stronger statement, common neighbors). If $x \nleftrightarrow y$, then x, y have a total of at least $n+k-2$ edges to the $n-2$ other vertices, which means they have at least k common neighbors (using $|A \cap B| = |A| + |B| - |A \cup B|$ for $A = N(x)$ and $B = N(y)$). Thus at least k vertices must be deleted to make some vertex unreachable from another.

Proof 2 (contradiction). If G is not k-connected, then the deletion of some $k - 1$ vertices S leaves a disconnected subgraph H. Consider $v \in V(H)$; since v has at most $k - 1$ neighbors in S, we have $d_H(v) \ge \delta(G) - k + 1 \ge (n - k)/2$. Therefore, each component of H has at least $1 + (n - k)/2$ vertices. Since H has at least two components, H has at least $n - k + 2$ vertices. However, $n = n(G) = n(H) + |S| \ge (n - k + 2) + (k - 1) > n$. The contradiction implies that G is k-connected.

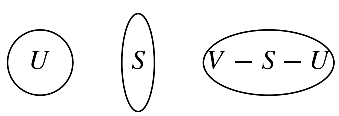
Proof 3 (induction on k). For $k = 1$, $\delta(G) \ge (n - 1)/2$ forces every pair of nonadjacent vertices to have degree-sum at least $n - 1$; hence they have a common neighbor among the remaining $n - 2$ vertices, and G is connected. For $k > 1$, let v be a vertex of a minimum separating set S.

Deleting v removes at most one edge to each other vertex, so $\delta(G-v) \ge [(n-1)+(k-1)-2]/2$. Using the induction hypothesis, we conclude that $G-v$ is $(k-1)$-connected. Since $S-v$ separates $G-v$, we have $|S-v| \ge k-1$ and hence $|S| \ge k$, and G is k-connected.

4.1.21. *If G is a simple n-vertex graph with $n \ge k+l$ and $\delta(G) \ge \frac{n+l(k-2)}{l+1}$, and $G-S$ has more than l components, then $|S| \ge k$.* Proof by contradiction. Suppose $G-S$ has more than l components and $|S| = k-1$ (if there is a smaller cut, we can add to it from components of the remainder that have at least 2 vertices until the cut reaches size $k-1$). Let H be a smallest component of $G-S$; we have $n(H) \le n-k+1)/(l+1)$. A vertex of H has at most $(n-k+1)/(l+1)-1$ neighbors in H and $k-1$ neighbors in S, which yields $\delta(G) \le \frac{n-(k-2)-1}{l+1} + (k-2) = \frac{n-l(k-2)-1}{l+1}$.

To prove that the result is best possible, partition $n-k+1$ vertices into $l+1$ sets of sizes $\lfloor \frac{n-k+1}{l+1} \rfloor$ and $\lceil \frac{n-k+1}{l+1} \rceil$. Place cliques on these sets, and form the join of this graph with K_{k-1}. The minimum degree is $\frac{n-l(k-2)-1}{l+1}$.

4.1.22. *a) If the vertex degrees $d_1 \le \cdots \le d_n$ of a simple graph G satisfy $d_j \ge j+k$ whenever $j \le n-1-d_{n-k}$, then G is $(k+1)$-connected.* Let S be a vertex cut; we will prove that $|S| \ge k+1$. Let U be the set of vertices in the smallest component of $G-S$, and let $j = |U|$. Only vertices of S can have degree exceeding $n-1-j$. Since there are at most $|S|$ such vertices, $d_{n-|S|} \le n-1-j$. If $|S| \le k$, then $j \le n-1-d_{n-|S|} \le n-1-d_{n-k}$, and the hypothesis applies. If $v \in U$, then $d_G(v) \le j-1+|S|$. Since this yields j vertices with degree at most $j-1+|S|$, we have $d_j \le j-1+|S|$. Since the hypothesis applies, $d_j \ge j+k$, and we conclude that $|S| \ge k+1$.

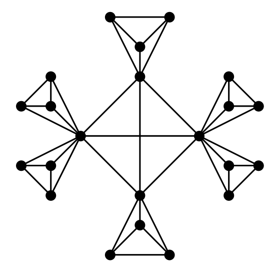


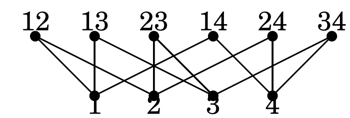
b) The result is sharp. Let $G = K_k \vee (K_j + K_i)$, where $i+j+k = n$; we may assum that $j \le i$. There are j vertices of degree $j+k-1$, i vertices of degree $i+k-1$, and k vertices of degree $n-1$. When i and j are positive, $\kappa(G) = k$. Since $j \le i$ and G has k vertices of degree $n-1$, we have $d_j = j+k-1$ and $d_{n-k} = n-j-1$. Thus the condition in part (a) does not hold. However, it just barely fails, since $d_{j'} = j+k-1 \ge j'+k$ for $j' < j$. Thus the result is sharp in the sense that it cannot be weakened by applying the requirement only when $j \le n-2-d_{n-k}$.

4.1.23. *If $n(G)$ is even, $\kappa(G) \ge r$, and G is $K_{1,r+1}$-free, then G has a 1-factor.* We verify Tutte's 1-factor condition. When $|S| = \varnothing$, the only component of $G-S$ has even order. When $1 \le |S| \le r-1$, there is only one component of $G-S$. For $|S| \ge r$, we prove that $G-S$ has at most $|S|$ components.

Each component H of $G-S$ sends edges to at least r distinct vertices in S, since $\kappa(G) = r$. For each such H, choose edges to r distinct vertices in S. Given $v \in S$, we have chosen at most one edge from v to each component of $G-S$. If $G-S$ has more than $|S|$ components, then we have chosen more than $r\,|S|$ edges to S. By the pigeonhole principle, some $x \in S$ appears in more than r of these edges. Since we chose at most one edge from x to each component of $G-S$, the chosen edges containing x have endpoints in distinct components of $G-S$, which creates the forbidden induced $K_{1,r+1}$.

This result is best possible: it is not enough to assume that G is r-edge-connected or that G is $r-1$-connected. Both graphs below have even order, no induced $K_{1,4}$, and no 1-factor (deleting a set of size 4 leaves 6 odd components). The graph on the left is 3-edge-connected, and the graph on the right is 2-connected.





4.1.24. *Degree conditions for $\kappa' = \delta$ in a simple n-vertex graph G.*

a) $\delta(G) \ge \lfloor n/2 \rfloor$ implies $\kappa'(G) = \delta(G)$, and this is best possible. If $\kappa'(G) < \delta(G)$ and F is a minimum edge cut, then the components of $G-F$ have more than $\delta(G)$ vertices (Proposition 4.1.10). Since $\delta(G) \ge \lfloor n/2 \rfloor$, this yields $n(G) \ge 2(\lfloor n/2 \rfloor + 1) \ge n+1$, which is impossible. Hence $\kappa'(G) = \delta(G)$.

To show that the inequality $\delta \ge \lfloor n/2 \rfloor$ cannot be weakened when $n \ge 3$, consider $G = K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$ (the disjoint union of two cliques). This G is disconnected, so $\kappa'(G) = 0$, and $\delta(G) = \lfloor n/2 \rfloor - 1$. The smallest case where this yields $\kappa'(G) < \delta(G)$ is $n = 4$, $\delta = 1$, $G = 2K_2$.

b) $\kappa'(G) = \delta(G)$ if each nonadjacent pair of vertices has degree sum at least $n-1$, and this is best possible. The example $G = K_{m+1} + K_{n-m-1}$ shows that $n-1$ cannot be replaced by $n-2$ in the hypothesis; the conclusion fails spectacularly with $\kappa'(G) = 0$ even though $d(x)+d(y) = m-1+n-m-1 = n-2$ when $x \nleftrightarrow y$. To prove the claim, suppose $[S, \overline{S}]$ is a minimum edge cut, with size $k = \kappa'(G) < \delta(G)$. This forces $|S|, |\overline{S}| > \delta(G)$ (Proposition 4.1.10). With degree-sum at least $n(G)-1$, any two nonadjacent vertices have a common neighbor. Hence if S has a vertex x with no neighbor in $\overline{S}$, then every vertex in $\overline{S}$ must have a neighbor in S. Now $|\overline{S}| > \delta(G)$ imples $k > \delta(G)$. Otherwise, every vertex of S has a neighbor in $\overline{S}$, and now $|S| > \delta(G)$ implies $k > \delta(G)$. We conclude that $k = \delta(G)$.

4.1.25. *$\kappa'(G)=\delta(G)$ for diameter 2.* Suppose that G is a simple graph with diameter 2. Let $[S,\overline{S}]$ be a minimum edge cut with $s=|S|\le|\overline{S}|$, and let $k=|[S,\overline{S}]|=\kappa'(G)$.

a) Every vertex of S has a neighbor in $\overline{S}$. If S has a vertex x with no neighbor in $\overline{S}$, then $d(x)\le s-1<n/2$, and $\operatorname{diam} G=2$ implies that every vertex of $\overline{S}$ has a neighbor in S. In this case $\delta(G)<n/2\le k$. Hence every vertex of S has a neighbor in $\overline{S}$.

b) $\kappa'(G)=\delta(G)$. Since $\kappa'(G)\le\delta(G)$ always, part (a) yields $s\le\delta(G)$. Each vertex of S has at least $\delta(G)-s+1$ neighbors in $\overline{S}$, so $k\ge s(\delta(G)-s+1)$. Since $\kappa'(G)\le\delta(G)$, we have $0\ge(s-1)(\delta(G)-s)$, which requires $s=1$ or $s\ge\delta(G)$. We conclude that $s=1$ or $s=\delta(G)$.

Consider the case $s=\delta(G)$. Since we have proved that each vertex of S has a neighbor in $\overline{S}$, from $k\le\delta(G)$ we conclude that each vertex of S has exactly one neighbor in $\overline{S}$. Hence each vertex of S has $\delta(G)-1$ neighbors in S. We conclude that S induces a clique and that $\kappa'(G)=\delta(G)$.

4.1.26. *A set F of edges in G is an edge cut if and only if F contains an even number of edges from every cycle in G. Necessity.* A cycle must wind up on the same side of an edge cut that it starts on, and thus it must cross the cut an even number of times.

Sufficiency. Given a set F that satisfies the intersection condition with every cycle, we construct a set $S\subseteq V(G)$ such that $F=[S,\overline{S}]$. Each component of $G-F$ must be all in $G[S]$ or all in $G[\overline{S}]$, but we must group them appropriately. Define a graph H whose vertices correspond to the components of $G-F$; for each $e\in F$, we put an edge in H whose endpoints are the components of $G-F$ containing the endpoints of e.

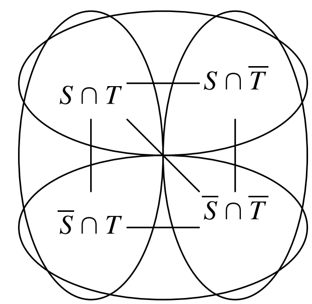
We claim that H is bipartite. From a cycle C in H, we can obtain a cycle C' in G as follows. For $v\in V(C)$ let e,f be the edges of C incident to v (not necessarily distinct), and let x,y be the endpoints of e,f in the component of $G-F$ corresponding to v. We expand v into an x,y-path in that component. Since C visits each vertex at most once, the resulting C' is a cycle in G. The number of edges of F in C' is the length of C. Hence the length of C is even.

We conclude that H is bipartite. Let S be the set of vertices in the components of $G-F$ corresponding to one partite set in a bipartition of H. Now F is the edge cut $[S,\overline{S}]$.

4.1.27. *Every edge cut is a disjoint union of bonds.* Using induction on the size of the cut, it suffices to prove that if $[S,\overline{S}]$ is not a bond, then $[S,\overline{S}]$ is a disjoint union of smaller edge cuts. First suppose G is disconnected, with components $G_1,\ldots,G_k$. If $[S,\overline{S}]$ cuts more than one component, we express $[S,\overline{S}]$ as a union of edge cuts that cut only one component: let the ith cut be $[S\cap V(G_i),\overline{S\cap V(G_i)}]$. This cut consists of the edges of $[S,\overline{S}]$ in G_i, because the vertices of the other components are all on one side of the cut. Hence we may assume that $[S,\overline{S}]$ cuts only one component of G or (equivalently) that G is connected.

An edge cut of a connected graph is a bond if and only if the subgraphs induced by the sets of the vertex partition are connected. Hence if $[S,\overline{S}]$ is not a bond, we may assume that $G[S]$ is not connected. Let $\{G_i\}$ be the components of the induced subgraph $G[S]$, and let $S_i=V(G_i)$. Since there are no edges between components of $G[S]$, $[S,\overline{S}]$ is the disjoint union of the edge cuts $[S_i,\overline{S}_i]$. Since G is connected, each of these is non-empty, so we have expressed $[S,\overline{S}]$ as a disjoint union of smaller edge cuts.

4.1.28. *The symmetric difference of two edge cuts is an edge cut.* The symmetric difference of $[S,\overline{S}]$ and $[T,\overline{T}]$ is $[U,\overline{U}]$, where $U=(S\cap T)\cup(\overline{S}\cap\overline{T})$ and $\overline{U}=(S\cap\overline{T})\cup(\overline{S}\cap T)$, as sketched below. The other edges of the union are those within U or within $\overline{U}$ and appear in both of the original edge cuts.



4.1.29. *A spanning subgraph H of G is a spanning tree of G if and only if $G-E(H)$ contains no bond of G and adding any edge of H creates a subgraph containing exactly one bond of G.*

Necessity: If H is a spanning tree, then H is connected, so $G-E(H)$ contains no edge cut. Also, $H-e$ is disconnected, with exactly two components having vertex sets S and $\overline{S}$. Let G' be the subgraph obtained by adding e to $G-E(H)$. Note that G' contains all of $[S,\overline{S}]$ (and perhaps additional edges). Since $S,\overline{S}$ induce connected subgraphs of G, $[S,\overline{S}]$ is a bond of G (Proposition 4.1.15). Since adding any edge of $[S,\overline{S}]$ to $H-e$ creates a spanning tree of G, an edge cut contained in G' must include all of $[S,\overline{S}]$, and hence $[S,\overline{S}]$ is the only bond in G'.

Sufficiency: Suppose that $G-E(H)$ contains no bond of G and each subgraph obtained by adding one edge of H contains exactly one bond. Now H is obtained by deleting a set of edges from G that does not disconnect G, and H is connected. Similarly, deleting any additional edge from G does contain a bond, so each $H-e$ is disconnected. Hence H is a tree.

4.1.30. *The graph with vertex set* $\{1, \ldots, 11\}$ *in which* $i \leftrightarrow j$ *if and only if* i *and* j *have a common factor bigger than 1 has six blocks.* Vertices $1, 7, 11$ are isolated and hence are blocks by themselves. The remaining vertices form a single component with blocks that are complete subgraphs. The vertex sets of these are $\{3, 6, 9\}$, $\{2, 4, 6, 8, 10\}$, and $\{5, 10\}$. Vertices 6 and 10 are cut-vertices.

4.1.31. *The maximum number of edges in a simple* n*-vertex cactus* G *is* $\lfloor 3(n-1)/2 \rfloor$. A cactus is a connected graph in which every block is an edge or a cycle. The bound is achieved by a set of $\lfloor (n-1)/2 \rfloor$ triangles sharing a single vertex, plus one extra edge to a leaf if n is even.

Proof 1 (induction on the number of blocks). Let k be the number of blocks. If $k = 1$, then $e(G) = n(G) - 1$ if $n(G) \le 2$, and $e(G) = n(G)$ if $n(G) > 2$. In either case, $e(G) \le \lfloor 3(n(G) - 1)/2 \rfloor$.

A graph that has more than one block is not a single block, so it has a cut-vertex v. Let S be the vertex set of one component of $G - v$. Let $G_1 = G[S \cup \{v\}]$, and let $G_2 = G - S$. Both G_1 and G_2 are cacti, and every block of G is a block in exactly one of $\{G_1, G_2\}$. Thus each has fewer blocks than G, and we can apply the induction hypothesis to obtain $e(G_i) \le \lfloor 3(n(G_i) - 1)/2 \rfloor$.

If $|S| = m$, then $n(G_1) = m + 1$ and $n(G_2) = n(G) - m$, since v belongs to both graphs. We thus have

$$e(G) = e(G_1) + e(G_2) \le \left\lfloor \tfrac{3(m+1-1)}{2} \right\rfloor + \left\lfloor \tfrac{3(n(G)-m-1)}{2} \right\rfloor \le \left\lfloor \tfrac{3(n(G)-1)}{2} \right\rfloor.$$

Proof 2 (summing over blocks). Let G be a simple n-vertex cactus with k blocks that are cycles and l blocks that are single edges. When we describe G by starting with one block and iteratively adding neighboring blocks, each time we add a block the number of vertices increases by one less than the number of vertices in the block, since one of those vertices (the shared cut-vertex) was already in the graph. If the blocks are $B_1, \ldots, B_{k+l}$, then $n(G) = (\sum n(B_i)) - (k + l - 1)$.

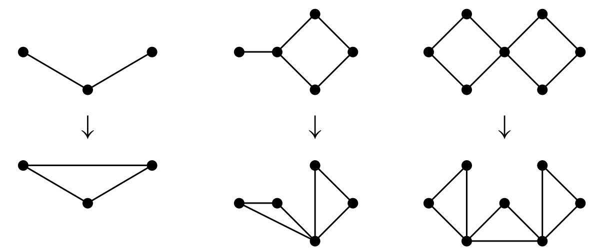
On the other hand, $e(G) = \sum e(B_i)$. We have $e(B_i) = n(B_i)$ if B_i is a cycle, and $e(B_i) = n(B_i) - 1$ if B_i is an edge. Therefore,

$$e(G) = \sum e(B_i) = (\sum n(B_i)) - l = n(G) + k - 1.$$

This implies that we maximize the number of edges by maximizing k, the number of blocks that are cycles. Viewing the cactus again as grown by adding blocks, observe that we add at least two vertices every time we add a block that is a cycle, since cycles have at least three vertices. Starting from a single vertex, the maximum number of cycles we can form is thus $\lfloor (n-1)/2 \rfloor$. This bound on k yields $e(G) \le \lfloor 3(n-1)/2 \rfloor$.

Proof 3 (local change). If the blocks are all triangles, except for at most one that is K_2 or one that is a 4-cycle, then the number of edges equals the given formula. Hence it suffices to show that a cactus not having this description cannot have the maximum number of edges.

If a block is a cycle of length more than 4, then deleting one edge e and replacing it with two edges joining the endpoints of e to another vertex on the cycle creates a new cactus on the same vertices having one more edge. If each of two blocks is a single edge or a 4-cycle, then the blocks can be rearranged by "cutting and pasting" so that the sizes of the blocks are the same as before, but these two special blocks share a vertex. Now a change can be made to increase the number of edges as shown below.



Proof 4 (spanning trees). An n-vertex cactus is a connected graph, so it has a spanning tree with $n - 1$ edges. Each additional edge completes a cycle using at least two edges in the tree. Each edge of the tree is used in at most one such cycle. Hence there are at most $(n-1)/2$ additional edges, and the total number of edges is at most $n + \lfloor (n-1)/2 \rfloor$.

4.1.32. *Every vertex of* G *has even degree if and only if every block of* G *is Eulerian. Sufficiency*. If every block is Eulerian, then each vertex receives even degree from each block containing it. The blocks partition the edges, so the total degree at each vertex is even.

Necessity. Since every block is connected, it suffices to show that each vertex has even degree in each block. Certainly this holds for a vertex appearing in only one block. For a cut-vertex v, let G' be the subgraph consisting of one component of $G - v$ together with its edges to v. Each block containing v appears in one such subgraph. Every vertex of G' other than v has even degree in G', since it retains all of its incident edges from G. By the Degree-Sum Formula, also v has even degree in G'. Hence v has even degree in the block of G containing v that is contained in G'.

4.1.33. *A connected graph is* k*-edge-connected if and only if each of its blocks is* k*-edge-connected.* We show that a set F of edges is a disconnecting set in a graph G if and only if it disconnects some block. If deleting F leaves each block of G connected, then the full graph remains connected. If deleting F disconnects some block B, then the remainder of G cannot contain a path between distinct components of $B - F$, because then B would not be a maximal subgraph having no cutvertex.

With this claim, the edge-connectivity of G is the minimum of the edge-connectivities of its blocks, which yields the desired statement.

4.1.34. *The block-cutpoint tree.* Given a graph G with connectivity 1, let $B(G)$ be the bipartite graph whose partite sets correspond to the blocks and the cut-vertices of G, with $x \leftrightarrow B$ if B is a block of G containing x.

a) $B(G)$ is a tree. If $G = K_2$, then $B(G) = K_1$. Otherwise G has at least two blocks, and every cut-vertex belongs to a block. Hence to show $B(G)$ is connected it suffices to establish a B, B'-path in $B(G)$, where B, B' are blocks of G. Since G is connected, G has a u, v-path, for any choice of vertices $u \in B$, $v \in B'$. This path visits some sequence of blocks from B to B', moving from one to the next via a cut-vertex of G belonging to both of them. This describes a B, B'-path in $B(G)$.

We prove by contradiction that $B(G)$ also has no cycles and hence is a tree. Suppose x is a cut-vertex of G on a cycle C in $B(G)$. Let B, B' be the neighbors of x on C. The B, B' path $C - x$ provides a route from $B - x$ to $B' - x$ without using x. This is impossible, since when B, B' are two blocks of G containing cut-vertex x, every path between $B - x$ and $B' - x$ in G must pass through x.

b) If G is not a block, then at least two blocks of G each contain exactly one cut-vertex of G. Each cut-vertex of G belongs to at least two blocks of G. Hence the leaves of $B(G)$ all arise from blocks of G, not cut-vertices of G. If G is not a block, then $B(G)$ has at least two leaves, and the leaves of $B(G)$ are the desired blocks in G.

c) G has exactly $k + \sum_{v\in V(G)}(b(v) - 1)$ blocks, where k is the number of components of G and $b(v)$ is the number of blocks containing v.

Proof 1 (explicit count). Since we can count the blocks separately in each component, it suffices to show that a connected graph has $1+\sum(b(v)-1)$ blocks. Select a block in a connected graph G and view it as a root; this corresponds to the 1 in the formula. Each vertex v in this block leads us to $b(v) - 1$ new blocks. For each new block, each vertex v other than the one that leads us there leads us to $b(v) - 1$ new blocks. This process stops when we have counted $b(v) - 1$ for each vertex of G.

This tree-like exploration gives the desired count of blocks as long as two facts hold: 1) no two blocks intersect in more than one vertex, and 2) no block can be reached in more than one way from the root. These guarantee that we don't count blocks more than once. If either happens, we get a cycle of blocks, $B_1, \ldots, B_n, B_1$, with $n \geq 2$, so that successive blocks share a vertex. Then there is no vertex whose deletion will disconnect the subgraph that is the union of these blocks, which is impossible since blocks are maximal subgraphs with no cut-vertex.

Proof 2 (induction on the number of blocks). We need only prove the formula for connected graphs, since both the number of blocks and the value of the formula are sums over the components of G. If G is a block, then every vertex of G appears in one block, and the formula holds.

If G has a cutvertex, then by part (a) this component has a block B containing only one cutvertex, u. Delete all vertices of $B - u$ to obtain a graph G'. The blocks of G' are the blocks of G other than u, u appears in one less block than before, and all other terms of the formula are the same except that for G' we have left out the value 0 for the other vertices of B. The induction hypothesis now yields

$$\begin{aligned}\#blocks(G) = \#blocks(G') + 1 &= [1 + \sum_{v\in V(G')} (b_{G'}(v) - 1)] + 1 \\ &= 1 + \sum_{v\in V(G')} (b(v) - 1).\end{aligned}$$

d) Every graph has fewer cut-vertices than blocks. In the formula of part (c), there is a positive contribution for each cut-vertex. Thus the number of blocks is bigger than the number of cut-vertices, each yielding a term that contributes at least one to the sum.

4.1.35. *If H and H' are distinct maximal k-connected subgraphs of G, then H and H' have at most $k - 1$ vertices in common.* Proof by contradiction; suppose H and H' share at least k vertices. Consider $F = H \cup H'$, and let S be an arbitrary subset of $V(F)$ with fewer than k vertices. It suffices to show that $F - S$ is connected, because then $H \cup H'$ is k-connected, contradicting the hypothesis that H and H' are *maximal* k-connected subgraphs. Since $|S| < k$ and H, H' are k-connected, $H - S$ and $H' - S$ are connected. If H, H' share at least k vertices, then some common vertex x remains, and every vertex that remains has a path to x in $H - S$ or $H' - S$.

4.1.36. *Algorithm 4.1.23 correctly computes blocks of graphs.* We use induction on $n(G)$. For K_2, the algorithm correctly identifies the single block (K_1 is a special case). For larger graphs, it suffices to show that the first set identified as a block is indeed a block B sharing one vertex w with the rest of the graph, since when $w \neq x$ the remaining blocks are the blocks of the graph obtained by deleting $B - w$ from G, and the sets identified as blocks in running the algorithm on $G_(B - w)$ are the sets identified as blocks in the remainder of running the algorithm on G.

When the vertex designated as ACTIVE is changed from v to its parent, w, we check whether any vertex in the subtree T' rooted at v has a neighbor above w. This is easy to do, given that in step 1B when we mark an edge to an ancestor explored, we record for the vertices on the path in T between them that there is an edge from a descendant to an ancestor. When w becomes active again, we check whether it was ever so marked.

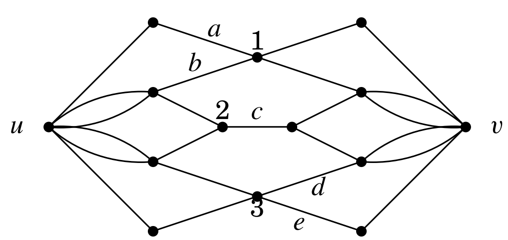
With v and w as above, in any rooted subtree of T' there is an edge from a descendant of the root to an ancestor of the root. Hence no proper subset of $T'(v)$ induces a block, because an additional vertex can be added via a path to an ancestor and then down through T, without introducing a cut-vertex. On the other hand, since there is no edge joining T' to an ancestor of w, then w is a cut-vertex, and hence $G[V(T') \cup \{w\}]$ is a maximal subgraph having no cut-vertex.

4.1.37. *An algorithm to compute the strong components of a digraph.* The algorithm is the same as Algorithm 4.1.23, except that all edges mentioned there are treated as directed edges, from tail to head in the order named there, and "block" changes to "strong component".

The proof that the algorithm works is essentially the same as Exercise 4.1.36. If there is a path from S to S that visits a vertex outside S, then S cannot be the vertex set of a strong component. When w becomes active from below with no edge from a descendant to an ancestor, all edges involving $V(T') \cup \{w\}$ and the remaining vertices are directed in toward $V(T') \cup \{w\}$. Thus a strong component is discovered.

4.2. k CONNECTED GRAPHS

4.2.1. *In the graph below, $\kappa(u, v) = 3$ and $\kappa'(u, v) = 5$.* Deleting the vertices marked 1, 2, 3 or the edges marked a, b, c, d, e makes v unreachable from u. These prove the upper bounds. Exhibiting a set of three pairwise internally disjoint u, v-paths proves $\kappa(u, v) \ge 3$, since distinct vertices must be deleted to cut the paths. Exhibiting a set of five pairwise edge-disjoint u, v-paths proves $\kappa'(u, v) \ge 5$, since distinct edges must be deleted to cut the paths. Lacking colors, we have not drawn these paths.

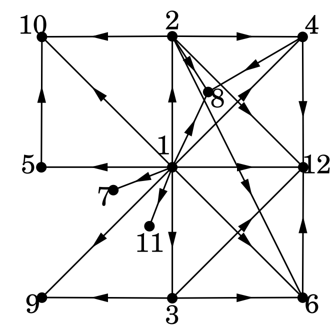


4.2.2. *If G is 2-edge-connected and G' is obtained from G by subdividing an edge of G, then G' is 2-edge-connected.* Let G' be obtained by subdividing an edge e, introducing a new vertex w. A graph is 2-edge-connected if and only if every edge lies on a cycle. This holds for G. If also holds for G', since every cycle in G containing e can be replaced with a cycle using the two edges incident to w instead of e.

Every graph having a closed-ear decomposition is 2-edge-connected. A cycle is 2-edge-connected; we show that adding ears and closed ears preserves 2-edge-connectedness. An ear or closed ear can be added by adding an edge joining existing endpoints or a double edge joining an old vertex to a new vertex, following by subdividing to lengthen the ear.

We have shown that subdivision preserves 2-edge-connectedness. The other operations preserve old cycles. When we add an edge, the new edge form a cycle with a path joining its endpoints. When we add two edges with the same endpoints, together they form a cycle. Hence the additions also preserve 2-edge-connectedness.

4.2.3. *An example of digraph connectivity.* In the digraph G with vertex set [12] defined by $i \to j$ if and only if i divides j, $\kappa(1, 12)$ is undefined and $\kappa'(1, 12) = 5$. Because $1 \to 12$, there is no way to make 12 unreachable from 1 by deleting other vertices. Because there are pairwise edge-disjoint paths from 1 to 12 through 2,3,4,6 and directly, it is necessary to delete at least five edges to make 12 unreachable from 1. Deleting the five edges entering 12 accomplishes this.



4.2.4. *If P is a u, v-path in a 2-connected graph G, then there need not be a u, v-path internally disjoint from P.* The graph $G = K_4 - uv$ with $V(G) = \{u, v, x, y\}$ is 2-connected (connected and no cut-vertex), but it has no u, v-path internally disjoint from the u, v-path P that visits vertices u, x, y, v in order.

4.2.5. *If G be a simple graph, and H is the graph with vertex set $V(G)$ such that $uv \in E(H)$ if and only if u, v appear on a common cycle in G, then H is* *a complete graph if and only if G is 2-connected.* A graph G is 2-connected if and only if for all $u, v \in V(G)$, there is a cycle containing u and v.

4.2.6. *A simple graph G is 2-connected if and only if G can be obtained from C_3 by a sequence of edge additions and edge subdivisions.* We have shown that edge addition and edge subdivision preserve 2-connectedness, so the condition is sufficient. For necessity, observe that every 2-connected graph has an ear decomposition. The initial cycle arises from C_3 by edge subdivisions, and then each ear addition consists of an edge addition followed by edge subdivisions.

4.2.7. *If xy is an edge in a digraph G, then $\kappa(G - xy) \ge \kappa(G) - 1$.* Since every separating set of G is a separating set of $G - xy$, we have $\kappa(G - xy) \le \kappa(G)$. Equality holds unless $G - xy$ has a separating set S that is smaller than $\kappa(G)$ and hence is not a separating set of G. Since $G - S$ is strongly connected, $G - xy - S$ has two induced subdigraphs $G[X]$ and $G[Y]$ such that $X \cup Y = V(G)$ and xy is the only edge from X to Y.

If $|X| \ge 2$, then $S \cup \{x\}$ is a separating set of G, and $\kappa(G) \le \kappa(G - xy) + 1$. If $|Y| \ge 2$, then again the inequality holds. In the remaining case, $|S| = n(G) - 2$. Since we have assumed that $|S| < \kappa(G)$, $|S| = n(G) - 2$ implies that $\kappa(G) \ge n(G) - 1$, which holds only when each ordered pair of distinct vertices is the head/tail for some edge. Thus $\kappa(G - xy) = n(G) - 2 = \kappa(G) - 1$, as desired.

4.2.8. *A graph is 2-connected if and only if for every ordered triple (x, y, z) of vertices, there is an x, z-path through y.* If G is 2-connected, then for any $y \in V(G)$ and set $U = \{x, z\}$, there is a y, U-fan. The two paths of such a fan together form an x, z-path through y. Conversely, if the condition holds, then clearly G is connected. Furthermore, G has no cut-vertex, because for any vertex x and any pair y, z, the condition as stated implies that $G - x$ has an y, z-path.

4.2.9. *A graph G with at least 4 vertices is 2-connected if and only if for every pair of disjoint sets of vertices $X, Y \subset V(G)$ with $|X|, |Y| \ge 2$, there exist two completely disjoint paths P_1, P_2 in G such that each path has an endpoint in X and an endpoint in Y and no internal point in X or Y.*

Sufficiency: If we apply the condition with X, Y being the endpoints of an arbitrary pair of edges, we find that every pair of edges lies on a cycle, so G is 2-connected. Alternatively, if G were disconnected or had a cut-vertex v, then we could select X and Y from separate components (of $G - v$), but then every path between X and Y passes through v.

Necessity: Form a graph G' be add an edge within X, if none exists, and within Y, if none exists. Since we only add edges, G' is still 2-connected. Hence there is a cycle containing an arbitrary pair of edges in G'; in particular, containing an edge within X and one within Y. For each portion of this cycle between the two edges, take the path between the last time it uses a vertex of X and the first time it uses a vertex of Y. This yields the desired completely disjoint paths in G.

4.2.10. (•) A *greedy ear decomposition* of a 2-connected graph is an ear decomposition that begins with a longest cycle and iteratively adds a longest ear from the remaining graph. Use a greedy ear decomposition to prove that every 2-connected claw-free graph G has $\lfloor n(G)/3 \rfloor$ pairwise-disjoint copies of P_3. (Kaneko–Kelmans–Nishimura [2000])

Comment. The proof takes many steps and several pages. It is too difficult for inclusion in this text, and the exercise will be deleted in the next edition.

4.2.11. *For a connected graph G with at least three vertices, the following are equivalent.*

A) G is 2-edge-connected.
B) Every edge of G appears in a cycle.
C) G has a closed trail containing any specified pair of edges.
D) G has a closed trail containing any specified pair of vertices.

A⇔B. A connected graph is 2-edge-connected if and only if it has no cut-edges. Cut-edges are precisely the edges belonging to no cycles.

A⇒D. By Menger's Theorem, a 2-edge-connected graph G has two edge-disjoint x, y-paths, where $x, y \in V(G)$. Following one path and returning on the other yields a closed trail containing x and y. (Without using Menger's Theorem, this can be proved by induction on $d(x, y)$.)

D⇒B. Let xy be an edge. D yields a closed trail containing x and y. This breaks into two trails with endpoints x and y. At least one of them, T, does not contain the edge xy. Since T is an x, y-walk, it contains an x, y-path. Since T does not contain xy, this path completes a cycle with xy.

B⇒C. Choose $e, f \in E(G)$; we want a closed trail through e and f. Subdivide e and f to obtain a new graph G', with x, y being the new vertices. Subdividing an edge does not destroy paths or cycles, although it may lengthen them. Thus G' is connected and has every edge on a cycle, because G has these properties. Because we have already proved the equivalence of B and D, we know that G' has a closed trail containing x and y. Replacing the edges incident to x and y on this trail with e and f yields a closed trail in G containing e and f.

C⇒D. Given a pair of vertices, choose edges incident to them. A closed trail containing these edges is a closed trail containing the original vertices.

4.2.12. *$\kappa(G) = \kappa'(G)$ when G is 3-regular, using Menger's Theorem.* By Menger's Theorem, for each x, y there are $\kappa'(G)$ pairwise edge-disjoint x, y-paths. Since G is 3-regular, these paths cannot share internal vertices (that would force four distinct edges at a vertex). Hence for each x, y there are $\kappa'(G)$ pairwise internally disjoint x, y-paths. This implies that $\kappa(G) \geq \kappa'(G)$, and it always holds that $\kappa(G) \leq \kappa'(G)$.

4.2.13. Given a 2-edge-connected graph G, define a relation R on $E(G)$ by $(e, f) \in R$ if $e = f$ or if $G - e - f$ is disconnected.

a) $(e, f) \in R$ *if and only if* e *and* f *belong to the same cycles.* Suppose that $(e, f) \in R$. If $e = f$, then e and f belong to the same cycles. If $G - e - f$ is disconnected, then f is a cut-edge in $G - e$, whence f belongs to no edges in $G - e$, and thus every cycle in G containing f must also contain e. Since similarly e is a cut-edge in $G - f$, we conclude also that f belongs to every cycle containing e. Thus e and f belong to the same cycles.

If $G - e - f$ is connected, then f is not a cut-edge in $G - e$ and thus belongs to a cycle in $G - e$; this is a cycle in G that does not contain e.

b) R *is an equivalence relation on* $E(G)$. The reflexive property holds by construction: $(e, e) \in R$ for all $e \in E(G)$. The symmetric property holds because $G - f - e$ is disconnected if $G - e - f$ is disconnected. The transitive property holds by part (a): if $(e, f) \in R$ and $(f, g) \in R$, then e and f belong to the same cycles, and f and g belong to the same cycles, and thus e and g belong to the same cycles (those containing f), and therefore $(e, g) \in R$.

c) Each equivalence class is contained in a cycle. We prove the stronger statement that a cycle contains an element of a class if and only if it contains the entire class. If some cycle contains some element e of the class and omits some other element f, then e and f do not belong to the same cycles, which contradicts (a).

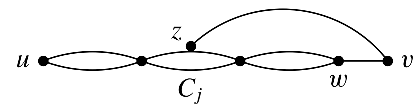
d) For each equivalence class F, $G - F$ *has no cut-edge.* If e is a cut-edge in $G - F$, then e lies in no cycle in $G - F$, so every cycle in G containing e contains some element of F. By the stronger statement in (c), every such cycle contains all of F. Deleting a single edge $f \in F$ breaks all cycles containing F. Thus $G - e - f$ is disconnected, which yields $(e, f) \in R$, which prevent e and f from being in different classes.

4.2.14. *A graph G is 2-edge-connected if and only if for all $u, v \in V(G)$ there is a u, v-necklace in G,* where a *u, v-necklace* is a list of cycles $C_1, \ldots, C_k$ such that $u \in C_1$, $v \in C_k$, consecutive cycles share one vertex, and nonconsecutive cycle are disjoint. The condition is sufficient, because a u, v-necklace has two edge-disjoint u, v-paths, and these cannot both be cut by deleting a single edge. Conversely, suppose that G is 2-edge-connected. We obtain a u, v-necklace.

Proof 1 (induction on $d(u, v)$). Basis step ($d(u, v) = 1$): A u, v-path in $G - uv$ combines with the edge uv to form a u, v-necklace in G.

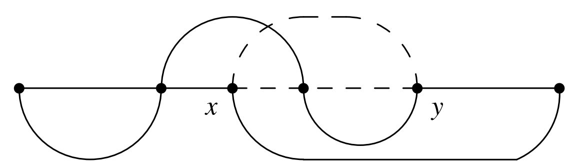
Induction step ($d(u, v) > 1$). Let w be the vertex before v on a shortest u, v-path; note that $d(u, w) = d(u, v) - 1$. By the induction hypothesis, G has a u, w-necklace. If v lies on this u, w-necklace, then the cycles up to the one containing v form a u, v-necklace.

Otherwise, let R be a u, v-path in $G - wv$; this exists since G is 2-edge-connected. Let z be the last vertex of R on the u, w-necklace; let C_j be the last cycle containing z in the necklace. The desired u, v-necklace consists of the cycles before C_j in the u, w-necklace together with a final cycle containing v. The final cycle consists of the remainder of R from z to v, the edge vw, a path from w to C_j in the u, v-necklace, and the path on C_j from there to z that contains the vertex of $C_j \cap C_{j-1}$. The choice of z guarantees that this is a cycle.



Comment. There is also a proof by induction on the number of ears in an ear decomposition, but showing that all pairs still have necklaces when an open ear is added still involves a discussion like that above. Another inductive proof involves showing that the union of a necklace from u to w and an necklace from w to v contains a necklace from u to v.

Proof 2 (extremality). Since G is 2-edge-connected, there exist two edge-disjoint u, v-paths. Among all such pairs of paths, choose a pair P_1, P_2 whose lengths have minimum sum. Let S be the set of common vertices of P_1 and P_2. If the vertices of S occur in the same order on P_1 and P_2, then $P_1 \cup P_2$ is a u, v-necklace. Otherwise, let x, y be the first vertices of P_1 in S that occur in the opposite order on P_2, with x before y in P_1 and after y in P_2. In the figure, P_1 is the straight path. Form two new u, v-paths: Q_1 consists of the portion of P_1 up to x and the portion of P_2 after x, and Q_2 consists of the portion of P_2 up to y and the portion of P_1 after y. Neither of Q_1, Q_2 uses any portion of P_1 or P_2 between x and y, so we have found edge-disjoint u, v-paths with shorter total length. This contradiction completes the proof.



4.2.15. *If G is a 2-connected graph and $v \in V(G)$, then v has a neighbor u such that $G - u - v$ is connected.*

Proof 1 (structure of blocks). Because G is 2-connected, $G - v$ is connected. If $G - v$ is 2-connected, then we may let u be any neighbor of v. If $G - v$ is not 2-connected, let B be a block of $G - v$ containing exactly one cutvertex of $G - v$, and call that cutvertex x. Now v must have a neighbor in $B - x$, else $G - x$ is disconnected, with $B - x$ as a component. Let u be a neighbor of v in $B - x$. Since $B - u$ is connected, $G - v - u$ is connected.

Proof 2 (extremality) If v has no such neighbor, then for every $u \in N(v)$, the graph $G - v - u$ is disconnected. Choose $u \in N(v)$ such that $G - v - u$ has as small a component as possible; let H be the smallest component of $G - v - u$. Since G is 2-connected, v and u have neighbors in every component of $G - v - u$. Let x be a neighbor of v in H. If $G - v - x$ is disconnected, then it has a component that is a proper subgraph of H. This contradicts the choice of u, so $G - v - x$ is connected.

4.2.16. *If G is a 2-connected graph, and T_1 and T_2 are two spanning trees of G, then T_1 transforms into T_2 by a sequence of operations in which a leaf is removed and reattached using another edge of G.* Let T be a largest tree contained in both T_1 and T_2; this is nonempty, since each single vertex is such a tree. We use induction on the number of vertices of G omitted by T. If none are omitted, then $T_1 = T_2$ and the sequence has length 0. If one vertex is omitted, then it is a leaf in both T_1 and T_2, and a single reattachment suffices.

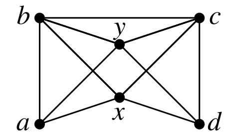
Otherwise, for $i \in \{1, 2\}$ let $x_i y_i$ be an edge of T_i with $x_i \in V(T)$ and $y_i \notin V(T)$. If $y_1 \neq y_2$, then enlarge $T + x_1y_1 + x_2y_2$ to a spanning tree T' of G. Since T' shares more with T_1 than T does, the induction hypothesis yields a sequence of leaf exchanges that turns T_1 into T'. Similarly, it yields a sequence that turns T' into T_2. Together, they complete the desired transformation.

Hence we may assume that $y_1 = y_2$ (this may be necessary even when T omits many vertices of G). We generate another edge x_3y_3 with $x_3 \in V(T)$ and $y_3 \in V(G - y_1)$ (this is possible since G is 2-connected). Now enlarge $T + x_1y_1 + x_3y_3$ to a spanning tree T' and $T + x_2y_2 + x_3y_3$ to a spanning tree T''. For each pair (T_1, T'), (T', T''), or (T'', T_2), there is now a common subtree consisting of T and one additional edge. Hence we can use the induction hypothesis to turn T_1 into T', then T' into T'', and finally T'' into T_2, completing the desired transformation.

(Note: Induction also yields the statement that the common subtree T is never changed during the transformation.)

4.2.17. *The smallest graph with connectivity 3 having a pair of nonadjacent vertices joined by 4 internally-disjoint paths.* "Smallest" usually means least number of vertices, and within that the least number of edges. Let x, y be the nonadjacent pair joined by 4 internally disjoint paths. Each such path has at least one vertex and two edges, so we have at least four more vertices $\{a, b, c, d\}$. We construct a graph achieving this. Since G must be 3-connected, $G - \{x, y\}$ is connected, so if we add no more vertices we

must have a tree on the other four vertices. We add the path a, b, c, d. To complete the prove, we need only show that the graph we have constructed has connectivity 3. Deleting $\{b, x, y\}$ separates a from $\{c, d\}$. To see that G is 3-connected, observed that for each v, $G - v$ contains a spanning cycle and hence is 2-connected, so G is 3-connected.



4.2.18. *If a graph G has no isolated vertices and no even cycles, then every block of G is an edge or a cycle.* A block with two vertices is an edge (if there are no even cycles, then there are no multiple edges). A block H with more than two vertices is 2-connected and has an ear decomposition. If H is not a single cycle, then the addition of the first ear to the first cycle creates a subgraph in which a pair of vertices is connected by three pairwise internally-disjoint paths. By the pigeonhole principle, two of the paths have length of the same parity (both odd or both even), and their union is an even cycle. Hence H must be a single cycle.

4.2.19. *Membership in common cycles.*

a) Two distinct edges lie in the same block of a graph if and only if they belong to a common cycle. Choose $e, f \in E(G)$. If e and f lie in a cycle, then this cycle forms a subgraph with no cut-vertex; by the definition of block, the cycle lies in a single block. Conversely, consider edges e and f in a block B. If e and f have the same endpoints, then they form a cycle of length 2. Otherwise, B has at least three vertices and is 2-connected. In a 2-connected graph, for every edge pair e, f, there is a cycle containing e and f.

b) If $e, f, g \in E(G)$, and G has a cycle through e and f and a cycle through f and g, then G also has a cycle through e and g. By part (a), e and f lie in the same block. By part (a), f and g lie in the same block. Since the blocks partition the edges, this implies that e and g lie in the same block. By part (a), this now implies that some cycle in G contains e and g.

4.2.20. *k-connectedness of the hypercube Q_k by explicit paths.* We use induction on k to show that for $x, y \in V(Q_k)$, there are k pairwise internally disjoint x, y-paths for each vertex pair $x, y \in V(Q_k)$. When $k = 0$, the claim holds vacuously.

For $k > 1$, consider vertex x and y as binary k-tuples. Suppose first that they agree in some coordinate. If they agree in coordinate j, then let Q be the copy of Q_{k-1} in Q_k whose vertices all have that value in coordinate j, and let Q' be the other copy of Q_{k-1}. By the induction hypothesis, Q

contains $k-1$ pairwise internally disjoint x,y-paths. Let x' and y' be the neighbors of x and y in Q'. Combining an x',y'-path in Q' with the edges xx' and yy' yields the kth path, since it has no internal vertices in Q.

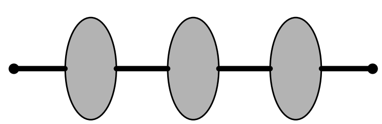
If x and y agree in no coordinate, then we define k paths explicitly as follows. The jth path begins from x by flipping the jth coordinate, then the $j+1$st, $j+2$nd, etc., cyclically (flipping the first coordinate after the kth). After k steps, the path reaches y. The vertices on the jth path agree with y for a segment of positions starting with coordinate j and agree with x for a segment ending with coordinate $j-1$, so the paths share no internal vertices.

4.2.21. *If G is $2k$-edge-connected and has at most two vertices of odd degree, then G has a k-edge-connected orientation.* It suffices to orient the edges so that at least k edges leave each nonempty proper subset of the vertices. When $k=0$, the statement is trivial, so we may assume that $k>0$.

Since G has at most two vertices of odd degree, G has an Eulerian trail. Choose an Eulerian trail T. Let D be the orientation obtained by orienting each edge of G in the direction in which T traverses it. Let $[S,\overline{S}]$ be an edge cut of G. When crossing the cut, the trail alternately goes from one side and then from the other, so it alternately orients edges leaving or entering S. Since G is $2k$-connected, $\left|[S,\overline{S}]\right| \ge 2k$, and the alternation means that at least k edges leave each side in the orientation.

4.2.22. *If $\kappa(G)=k$ and $\operatorname{diam} G = d$, then $n(G) \ge k(d-1)+2$ and $\alpha(G) \ge \lceil (1+d)/2 \rceil$, and these bounds are best possible.* Let G be a k-connected graph with diameter d, in which $d(x,y)=d$. Since G is k-connected, Menger's Theorem guarantees k pairwise internally disjoint x,y-paths in G. With x and y, these paths form a set of $k(d-1)+2$ vertices in G. The vertices consisting of all vertices having even distance from x along a shortest x,y-path form an independent set of size $\lceil (1+d)/2 \rceil$.

For optimality of the bounds, let $V_0,\ldots,V_d$ be "level sets" of size k, except that $|V_0| = |V_d| = 1$. Form G on these $k(d-1)+2$ vertices by making each vertex adjacent to the vertices in its own level and the two neighboring levels. The graph G has order $k(d-1)+2$ and diameter d. Also it is k-connected; if fewer than k vertices are deleted, then each internal set still has an element, so paths remain from each remaining vertex to each neighboring layer. The vertex set is covered by $\lceil (1+d)/2 \rceil$ cliques (each consisting of two consecutive levels), so $\alpha(G) \le \lceil (1+d)/2 \rceil$.



4.2.23. *König-Egerváry from Menger.* Let G be an X,Y-bigraph. Form a digraph G' by adding a vertex x with edges to X and a vertex y with edges from Y, and direct the edges of G from X to Y in G'. The idea is that internally disjoint x,y-paths in G' correspond to edges of a matching in G. Menger's Theorem states that the condition for having a set of k internally disjoint x,y-paths in G' (and hence a matching of size k in G) is that every x,y-separating set R has size at least k.

If we delete the endpoints from a set of internally disjoint x,y-paths in G', we obtain a set of edges in G with no common endpoints. Hence $\alpha'(G) \ge \lambda_{G'}(x,y)$.

An x,y-separating set R in G' consists of some vertices in X and some vertices in Y. In order to break all x,y-paths in G', such a set must contain an endpoint of every edge in G. Hence R is a vertex cover in G. Applying this to a smallest x,y-separating set yields $\kappa_{G'}(x,y) \ge \beta(G)$.

By Menger's Theorem, we now have $\alpha'(G) \ge \lambda_{G'}(x,y) = \kappa_{G'}(x,y) \ge \beta(G)$. Since weak duality yields $\alpha'(G) \le \beta(G)$ for every graph G, we have $\alpha'(G) = \beta(G)$ (König-Egerváry Theorem).

4.2.24. *If G is k-connected, and S,T are disjoint subsets of $V(G)$ with size at least k, then there exist k pairwise disjoint S,T-paths.* By the Expansion Lemma, we can add a vertex x adjacent to each vertex of S and a vertex y adjacent to each vertex of T, and the resulting graph will also be k-connected. Menger's Theorem then yields k disjoint x,y-paths, and since x is adjacent to all X and y to all Y we may assume each path has only one vertex of X and only one vertex of Y. If we delete x and y from these paths, we obtain k pairwise disjoint S,T-paths in G.

4.2.25. *Dirac's Theorem that every k vertices in a k-connected graph lie on a cycle is best possible.* $K_{k,k+1}$ is a k-connected graph where the $k+1$ vertices of the larger partite set do not lie on a cycle.

4.2.26. *For $k \ge 2$, a graph G with at least $k+1$ vertices is k-connected if and only if for every $T \subseteq S \subseteq V(G)$ with $|S| = k$ and $|T| = 2$, there is a cycle in G that contains T and avoids $S-T$.*

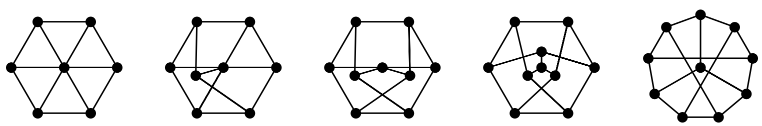
Necessity. If G is k-connected, then $G-(S-T)$ is 2-connected, since $|S-T| = k-2$. In a 2-connected graph, every pair of vertices (such as T) lies on a cycle. Since $S-T$ has been discarded, the cycle avoids it.

Sufficiency. We prove the contrapositive. If G is not k-connected, then G has a separating set U of size $k-1$. Let T consist of one vertex from each of two components of $G-U$. Let $S = T \cup U$. The condition now fails, since deleting $S-T$ leaves no cycle through both vertices of T.

4.2.27. A *vertex k-split* of a graph G is a graph H obtained from G by replacing one vertex $x \in V(G)$ by two adjacent vertices x_1, x_2 such that $d_H(x_i) \ge k$ and that $N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\}$.

a) If G is a k-connected graph, and G' is a graph obtained from G by replacing one vertex $x \in V(G)$ with two adjacent vertices x_1, x_2 such that $N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\}$ and $d_H(x_i) \ge k$, then G is k-connected. Suppose S is a separating j-set of H, where $j < k$, and let $X = \{x_1, x_2\}$. Note that $H - S$ cannot have x_1, x_2, or X as a component, because $d_H(x_i) \ge k$ and X has edges to at least k distinct vertices of $H - X$. If $|S \cap X| = 2$, then $H - S = G - (S - X \cup x)$, and $S - X \cup x$ is a separating $j-1$-set of G. If $|S \cap X| = 1$, then $S \cup X$ separates H, since $X - S$ is not a component of $H - S$. Hence $S - X \cup x$ is a separating j-set of G, which requires $j \ge k$. Finally, suppose $S \cap X = \varnothing$. Now $\{x_1, x_2\}$ must belong to the same component of $H - S$. Contracting an edge of a component in a disconnected graph leaves a disconnected graph, so in this case S separates G.

b) Every graph obtained from a "wheel" $W_n = K_1 \vee C_{n-1}$ by a sequence of edge additions and vertex 3-splits on vertices of degree at least 4 is 3-connected. Since wheels are 3-connected, part (a) implies that every graph arising from wheels by 3-splits and edge additions is also 3-connected. The Petersen graph arises by successively splitting off vertices from the central vertex of the wheel $K_1 \vee C_6$. Each newly-split vertex acquires two neighbors on the outside and remains adjacent to the central vertex.



4.2.28. *If X and Y are disjoint vertex sets in a k-connected graph G and are assigned nonnegative integer weights with $\sum_{x\in X} u(x) = \sum_{y\in Y} w(y) = k$, then G has k pairwise internally disjoint X, Y-paths from X to Y such that $u(x)$ of them start at x and $w(y)$ of them end at y.* We may assume that all weights are positive, since otherwise we delete vertices of weight 0 from X and Y and apply the argument to the sets that remain.

We construct a related G' and apply Menger's Theorem. Add copies of vertices in X and Y, with each new vertex having the same neighborhood as the vertex it copies. Since G is k-connected, these neighborhoods have size at least k, and by the Expansion Lemma the new graph is k-connected. We do this until there are $u(x)$ copies of each x and $w(y)$ copies of each y.

Next add two additional vertices s and t joined to the copies of all $x \in X$ and the copies of all $y \in Y$, respectively. Note that s and t each have degree k in this final graph G'. By the Expansion Lemma, G' is k-connected. By Menger's Theorem, there are k pairwise internally disjoint s, t-paths in G'.

These must depart s via its k distinct neighbors and reach t via its k distinct neighbors, so each path connects a copy of some $x \in X$ to a copy of some $y \in Y$, and no x or y appears in one of these paths except at endpoints. Collapsing G' to G by identifying the copies of each original vertex turns these into the desired paths, since there are $u(x)$ copies of each x and $w(y)$ copies of each y and one path at the original vertex arising from each copy of it in G'.

4.2.29. *Graph connectivity from connectivity in the corresponding symmetric digraph.* From a graph G, we form D be by replacing each edge with two oppositely-directed edges. Given two vertices a, b on a path P, let $P(a, b)$ denote the a, b-path along P.

If $\kappa'_D(x, y) = \lambda'_D(x, y)$, then $\kappa'_G(x, y) = \lambda'_G(x, y)$. It suffices to prove that $\lambda'_G(x, y) \ge \lambda'_D(x, y)$ and $\kappa'_G(x, y) \le \kappa'_D(x, y)$, since the weak duality $\lambda'_G(x, y) \le \kappa'_G(x, y)$ holds always.

Let $\mathbf{F}$ be a family of $\lambda'_D(x, y)$ pairwise edge-disjoint x, y-paths in D. If there is some vertex pair u, v such that uv appears in a path P in $\mathbf{F}$ and vu appears in another path Q in $\mathbf{F}$, then we modify $\mathbf{F}$. Let P' be path consisting of $P(x, u)$ followed by $Q(u, y)$, and let Q' be the path consisting of $Q(x, v)$ followed by $P(v, y)$. Replacing P and Q with P' and Q' in $\mathbf{F}$ reduces the number of edges that used in both directions. Repeating this replacement yields a family $\mathbf{F}'$ with no such doubly-used pair. Now $\mathbf{F}'$ becomes a family of $\lambda'_D(x, y)$ pairwise edge-disjoint x, y-paths in G using the same succesion of vertices, and hence $\lambda'_G(x, y) \ge \lambda'_D(x, y)$.

Let R be a set of $\kappa'_D(x, y)$ edges in D whose removal makes y unreachable from x. By the construction of D from G, every x, y-path in G must use an edge having a copy in R. Hence the corresponding edges in G form an x, y-disconnecting set, and $\kappa'_G(x, y) \le \kappa'_D(x, y)$.

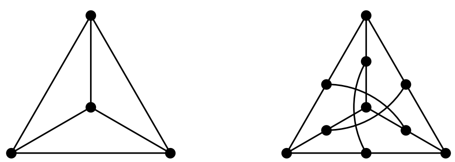
If $x \not\to y$ in D and $\kappa_D(x, y) = \lambda_D(x, y)$, then $\kappa_G(x, y) = \lambda_G(x, y)$. It suffices to prove that $\lambda_G(x, y) \ge \lambda_D(x, y)$ and $\kappa_G(x, y) \le \kappa_D(x, y)$, since the weak duality $\lambda_G(x, y) \le \kappa_G(x, y)$ holds always.

Let $\mathbf{F}$ be a family of $\lambda_D(x, y)$ pairwise internally-disjoint x, y-paths in D. Since these pairs pairwise share no vertices other than their endpoints, there is no pair u, v such that the edges uv and vu are both used. In particular, the paths (listed by vertices) in $\mathbf{F}$ also form a family of $\lambda_D(x, y)$ pairwise internally-disjoint x, y-paths in G, and $\lambda_G(x, y) \ge \lambda_D(x, y)$.

Let R be a set of $\kappa_D(x, y)$ vertices in D whose removal makes y unreachable from x. By the construction of D from G, every x, y-path in G uses a vertex of R. Hence R is an x, y-separating set in G, and $\kappa_G(x, y) \le \kappa_D(x, y)$.

4.2.30. *Expansion preserves 3-connectedness.* Suppose that G' is obtained from G by expansion (subdividing xy and wz and adding an edge st joining the two new vertices). It suffices to show that if G is 3-connected, then deleting a vertex from G' always leaves a 2-connected graph. If $v \in V(G)$, then we can obtain an ear decomposition of $G' - v$ from an ear decomposition of $G - v$ by making the ear a bit longer when the edge xy or wz is added and adding the edge st at the end. To obtain an ear decomposition of $G' - t$, observe that $G - wz$ is 2-connected (deleting an edge reduces connectivity by at most 1). Use an ear decomposition of $G - wz$, lengthening the ear when xy is added, and then add two ears through t. (There are many other proofs.)

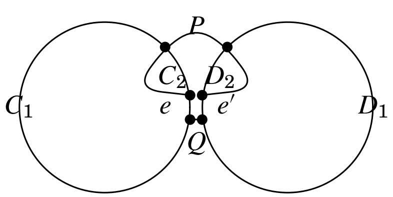
To obtain the Petersen graph from K_4 by expansions, perform expansion on the three pairs of nonincident edges in K_4, independently.

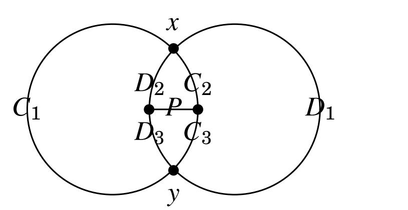


4.2.31. *Longest cycles in k-connected graphs.*

a) In a k-connected graph (for $k = 2, 3$), any two longest cycles have at least k vertices in common. (The claim is false for $k = 1$, as shown by two cycles joined by a single cut edge.) Let $l(H)$ denote the length of a cycle or path H, let C, D be two longest cycles, and let $S = V(C) \cap V(D)$. The proof is by contradiction; if $|S| < k$, it suffices to construct two other cycles C', D' such that $l(C') + l(D') > l(C) + l(D)$, because then C and D are not longest cycles in G.

Consider $k = 2$. Let e be an edge of C, and e' an edge of D, chosen to share the vertex of S if $|S| = 1$. Since G is 2-connected, there is a cycle R containing both e and e'. The two portions of R between e and e' contain paths P, Q that travel from $V(C)$ to $V(D)$ with no vertices of $V(C) \cup V(D)$ along the way. (If $|S| = 1$, then one of these paths is a single vertex and has length 0.) Note that since R is a cycle, P and Q are disjoint. The vertices where P and Q intersect C and D partition C and D into paths C_1, C_2 and D_1, D_2, respectively. Let $C' = C_1 \cup P \cup D_1 \cup Q$ and $D' = C_2 \cup P \cup D_2 \cup Q$; we have $l(C') + l(D') = l(C) + l(D) + 2l(P) + 2l(Q) > l(C) + l(D)$.





Consider $k = 3$. Since G is also 2-connected, we may assume by the

argument above that $|S| = 2$. Now $G - S$ is connected and has a shortest path P between $C - S$ and $D - S$. The vertices where P meets C and D, together with the vertices $S = \{x, y\}$, partition C and D into three paths C_1, C_2, C_3 and D_1, D_2, D_3, where C_1, D_1 are y, x-paths, C_2, D_2 are $x, V(P)$-paths, and C_3, D_3 are $y, V(P)$-paths. Let $C' = C_1 \cup C_2 \cup P \cup D_3$ and $D' = D_1 \cup D_2 \cup P \cup C_3$. Now $l(C') + l(D') = l(C) + l(D) + 2l(P) > l(C) + l(D)$.

b) For $k \ge 2$, one cannot guarantee more than k common vertices. The graph $K_{k,2k}$ is k-connected and has two cycles sharing only the smaller partite set.

4.2.32. *Given $k \ge 2$, let G_1 and G_2 be disjoint k-connected graphs, with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. If B is a bipartite graph with parts $N_{G_1}(v_1)$ and $N_{G_2}(v_2)$ that has no isolated vertex and has a matching of size at least k, then $(G_1 - v_1) \cup (G_2 - v_2) \cup B$ is k-connected.* Let $G = (G_1 - v_1) \cup (G_2 - v_2) \cup B$. It suffices to show that for distinct vertices $x, y \in V(G)$, there is a family of k independent x, y-paths.

If $x, y \in V(G_1)$, then there are k such paths from G_1, except that one of them may pass through v. If x' and y' are the neighbors of v along this path, then we replace $\langle x', v, y' \rangle$ with a path through G_2, using edges in B incident to x' and y'. The argument is symmetric when $x, y \in V(G_2)$.

If $x \in V(G_1)$ and $y \in V(G_2)$, then let $X \subseteq N_{G_1}(v_1)$ and $Y \subseteq N_{G_2}(v_2)$ be the partite sets of a matching M of size k in B. Deleting v_1 from k independent x, v_1-paths in G_1 leaves an x, X-fan. Similarly, deleting v_2 from k independent y, v_2-paths in G_2 leaves an y, Y-fan. Combining M with these two fans yields the desired x, y-paths.

The claim fails for $k = 1$. If G_1 and G_2 are stars, with centers v_1 and v_2, then the resulting graph G is simply the bipartite graph B. The only requirement on B is that it have no isolated vertices. In particular, it need not be connected.

4.2.33. *Ford-Fulkerson CSDR Theorem implies Hall's Theorem.* Given an X, Y-bigraph G with $X = \{x_1, \ldots, x_m\}$, let $A_i = B_i = N(x_i)$. If the systems $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ have a common CSDR, then $A_1, \ldots, A_m$ has an SDR, and thus G has a matching saturating X. Thus it suffices to show that Hall's Condition on G implies the Ford-Fulkerson condition for these systems.

Let $I, J \subseteq [m]$ be sets of indices. Since $\bigcup_{j \in J} B_j = \bigcup_{j \in J} A_j$, we have

$$\left|\bigcup_{i \in I} A_i \cap \bigcup_{j \in J} B_j\right| = \left|(\bigcup_{i \in I} A_i) \cap \bigcup_{j \in J} A_j)\right| \ge \left|\bigcup_{i \in I \cap J} A_i\right|.$$

By Hall's Condition, $\left|\bigcup_{i \in I \cap J} A_i\right| \ge |I \cap J| \ge |I| + |J| - m$. Thus the Ford-Fulkerson condition holds in G, as desired.

If $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ are partitions of a set E into sets of size s, then the two systems have a CSDR. It suffices to show that the systems satisfy the Ford-Fulkerson condition. By the defining condition, $\left|\bigcup_{i\in I} A_i\right| = |I|\,s$ and $\left|\bigcup_{j\in J} B_j\right| = |J|\,s$. Thus

$$\begin{aligned}\left|\bigcup_{i\in I} A_i \cap \bigcup_{j\in J} B_j\right| &\ge \left|\bigcup_{i\in I} A_i\right| + \left|\bigcup_{j\in J} B_j\right| - ms = |I|\,s + |J|\,s - ms \\ &= s(|I| + |J| - m) \ge |I| + |J| - m.\end{aligned}$$

4.2.34. *Every minimally 2-connected graph has a vertex of degree 2.* Consider an ear decomposition of a minimally 2-connected graph G. If the last ear adds just one edge e, then $G - e$ also has an ear decomposition and is 2-connected. Hence the last ear added contains a vertex of degree 2.

A minimally 2-connected graph G with at least 4 vertices has at most $2n(G)-4$ edges, with equality only for $K_{2,n-2}$. The graph $K_{2,n-2}$ is minimally 2-connected and has $2n-4$ vertices. For the upper bound, we use induction on $n(G)$. When $n(G) = 4$, $K_{2,2}$ is the only minimally 2-connected graph. When $n(G) > 4$, consider an ear decomposition of G. If G is only a cycle, then the bound holds, with strict inequality. Otherwise, delete the last added ear from G to obtain G'. This deletes k vertices and $k+1$ edges, where $k \ge 1$ as observed above.

The graph G' is also minimally 2-connected, since if $G' - e$ is 2-connected, then also $G - e$ is 2-connected. Hence $e(G') \le 2n(G') - 4$, by the induction hypothesis. In terms of G, this states that $e(G) - k - 1 \le 2n(G) - 2k - 4$, which simplifies to $e(G) \le 2n(G) - k - 3 \le 2n(G) - 4$. Equality requires $k = 1$, and by the induction hypothesis also $G' = K_{2,n-3}$. The only way to add an ear of length two to $K_{2,n-3}$ and obtain a minimally 2-connected graph is to add it connecting the two vertices of high degree.

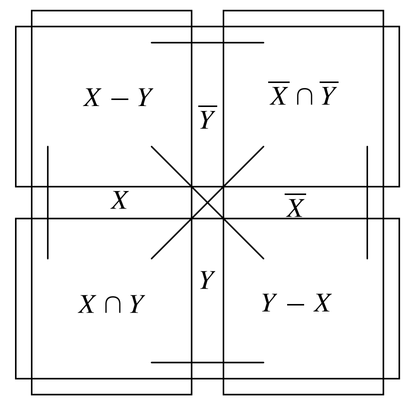
4.2.35. *A 2-connected graph is minimally 2-connected if and only if no cycle has a chord.* Suppose that G is 2-connected. We show that $G - xy$ is 2-connected if and only if x and y lie on a cycle in $G - xy$. If $G - xy$ is not 2-connected, then there is a vertex v whose deletion separates x and y, and thus all x, y-paths in $G - xy$ pass through v and $G - xy$ has no cycle containing x and y. Conversely, if $G - xy$ is 2-connected, then every pair of vertices (including x, y) lies on a cycle.

If a cycle in G has a chord x, y, then this argument shows that $G - xy$ is still 2-connected, and hence G is not minimally 2-connected. If no cycle has a chord, then for any edge xy, the graph $G - xy$ has no cycle containing x and y, and so $G - xy$ is not 2-connected.

4.2.36. If $X, Y \subseteq V(G)$, then $d(X \cap Y) + d(X \cup Y) \le d(X) + d(Y)$, where $d(S)$ is the number of edges leaving S. With respect to the sets X, Y, there

are four types of vertices, belonging to none, either, or both of the two sets. Between pairs of the four sets $X\cap Y$, $X - Y$, $Y - X$, $\overline{X}\cap\overline{Y}$, there are six types of edges. We list the contribution of each type to the counts on both sides of the desired inequality. Each edge contributes at least as much to the right side as to the left side of the inequality. This proves the inequality; note that equality holds if and only if $[X - Y, Y - X]$ is empty.

endpoints	$d(X\cap Y)$	$d(X\cup Y)$	$d(X)$	$d(Y)$
$X\cap Y,\ X-Y$	•			•
$X\cap Y,\ Y-X$	•		•	
$X\cap Y,\ \overline{X}\cap\overline{Y}$	•	•	•	•
$X-Y,\ \overline{X}\cap\overline{Y}$		•	•	
$Y-X,\ \overline{X}\cap\overline{Y}$		•		•
$X-Y,\ Y-X$			•	•



4.2.37. *Every minimally k-edge-connected graph G has a vertex of degree k.* Let $d(X) = \left|[S, \overline{S}]\right|$. If $d(X) > k$ whenever $\varnothing \ne X \subset V(G)$, then $G - e$ is k-edge-connected for each $e \in E(G)$, and G is not minimally k-edge-connected. Hence we may assume that $d(X) = k$ for some set X.

Suppose that $G[X]$ has an edge xy. Since $G - xy$ is not k-edge-connected, there is a nonempty $Z \subset V(G)$ (containing exactly one of $\{x, y\}$) such that $k - 1 \ge d_{G-xy}(Z) = d(Z) - 1$. Since G is k-edge-connected, $d(Z) \ge k$, so equality holds.

Now k-edge-connectedness of G and submodularity of d (the result of Exercise 4.2.36 yield

$$k + k \le d(X\cap Z) + d(X \cup Z) \le d(X) + d(Z) = k + k.$$

Since G is k-edge-connected, we obtain $d(X \cap Z) = k$. Since Z contains exactly one of $\{x, y\}$, the set $X \cap Z$ is smaller than X.

Hence a minimal set X such that $d(X) = k$ must be an independent set. Since each vertex of X has at least k incident edges leaving X, we have $|X| = 1$, and this is the desired vertex of degree k.

4.2.38. *Every $2k$-edge-connected graph has a k-edge-connected orientation.* To prove this theorem of Nash-Williams, we are given Mader's Shortcut Lemma: "If z is a vertex of a graph G such that $d_G(z) \notin \{0,1,3\}$ and z is incident to no cut-edge, then z has neighbors x and y such that $\kappa_{G-xz-yz+xy}(u,v) = \kappa_G(u,v)$ for all $u, v \in V(G) - \{z\}$."

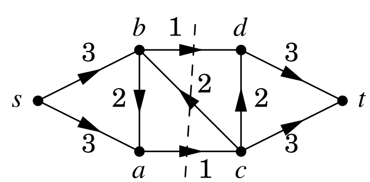
We use induction on $n(G)$. For the basis step, consider two vertices joined by at least $2k$ edges, and orient at least k in each direction.

For the induction step, let G be a $2k$-edge-connected graph with $n(G) > 2$. We discard edges to obtain a minimal $2k$-edge-connected graph; we may later orient the deleted edges arbitrarily. By Exercise 4.2.37, the resulting graph has a vertex z of degree $2k$, which is even. Mader's Shortcut Lemma iteratively finds shortcuts of z until we reduce the degree of z to 0. Throughout this process, we maintain $2k$-edge-connectedness for pairs of points not including z. At the end, we delete z to obtain a $2k$-edge-connected graph G' with $n(G) - 1$ vertices.

By the induction hypothesis, G' has a k-edge-connected orientation. Orient G by replacing each shortcut edge uv with the path u, z, v or v, z, u, oriented consistently with uv in G'. For $X \neq \{z\}$, lifting uv preserves $d(X) \geq k$ in the orientation; the only edge lost is uv, and if uv leaves X, then uz or zv is a new edge leaving X, depending on whether $z \in X$. The set $X = \{z\}$ itself reaches $d(X) = k$ after all k lifts.

4.3. NETWORK FLOW PROBLEMS

4.3.1. *Listing of feasible integer s,t-flows in a network.* This problem demonstrates the value of integer min-max relations in escaping exhaustive computation.



A feasible flow is an assignment of a flow value to each edge. It is not an assignment of flow paths. Every network has a feasible flow of value 0. In this network, there is a cycle ba, ac, cb with positive capacity, which makes it possible to "add" to a flow without adding

to the value of the flow. In particular, there are two feasible integer flows of value 0, eight of value 1, and four of value 2. We can specify each flow by the vector of values on the edges. We list these as $(f(sa), f(sb), f(ba), f(ac), f(bd), f(cb), f(cd), f(ct), f(dt))$, with each column of the matrix below corresponding to one flow.

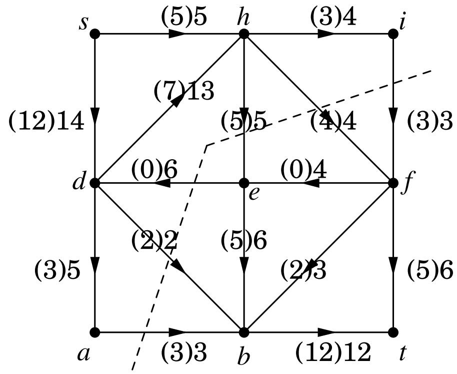
sa	0	0	1	1	1	0	0	0	0	0	1	0	1	0
sb	0	0	0	0	0	1	1	1	1	1	1	2	1	2
ba	0	1	0	0	0	1	1	1	0	1	0	1	0	1
ac	0	1	1	1	1	1	1	1	0	1	1	1	1	1
bd	0	0	0	0	1	0	0	1	1	1	1	1	1	1
cb	0	1	0	0	1	0	0	1	0	1	0	0	0	0
cd	0	0	0	1	0	0	1	0	0	0	0	0	1	1
ct	0	0	1	0	0	1	0	0	0	0	1	1	0	0
dt	0	0	0	1	1	0	1	1	1	1	1	1	2	2
value	0	0	1	1	1	1	1	1	1	1	2	2	2	2

Since the network has four internal vertices, there are $2^4 = 16$ ways to specify a source-sink cut $[S, T]$. In general, the resulting edge cuts might not be distinct as sets of edges, but for this network they are distinct. Incidence vectors for the cuts appear in the columns below; a 1 for edge e in column $[S, T]$ means that e belongs to the cut $[S, T]$. The cut $[sab, cdt]$ with capacity equal to the maximum flow value is suggested by dashed lines in the figure. Exhibiting a flow and a cut of equal value proves that the flow value is maximal and the cut capacity is minimal; this is a shorter and more reliable proof of flow optimality than listing all feasible flows.

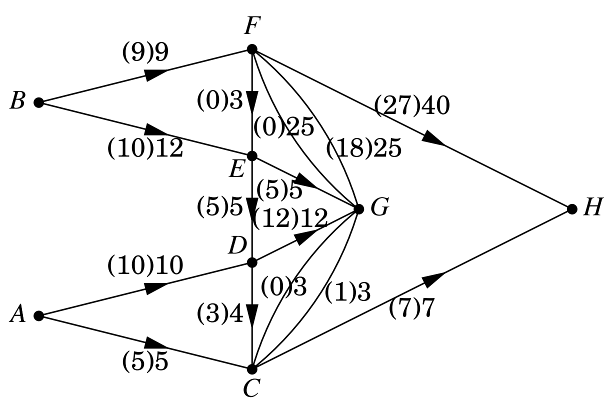
S	*s*	*sa*	*sb*	*sc*	*sd*	*sab*	*sac*	*sad*	*sbc*	*sbd*	*scd*	*sabc*	*sabd*	*sacd*	*sbcd*	*sabcd*
T	*abcdt*	*bcdt*	*acdt*	*abdt*	*abct*	*cdt*	*bdt*	*bct*	*adt*	*act*	*abt*	*dt*	*ct*	*bt*	*at*	*t*
sa	1	0	1	1	1	0	0	0	1	1	1	0	0	0	1	0
sb	1	1	0	1	1	0	1	1	0	0	1	0	0	1	0	0
ba	0	0	1	0	0	0	0	0	1	1	0	0	0	0	1	0
ac	0	1	0	0	0	1	0	1	0	0	0	0	1	0	0	0
bd	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0
cb	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0
cd	0	0	0	1	0	0	1	0	1	0	0	1	0	0	0	0
ct	0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
dt	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1
capac	6	4	6	13	9	2*	10	7	11	8	14	6	4	11	11	6

4.3.2. *In the network with edge capacities as indicated below, the flow values listed in parentheses form a maximum feasible flow.*

By inspection, they satisfy the capacity constraints and the conservation constraints. The value of the flow is 17. To prove that the flow is optimal, we exhibit a solution to the dual (min cut) problem that has the same value. This proves optimality because the value of every feasible flow is at most the capacity of this cut. The cut has source set $\{s, h, i, d, a\}$ and sink set $\{e, f, b, t\}$. The edges of the cut are $\{if, hf, he, db, ab\}$ with total capacity 17; the edge ed does not belong to this cut.



4.3.3. *A maximum flow problem.* Add a source X with edges of infinite capacity to the true sources A and B. The optimal net flow from source to sink is 34. This is optimal because the cut $[XABE, CDFGH]$ has capacity 9+5+5+10+5=34, and no flow can have value larger than the capacity of any cut. The optimal flow values on the various edges are indicated in parentheses on the edges below. (The edges from $XABE$ to $CDFGH$ are saturated; those going back have zero flow.)



4.3.4. *Maximum flow in a network with lower bounds.* When the lower bounds equal 0, we have an ordinary network, and the ordinary Ford-Fulkerson labeling algorithm begins with the feasible 0 flow. In a network

with lower bound $l(uv)$ on the flow in each edge uv (and upper bound $c(uv)$), the labeling algorithm generalizes when we are given a feasible flow.

We seek an augmenting path to obtain a flow of larger value. In the statement of the algorithm, we still use the same definition of excess capacity when exploring vw from v. For an edge uv entering v, the requirement for reducing flow along uv to extend the potential augmenting path to u is "excess flow": $f(uv) > l(uv)$. Under either condition, we place the other endpoint of the edge involving v in the set R of "reached vertices". If we reach the sink, we have an augmenting path, and we let ε be the minimum value of the excess capacities $(c(uv) - f(uv))$ along the forward edges in the path and the excess flows $(f(vu) - l(vu))$ along the backward edges in the path. We adjust the values of f along the path by ε (up for forward edges, down for backward edges), again obtaining a feasible flow with value ε larger than f.

If we do not reach the sink after searching from all vertices of R, then the final searched set S provides a source/sink cut $[S, \overline{S}]$ that proves there is no larger feasible flow. Proving this needs a more general definition of cut capacity. The capacity of a source/sink partition $(S, \overline{S})$ is $\sum_{vw\in[S,\overline{S}]} c(vw) - \sum_{uv\in[\overline{S},S]} l(uv)$. This is an upper bound on the net flow from S to $\overline{S}$. Termination without reaching the sink in the algorithm above requires that the flow equals $c(vw)$ whenever $vw \in [S, \overline{S}]$ and equals $l(uv)$ whenever $uv \in [\overline{S}, S]$. Hence the net flow across this cut equals the generalized capacity of the cut.

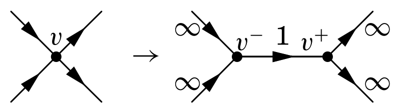
The conservation constraints force the value of a flow f to equal the net flow across any source/sink cut. Thus the value of the final flow equals the generalized capacity of the resulting cut. Since edge cut establishes an upper bound on the value of each feasible flow, this equality shows that both the flow and the cut are optimal.

4.3.5. *Menger for vertices in digraphs, from Ford-Fulkerson.* Consider a digraph G containing vertices x, y, with $xy \notin E(G)$. As usual, the definitions of $\kappa(x, y)$ and $\lambda(x, y)$ yield $\kappa(x, y) \ge \lambda(x, y)$ (weak duality), so the problem is to use the Ford-Fulkerson theorem to prove the opposite inequality. We want to design a suitable network G' so that

$$\lambda(x, y) \ge \max \operatorname{val} f = \min \operatorname{cap}(S, T) \ge \kappa(x, y).$$

In designing a suitable network G', we want to obtain pairwise internally-disjoint x, y-paths in G from units of flow in G'. Thus we have the problem of limiting the total flow through a vertex to 1. Since we can only limit flow via edge capacities, we expand vertex v into two vertices v^- and v^+ joined by an edge v^-v^+ of capacity 1 (for $v \notin \{x, y\}$). Call these the *intra-vertex* edges.

To complete the network, v^- inherits the edges entering v and v^+ inherits those leaving v. More precisely, an edge uv in G becomes an edge u^+v^- in G' (we think of the source x as x^+ and the sink y as y^-. The network also needs capacities on these edges. To simplify our later discussion of the cut, we assign huge capacities to these edges. We may view this as infinite capacity; any integer larger than $n(G)$ suffices.



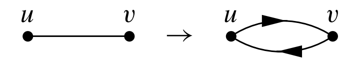
The equality is by the max-flow min-cut theorem. Let k be the common optimal value for the flow problem and the cut problem. For the first inequality, we convert a maximum flow of value k into k pairwise internally-disjoint x, y-paths in G; thus $\lambda(x, y) \geq k$. The integrality theorem breaks the x, y-flow into x, y-paths of unit flow, and these correspond to x, y-paths in G when we shrink the intra-vertex edges. Since each intra-vertex edge has capacity is 1, each vertex of G appears in at most one such path.

For the final inequality, we convert a minimum source/sink cut $[S, T]$ of capacity k into k vertices in G that break all x, y-paths; these yields $\kappa(x, y) \leq k$. If $S = x \cup \{v^-: v \neq x, y\}$ and $T = y \cup \{v^+: v \neq x, y\}$, then $cap(S, T) = n - 2 < n(G)$. Thus no minimum capacity cut has an edge from S to T that is not an intra-vertex edge (this is the reason for assigning the other edges large capacity). As a result, the capacity of every minimum cut equals the number of vertices $v \in V(G)$ such that the intra-vertex edge for v belongs to $[S, T]$. Since deleting the edges of the cut leaves no capacity from S to T, these edges break all x, y-paths in G', and thus the corresponding k vertices form an x, y-separating set in G.

4.3.6. *Menger for edge-disjoint paths in graphs, from Ford-Fulkerson.* Consider a graph G containing vertices x, y. The definitions of $\kappa'(x, y)$ and $\lambda'(x, y)$ yield $\kappa'(x, y) \geq \lambda'(x, y)$ (weak duality), so the problem is to use the Ford-Fulkerson theorem to prove the opposite inequality. We design a suitable network G' so that

$$\lambda'(x, y) \geq \max \operatorname{val} f = \min \operatorname{cap}(S, T) \geq \kappa'(x, y).$$

In designing a suitable network G' with source x and sink y, we want to obtain pairwise edge-disjoint x, y-paths in G from units of flow in G'. An edge can be used in either direction. Thus we obtain G' from G by replacing each undirected edge uv with a pair of oppositely directed edges with endpoints u and v, as suggested below. We give each capacity 1 to each resulting edge.

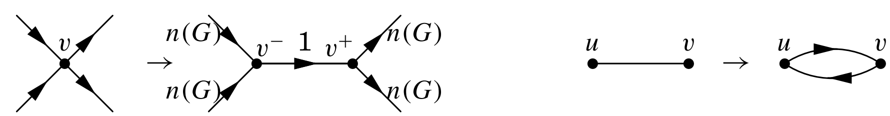


The Integrality Theorem guarantees a maximum flow from x to y in which all values are integers. If this assigns nonzero flow to two oppositely directed edges, then it assigns 1 to each. Replacing these values with 0 preserves the conservation conditions and does not change the value of the flow. Hence we may assume that in our maximum flow each edge from G is used in at most one direction. Now the Integrality Theorem breaks the flow into units of flow from x to y. These yield $\operatorname{val} f$ pairwise edge-disjoint x, y-paths in G, thereby proving the first part of the displayed inequality.

For any cut $[S, T]$ in G' each edge of G between S and T is counted exactly once, in the appropriate direction. Hence $\operatorname{cap}(S, T) = |[S, T]|$. Since $[S, T]$ is an edge cut, the last part of the displayed inequality also holds. We have proved the needed inequality $\lambda'(x, y) \geq \kappa'(x, y)$.

4.3.7. *Menger's Theorem for nonadjacent vertices in graphs:* $\kappa(x, y) = \lambda(x, y)$. Let x and y be vertices in a graph G. An x, y-separating set has a vertex of each path in a set of pairwise internally-disjoint x, y-paths, so $\kappa(x, y) \geq \lambda(x, y)$. It suffices to show that some x, y-separating set and some set of pairwise internally-disjoint x, y-paths have the same size.

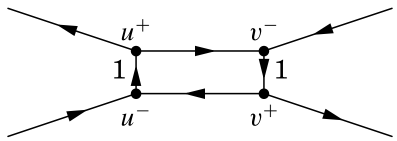
Starting with G, first replace each edge uv with two directed edges uv and vu, as on the right below. Next, replace each vertex w outside $\{x, y\}$ with two vertices w^- and w^+ and an edge of unit capacity from w^- to w^+, as on the left below; call these *internal edges*. Every edge that had the form uv before this split now is replaced with the edge u^+v^-, having capacity $n(G)$. Let D be the resulting network, with source x and sink y. (We often write "∞" as a capacity to mean a sufficiently large capacity to keep those edges out of minimum cuts. Here $n(G)$ is enough.)



By the Max-Flow Min-Cut Theorem, the maximum value of a feasible flow in D equals the minimum value of a source/sink cut in D. Let k be the common value. We show that G has k pairwise internally disjoint x, y-paths and has an x, y-separating set of size k.

By the Integrality Theorem, there is a flow of value k that has integer flow on each edge. Since only the internal edge leaves v^-, with capacity 1, at most one edge into v^- has nonzero flow, and that flow would be 1. Since only the internal edge enters v^+, with capacity 1, at most one edge leaving

v^+ has nonzero flow, and that flow would be 1. Hence the k units of flow transform back into k x, y-paths in G, and the restriction of capacity 1 on v^-v^+ ensures that these paths are internally disjoint. (This includes the observation that we cannot have one path use the edge from u to v and another from v to u; one can see explicitly that the capacity of 1 on the internal edges directly prevents this, as illustrated below.)



Since the capacity of every edge of the form v^+w^- is $n(G)$, every source/sink cut $[S, T]$ that has some such edge has capacity at least $n(G)$. On the other hand, the cut that has x and all internal vertices of the form u^- in S and has y^- and all internal vertices of the form u^+ in T has capacity $n(G) - 2$. Therefore, in every cut with minimum capacity the only edges from S to T are edges of the form u^-u^+. If such a set of edges $[S, T]$ breaks all x, y-paths in D, then $\{u \in V(G)\colon u^-u^+ \in [S, T]\}$ is a set of k vertices in G that breaks all x, y-paths in G.

4.3.8. *Networks to model vertex capacities.* Let G be a digraph with source x, sink y, and vertex capacities $l(v)$ for $v \in V(G)$. To find maximum feasible flow from x to y in G, we define an ordinary network N and use the maximum flow labeling algorithm. For each $v \in V(G)$, create two vertices v^-, v^+ in N, with an edge from v^- to v^+ having capacity $l(v)$. For each $uv \in E(G)$, create an edge $u^+v^- \in E(N)$ with infinite capacity.

Consider a maximum x^+, y^--flow in N, where x, y are the source and sink of G. Contracting all edges of the form v^-v^+ in N transforms any feasible flow in N into a vertex-feasible flow in G with the same value. Similarly, any feasible flow in the vertex-capacitated network G "expands" into a feasible flow in N with the same value. Therefore, the max flow algorithm in N solves the original problem.

4.3.9. *Use of Network Flow to characterize connected graphs.* Given a graph G, form a digraph D by replacing each edge uv of G with the directed edges uv and vu, and give each edge capacity 1. Then G has an x, y-path if and only if the network D with source x and sink y has a flow of value at least 1. By the Max Flow = Min Cut Theorem, this holds if and only every cut S, T with $x \in S$ and $y \in T$ has capacity at least 1, i.e. an edge from S to T. If all partitions S, T have such an edge in G, then for every choice of x and y there is an x, y-path. If for every pair x, y there is a path, the to explore the partition S, T we choose $x \in S$ and $y \in T$, and then the corresponding network problem guarantees that the desired edge exists.

4.3.10. *König-Egerváry from Ford-Fulkerson.* Let G be a bipartite graph with bipartition X, Y. Construct a network N by adding a source s and sink t, with edges of capacity 1 from s to each $x \in X$ and from each $y \in Y$ to t. Orient each edge of G from X to Y in N, with infinite capacity. By the integrality theorem, there is a maximum flow f with integer value at each edge. The edges of capacity one then force the edges between X and Y receiving nonzero flow in f to be a matching. Furthermore, $\mathrm{val}(f)$ is the number of these edges, since the conservation constraints require the flow along each such edge to extend by edges of capacity 1 from s and to t. We have constructed a matching of size $\mathrm{val}(f)$, so $\alpha'(G) \ge \mathrm{val}(f)$.

A minimum cut must have finite capacity, since $[s, V(N) - s]$ is a cut of finite capacity. Let $[S, T]$ be a minimum cut in N, and let $X' = S \cap X$ and $Y' = T \cap Y$. A cut of finite capacity has no edge of infinite capacity from S to T. Hence G has no edge from $S \cap X$ to $T \cap Y$. This means that $(X - S) \cup (Y - T)$ is a set of vertices in G covering every edge of G. Furthermore, the cut $[S, T]$ consist of the edges from s to $X \cap T = X - S$ and from $Y \cap S = Y - T$ to t. The capacity of the cut is the number of these edges, which equals $|(X - S) \cup (Y - T)|$. We have constructed a vertex cover of size $\cap(S, T)$, so $\beta(G) \le \cap(S, T)$.

By the Max flow-Min cut Theorem, we now have $\beta(G) \le \cap(S, T) = \mathrm{val}(f) \le \alpha'(G)$. But $\alpha'(G) \le \beta(G)$ in every graph, so equality holds throughout, and we have $\alpha'(G) = \beta(G)$ for every bipartite graph G.

4.3.11. *The Augmenting Path Algorithm for bipartite graphs (Algorithm 3.2.1) is a special case of the Ford-Fulkerson Labeling Algorithm.* Call these algorithms AP and FF, respectively.

Given an X, Y-bigraph G, construct a network N by directing each edge of G from X to Y, adding vertices s and t with edges sx for all $x \in X$ and yt for all $y \in Y$, and making all capacities 1. A matching M in G determines a flow f in N by letting $f(sx) = f(xy) = f(yt) = 1$ if $xy \in M$, and f be 0 on all other edges.

We run AP on a matching M and FF on the corresponding flow f. AP starts BFS from the unsaturated vertices $U \subseteq X$. Similarly, FF starts by adding U to the reached set $\{s\}$, since the sx arcs for $x \in U$ are exactly those leaving s with flow less than capacity.

Let W be the set of all $y \in Y$ adjacent to some $x \in U$. AP next reaches all of W. Similarly, FF adds W to R (since again the edges reaching them have flow less than capacity) and moves U to S. Next AP moves back to X along edges of M. FF will do so also when searching from W, because these are entering backward edges with flow equal to capacity. Note that FF cannot move forward to t since vertices of W are saturated by M, and thus the edges from W to t have flow equal to capacity.

Iterating this argument shows that AP and FF continue to search the same vertices and edges until one of two things happens. If AP terminates by reaching an unsaturated vertex $y \in Y$ and returns an M-augmenting path, then when FF searches y it finds that $f(yt) = 0$ and reaches t. It also stops and returns the corresponding f-augmenting path.

If AP terminates without finding an augmenting path and instead returns a minimal cover Q, then FF terminates at this time and returns an s,t-cut $[S, T]$ with capacity $|Q|$. It suffices to show that each $v \in G$ contributes 0 to both $|Q|$ and $\text{cap}\,[S, T]$ or contributes 1 to both.

Let F be the search forest created by AP. By what we have shown, the corresponding tree F' of potential f-augmenting paths for FF is F with s attached as a root to the vertices of U. Consider $v = x \in X$. If $x \in V(F)$, then $x \notin Q$ and x contributes 0 to $|Q|$. Since $x \in V(F')$, also $x \in S$. There is no edge xy with $y \in Y \cap T$, since if there were then FF would have entered x from y. It follows that x contributes nothing to $\text{cap}\,[S, T]$.

If on the other hand $x \notin V(F)$, then $x \in Q$, and x contributes 1 to $|Q|$. Also $x \notin V(F')$, so $x \in T$. Hence the only arc from S to T ending at x is sx, and x contributes 1 to $\text{cap}\,[S, T]$. Similar considerations prove the claim when $v = y \in Y$.

4.3.12. Let $[S, \overline{S}]$ and $[T, \overline{T}]$ be source/sink cuts in a network N.

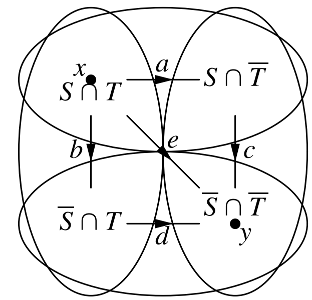
a) $\text{cap}(S \cup T, \overline{S \cup T}) + \text{cap}(S \cap T, \overline{S \cap T}) \le \text{cap}([S, \overline{S}]) + \text{cap}(T, \overline{T})$. Consider the contributions to the two new cuts, as suggested in the diagram below. Let a, b, c, d, e, f, g be the total capacities of the edges in $[S\cap T, S\cap\overline{T}]$, $[S \cap T, \overline{S} \cap T]$, $[S \cap \overline{T}, \overline{S} \cap \overline{T}]$, $[\overline{S} \cap T, \overline{S} \cap \overline{T}]$, $[S \cap T, \overline{S} \cap \overline{T}]$, $[S \cap \overline{T}, \overline{S} \cap T]$, and $[\overline{S} \cap T, S \cap \overline{T}]$, respectively. We have

$$\text{cap}[S \cap T, \overline{S \cap T}] + \text{cap}[S \cup T, \overline{S \cup T}] = (a + b + e) + (c + d + e),$$

$$\text{cap}[S, \overline{S}] + \text{cap}[T, \overline{T}] = (b + c + e + f) + (a + d + e + g).$$

Hence the desired inequality holds.

b) If $[S, \overline{S}]$ and $[T, \overline{T}]$ are minimum cuts, then $[S \cup T, \overline{S \cup T}]$ and $[S \cap T, \overline{S \cap T}]$ are also minimum cuts, and no edge between $S - T$ and $T - S$ has positive capacity. When $[S, \overline{S}]$ and $[T, \overline{T}]$ are minimum cuts, we obtain equality in the inequality of part (a). Neither summand on the left can be smaller, so both must equal the minimum. As shown in part (a), the difference between the two sides is $f + g$, which equals $\text{cap}[S \cap \overline{T}, \overline{S} \cap T] + \text{cap}[\overline{S} \cap T, S \cap \overline{T}]$. Equality requires that the difference be 0, so no edge between $S - T$ and $T - S$ has positive capacity.



4.3.13. *Modeling by network flows.* Several companies send delegates to a meeting; the ith company sends m_i delegates. The conference features simultaneous networking groups; the jth group can accommodate up to n_j delegates. The organizers want to schedule all delegates into groups, but delegates from the same company must be in different groups. The groups need not all be filled.

a) Use of network flow to test feasibility. Establish a network with a source s, sink t, vertex x_i for the ith company, and vertex y_j for the jth networking group. For each i, add an edge from s to x_i with capacity m_i. For each j, add an edge from y_j to t with capacity n_j. For each i, j, add $x_i y_j$ with capacity 1.

With integer capacities, the integrality theorem guarantees that some maximum flow breaks into paths of unit capacity. All s,t-paths have the form s, x_i, y_j, t and thus correspond to sending a delegate from company i to group j. The capacity on sx_i limits the ith company to m_i delegates. The capacity on $y_j t$ limits the jth group to n_j delegates. The capacity on $x_i y_j$ ensures that only one delegate from company i attends group j. The conditions of the problem are satisfiable if and only if this network has a flow of value $\sum m_i$. A flow of that value assigns, for each i, m_i delegates from company i to distinct groups.

b) A necessary and sufficient condition for successful construction is $k(q - l) + \sum_{j=1}^{l} n_j \ge \sum_{i=1}^{k} m_i$ for all $0 \le k \le p$ and $0 \le l \le q$, where $m_1 \ge \cdots \ge m_p$ and $n_1 \le \cdots \le n_q$.

Proof 1 (network flows). By the Max-flow/min-cut Theorem, there is a flow of value $\sum m_i$ if and only if there is no cut of capacity less than $\sum m_i$. Let $[S, T]$ be a source/sink cut, with $k = |S \cap \{x_1, \ldots, x_m\}|$ and $l = |S \cap \{y_1, \ldots, y_n\}|$. The capacity of the cut is $\sum_{i:\, x_i \in T} m_i + \sum_{j:\, y_j \in S} n_j + k(q - l)$. The network has a flow of value $\sum m_i$ if and only if this sum is at least $\sum m_i$ for each cut $[S, T]$. This will be true if and only if it is true when T has the $p - k$ companies with fewest participants and S has the l smallest groups. That is,

$$\sum_{i=k+1}^{p} m_i + \sum_{j=1}^{l} n_j + k(q-l) \ge \sum_{i=1}^{p} m_i,$$

which is equivalent to the specified inequality.

Proof 2 (bigraphic lists). The assignment of delegates to groups can be modeled by a bipartite graph. We may assume that $\sum n_j \ge \sum m_i$, since this is necessary to accommodate all the delegates. Let $t = \sum_j n_j - \sum_i m_i$. We add t phantom companies with one delegate each to absorb the excess capacity in the groups. Now there is a feasible assignment of delegates if and only if the pair (n, m) of lists is bigraphic, since each company sends at most one delegate to each group.

Note first that the given condition holds for all l if and only if it holds when $n_l \le k$ and $n_{l+1} \ge k$. The reason is that reducing l will cause terms smaller than k to contribute k and increasing l will cause contribution exceeding k from terms that contributed k.

By the Gale–Ryser Theorem, (n,m) is bigraphic if and only if $\sum_{i=1}^{q} \min\{n_i, k\} \ge \sum_{j=1}^{k} m_j$ for all $0 \le k \le p+t$. For $k > p$, we gain 1 with each increase in k on the right and at least 1 on the left unless we already have everything, so the inequality holds for all k if and only if it holds for $0 \le k \le p$. Since we have indexed $n_1, \ldots, n_q$ in increasing order, the left side equals $k(q-l) + \sum_{j=1}^{l} n_j$ when $n_l \le k$ and $n_{l+1} \ge k$. Thus the specified condition is equivalent to the condition in the Gale-Ryser Theorem and is necessary and sufficient for the existence of the bipartite graph and the assignment of delegates.

4.3.14. *A network flow solution to choosing $k/3$ assistant professors, $k/3$ associate professors, and $k/3$ full professors, one to represent each department.* We design a maximum flow problem with a node for each department, each professor, and each professorial rank. Let *unit edges* be edges of capacity 1. The source node s sends a unit edge to each departmental node. Each departmental node sends an edge to each of its professors' nodes; these may have infinite capacity. Each professorial node sends a unit edge to the node for that professor's rank. Finally, there is an edge of capacity $k/3$ from each rank to the sink t.

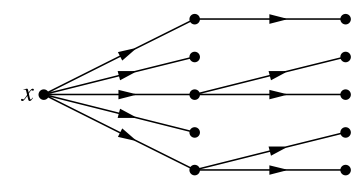
Each unit of flow selects a professor on the committee. The edges from the source to the departments ensure that each department is represented at most once. Since capacity one leaves each professor, the professor can represent only one department. The capacities on the three edges into the sink enforce balanced representation across ranks. The desired committee exists if and only if the network has a feasible flow of value k.

The network has a feasible flow of value k if and only if every source/sink cut has capacity at least k. Using edges with infinite capacity simplifies the analysis of finite cuts; such cuts $[S, T]$ cannot have an edge of infinite capacity from S to T. Any capacity at least 1 on the edges from a department to its professors yields the same feasible flows.

4.3.15. *Spanning trees and cuts.* The *value* of a spanning tree T is the minimum weight of its edges, and the *cap* from an edge cut $[S, \overline{S}]$ is the maximum weight of its edges. Since every spanning tree contains an edge from every cut, the value of a tree T is at most the cap from $[S, \overline{S}]$.

Let m be the minimum cap from edge cuts in a connected graph G; thus every edge cut has an edge with weight at least m. Let H be the subgraph of G consisting of all edges with weight at least m. Since H has an edge from every edge cut in G, H is a spanning connected subgraph of G. Let T be a spanning tree of H. Since every edge in H has weight at least m, the minimum edge weight in T is at least m. Also T is a spanning tree of G. Hence equality holds between maximum value of a spanning tree and minimum cap from edge cuts.

4.3.16. *If x is a vertex of maximum outdegree in a tournament G, then G has a spanning directed tree rooted at x such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2, as sketched below.*



We create a network with source x. We keep the edges of G from x to $N^+(x)$ and edges from $N^+(x)$ to $N^-(x)$, and then we add a sink z and edges from $N^-(x)$ to z. The edges leaving x have capacity 2, the edges from $N^+(x)$ to $N^-(x)$ have infinite capacity, and the edges from $N^-(x)$ to z have capacity 1.

The Integrality Theorem yields an integer-valued maximum flow consisting of x,z-paths with unit flow, arriving at z from distinct vertices of $N^-(x)$. Since the capacity on edges out of x is 2, each successor of x is on at most two such paths. Hence a flow of value $d_G^-(x)$ yields the desired spanning tree, since successors of x belonging to none of the paths can be added freely as leaves of the tree.

The Max flow–Min cut Theorem guarantees that such a flow exists if we show that every cut has value at least $d_G^-(x)$. A cut of finite value has no edge from $N^+(x)$ to $N^-(x)$. Consider a source/sink cut $[S, T]$, and let $T' = T \cap N^-(x)$. Let Q be the set of vertices in $N^+(x)$ having successors in $T \cap N^-(x)$; such vertices must also be in T. Let $q = |Q|$ and $t = |T'|$. The capacity of this cut is at least $2q + d^-(x) - t$.

Every vertex in T' has as successors x and all of $N^+(x) - Q$; a total of $d^+(x) - q + 1$ vertices. Also, some vertex of T' has outdegree at least $(t-1)/2$ in the subtournament induced by T'. Since x has maximum outdegree, we thus have $d^+(x) - q + 1 + (t-1)/2 \le d^+(x)$. This yields $2q - t \ge 1$ when T' is nonempty. Hence every cut other than the trivial cut $[S, T]$ that isolates z has capacity strictly greater than $d^-(x)$, and the desired flow exists.

Comment: Because the nontrivial cuts have capacity strictly greater than $d^-(x)$ in this argument, we still obtain a spanning tree of the desired form even under the additional restricted that any one desired successor of x be required to have outdegree at most 1.

4.3.17. *There is no simple bipartite graph for which the vertices in each partite set have degrees (5,4,4,2,1).* In any bipartite graph, the ith vertex on the side with degrees $\{p_i\}$ has at most $\min\{p_i, k\}$ neighbors among any set of k vertices on the other side. If we take the k largest degrees on the other side, their incident edges must come from somewhere, so $\sum \min\{p_i, k\} \ge \sum_{j=1}^{k} q_j$. The example given here violates this necessary condition when $k = 3$, because $3 + 3 + 3 + 2 + 1 = 12 < 13 = 5 + 4 + 4$.

4.3.18. *Given lists $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$, there is a digraph D with vertices $v_1, \ldots, v_n$ such that each ordered pair occurs at most once as an edge and $d^+(v_i) = r_i$ and $d^-(v_i) = s_i$ for all i if and only if $\Sigma r_i = \Sigma s_j$ and, for $1 \le k \le n$, the sum $\sum_{i=1}^{n} \min\{r_i, k\}$ is at least the sum of the largest k values in $s_1, \ldots, s_n$.*

We transform this question into that of realization of degree lists by a simple bipartite graph. Splitting each vertex v into two vertices v^- and v^+ such that v^- inherits edges leaving v and v^+ inherits edges entering v turns such a digraph into a simple bipartite graph with degree lists $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ for the partite sets.

Conversely, a simple bipartite graph with vertices $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ such that $d(u_i) = r_i$ and $d(w_j) = s_j$ becomes a digraph as described if we orient each edge $u_i w_j$ from u_i to w_j and then merge u_i and w_i into v_i for each i.

Thus the desired condition is the necessary and sufficient condition for $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ to be bigraphic. The condition is the Gale–Ryser condition, found in Theorem 4.3.18.

4.3.19. *A consistent rounding of the data in the matrix A below appears in matrix B.* Every row permutation of B is a consistent rounding of A, as are some matrices with larger total sum, so the answer is far from unique.

$$A = \begin{pmatrix} .55 & .6 & .6 \\ .55 & .65 & .7 \\ .6 & .65 & .7 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

4.3.20. *Every 2-by-2 matrix can be consistently rounded.* We need only consider the fractional part of each entry. That is, we may assume that entries are at least 0 and are less than 1. A consistent rounding is now obtained by rounding each entry other than .5 to the nearest integer and, for each entry equal to .5, rounding down when the sum of the indices is odd and up when the sum of the indices is even. The resulting change in a column or row total is strictly less than 1. Hence it cannot be moved far enough to make the resulting total not be a rounding of the original total.

4.3.21. *If every entry in an n-by-n matrix is strictly between $1/n$ and $1/(n-1)$, then the possible consistent roundings are the $0, 1$-matrices of order n with one or two 1s in each row and column.* Each entry in the rounding must be 0 or 1. Each row and column sum in the original is larger than 1 and less than $n/(n-1)$.

4.3.22. *A network D with conservation constraints at every node has a feasible circulation if and only if $\sum_{e\in[S,\overline{S}]} l(e) \le \sum_{e\in[\overline{S},S]} u(e)$ for every $S \subseteq V(D)$.* We have lower and upper bounds $l(e)$ and $u(e)$ for the flow $f(e)$ on each edges e. The conservation constraints require that the net flow out of each vertex is 0, and hence the net flow across any cut is 0. Thus

$$\sum_{e\in[S,\overline{S}]} l(e) \le \sum_{e\in[S,\overline{S}]} f(e) = \sum_{e\in[\overline{S},S]} f(e) \le \sum_{e\in[\overline{S},S]} u(e),$$

and hence the condition is necessary.

For sufficiency, we convert the circulation problem into a transportation network as in Solution 4.3.20. Let $b(v) = l^-(v) - l^+(v)$, where $l^-(v)$ and $l^+(v)$ are the totals of the lower bounds on edges entering and departing v, respectively; note that $\sum_v b(v) = 0$. We require $l(e) \le f(e) \le u(e)$ on each edge e; we transform this to $0 \le f'(e) \le c(e)$, where $c(e) = u(e) - l(e)$. Since the flow on each edge is adjusted in passing between f and f', the difference between the net out of v under these two functions is $b(v)$. If $b(v)$ is positive, then f sends $b(v)$ more into v than f' does when we alter f' by adding $l(e)$ on each edge e. Therefore, we want f' to produce net outflow $b(v)$ from v.

If $b(v) \ge 0$, then we make v a source and set $\sigma(v) = b(v)$; otherwise, we make v a sink and set $\partial(v) = -b(v)$. Since $\sum b(v) = 0$, the only way to satisfy all the demands at the sinks is to use up all the supply at the sources. If f' solves this transportation problem, then adding $l(e)$ to $f'(e)$ to obtain $f(e)$ on each edge e will solve the circulation problem with net outflow 0 at each node.

By Theorem 4.3.17, a transportation network with source set X and sink set Y is feasible if for every set T of vertices, the capacity of edges entering T is at least $\partial(Y\cap T)-\sigma(X\cap T)$; that is, the demand of T minus the supply in T.

Given a set T of vertices, the total capacity on entering edges is

$$c(\overline{T},T)=\sum_{e\in[\overline{T},T]}c(e)=\sum_{e\in[\overline{T},T]}[u(e)-l(e)].$$

For the supplies and demands, we compute

$$\begin{aligned}\partial(Y\cap T)-\sigma(X\cap T)&=\sum_{v\in Y\cap T}[l^+(v)-l^-(v)]-\sum_{v\in X\cap T}[l^-(v)-l^+(v)]\\&=\sum_{v\in T}[l^+(v)-l^-(v)]=\sum_{e\in[T,\overline{T}]}l(e)-\sum_{e\in[\overline{T},T]}l(e)\end{aligned}$$

The given condition $\sum_{e\in[S,\overline{S}]}l(e)\le\sum_{e\in[\overline{S},S]}u(e)$ for every $S\subseteq V(D)$ now implies that the condition for feasibility of the transportation problem holds. Hence there is a feasible solution to the transportation problem, and we showed earlier that such a solution produces a circulation in the original problem.

4.3.23. *A $(k+l)$-regular graph is (k,l)-orientable (it has an orientation in which each in-degree is k or l) if and only if there is a partition X,Y of $V(G)$ such that for every $S\subseteq V(G)$,*

$$(k-l)(|X\cap S|-|Y\cap S|)\le\left|[S,\overline{S}]\right|.$$

Note first that the characterization implies that every (k,l)-orientable with $k>l$ is also $(k-1,l+1)$-orientable, since $(k-1)-(l+1)\le k-l$; that is, the condition becomes easier to fulfill.

Note also that when $k=l$ the condition is always satisfied, and a consistent orientation of an Eulerian circuit is an orientation with the desired property. Hence we may assume that $k>l$.

Necessity. Given that G is (k,l)-orientable, let D be a suitable orientation of G. Let $X=\{v\in V(G)\colon d_D^-(v)=l\}$ and $Y=\{v\in V(G)\colon d_D^-(v)=k\}$. For a given set $S\subseteq V(D)$, the total indegree of the vertices in S is $l\,|X\cap S|+k\,|Y\cap S|$. The total outdegree in S is $k\,|X\cap S|+l\,|Y\cap S|$. Of the total outdegree, the amount generated by edges within S is at most the total outdegree minus the total indegree in S. Thus the left side of the displayed inequality is a lower bound on the number of edges in D that depart from S. The right side is an upper bound on that quantity.

Sufficiency. We are given a partition X,Y of $V(G)$ such that the displayed inequality holds for every $S\subseteq V(G)$. We create a transportation problem in which a feasible flow will provide the desired orientation. Replace each edge of G with a pair of opposing edges, each with unit capacity. Let X be the set of sources, and let Y be the set of sink. Let each supply and demand value be $k-l$. The given condition is now precisely the necessary and sufficient condition in Theorem 4.3.17 for the existence of a feasible solution to the transportation problem.

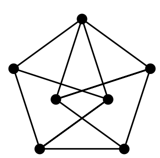
Since such a feasible solution is found by network flow methods, we may assume from the Integrality Theorem that there is a feasible solution in which the flow on each edge is 0 or 1. Also, we can cancel flows in opposing edges if they both equal 1. The edges that now have flow 1 specify an orientation of a subgraph of G. In this subgraph, $d^+(v)-d^-(v)=k-l$ for $v\in X$, and $d^+(v)-d^-(v)=l-k$ for $v\in Y$.

The degree remaining at a vertex v is $k+l-(d^+(v)+d^-(v))$. Always this value is even. A consistent orientation of an Eulerian circuit of each component of the remaining graph completes the desired orientation of G.

5. COLORING OF GRAPHS

5.1. VERTEX COLORING & UPPER BOUNDS

5.1.1. *Clique number, independence number, and chromatic number of the graph G below.* We have $\omega(G) = 3$ (no triangle extends to a 4-clique), $\alpha(G) = 2$ (every nonadjacent pair dominates all other vertices), and $\chi(G) = 4$ (a proper 3-coloring would give the top vertex the same color as the bottom two, but they are adjacent). Since there are seven vertices $\chi(G) \geq n(G)/\alpha(G)$ yields $\chi(G) \geq 4$. The graph is color-critical; checking each edge e shows that every $\chi(G-e)$ has a proper 3-coloring. By symmetry, there are only four types of edges to check.



5.1.2. *The chromatic number of a graph equals the maximum of the chromatic numbers of its components.* Since there are no edges between components, giving each component a proper coloring produces a proper coloring of the full graph. On the other hand, every proper coloring of the full graph must restrict to a proper coloring on each component.

5.1.3. *The chromatic number of a graph is the maximum of the chromatic number of its blocks.* We use induction on the number of blocks in G. If G has only one block, then the claim is immediate. Otherwise G is disconnected or has a cut-vertex v. In either case, we have subgraphs H_1, H_2 whose union is G, such that H_1, H_2 are disjoint (if G is disconnected) or share only the vertex v (if v is a cut-vertex).

The blocks of G are precisely the blocks of H_1 and H_2. Each has fewer blocks than G. Thus the induction hypothesis implies that $\chi(H_i)$ is the maximum of the chromatic numbers of the blocks in H_i. To complete the proof, it suffices to show that $\chi(G) = \max\{\chi(H_1), \chi(H_2)\}$.

The lower bound holds because both H_1 and H_2 are subgraphs of G. For the upper bound, assume by symmetry that $\chi(H_1) \geq \chi(H_2)$. Starting with an optimal coloring of H_1, we can incorporate an optimal coloring of H_2 by switching a pair of color names to make the coloring agree at v (if G is connected). This produces a proper coloring of G.

5.1.4. *The 5-cycle is a graph G with a vertex v so that $\chi(G-v) < \chi(G)$ and $\chi(\overline{G}-v) < \chi(\overline{G})$.* The 5-cycle is self-complementary and 3-chromatic, but deleting any vertex from C_5 (or $\overline{C}_5$) yields P_4, which is 2-colorable.

5.1.5. *Always $\chi(G+H) = \max\{\chi(G), \chi(H)\}$ and $\chi(G \vee H) = \chi(G) + \chi(H)$.* A coloring is a proper coloring of $G+H$ if and only if it restricts to a proper coloring on each of $\{G, H\}$, so the number of distinct colors needed is the maximum of $\chi(G)$ and $\chi(H)$.

In a proper coloring of $G \vee H$, the set of colors used on $V(G)$ must be disjoint from the set of colors used on $V(H)$. On the other hand, proper colorings of G and H that use disjoint sets of colors combine to form a proper coloring of $G \vee H$, so the number of colors needed is the sum of the numbers needed on G and H.

5.1.6. *If $\chi(G) = \omega(G)+1$, and $H_1 = G$ and $H_k = H_{k-1} \vee G$ for $k > 1$, then $\chi(H_k) = \omega(H_k)+k$.* The union of a clique in F and a clique in H is a clique in $F \vee H$; hence $\omega(F \vee H) = \omega(F)+\omega(H)$. Since distinct colors must be used on $V(F)$ and $V(H)$ in a proper coloring of $F \vee H$, also $\chi(F \vee H) = \chi(F)+\chi(H)$.

Now we can prove the claim by induction on k. For $k = 1$, we are given $\chi(H_1) = \chi(G) = \omega(G)+1 = \omega(H_1)+1$. For $k > 1$, we compute

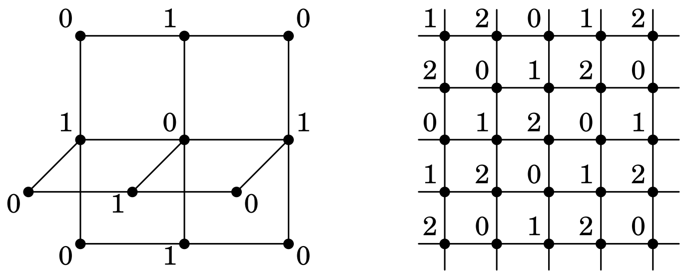
$$\begin{aligned}\chi(H_k) = \chi(H_{k-1} \vee G) = \chi(H_{k-1} + \chi(G) &= \omega(H_{k-1} + (k-1) + \omega(G) + 1 \\ &= \omega(H_{k-1} + \omega(G) + k = \omega(H_k) + k\end{aligned}$$

5.1.7. *The graph P_4 is neither a complete graph nor an odd cycle but has a vertex ordering relative to which greedy coloring uses $\Delta(P_4)+1$ colors.* Although P_4 is bipartite, with maximum degree 2, coloring the endpoints first greedily with color 1 forces us to use colors 2 and 3 on the center.

5.1.8. *Comparison of $\chi(G) \leq 1+\Delta(G)$ and $\chi(G) \leq 1+\max_{H\subseteq G}\delta(H)$.* Let H' be a subgraph of G for which the minimum degree attains its maximum value. We have $\max_{H\subseteq G}\delta(H) = \delta(H') \leq \Delta(H') \leq \Delta(G)$. Hence the second bound is always at least as good as the first bound.

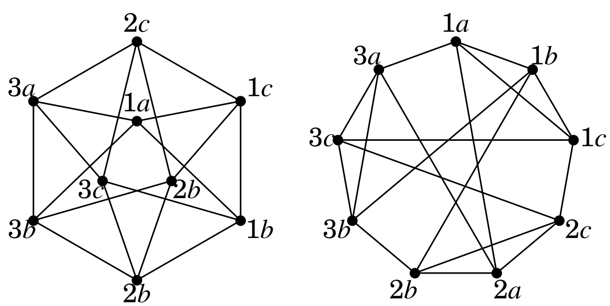
In order for equality to hold, we must have $\delta(H') = \Delta(H') = \Delta(G)$. Hence H' is k-regular, where $k = \Delta(G)$. This requires that no vertex of H' has a neighbor outside H'. Thus equality holds if and only if G has a component that is $\Delta(G)$-regular.

5.1.9. *Optimal (equitable) colorings of $K_{1,3} \square P_3$ and $C_5 \square C_5$.* The edges in the second figure wrap around to complete the 5-cycles in $C_5 \square C_5$.

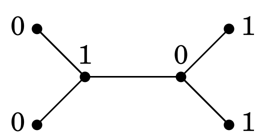


5.1.10. *The cartesian product graph $G \square H$ decomposes into a copies of H and b copies of G, where $n(G) = a$ and $n(H) = b$.* By the definition of cartesian product, $G \square H$ has two types of edges: those whose vertices have the same first coordinate, and those whose vertices have the same second coordinate. The edges joining vertices with a given value of the first coordinate form a copy of H, so the edges of the first type form aH. Similarly, the edges of the second type form bG, and the union is $G \square H$.

5.1.11. *Each graph below is isomorphic to $C_3 \square C_3$.* We label the vertices with $\{1, 2, 3\} \times \{a, b, c\}$ so that vertices are adjacent if and only if their labels agree in one coordinate and differ in the other.



5.1.12. *Every k-chromatic graph G has a proper k-coloring in which some color class has $\alpha(G)$ vertices—FALSE.* In the bipartite graph G below, every proper 2-coloring has three vertices in each color class, but $\alpha(G) = 4$.



5.1.13. *If $G = F \cup H$, then $\chi(G) \le \chi(F) + \chi(H)$—FALSE.* The complement of $K_{3,3}$ is $2K_3$. Hence we can express K_6 as the union of $K_{3,3}$ and $2K_3$. However, $\chi(K_6) = 6 > 5 = \chi(K_{3,3}) + \chi(2K_3)$.

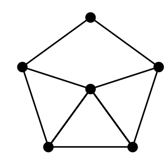
5.1.14. *For every graph G, $\chi(G) \le n(G) - \alpha(G) + 1$—TRUE.* We can produce a proper coloring by giving color 1 to a maximum independent set and giving distinct colors other than 1 to the remaining $n(G) - \alpha(G)$ vertices.

5.1.15. *It need not hold that $\chi(G) \le 1 + d$, where $d = 2e(G)/n(G)$ and G is a connected graph.* Form G by adding one edge joining a vertex of K_r to an endpoint of P_s. The graph G is connected, and $\chi(G) = r$. If $s > r$, then the average vertex degree is less than $(r+1)/2$. If also $r > 2$, then $r \ge (r+3)/2 > 1 + d$.

5.1.16. *Every tournament has a spanning path.* A n-vertex tournament D is an orientation of $G = K_n$, which has chromatic number n. By the Gallai–Roy Theorem, D has a path of length at least $\chi(G) - 1$, which equals $n - 1$. This is a spanning path.

5.1.17. *Chromatic number by critical subgraphs.* A graph with chromatic number at least 5 has a 5-critical subgraph, which has minimum degree at least 4. Since the graph below has only one vertex of degree at least 4, it has no subgraph with minimum degree at least 4.

A graph with chromatic number at least 4 has a 4-critical subgraph, which has minimum degree at least 3. Such a graph has at least 4 vertices. Deleting the one vertex with degree less than 2 from the graph below leaves only three vertices of degree at least 3. Hence there is no 4-critical subgraph, and $\chi(G) \le 3$.



5.1.18. *the number of colors needed to label $V(K_n)$ such that each color class induces a subgraph with maximum degree at most k is $\lceil n/k \rceil$.* With this many classes, we can partition the vertices into sets of size at most k.

5.1.19. *A false argument for Brooks' Theorem.* "We use induction on $n(G)$; the statement holds when $n(G) = 1$. For the induction step, suppose that G is not a complete graph or an odd cycle. Since $\kappa(G) \le \delta(G)$, the graph G has a separating set S of size at most $\Delta(G)$. Let $G_1, \ldots, G_m$ be the components of $G - S$, and let $H_i = G[V(G_i) \cup S]$. By the induction hypothesis, each H_i is

$\Delta(G)$-colorable. Permute the names of the colors used on these subgraphs to agree on S. This yields a proper $\Delta(G)$-coloring of G."

Since $G[S]$ need not be a complete graph, it may not be possible to make the colorings of $H_1, \ldots, H_m$ agree on S. When x, y are nonadjacent vertices in S, they may have the same color in all proper $\Delta(G)$-colorings of H_i but have different colors in all proper $\Delta(G)$-colorings of H_j.

5.1.20. *If the odd cycles in G are pairwise intersecting, then* $\chi(G) \le 5$.

Proof 1 (direct). If G has no odd cycle, then $\chi(G) \le 2$, so we may assume that G has an odd cycle. Let C be a shortest odd cycle in G. If $\chi(G - V(C)) \ge 3$, then we have an odd cycle disjoint from C. Hence $\chi(G - V(C)) \le 2$. Since C is a shortest odd cycle, it has no chords, and the subgraph induced by C is 3-colorable. Thus we can combine a 2-coloring of $G - V(C)$ with a 3-coloring of C to obtain a 5-coloring of G.

Proof 2 (contrapositive). If $\chi(G) \ge 6$, consider an optimal coloring. The subgraph induced by vertices colored 1,2,3 coloring must have an odd cycle, else it would be bipartite and we could replace these three colors by two. Similarly, the subgraph induced by vertices colored 4,5,6 in the optimal coloring has an odd cycle, and these two odd cycles are disjoint.

5.1.21. *If every edge of a graph G appears in at most one cycle, then every block of G is an edge, a cycle, or an isolated vertex.* A block B with at least three vertices is 2-connected and has a cycle C. We show that $B = C$.

Proof 1. If B has an edge e not in C, then the properties of 2-connected graphs imply that e and an edge e' of C lie in a common cycle (Theorem 4.2.4). Now e' lies in more than one cycle.

Proof 2. Every 2-connected graph has an ear decomposition. If B is not a cycle, then adding the next ear completes two cycles sharing a path.

Proof 3. If B has a vertex x of degree at least 3, then consider $u, v, w \in N(x)$. Since $G - x$ is connected, it has a u,v-path and a v,w-path. These complete two cycles containing the edge uv.

For such a graph G, $\chi(G) \le 3$.

Proof 1 (structural property). By Exercise 5.1.3, $\chi(G)$ equal the largest chromatic number of its blocks. Here the blocks are edges or cycles and have chromatic number at most 3.

Proof 2 (induction on the number of blocks). If G has one block, then $\chi(G) \le 3$ since G is a vertex, an edge, or a cycle. Otherwise, we decompose G into G_1 and G_2 sharing a cut-vertex x of G. The blocks of G_1 and G_2 are the blocks of G. Using 3-colorings of G_1 and G_2 given by the induction hypothesis, we can permute colors in G_2 so the colorings agree at x.

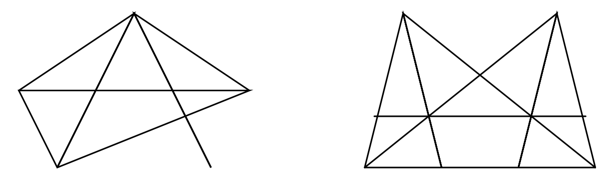
Proof 3 (subdivisions). Theorem 5.2.20 states that if G is not 3-colorable, then G contains a subdivision of K_4. Edges in such a subgraph appear in more than one cycle.

5.1.22. *The segment graph of a collection of lines in the plane with no three intersecting at a point is 3-colorable.* The vertices of G are the points of intersection of a family of lines; the edges are the segments on the lines joining two points of intersection.

Proof 1. By tilting the plane, we can ensure that no two vertices have the same x-coordinate. On each line, a vertex v has at most one neighbor with smaller x-coordinate. Thus each vertex has at most two earlier neighbors when $V(G)$ is indexed in increasing order of x-coordinates. Applying the greedy algorithm to this ordering uses at most three colors.

Proof 2. If $H \subseteq G$, the vertex of H with largest x-coordinate has degree at most 2 in H, for the same reason as above; on each line through that vertex it has at most one neighbor with smaller x-coordinate and none with larger x-coordinate. By the Szekeres–Wilf Theorem, $\chi(G) \le 1 + \max_{H \subseteq G} \delta(H) \le 3$.

The configurations below illustrate that the bound does not hold when more than two lines are allowed to meet at a point. The configuration on the left has seven lines, of which four meet at a point. The configuration on the right has eight lines, without four meeting at a point. In each case the resulting graph is 4-chromatic.



5.1.23. *The chromatic number of the graph $G_{n,k}$ obtained by joining each of n points on a circle to the 2k points nearest to it is $k+1$ if $k+1$ divides n and $k+2$ otherwise, if $n \ge k(k+1)$.* Every set of $k+1$ consecutive points forms a clique, so $\chi(G_{n,k}) \ge k+1$. If there is a $(k+1)$-coloring, each string of $k+1$ points must get distinct colors. Hence the coloring without loss of generality reads $123\cdots k(k+1)123\cdots k(k+1)123\cdots$ in order around the circle, since the new point must have the same color as the point just dropped from the most recent clique to avoid introducing a new color. The coloring will be proper if and only if the last vertices have colors $123\cdots k(k+1)$ before starting over, so $\chi(G_{n,k}) = k+1$ if and only if $k+1$ divides n.

If not, then one more color suffices if $n \ge k(k+1)$. Suppose $n = q(k+1) + r$, where $1 \le r < k+1$. After q complete stretches of $123\cdots k(k+1)$ in the scheme suggested above there are r vertices remaining to be colored. If $q \ge r$, then inserting color $k+2$ after $k+1r$ times will swell the sequence to fill up all the vertices with a proper coloring. In other words, expressing n as $r(k+2) + (q-r)(k+1)$, we can use $123\cdots(k+1)$ in order $q-r$ times and then $123\cdots(k+2)$ in order r times. If $n \ge k(k+1)$, then $q \ge k \ge r$, so the construction works.

If $n = k(k+1)-1$, then $\chi(G_{n,k}) > k+2$. If only $k+2$ colors are available, then some color must be used k times, since $(k-1)(k+2) = k(k+1)-2$. Following the n steps around the circle, the minimum separation between consecutive appearances among the k appearances of this color is less than $k+1$, since the total distance is $k(k+1)-1$. Since vertices at most k apart are adjacent, this prohibits a proper $(k+2)$-coloring.

5.1.24. *If G is a 20-regular graph with 360 vertices spaced evenly around a circle so that vertices separated by 1 or 2 angular degrees are nonadjacent and vertices separated by 3, 4, 5 or 6 degrees are adjacent, then $\chi(G) \le 19$.* We number the vertices 1 through 360 consecutively around the circle, and show that the greedy coloring algorithm uses at most 19 colors with respect to that order. Vertices $1, 2, 3$ receive color 1, and vertices $4, 5, 6$ receive color 2. Each of vertices 1 through 356 has at most 18 of its predecessors among its 20 neighbors, so color 20 will not have been assigned to any of the first 356 vertices.

Vertex 357 has 19 of its predecessors among its neighbors, but among those, vertices $1, 2, 3$ have the same color. Hence vertex 357 is assigned a color 19 or lower, having at most 17 differently colored predecessors. Similarly, vertex 358 has at most 18 differently colored predecessors ($1, 2, 3$ have the same color), 359 has at most 18 ($2, 3$ and $4, 5$ are pairs with the same color), and 360 has at most 18 ($4, 5, 6$ have the same color), so their assigned colors are 19 or lower.

5.1.25. *The unit-distance graph in the plane has chromatic number greater than 3 and at most 7.* For the lower bound, suppose the graph has a proper 3-coloring. Consider two equilateral triangles of side-length one that share an edge. The corners not on the shared edge must have the same color. The distance between these two points is $\sqrt{3}$. Hence in a proper 3-coloring, any two points $\sqrt{3}$ apart must have the same color. If C is a circle of radius $\sqrt{3}$, every point on C must have the same color as the center. This cannot be a proper coloring, since C contains two points that are distance 1 apart.

A 7-coloring can be obtained using regions in a tiling of the plane. Consider a tiling by hexagons of diameter 1, where each hexagon has two parallel horizontal edges and the hexagons lie in vertical columns. The interior of each hexagon receives a single color, along with the top half of the boundary (including the top two corners but not the middle two corners).

The rest of the boundary is colored as part of the top half of neighboring hexagons. In a single region, the distance between any pair of points is less than 1; we need only assign colors to regions so that no pair of regions with the same color contain pairs of points at distance 1. This we achieve

by using colors 1,2,3,4,5,6,7 cyclically in order on the regions in a column, with the region labeled 1 in a given column nestled between regions labeled 3 and 4 in the column to its left.

The closest points in two regions with the same color are opposite endpoints of a zig-zag of three edges in the tiling; the distance between these is greater than one. (An 8-coloring can be obtained using a grid of squares of diameter 1, with colors 1,2,3,4 on the odd columns and colors 5,6,7,8 on the even columns, cyclically in order, where the pattern in the odd rows repeats 1,5,3,7 and the pattern in the even rows repeats 2,6,4,8.)

5.1.26. *Chromatic number of a special graph.* Given finite sets $S_1, \ldots, S_m$, let $V(G) = S_1 \times \cdots \times S_m$, and define $E(G)$ by putting $u \leftrightarrow v$ if and only if u and v differ in every coordinate.

The chromatic number is $\min_i |S_i|$. Let $k = \min_i |S_i|$. We may assume that $S_i = \{1, \ldots, k\}$. We obtain a clique of size k by letting $v_i = (i, \ldots, i)$ for $1 \le i \le k$ (when $i \ne j$, v_i and v_j differ in every coordinate). Hence $\chi(G) \ge k$.

To obtain a proper k-coloring, we color each vertex v with its value in a coordinate j such that $|S_j| = k$. The vertices having value i in coordinate j form an independent set, so this is a proper k-coloring, and $\chi(G) \le k$.

5.1.27. *The complement of the graph in Exercise 5.1.26 has chromatic number $\prod_{i=1}^{m} |S_i| / \min_i |S_i|$.* Nonadjacency means differing in every coordinate.

Let j be a coordinate such that $|S_j| = \min_i\{|S_i|\}$. The vertices with a fixed value in coordinate j form a clique of the specified size.

Let $k = |S_j|$. To obtain the desired coloring, we partition the vertices into independent sets of size k. Each must have a vertex with each value in coordinate j. The vertices with value 1 in coordinate j lie in different independent sets; use the remainder of the name of each such vertex as the name for its independent set. The ith vertex in this independent set has value i in coordinate j. Its value in coordinate r is obtained by adding $i-1$ (modulo $|S_r|$) to the value in coordinate r of the naming vertex.

To find the name of the independent set containing a vertex v, we let i be the value it has in coordinate j and subtract $i-1$ (modulo $|S_r|$) from the value in the rth coordinate, for each r.

5.1.28. *The traffic signal controlled by two switches is really controlled by one of the switches.* Each switch can be set in n positions. For each setting of the switches, the traffic signal shows one of n possible colors. Whenever the setting of *both* switches changes, the color changes.

Since the color changes when both coordinates change, assigning the color that shows to the vector of positions yields a proper n-coloring of the graph defined in Exercise 5.1.26, where $m = 2$ and both sets have size n. Since $\{(i, i) \colon 1 \le i \le n\}$ is a clique of size n, this is an optimal coloring.

Proof 1 (characterization of maximum independent sets). The vertices having value i in one coordinate form an independent set. This defines a proper n-coloring. We claim that every proper n-coloring has this form, and hence the color is controlled by the value in one coordinate. Every independent set has size at most n, since $n+1$ vertices cannot have distinct values among the n possible values in the first coordinate. In order to obtain an n-coloring of the n^2 vertices when each independent set has size at most n, we must use n independent sets of size n.

We claim that every independent set of size n is fixed in one coordinate. Let S be an independent set in which distinct values r and s appear in the first coordinate. Since these vertices in S are nonadjacent, they must agree in the second coordinate, so we now have $(r,t),(s,t) \in S$. If S has some vertex not having value t in the second coordinate, then its value in the first coordinate must equal both r and s, since it is nonadjacent to these two vertices. This is impossible, so the vertices S must agree in one coordinate.

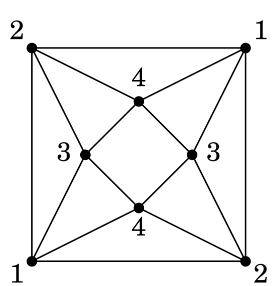
To partition $V(G)$ into n independent sets of size n, the sets must be parwise disjoint. Hence we cannot used one set fixed in the first coordinate and another set fixed in the second coordinate. Hence all the sets used in the coloring are constant in the same coordinate, and the color is controlled by the position in that coordinate.

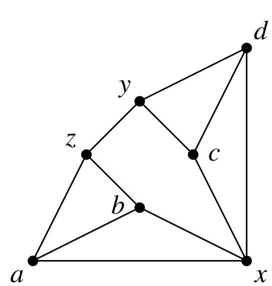
Proof 2 (induction on n). The claim is trivial for $n=1$. Let G_n be the product graph. For the induction step, note that $N_{G_n}((n,n))$ induces G_{n-1}. Also, the color n used on (n,n) cannot be used on that subgraph, so the coloring of G_n restricts to a proper $(n-1)$-coloring on that subgraph. By the induction hypothesis, it is determined by one coordinate. By symmetry, we may assume that (i,j) has color i when $i,j \in [n-1]$.

Now (n,j) for $1 \le j \le n-1$ is adjacent to vertices of the first $n-1$ colors, so it has color n. Now (i,n) is adjacent to vertices of all n colors except color i, so it has color i.

5.1.29. *A 4-critical subgraph in a 4-chromatic graph.*

The figure on the left illustrates a proper 4-coloring. On the right we show a 4-critical subgraph. Verifying that this is 4-critical also proves the lower bound to show that the full graph is 4-chromatic.





In any proper 3-coloring of $K_4 - e$, the nonadjacent vertices have the same color. Thus in a proper 3-coloring of the graph F on the right, y and z have the same color as x, which is forbidden because they are adjacent. 4-criticality is easy to verify using symmetry; there are only four "types" of edges. Thus the solution is completed by exhibiting proper 3-colorings of $F - yz$, $F - zb$, $F - ab$, and $F - ax$. If we remove any edge of the outer 5-cycle, then we can 2-color its vertices and use the third color on the two inner vertices. The analogous argument works for the inner 5-cycle. This leaves only $F - ab$ to consider, where we can obtain a proper 3-coloring by giving all of $\{a,b,x,y\}$ the same color.

5.1.30. *The chromatic number of the shift graph G_n is $\lceil \lg n \rceil$.* Here $V(G_n) = \binom{[n]}{2}$ and $E(G_n) = \{(ij, jk) \colon i < j < k\}$. It suffices to show that G_n is r-colorable if and only if $[r]$ has at least n distinct subsets.

Given a map $f \colon V(G) \to [r]$, define $T_j = \{f(jk) : k > j\}$. The labeling f is a proper coloring if and only if $f(ij) \notin T_j$ for all $i < j$. In particular, if f is proper, then $T_i \ne T_j$ for all $i < j$, and thus r must be large enough so that $[r]$ has n distinct subsets.

Conversely, if $[r]$ has n distinct subsets, we index n such subsets so that $A_j \not\subseteq A_i$ for $j > i$ (start with n and work back, always choosing a minimal set in the collection of subsets that remain). Now f can be defined by naming $f(ij)$ for each $i < j$ to be an element of $A_i - A_j$. This ensures that ij and jk receive distinct colors when $i < j < k$.

5.1.31. *A graph G is m-colorable if and only if $\alpha(G \square K_m) \ge n(G)$.*

Proof 1 (direct construction). Let $V(G) = \{v_i\}$ and $V(K_m) = \{1, \dots, m\}$, so the vertices of $G \square K_m$ are $\{(v_i, j) \colon 1 \le i \le n,\ 1 \le j \le m\}$. If G is m-colorable, let $C_1, \dots, C_m$ be the independent sets in a proper m-coloring of G. Then $\{(v, j) \colon v \in C_j\}$ is an independent set in $G \square K_m$ of size $n(G)$ (it contains one copy of each vertex of G).

Conversely, if $G \square K_m$ has an independent set S of size $n(G)$, then S can only contain one copy of each vertex of G (since (v,i) and (v,j) are adjacent), and the elements of S whose pairs use a single vertex of K_m must be an independent set in G. Hence from S we obtain a covering of $V(G)$ by m independent sets.

Proof 2 (applying known inequalities). If G is m-colorable, then $\chi(G \square K_m) = \max\{\chi(G), \chi(K_m)\} = m$. Because $\chi(H) \ge n(H)/\alpha(H)$ for every graph H, and $n(G \square K_m) = n(G)m$, we obtain $\alpha(G \square K_m) \ge n(G)m/m = n(G)$.

Conversely, $\alpha(G \square K_m) \ge n(G)$ yields $\alpha(G \square K_m) = n(G)$, since an independent set has at most one vertex in each copy of K_m. The vertices of a maximum independent set S have the form (v,i), where $v \in V(G)$ and $i \in [m]$. By the definition of cartesian product, adding 1 (modulo m) to the second coordinate in each vertex of S yields another independent set of size $n(G)$. Doing this m times yields m pairwise disjoint independent sets covering all the vertices of $G \,\square\, K_m$. Therefore, $G \,\square\, K_m$ is m-colorable. Since G is a subgraph of $G \,\square\, K_m$, also G is m-colorable.

5.1.32. *A graph G is 2^k-colorable if and only if G is the union of k bipartite graphs.* View the colors as binary k-tuples. If G has a proper 2^k-coloring f, let X_i be the set of all $v \in V(G)$ such that the binary expansion of $f(v)$ has a 0 in the ith coordinate, and let $Y_i = V(G) - X_i$. Define a bipartite subgraph B_i of G with bipartition X_i, Y_i and edge set $[X_i, Y_i]$. By construction, each such graph is bipartite. For every edge e in G, the endpoints of e have different colors in f, so their binary expansions differ in some coordinate, and thus e appears in one of these subgraphs.

Conversely, suppose that G is the union of k bipartite graphs, with X_i, Y_i being the bipartition of the ith subgraph. We use binary k-tuples as colors. Assign v the k-tuple that is 0 in the ith coordinate if $v \in X_i$, or 1 if $v \in Y_i$. Since each edge is in one of the bipartite graphs, the k-tuples assigned to its endpoints are distinct, and this is a proper 2^k-coloring.

5.1.33. *For each graph G, there is an ordering of $V(G)$ where the greedy algorithm uses only $\chi(G)$ colors.* Consider an optimal coloring f. Number the vertices of G as $v_1, \ldots, v_n$ as follows: start with the vertices of color 1 in f, then those of color 2, and so on. By induction on i, we prove that the greedy algorithm assigns v_i a color at most $f(v_i)$.

Certainly v_1 gets color 1. For $i > 1$, the induction hypothesis says that v_j has received color at most $f(v_j)$, for every $j < i$. Furthermore, the only such vertices v_j with $f(v_j) = f(v_i)$ are those in the same color class with v_i in the optimal coloring, and these are not adjacent to v_i. Hence the colors used on earlier neighbors of v_i are in the set $\{1, \ldots, f(v_i) - 1\}$, and the algorithm assigns color at most $f(v_i)$ to v_i.

5.1.34. *There is a tree T_k with maximum degree k having a vertex ordering such that the greedy algorithm uses $k+1$ colors.* There are at least three ways to construct the same tree T_k and essentially the same ordering. In each, we construct T_k by induction on k along with a vertex ordering such that the last vertex has degree k and receives color $k+1$ under the greedy algorithm. In each, the tree K_1 works as T_0 when $k = 0$.

Construction 1. For $k > 0$, T_k consists of copies of $T_0, \ldots, T_{k-1}$, with one additional vertex x joined to the vertex of maximum degree in each T_i.

By the induction hypothesis, each old vertex has degree at most $k-1$, and the only one that attains degree k, along with x, is the vertex of maximum degree in T_{k-1}. The vertex ordering uses $V(T_i), \ldots, V(T_{k-1})$ in order and puts x last. The ordering within $V(T_i)$ is the ordering guaranteed for it by the induction hypothesis. The coloring of each T_i happens independently

according to the order for that subtree, because the only edge leaving the copy of T_i goes to x. By the induction hypothesis, the neighbor of x in T_i, which is the last vertex of T_i, gets color $i+1$. Thus x has earlier neighbors of colors $1, \ldots, k$ and receives color $k+1$.

Construction 2. Build two copies of T_{k-1} (T' and then T''), with the vertex orderings given by the induction hypothesis. Include in the induction hypothesis the statement that the last vertex has degree $k-1$ and receives color k under the greedy coloring. When the last vertex x of T'' is created, make it also adjacent to the last vertex y of T'. Hence x and y have degree k in the resulting tree T.

When y is reached in the ordering, it receives color k, by the induction hypothesis. For the same reason, x is adjacent in T to vertices that have received colors $1, \ldots, k-1$ (in T'') and also to y. Hence x receives color $k+1$, as desired.

Construction 3. Given T_{k-1} and its ordering, form T_k by appending a leaf to each vertex. In the ordering, place all these leaves first. These form an independent set and receive color 1. After this independent set, use the ordering for T_{k-1} on the vertices in the copy of T_{k-1}. Since each of these vertices already has a neighbor with color 1, the colors assigned are 1 higher than the colors assigned under the ordering of T_{k-1}. Also the degree of each vertex is larger by 1. Hence this T_k has maximum degree k, and the given ordering assigns color $k+1$ to the last vertex.

Explicit Construction. The tree T_k can be described by letting the vertices be $\{0, \ldots, 2^k - 1\}$. Make i and j adjacent whenever $i \ge 2^k - 2^{r+1}$ and $j = i + 2^r$ for some r. This produces the same tree as above (it can be proved by induction on k that it is the same tree as in Construction 3), and the vertex ordering that puts color $k+1$ on $2^k - 1$ is $0 \ldots, 2^k - 1$.

5.1.35. *In a P_4-free graph, every greedy coloring is optimal.* Consider an ordering $v_1, \ldots, v_n$, and suppose that greedy coloring with respect to this ordering uses k colors. Let i be the smallest integer such that G has a clique consisting of vertices assigned colors i through k in this coloring. Proving that $i = 1$ yields a k-clique in G, which proves that the coloring is optimal.

Let $Q = \{u_i, \ldots, u_k\}$ be such a clique. If $i > 1$, then by the greedy procedure each element of Q has an earlier neighbor with color $i-1$. If some such vertex were adjacent to all of Q, then we could reduce i. Let x be a vertex with color $i-1$ that is adjacent to the most vertices of Q. Let z be a nonneighbor of x in Q. Let w be a neighbor of z with color $i-1$. By the choice of x, w is not adjacent to all neighbors of x in Q; choose $y \in (N(x) \cap Q) - N(w)$. Since x and w have the same color, they are nonadjacent. Now x, y, z, w induces P_4. The contradiction implies that $i = 1$.

5.1.36. *The ordering σ that minimizes the greedy coloring bound $f(\sigma) = 1+\max_i d_{G_i}(x_i)$ is the "smallest-last" ordering σ^* in which, for i from n down to 1, x_i is chosen to be a vertex of minimum degree in G_i.* Furthermore, $f(\sigma^*) = 1 + \max_{H\subseteq G} \delta(H)$. Let H^* be a subgraph of G maximizing $\delta(H)$. For a vertex ordering σ, let i be the position in σ where the last vertex of H^* is included. We have $d_{G_i}(x_i) \ge \delta(H^*)$, and thus $f(\sigma) \ge 1 + \delta(H^*) = 1 + \max_{H\subseteq G} \delta(H)$.

When greedy coloring is run with respect to σ^*, each v_i is a vertex of minimum degree in G_i. Thus $f(\sigma^*) = 1 + \max_i \delta(G_i) \le 1 + \max_{H\subseteq G} \delta(H)$. By the first paragraph, equality holds.

5.1.37. *The vertices of a simple graph G can be partitioned into $1 + \max_{H\subseteq G} \delta(H)/r$ classes such that the subgraph induced by the each class has a vertex of degree less than r.* Let v_n be a vertex of minimum degree in G, and for $i < n$ let v_i be a vertex of minimum degree in $G - \{v_{i+1}, \ldots, v_n\}$. Place the vertices $v_1, \ldots, v_n$ in order into the partition. Place v_i into the least-indexed set in which it has fewer than r neighbors already placed. This produces a partition of the desired form. Let $k = \max_{H\subseteq G} \delta(H)$. Since the degree of v_i in the subgraph induced by $v_1, \ldots, v_i$ is at most k, v_i has r neighbors each in at most k/r classes, and therefore $1 + k$ classes suffice.

5.1.38. *If H is bipartite, then $\chi(\overline{H}) = \omega(\overline{H})$.* If H has isolated vertices, then in $\overline{H}$ they increase the clique number, and we may color them with extra colors. Hence we may assume that H has no isolated vertices.

Proof 1 (min-max relations). Because every independent set induces a clique in the complement and vice versa, we have $\omega(\overline{H}) = \alpha(H)$. Also $\chi(\overline{H})$ is the number of cliques in H needed to cover $V(H)$. If H is bipartite, then these cliques must be edges. Hence for a bipartite graph H with no isolated vertices, we have $\chi(\overline{H}) = \beta'(H) = \alpha(H) = \omega(\overline{H})$, using König's Theorem that $\beta'(H) = \alpha(H)$ in a bipartite graph with no isolated vertices.

We could also argue that each color in a proper coloring of $\overline{H}$ is used once or twice, since $\alpha(\overline{H}) = 2$. If k colors are used twice, then $k + (n - 2k) = n - k$ colors are used. The colors used twice color the edges of a matching in H, so $\chi(\overline{H}) = n - \alpha'(H) = \beta'(H)$ as before.

Proof 2 (construction). Let T be a maximum independent set in H, and let $A = X \cap T$ and $B = Y \cap T$, where H is an X,Y-bigraph. It suffices to find a matching of $Y - B$ into A and a matching of $X - A$ into B, because the edges of the matching disappear in $\overline{H}$, and this yields a covering of $V(\overline{H})$ using $|T|$ independent sets of sizes 1 and 2. To show that the matching exists, consider any $S \subseteq Y - B$ (the same argument works for $X - A$). Because $A - N(S) \cup S \cup B$ is a independent set and $A \cup B$ is a maximum independent set, we have $|N(S)| \ge |S|$, which by Hall's Theorem guarantees the desired matching.

5.1.39. *Every k-chromatic graph has at least $\binom{k}{2}$ edges.*

Proof 1. Consider a k-coloring of a k-chromatic G. If $e(G) < \binom{k}{2}$, then for some pair i, j of colors, no edge has colors i and j. Thus the vertices with colors i and j form a single independent set, and $V(G)$ is covered by $k - 1$ independent sets.

Proof 2. A k-chromatic graph G contains a k-critical subgraph G'. A k-critical graph has minimum degree at least $k - 1$, and since G' requires k colors it has at least k vertices. Hence

$$e(G) \ge e(G') \ge n(G')\delta(G')/2 \ge k(k-1)/2 = \binom{k}{2}.$$

If G is the union of m cliques of order m, then $\chi(G) \le 1 + m\sqrt{m-1}$. The construction of G implies $e(G) \le m\binom{m}{2}$. If $\chi(G) = k$, then $\binom{k}{2} \le m\binom{m}{2}$, or $k^2 - k - m^2(m-1) \le 0$. Using the quadratic formula and $\sqrt{x+1} < \sqrt{x} + 1$ (for $x > 0$), we have

$$k \le \tfrac{1}{2}(1 + \sqrt{1 + 4m^2(m-1)}) < \tfrac{1}{2}(2 + \sqrt{4m^2(m-1)} = 1 + m\sqrt{(m-1)}.$$

5.1.40. $\chi(G)\cdot\chi(\overline{G}) \ge n(G)$. If a proper coloring partitions n vertices into k color classes, there must be at least n/k vertices in some class, by the pigeonhole principle. These vertices form a clique in the complement, which forces $\chi(\overline{G}) \ge n/k$. Hence $\chi(\overline{G}) \ge n/\chi(G)$, or $\chi(G)\cdot\chi(\overline{G}) \ge n$.

$\chi(G) + \chi(\overline{G}) \ge 2\sqrt{n(G)}$. Two numbers with a fixed product x have smallest sum when they are equal; then their sum is $2\sqrt{x}$. Hence the first inequality implies this bound.

For $n = m^2$, the bound is achieved by $G = mK_m$, a disjoint union of $\sqrt{n}$ cliques of size $\sqrt{n}$. Since the complement is a complete $\sqrt{n}$-partite graph, both graphs have chromatic number $\sqrt{n}$.

5.1.41. $\chi(G) + \chi(\overline{G}) \le n(G) + 1$ *for every graph G.*

Proof 1 (induction on $n(G)$). The inequality holds (with equality) if $n = 1$. For $n > 1$, choose $v \in V(G)$, and let $G' = G - v$. By the induction hypothesis, $\chi(G') + \chi(\overline{G'}) \le n$. When we replace v to obtain G and $\overline{G}$, each chromatic number increases by at most 1. We have the desired bound unless they both increase.

If both increase, then v must have at least $\chi(G')$ neighbors in G (else we could augment a proper coloring of G' to include v) and similarly at least $\chi(\overline{G'})$ neighbors in $\overline{G}$. Since v has altogether $n-1$ neighbors in G and $\overline{G}$, we conclude that in this case $\chi(\overline{G'}) + \chi(G') \le n - 1$, and adding 2 again yields the desired bound $\chi(\overline{G}) + \chi(G) \le n + 1$.

Proof 2 (greedy coloring bound). When the vertices are colored greedily in nonincreasing order of degree, the color used on the ith vertex is at most $\min\{d_i + 1, i\}$. Let k be the index where the maximum of $\min\{d_i + 1, i\}$

is achieved, so that $d_i + 1 \le k$ for $i \ge k$ and $d_i + 1 > k$ for $i < k$. Greedy coloring yields $\chi(G) \le k$.

Let d'_j denote the jth largest vertex degree in $\overline{G}$. Since $d'_j = n-1-d_{n-j}$, we have $d'_j < n-k$ for $n-j < k$ and $d'_j \ge n-k$ for $n-j \ge k$. This becomes $d'_j \le n-k$ for $j \ge n-k+1$ and $d'_j > n-k$ for $j < n-k+1$. Therefore $\max_j \min\{d'_j+1, j\} = n-k+1$, so $\chi(G)+\chi(\overline{G}) \le k+(n-k+1) = n+1$.

Proof 3 (degeneracy). By the Szekeres–Wilf Theorem, it suffices to show that $\max_{H\subseteq G}\delta(H) + \max_{H\subseteq\overline{G}}\delta(H) \le n-1$. Let H_1 and H_2 be subgraphs of G and $\overline{G}$ achieving the maximums. Let $k_i = \delta(H_i)$. Note that $n(H_i) \ge k_i+1$. If $k_1+k_2 \ge n$, then H_1 and H_2 have a common vertex v. Now v must have at least k_i neighbors in H_i, for each i, but only $n-1$ neighbors are available in total.

5.1.42. *Analysis of the ratio of $\chi(G)$ to $n(G)/\alpha(G)$.*

a) $\chi(G)\cdot\chi(\overline{G}) \le (n(G)+1)^2/4$, and the ratio of $\chi(G)$ to $(n+1)/\alpha(G)$ is at most $(n+1)/4$. Two numbers with a fixed sum x have largest product when they are equal, in which case their product is $x^2/4$. Hence the previous exercise implies this bound. The ratio of $\chi(G)$ to $(n+1)/\alpha(G)$ equals $\chi(G)\alpha(G)/(n+1)$. Since $\alpha(G) = \omega(\overline{G})$, we have $\chi(\overline{G}) \ge \alpha(G)$. Hence $\chi(G)\alpha(G)/(n+1) \le \chi(G)\cdot\chi(\overline{G})/(n+1) \le (n+1)/4$.

b) Construction for equality when n is odd. Let G be the join of a clique of order $(n-1)/2$ and a independent set of order $(n+1)/2$. Since the independent set can receive a single color and G has cliques of order $(n+1)/2$, $\chi(G) = (n+1)/2$. Also $\alpha(G) = (n+1)/2$, and equality holds in the bound $\chi(G)\chi(\overline{G}) \le (n+1)^2/4$.

5.1.43. *Paths and chromatic number in digraphs.*

a) $\chi(F\cup H) \le \chi(F)\chi(H)$. Let $G = F\cup H$. We may assume that $V(F) = V(H)$, because otherwise we can add vertices in exactly one of these digraphs as isolated vertices in the other without affecting any of the chromatic numbers. The chromatic number of a digraph is taken to be the same as the chromatic number of the underlying undirected graph.

Proof 1 (producing a coloring)

Assign to each vertex of G the "color" that is the pair of colors it gets in optimal colorings of F and H. Since every edge of G comes from F or H, no pair of adjacent vertices in G get the same color pair. Since there are $\chi(F)\chi(H)$ possible pairs, we have a proper $\chi(F)\chi(H)$-coloring of G.

Proof 2 (covering by independent sets) Let $U_1,\dots,U_r$ be the color classes in an optimal coloring of F, and let $W_1,\dots,W_s$ be the color classes in an optimal coloring of H. Each vertex belongs to exactly one class in each graph, so it belongs to $U_i\cap W_j$ for exactly one pair (i,j). Furthermore, $U_i\cap W_j$ is an independent set in G, since it is independent in both F and H.

Now the sets of the form $U_i\cap W_j$ for $1\le i\le r=\chi(F)$ and $1\le j\le s=\chi(H)$ partition $V(G)$ into the desired number of independent sets.

b) If D is an orientation of G, and $\chi(G) > rs$, and each $v\in V(G)$ is assigned a real number $f(v)$, then D has a path $u_0,\dots,u_r$ with $f(u_0)\le\cdots\le f(u_r)$ or a path $v_0,\dots,v_s$ with $f(v_0) > \cdots > f(v_s)$. Obtain from D two digraphs F and H defined as follows. Given the edge xy in D, put xy in F if $f(x)\le f(y)$, and put xy in H if $f(x) > f(y)$. If D has no nondecreasing path of length r and no decreasing path of length s, then F has no path of length r and H has no path of length s. By the Gallai–Roy Theorem, this implies $\chi(F)\le r$ and $\chi(H)\le s$. By part (a), we have $\chi(G)\le rs$, where $G = F\cup H$, but this contradicts the hypothesis on G. Hence one of the specified long paths exists.

c) Every sequence of $rs+1$ distinct real numbers has an increasing subsequence of size $r+1$ or a decreasing subsequence of size $s+1$. Let D be the tournament with vertices $v_1,\dots,v_{rs+1}$ and $v_i\to v_j$ if $i > j$, and let $f(v_i)$ be the ith value in the sequence σ. Every path in D corresponds to a subsequence of σ, where the vertex labels are the values in σ. Because $\chi(D) = rs+1$, part (b) guarantees an increasing path with $r+1$ vertices or a decreasing path with $s+1$ vertices.

5.1.44. *Minty's Theorem.* Given an acyclic orientation D of a connected graph G, let $r(D) = \max_C \lceil a/b\rceil$, where a counts the edges of C that are forward in D and b counts those that are backward in D. Fix a vertex $x\in V(G)$, and let W be a walk from x in G. Let $g(W) = a - b\cdot r(D)$, where a counts the steps along W followed forward in D and b counts those followed backward in D. For $y\in V(G)$, let $g(y) = \max\{g(W)\colon W$ is an x,y-walk$\}$.

a) $g(y)$ is finite and thus well-defined, and G is $1+r(D)$-colorable. By the definition of r, every cycle with a forward edges has at least ra backward edges. Hence traversing a cycle makes no positive contribution to $g(W)$, and $g(y) = g(W)$ for some x,y-path W. Thus there are only finitely many paths to consider, and $g(y)$ is well-defined.

To obtain a proper coloring of G, let the color of y be the congruence class of $g(y)$ modulo $1+r(D)$. If $u\to v$ in D, then $g(v)\ge g(u)+1$, since uv can be appended to an x,u-walk. On the other hand $g(u)\ge g(v)-r(D)$, since vu can be appended to an x,v-walk. Thus $g(u)+1\le g(v)\le g(u)+r(D)$ when u and v are adjacent in G, which means that $g(u)$ and $g(v)$ do not lie in the same congruence class modulo $r(D)+1$.

b) $\chi(G) = \min_{D\in\mathbf{D}} 1+r(D)$, where $\mathbf{D}$ is the set of acyclic orientations of G. The upper bound follows immediately from part (a). For the lower bound, we present an acyclic orientation D such that $r(D)\le\chi(G)-1$. Given an optimal coloring f with colors $1,\dots,\chi(G)$, orient each edge xy in the direction of the vertex with the larger color. Since colors increase strictly along every path, the orientation is acyclic and has maximum path length at most $\chi(G) - 1$.

5.1.45. *Gallai-Roy Theorem from Minty's Theorem.* We first prove that $1 + l(D)$ is minimized by an acyclic orientation, to which we can then apply Minty's Theorem. If D is an arbitrary orientation, let D' be a maximal acyclic subgraph of D. Let xy be an edge of $D - D'$. Since adding xy to D' creates a cycle, D' contains a y,x-path.

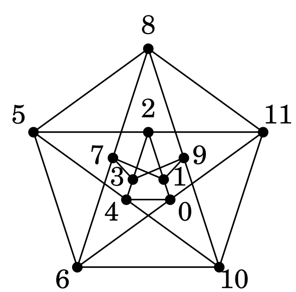
Let D^* be the orientation of G obtained from D by reversing the orientation on each edge of $D - D'$. If D^* contains a cycle C, then for each reversed edge yx on C corresponding to an edge xy of $D - D'$, we replace yx with a y,x-path that exists in D'. The result is a closed (directed) walk in D'. This yields a cycle in D', because a shortest closed walk in a digraph that has a closed walk is a cycle. Since by construction D' is acyclic, we conclude that D^* is acyclic.

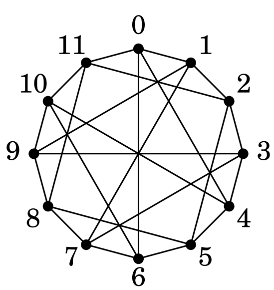
We also claim that $l(D^*) \le l(D')$. Let P be a u,v-path in D^*; some edges of P may have opposite orientation in D and D^*. For such an edge $yx \in E(P)$, there is a y,x-path in D'. When we replace all such edges of $D^* - D$ in P by paths in D', we obtain a u,v-walk in D'. This must in fact be a u,v-path in D', because D' is acyclic. Finally, the path we have found in D' is at least as long as P, because we replaced each edge of P not in D' with a nontrivial path in D'.

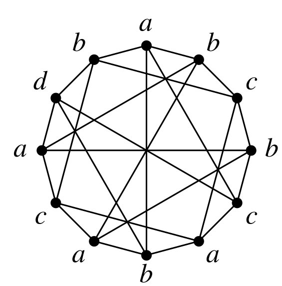
Since $D' \subseteq D$, also $l(D') \le l(D)$, so $l(D^*) \le l(D)$, and maximum path length is minimized over all orientations by an acyclic orientation.

With D^* an acyclic orientation minimizing the maximum path length, Minty's Theorem yields $\chi(G) \le 1 + r(D^*)$, where $r(D^*)$ is the floor of the maximum ratio of forward edges of D^* to backward edges of D^* when traversing a cycle of G. If a cycle of G achieving the maximum has k backward edges, then it has at least $kr(D^*)$ forward edges, and by the pigeonhole principle it has $r(D^*)$ consecutive forward edges. Hence $l(D^*) \ge r(D^*)$, and we obtain the desired inequality $\chi(G) \le 1 + r(D^*) \le 1 + l(D^*) \le 1 + l(D)$, where D is any orientation.

5.1.46. *4-regular triangle-free 4-chromatic graphs.* The graph on the left is isomorphic to the graph in the middle and properly 4-colored on the right.







We show that there is no proper 3-coloring. In a proper 3-coloring, the largest independent set has size at least 4, and the remaing vertices induce a bipartite subgraph. Thus it suffices to show that deleting a maximal independent set of size at least 4 always leaves a 3-cycle.

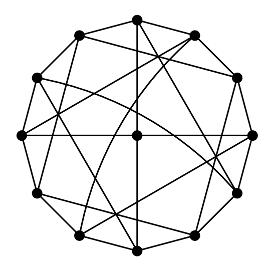
Rotational and reflectional symmetries partition the vertices into two orbits. Using the labeling on the left, class 1 is $\{2,5,8,11\}$, and class 2 is the rest. The subgraph G_1 induced by class 1 is a 4-cycle, with the maximal independent sets consisting of opposite vertices. The subgraph G_2 induced by class 2 is an 8-cycle plus chords joining opposite vertices. The maximal independent sets have size 3. By symmetries and flips, all such sets are equivalent by isomorphisms to $\{0,7,9\}$. This set is adjacent to all vertices of Class 1 except 2 and 5, and we can add just one of those two.

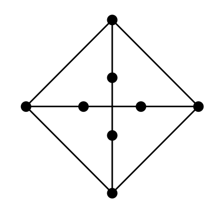
Therefore, when we use two vertices of Class 1, we can add only two from class 2. By symmetry, we may assume that 2 and 8 are used from Class 1. This eliminates all vertices of Class 1 except $\{4,6,10,0\}$. The maximal additions are $\{4,6\}$ and $\{10,0\}$, equivalent by symmetry.

Below we list in the first column a representative for each type of maximal independent set of size at least 4. The second column gives an odd cycle among the remaining vertices. Hence there is no proper 3-coloring.

$\{0,7,9,2\}$	$(10,11,8,5,6)$
$\{0,7,9,5\}$	$(10,11,2,3,4)$
$\{0,2,8,10\}$	$(3,4,5,6,7)$

Another 4-chromatic graph. This graph is obtained from the graph above by deleting the 0-6 and 3-9 edges and replacing them with a new vertex z adjacent to 0, 3, 6, 9. Hence we obtain a proper 4-coloring from the coloring of the previous graph by using color d on z.





To show that there is no proper 3-coloring, consider the previous graph. Vertices 0 and 6 together dominate all but 2, 3, 8, 9, which now induce $2K_2$. By rotational symmetry, 3 and 9 also belong to no independent 5-set of old vertices. Since the new graph has 13 vertices, 3-coloring requires a color class of size 5, and this must include the new vertex z.

The subgraph induced by the nonneighbors of z is shown on the right above. The four vertices of degree 2 are not independent, so a vertex of degree 3 is needed to form an independent 4-set. However, deleting a vertex of degree 3 leaves $2K_2$, and only two additional vertices can be chosen.

We have shown that the 13-vertex graph has no independent 5-set, so its chromatic number is 4.

5.1.47. *Brooks' Theorem and the following statement (*) are equivalent: every $(k-1)$-regular k-critical graph is a complete graph or an odd cycle.* Suppose first that Brooks' Theorem is true. Let G be a $(k-1)$-regular k-critical graph. Thus $\Delta(G) = k-1$ and $\chi(G) = k$. By Brooks' Theorem, G must be a complete graph or an odd cycle. Hence (*) follows.

Conversely, assume (*): every $(k-1)$-regular k-critical graph is a complete graph or an odd cycle. Let G be a connected graph with chromatic number k. In order to prove Brooks' Theorem, we must show that $\Delta(G) \ge k$ unless G is a complete graph or an odd cycle.

Let H be a k-critical subgraph of G. Since H is k-critical, $\delta(H) \ge k-1$. If $\Delta(G) < k$, then $k-1 \le \delta(H) \le \Delta(H) \le \Delta(G) < k$, which requires H to be $(k-1)$-regular. By (*), H is a complete graph or an odd cycle. If also $\Delta(G) = k-1$, then no vertex of H has an additional incident edge in G. This means that H is a component of G, so it is all of G, since G is connected. We have shown that if $\Delta(G) < \chi(G)$, then G is a complete graph or an odd cycle. Hence (*) implies Brooks' Theorem.

5.1.48. *A simple graph G with maximum degree at most 3 and no component isomorphic to K_4 has a bipartite subgraph with at least $e(G) - n(G)/3$ edges.* By Brooks' Theorem, G is 3-colorable. In a proper 3-coloring, let red be the smallest color class, with green and blue being the other two. By the pigeonhole principle, there are at most $n(G)/3$ red vertices.

Each red vertex v has three neighbors. By the pigeonhole principle, blue or green appears at most once in $N(v)$. If blue appears at most once, then we delete the edge from v to its blue neighbor (if it has one) and change the color of v to blue. If green appears at most once, then we delete the edge to the green neighbor (if it has one) and make v green.

This alteration deletes at most $n(G)/3$ edges and eliminates the color red. Thus it produces the desired bipartite subgraph.

5.1.49. *The Petersen graph can be 2-colored so that each color class induces only isolated edges and vertices.* Such a coloring appears on the front cover of the text. One color class is an independent set of size 4. Deleting an independent 4-set from the Petersen graph leaves $3K_2$.

5.1.50. *Improvement of Brooks' Theorem.*

a) For any graph G and parameters $D_1, \ldots, D_t$ such that $\sum D_i \ge \Delta(G) - t + 1$, $V(G)$ can be partitioned into t classes such that the subgraph G_i induced by the ith class has $\Delta(G_i) \le D_i$.

Proof 1 (extremality). Given a partition $V_1, \ldots, V_t$ of $V(G)$, let $e_i = e(G[V_i])$ for each i, and let $d_i(x) = |N(x) \cap V_i|$. We claim that a partition minimizing $f = \sum e_i/D_i$ has the desired property. If $d_i(x) > D_i$ for some $x \in V_i$, then $|N(x) - V_i| \le d(x) - D_i \le \Delta(G) - D_i \le \sum_{j \ne i}(D_j + 1)$. Thus for some j other than i, we have $d_j(x) \le D_j$. Moving x from V_i to V_j reduces f by $d_i(x)/D_i - d_j(x)/D_j$, which is positive. Thus when f is minimized each induced subgraph meets its degree bound.

Proof 2 (induction on t). For $t = 2$, we claim that the partition minimizing $D_1 e(G_2) + D_2 e(G_1)$ satisfies the desired bounds. If not, then there is a vertex x, say $x \in V(G_1)$, such that $d_{G_1}(x) > D_1$; x has at most $D_2 - 1$ neighbors in $V(G_2)$. Moving x to the other part gains less than $D_1 D_2$ and loses more than $D_2 D_1$, which contradicts the optimality of the original partition.

For $t > 2$, let $D = D_1 + \cdots D_{t-1} + (t-2)$. We have $D + D_t \ge \Delta(G) - 1$, and the hypothesis for 2 parameters guarantees a vertex partition where the induced subgraphs have maximum degrees bounded by D and D_t. Since $D_1 + \cdots + D_{t-1} = D - t + 2$, the hypothesis for $t-1$ parameters yields the rest of the desired partition.

b) (general case). If G contains no r-clique, where $4 \le r \le \Delta(G)+1$, then $\chi(G) \le \lceil \frac{r-1}{r}(\Delta(G)+1) \rceil$. Let $D_1 = \cdots = D_{t-1} = r-1$, and require $D_t \ge r-1$ and $\sum D_i = \Delta(G) - t + 1$. Thus $t = \lfloor (\Delta(G)+1)/r \rfloor$. By (a), $V(G)$ partitions into t classes such that $\Delta(G_i) \le D_i$. Since G has no r-clique, Brooks' Theorem implies $\chi(G_i) \le D_i$. Coloring the subgraphs with disjoint color sets, we have $\chi(G) \le \sum \chi(G_i) \le \sum D_i = \Delta(G) + 1 - t$.

b) (special case). If G contains no K_4 and $\Delta(G) = 7$, then $\chi(G) \le 6$. Letting $t = 2$ and $D_1 = D_2 = 3$, we have $D_1 + D_2 = \Delta(G) - t + 1$ Applying part (a), we are guaranteed a partition of $V(G)$ into two sets such that the subgraph induced by each set has maximum degree at most 3. Since G has no K_4, Brooks' Theorem guarantees that both subgraphs are 3-colorable. Coloring the two subgraphs with disjoint color sets, we have $\chi(G) \le 6$.

5.1.51. *If G is a k-colorable graph, and P is a set of vertices in G such that $d(x,y) \ge 4$ whenever $x, y \in P$, then every coloring of P with colors from $[k+1]$ extends to a proper $k+1$ coloring of G.* Let $c\colon P \to [k+1]$ be the coloring on the precolored set P, and let $f\colon V(G) \to [k]$ be a proper k-coloring of G. We define an $(k+1)$-coloring g of G by

$$g(u) = \begin{cases} c(u) & u \in P \\ k+1 & u \in N(v),\ v \in P, \text{ and } f(u) = c(v) \\ f(u) & \text{otherwise} \end{cases}$$

Since the neighbors of $v \in P$ having color $c(v)$ under f must form an independent set and vertices of P are separated by distance at least 4,

$\{u: g(u) = r + 1\}$ is independent. A color among the first r is used only on P or on vertices receiving that color under f. The latter type form an independent set, and when a color is used on a vertex $v \in P$, all neighbors of P explicitly receive a color different from that. Hence each color class in g is an independent set, and g is a proper coloring that extends c.

5.1.52. *Every graph G can be $\lceil(\Delta(G)+1)/j\rceil$-colored so that each color class induces a subgraph having no j-edge-connected subgraph.* A j-edge-connected subgraph has minimum degree at least j. Hence it suffices to color $V(G)$ so that when each vertex is assigned a color, it has fewer than j neighbors among vertices already colored. In this way, no j-edge-connected subgraph is ever completed within a color class.

Color vertices according to some order σ. When a vertex is reached, it has at most $\Delta(G)$ earlier neighbors. Since $\lceil(\Delta(G)+1)/j\rceil$ colors are available, by the pigeonhole principle some color has been used on fewer than j earlier neighbors. We assign such a color to the new vertex.

No smaller number of classes suffices if G is an j-regular j-edge-connected graph or an n-clique with $n \equiv 1 \pmod j$ (or an odd cycle when $j = 1$). If G is a j-regular j-edge-connected graph, then two colors are needed. If $G = K_n$ with $n \equiv 1 \pmod j$, then we can give each color to at most j vertices, and thus $\lceil n/j\rceil$ colors are needed. If $j = 1$ and G is an odd cycle, then 3 colors are needed. In each case, the needed number of colors equals $\lceil(\Delta(G)+1)/j\rceil$.

5.1.53. *Relaxed colorings of the $2k$-regular graph $G_{n,k}$ of Exercise 5.1.23.* For $k \le 4$, we seek n such that there is a 2-coloring in which each color class induces a subgraph with maximum degree at most k.

Of the $2k$ neighbors of a vertex v, at most half can have the same color as v. When n is even, alternating colors works, and when n is odd we can insert one additional vertex with either color. This solution is trivial because the problem was improperly stated: “at most k” should be “less than k”. Say that a 2-coloring is *good* if each vertex has at most $k-1$ neighbors of its own color. We solve the intended problem.

If n is even and k is odd, then alternating the colors around C gives an good 2-coloring of $G_{n,k}$, since each vertex has exactly $(k-1)/2$ neighbors with its own color in each direction. More generally, let n be a multiple of $2j$, and 2-color $G_{n,k}$ using runs of j consecutive vertices with the same color. Suppose that $k = q\cdot 2j + r$, where $j \le r < 2j$. Consider the ith vertex in a run. Following it, this vertex has $(j-i)+qj+[r-j-(j-i)]$ neighbors of its own color; preceding it, this vertex has $(i-1)+qj+[r-j-(i-1)]$ neighbors of its own color. Altogether, then, every vertex has fewer than k neighbors of its own color, since $q(2j)+2r-2j = k+r-2j < k$. Thus there is a good 2-coloring when n is a multiple of $2j$ and j is a positive integer

such that the k is congruent modulo $2j$ to a value in $\{j,\ldots,2j-1\}$. For $k \in \{1,3\}$, this permits all even n. For $k = 2$, it permits multiples of 4. For $k = 4$, it permits multiples of 6 or 8. Let T denote this set of values of n.

When $k \le 4$, we show that the set T consists of all values of n that permit good 2-colorings. Consider a good 2-coloring of $G_{n,k}$ for n not in this set. Obviously there is no run of length at least $k+1$ in the same color; each vertex neighbors all others in a run of length $k+1$. If there is a run of length k, then since n is not a multiplie of $2k$ there is a run of length k followed by a shorter run. Now the last vertex of the k-run has $k-1$ neighbors of its own color in that run and another neighbor after the subsequent run, which is forbidden.

Hence we may assume that all runs have length less than k. If they all have the same length j, then n is a multiple of $2j$. If $k = q\cdot 2j + r$ with $0 \le r < j$, then the last vertex of a run has qj neighbors of its own color following it and $qj-1+r+1$ neighbors of its own color preceding it. These sum to k, so such a coloring is not good.

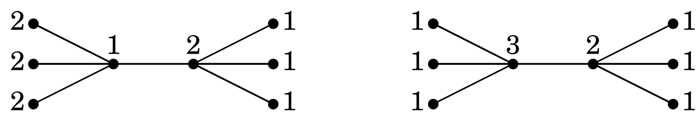
Hence we may assume that the coloring has adjacent runs of different sizes, each each less than k. For $k \le 2$, this is impossible. If the coloring has a $(k-1)$-run A, then the absence of k-runs implies that the first vertex has of A has an earlier neighbor in its own color. Since it also has $k-2$ neighbors in A, the run following A must have length at least 2. Thus runs of length $k-1$ are surrounded by runs of length at least 2. For $k = 3$, this forbids runs of distinct length and therefore completes the proof.

Now consider $k = 4$. If the coloring has a 3-run, then the preceding argument and the requirement of having runs of different lengths allows us to assume colors *abbbaab* in order on $v_0,\ldots,v_6$. Now successively examining the neighborhoods of v_3 and v_4 allows us to conclude that $f(v_7) = a$ and then $f(v_8) = b$. Next the neighborhood of v_6 leads us to $f(v_9) = f(v_{10}) = a$, but now v_7 has four neighbors of its own color.

The only remaining possibility is that the largest run has size 2. With runs of different lengths, we may assume collors *abbab* on $v_0,\ldots,v_4$. Since v_5 has a later neighbor of its own color, $f(v_5) = a$. Since there is no 3-run, v_2 has two earlier neighbors of its own color b, and thus $f(v_6) = a$. Since there is no 3-run, $f(v_7) = a$. Now v_4 has three neighbors in its own color, which forces $f(v_8) = a$. We have produced the pattern of $f(v_0),\ldots,f(v_4)$ on vertices $v_4,\ldots,v_8$, with the colors exchanged. Hence the pattern *abba baab abba baab*, continues. This is a good coloring, but it requires n to be a multiple of 8, and these values of n are already in T.

Comment. West–Weaver [1994] conjectured that T contains all n for which $G_{n,k}$ has a good coloring is not valid for larger k. This was disproved by Brad Friedman, who discovered other values of n with good colorings for all larger k.

5.1.54. Let f be a proper coloring of a graph G in which the colors are natural numbers. The *color sum* is $\sum_{v\in V(G)} f(v)$. Minimizing the color sum may require using more than $\chi(G)$ colors. In the tree below, for example, the best proper 2-coloring has color sum 12, while there is a proper 3-coloring with color sum 11. Construct a sequence of trees in which the kth tree T_k use k colors in a proper coloring that minimizes the color sum. (Kubicka–Schwenk [1989])



5.1.55. *Chromatic number is bounded by one plus longest odd cycle length.*

a) If G is a 2-connected non-bipartite graph containing an even cycle C, then there exist vertices x, y on C and an x, y-path P internally disjoint from C such that $d_C(x, y) \not\equiv d_P(x, y)$ (mod 2). Let C' be an odd cycle in G. Since G is 2-connected, G has a cycle containing an edge of C and an edge of C'. Using edges of this and C', we can form an odd cycle D containing at least two vertices of C. Let $x_1, \ldots, x_t$ be the common vertices of C and D, indexed in order of their appearance on D. Letting $x_{t+1} = x_1$, we have $\sum_{i=1}^{t} d_D(x_i, x_{i+1}) \equiv 1 \pmod 2$, since D is an odd cycle. On the other hand, since C is an even cycle, it has a bipartition, and $d_C(x_i, x_{i+1})$ is even if x_i and x_{i+1} are on the same side of the bipartition of C, odd if they are on opposite sides. Hence $\sum_{i=1}^{t} d_C(x_i, x_{i+1}) \equiv 0 \pmod 2$. Therefore, for some value of i we have $d_D(x_i, x_{i+1}) \not\equiv d_C(x_i, x_{i+1})$, and we use this portion of D as P.

b) If $\delta(G) \ge 2k$ and G has no odd cycle longer than $2k-1$, then G has a cycle of length at least $4k$. Let $P = x_1, \ldots, x_t$ be a maximal path in G, so $N(x_1) \subseteq V(P)$. Let x_r be the neighbor of x_1 farthest along P; $d(x_1) \ge 2k$ implies $r \ge 2k+1$. By the odd cycle condition, r is even, and neither x_{2i+1} nor x_{r-2i+1} can belong to $N(x_1)$ if $i \ge k$. If $\{2i+1\colon i \ge k\}$ and $\{x_{r-2i+1}\colon 2 \ge k\}$ are disjoint, then together with $N(x_1)$ we have at least $r - 2k + 2k = r$ vertices with indices from 2 to r. This is impossible, so we must have $2k+1 \le r - 2k + 1$, implying $r \ge 4k$.

c) If G is a 2-connected graph having no odd cycle longer than $2k-1$, then $\chi(G) \le 2k$. We use induction on $n(G)$. For $n(G) = 2$, the claim holds using $k = 1$. For the induction step, suppose $n(G) = n > 2$ and the claim holds for graphs with fewer than n vertices. Since $\chi(G)$ is the maximum chromatic number of its blocks, we may assume G is 2-connected. Suppose the longest odd cycle in G has length $2k-1$, but $\chi(G) > 2k$. For any $x \in V(G)$, the induction hypothesis implies $\chi(G - x) \le 2k$. Hence G is vertex-$(2k+1)$-critical, which implies $\delta(G) \ge 2k$. By part (b), G has a cycle C of length at least $4k$. By part (a), G has a path P joining two vertices x, y

of C such that P together with either x, y-path along C forms an odd cycle. The sum of the lengths of these two odd cycles is at least $4k+2$. Hence one of them has length at least $2k+1$, contradicting the hypothesis. The contradiction yields $\chi(G) \le 2k$.

5.2. STRUCTURE OF k CHROMATIC GRAPHS

5.2.1. *If $\chi(G - x - y) = \chi(G) - 2$ for all pairs x, y of distinct vertices, then G is a complete graph.* If $x \not\leftrightarrow y$, then a proper coloring of $G - x - y$ can be augmented with one new color on x and y to obtain a proper coloring of G. This yields $\chi(G) \le \chi(G - x - y) + 1$, so the given condition forces $x \leftrightarrow y$ for all $x, y \in V(G)$.

5.2.2. *A simple graph is a complete multipartite graph if and only if it has no induced three-vertex subgraph with one edge.* If a connected graph is not a clique, then the shortest of all paths between nonadjacent pairs of vertices has length two, and the three vertices of this path induce a subgraph with exactly two edges. Hence each successive pair of the following statements are equivalent: (1) G has no induced 3-vertex subgraph with one edge. (2) $\overline{G}$ has no induced 3-vertex subgraph with two edges. (3) Every component of $\overline{G}$ is a clique. (4) G is a complete multipartite graph.

5.2.3. *The smallest k-critical graphs.*

a) If x, y are vertices in a color-critical graph G, then $N(x) \subseteq N(y)$ is impossible, and hence there is no k-critical graph with $k+1$ vertices. If G is k-critical, then $G - x$ is $(k-1)$-colorable, but $N(x) \subseteq N(y)$ would allow us to return x with the same color as y to obtain a $(k-1)$-coloring of G. If $n(G) = k+1$, then we have $\delta(G) < k$ since K_{k+1} is not k-critical, and we have $\delta(G) \ge k-1$ by the properties of k-critical graphs. Hence $\delta(G) = k-1$, which implies that nonadjacent vertices x, y have the same set of neighbors (the remaining $k-1$ vertices), which contradicts the statement just proved. Hence there is no k-critical graph with $k+1$ vertices.

b) $\chi(G \vee H) = \chi(G) + \chi(H)$, and $G \vee H$ is color-critical if and only if both G and H are color-critical, and hence there is a k-critical graph with $k+2$ vertices. Coloring G and H optimally from disjoint sets yields a proper coloring of $G \vee H$, so $\chi(G \vee H) \le \chi(G) + \chi(H)$. The colors used on the subgraph of $G \vee H$ arising from G must be disjoint from the colors on the copy of H, since each vertex of the former is adjacent to each of the latter; hence $\chi(G \vee H) \ge \chi(G) + \chi(H)$.

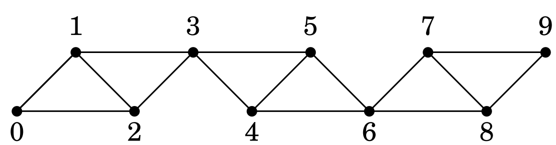
For criticality, consider an arbitrary edge $xy \in E(G \vee H)$. If $xy \in E(G)$, then $(G \vee H) - xy = (G - xy) \vee H$, and hence $\chi(G \vee H) - xy =$

$\chi(G \vee H) - 1$ if and only if $\chi(G - xy) = \chi(G) - 1$. Similarly for $xy \in E(H)$. Hence $G \vee H$ being color-critical implies that G and H are color-critical. For the converse, assume that G and H are color-critical. We have already considered $G \vee H - xy$ for $xy \in E(G) \cup E(H)$; we must also consider $xy \in E(G \vee H)$ with $x \in V(G)$ and $y \in V(H)$. By the properties of color-critical graphs, we know that G and H have optimal colorings in which x and y, respectively, are the only vertices in their color classes. In $G \vee H - xy$, we use these colorings but change the color of y to agree with x. This uses $\chi(G) + \chi(H) - 1$ colors.

Since C_5 is 3-critical and K_{k-3} is $(k-3)$-critical, we conclude that $C_5 \vee K_{k-3}$ is a k-critical graph with $k+2$ vertices.

5.2.4. *Blocks and coloring in a special graph.* Let G be the graph with vertex set $\{v_0, \ldots, v_{3n}\}$ defined by $v_i \leftrightarrow v_j$ if and only if $|i-j| \le 2$ and $i+j$ is not divisible by 6.

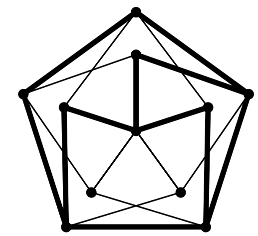
a) The blocks of G. Because consecutive integers sum to a number that is odd and hence not divisible by 6, the vertices $v_0, \ldots, v_{3n}$ form a path in order. Edges of the form $\{v_i, v_{i+2}\}$ are added when i is congruent to one of $\{0, 1, 3, 4\}$ modulo 6, but not when i is congrent to 2 or 5 modulo 6. Thus G is the graph below, and there are n blocks. The blocks are the subgraphs induced by $\{v_{3i-3}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \le i \le n$.



b) Adding the edge v_0v_{3n} to G creates a 4-critical graph. In a proper 3-coloring of G, the induced kites force successive vertices whose indices are multiples of 3 to have the same color. When the edge v_0v_{3n} is added to form G', the graph is no longer 3-colorable.

If an edge in the ith kite is deleted, then giving its endpoints the same color permits properly 3-coloring the remainder of the subgraph induced by $\{v_{3i-3}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ so that v_{3i-3} and v_{3i} have different colors. Continuing the proper 3-coloring in both directions gives v_0 the color of v_{3i-3} and gives v_{3n} the color of v_{3i}. Thus the edge v_0v_{3n} is also properly colored. We have shown that for each edge e, the graph $G' - e$ is 3-colorable, so G' is 4-critical.

5.2.5. *A subdivision of K_4 in the Grötzsch graph.* The subgraph in bold below is a subdivision of K_4.



5.2.6. *The minimum number of edges in a connected n-vertex graph with chromatic number k is $\binom{k}{2} + n - k$.* Equality holds for the graph obtained by identifying a vertex of K_k with an endpoint of P_{n-k+1}. The desired lower bound on $e(G)$ when $k = 2$ is $n - 1$ and holds trivially for connected graphs, so we may assume that $k \ge 3$.

Proof 1 (critical subgraph). Let G be a connected k-chromatic n-vertex graph. Let H be a k-critical subgraph of G. If H has t vertices, then $e(H) \ge (k-1)t/2$, since $\delta(H) \ge k - 1$. With H and the remaining $n - t$ vertices of G as $n - t + 1$ components, we must add at least $n - t$ additional edges to reduce the number of components to 1. Hence

$$e(G) \ge (k-1)t/2 + n - t = (k-3)t/2 + n.$$

Since $n \ge t \ge k$, this is minimized when $t = k$, yielding the desired value.

Proof 2 (induction on n). For $n = k$, the bound again is trivial. For $n > k$, let G be a minimal connected k-chromatic n-vertex graph. By the choice of G, deletion of any edge disconnects G or reduces k.

If $G - e$ is disconnected for some $e \in E(G)$, then it has two components. At least one of these must be k-chromatic, else we can recolor G with fewer than k colors. Letting l be the number of vertices in a k-chromatic component of $G - e$, the induction hypothesis yields

$$e(G) \ge \left[\binom{k}{2} + l - k\right] + 1 + (n - l - 1) = \binom{k}{2} + n - k,$$

where the additional terms count e itself and the edges of a spanning tree of the other component.

In the remaining case, $\chi(G - e) < k$ for all $e \in E(G)$. Hence G is k-critical, which requires $\delta(G) \ge k - 1$. Hence

$$e(G) \ge n(k-1)/2 = n + n(k-3)/2 > n + k(k-3)/2 = n - k + \binom{k}{2}.$$

5.2.7. *In an optimal coloring of a graph, for each color there is a vertex of that color that is adjacent to vertices of all other colors.* Let C be the set of vertices of color i, and consider $v \in C$. If v has no neighbor of color j, then we can switch the color of v to j. Since we are changing colors only for vertices in C, moving several of them to color j in this way creates no conflicts, since C is an independent set. After relabeling all vertices of C, we have obtained a proper coloring without using color i. Hence C must have some "unmovable" vertex, adjacent to vertices of every other color.

5.2.8. *Critical subgraph approach to $\chi(G) \le \max_i \min\{d_i+1, i\}$.* If $\chi(G) = k$, then G has a k-critical subgraph, which has at least k vertices of degree at least $k-1$. These vertices also have degree at least k in G, so $d_k \ge k-1$. Hence $\chi(G) = k = \min\{d_k+1, k\} \le \max_i \min\{d_i+1, i\}$.

5.2.9. *If G is a color-critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical.* We use properties of a k-critical graph G obtained in Proposition 5.2.13a: a) For $v \in V(G)$, there is a proper k-coloring of G in which color k appears only at v, and b) For $e \in E(G)$, every proper $(k-1)$-coloring of $G-e$ uses the same color on the endpoints of e.

Given $V(G) = \{v_1, \ldots, v_n\}$, let G' be as in Mycielski's construction, with $V(G') = V(G) \cup \{u_1, \ldots, u_n\} \cup \{w\}$. Suppose that G is k-critical. The proof of Theorem 5.2.2 yields $\chi(G') = k+1$; thus it suffices to show that $\chi(G'-e) = k$ for $e \in E(G')$.

For $e = wu_j$, let f be a proper k-coloring of G with color k appearing only on v_j. Extend f to $G'-e$ by setting $f(u_i) = f(v_i)$ for $1 \le i \le n$ and $f(w) = f(v_j)$.

For $e = v_r v_s$, let f be a proper $(k-1)$-coloring of $G-e$, which exists because G is k-critical. Extend f to $G'-e$ by letting $f(u_i) = k$ for $1 \le i \le n$ and $f(w) = 1$.

For $e = v_r u_s$, let f be a proper $(k-1)$-coloring of $G - v_r v_s$. By Proposition 5.2.13b, $f(v_r) = f(v_s)$. Extend f to $G'-e$ by letting $f(u_i) = f(v_i)$ for $1 \le i \le n$ and $f(w) = k$. This uses k colors and uses color k only on w, but this is not a proper coloring of $G'-e$, because the endpoints of the edges $v_r v_s$ and $u_r v_s$ have received the same color. We correct this to a proper coloring by changing the color of v_s to k.

5.2.10. *Given $H \subseteq G$ with $V(G) = \{v_1, \ldots, v_n\}$, if G'' is obtained from G by applying Mycielski's construction and adding the edges $\{u_i u_j : v_i v_j \in E(H)\}$, then $\chi(G'') = \chi(G)+1$ and $\omega(G'') = \max\{\omega(G), \omega(H)+1\}$.* Since G'' is a supergraph of the result G' of Mycielski's construction, $\chi(G'') \ge \chi(G') = \chi(G)+1$. On the other hand, the proper coloring of G' that uses a proper $\chi(G)$-coloring on $v_1, \ldots, v_n$, copies the color of v_i onto u_i for each i, and assigns a new color to w, is still a proper coloring of G'', so $\chi(G'') = \chi(G)+1$.

Since u_i and v_i remain nonadjacent for all i, every complete graph induced by $\{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\}$ is a copy of a complete subgraph of G, using at most one of $\{v_i, u_i\}$ for each i. Every complete graph involving w is an edge wu_i or consists of w together with a complete subgraph in H. Hence $\omega(G'') = \max\{\omega(G), \omega(H)+1\}$.

5.2.11. *If G has no induced $2K_2$, then $\chi(G) \le \binom{\omega(G)+1}{2}$.* To prove the upper bound, we define $k + \binom{k}{2}$ independent sets that together cover $V(G)$.

Let $Q = \{v_1, \ldots, v_k\}$ be a maximum clique in G. Let S_i be the set of vertices in G that are adjacent to all of Q except v_i. This set is independent, since two adjacent vertices in S_i would form a $(k+1)$-clique with $Q - \{v_i\}$.

Let $T_{i,j}$ be the set of vertices in G that are adjacent to neither of $\{v_i, v_j\}$. This set is independent, since two adjacent vertices in $T_{i,j}$ would form an induced $2K_2$ with $\{v_i, v_j\}$.

Every vertex of G has at least one nonneighbor in Q, since Q is a maximum clique. Thus every vertex of G is in some S_i or in some $T_{i,j}$, and we have covered $V(G)$ with the desired number of independent sets.

5.2.12. *Zykov's construction.* Let $G_1 = K_1$. For $k > 1$, construct G_k from $G_1, \ldots, G_{k-1}$ by taking the disjoint union $G_1 + \cdots + G_{k-1}$ and adding a set T of $\prod_{i=1}^{k-1} n(G_i)$ additional vertices, one for each way to choose exactly one vertex v_i from each G_i. Let the vertex of T corresponding to a particular choice of $v_1, \ldots, v_{k-1}$ have those $k-1$ vertices as its neighborhood.

a) $\omega(G_k) = 2$ and $\chi(G_k) = k$. Giving all of T a single color k and using an i-coloring from $\{1, \ldots, i\}$ on each copy of G_i yields a proper k-coloring of G_k. Since the neighbors of each vertex of T are in distinct components of $G_k - T$, the edges to T introduce no triangle.

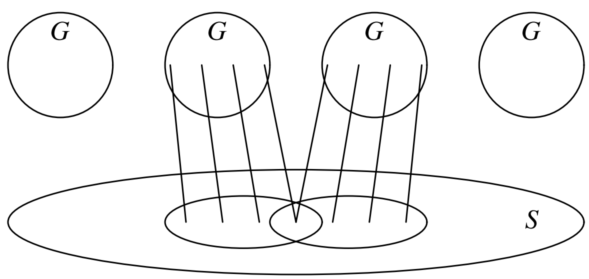
Suppose that G_k has a proper $(k-1)$-coloring. Because $\chi(G_i) = i$, some color is used on G_1, some other color is used on G_2, some third color used on G_3, and so on. Thus vertices can be selected from the subgraph $G_1 + \cdots + G_{k-1}$ having all $k-1$ colors. By the construction of G_k, some vertex of T has these as neighbors, and the proper coloring cannot be completed.

b) Zykov's construction produces color-critical graphs. We must show that, for any edge xy, $G_k - xy$ has a proper $(k-1)$-coloring. Suppose this has been shown for $G_1, \ldots, G_{k-1}$, and consider an edge xy of G_k. If $x, y \notin T$, then xy is an edge of G_t for some $t < k$. Color each G_i with colors $1, \ldots, i$ for $i \ne t$, but color $G_t - xy$ with colors $1, \ldots, t-1$. Each vertex of T has $k-1$ neighbors, but it is not possible for each of the colors $1, \ldots, k-1$ to appear among the neighbors of any vertex of T, because its neighbors in $\{G_1, \ldots, G_t\}$ have received only $t-1$ colors. Hence there is a color in $\{1, \ldots, k-1\}$ available for any vertex of T.

Finally, suppose we delete an edge xy with $x \in T$, $y \notin T$, and let $S = N(x)$. By the criticality of $G_1, \ldots, G_{k-1}$, each G_i has a proper i-coloring with colors $1, \ldots, i$ in which the only vertex of color i is the neighbor of x in G_i; use these colorings to color $G_k - T$. In order to choose one vertex from each G_i and obtain a set with colors $1, \ldots, k-1$, we must choose the vertex with color i from G_i; the only way to do this is to choose S. Since x is the only vertex of T with these neighbors, for every other vertex of T there is a color in $\{1, \ldots, k-1\}$ available for it. Finally, if the other endpoint y of the deleted edge xy is the neighbor of x in G_i, we can give color i to x to complete the proper $(k-1)$-coloring of $G - xy$.

5.2.13. *Inductive construction of k-chromatic graphs of girth at least six.* Given G with girth at least 6 and $\chi(G) = k$, form G' by taking $\binom{kn(G)}{n(G)}$ copies of G and an independent set S with $kn(G)$ vertices, with each subset of n vertices in S joined by a matching to one copy of G (distinct subsets match to different copies of G). Since G is k-colorable, G' has a proper $(k+1)$-coloring where all of S has color $k+1$. If G' is k-colorable, then any proper k-coloring of G' gives the same color to at least $n(G)$ vertices of S with the same color, by the pigeonhole principle. This color is forbidden from the copy of G matched to this $n(G)$-subset of S. Now the coloring cannot be completed, since proper coloring of this copy of G require at least k colors.

Since G has girth at least 6, any cycle of length less than 6 must use at least two vertices of S. However, S is independent, and vertices of S have no common neighbors, so it must take at least 3 edges to go from one vertex of S to another.



5.2.14. *Chromatic number and cycle lengths.*

a) If v is a vertex in a graph G, and T is a spanning tree that maximizes $\sum_{u \in V(G)} d_T(u, v)$, then every edge of G joins vertices belonging to a path in T starting at v. View v as the root of T. If u is on the v,w-path in T, then w is a *descendant* of u. Suppose that xy is an edge in G such that neither of $\{x, y\}$ is a descendant of the other in T. We may assume that $d_T(v, x) \le d_T(v, y)$. Now cutting the edge reaching x on the v,x-path in T and replacing it with yx increases the distance from v to x and to all its descendants. This contradicts the choice of T, so there is no such edge.

b) If $\chi(G) > k$, then G has a cycle whose length is one more than a multiple of k. Define T as in part (a). Define a coloring of G by letting the color assigned to each vertex x be the congruence class modulo k of $d_T(v, x)$. This is a proper coloring unless G has an edge xy outside T that joins vertices of the same color. By part (a), x or y is a descendant of the other, and the length of the x,y-path in T is a multiple of k. If G has no cycle with length one more than a multiple of k, then there is no such edge, and the coloring is proper. We have proved the contrapositive of the claim.

5.2.15. *Every triangle-free n-vertex graph G has chromatic number at most $2\sqrt{n}$.* Since G is triangle-free, every vertex neighborhood is an independent set. Iteratively use a color on a largest remaining vertex neighborhood and delete those vertices. After $\lfloor \sqrt{n} \rfloor$ iterations, the maximum degree in the remaining subgraph is less than $\sqrt{n}$. Otherwise, we have deleted at least $\sqrt{n}$ vertices $\lfloor \sqrt{n} \rfloor$ times, and there are at most $\sqrt{n}$ vertices remaining. Since the maximum degree of the remaining subgraph is less than $\sqrt{n}$, we can use $\sqrt{n}$ additional colors to properly color what remains.

5.2.16. *A simple n-vertex graph with no K_{r+1} has at most $(1 - \frac{1}{r})\frac{n^2}{2}$ edges.*

Proof 1 (induction on r). Basis step: If $r = 1$, then G has no edges, as claimed. Induction step: For $r > 1$, let x be vertex of maximum degree, with $d(x) = k$. Since G has no $(r+1)$-clique, the subgraph G' induced by $N(x)$ has no r-clique. Hence G' has at most $\frac{r-2}{r-1}k^2/2$ edges, by the induction hypothesis. The remaining edges are incident to the remaining $n - k$ vertices; since each such vertex has degree at most k, there are at most $k(n-k)$ such edges. Summing the two types of contributions, we have $e(G) \le k(n - ak)$, where $a = r/(2r-2)$. The function $k(n - ak)$ is maximized by setting $k = \frac{n}{2a}$, where it equals $\frac{n^2}{4a}$. Hence $e(G) \le \frac{n^2}{4a} = (1 - \frac{1}{r})\frac{n^2}{2}$.

Proof 2 (by Turán's Theorem). By Turán's Theorem, the maximum number of edges in a graph with no $(r+1)$-clique is achieved by the complete r-partite graph with no two part-sizes differing by more than one. If the part-sizes are $\{n_i\}$, the degree-sum is $\sum_{i=1}^r n_i(n - n_i) = n^2 - \sum_{i=1}^r n_i^2$.

By the convexity of the squaring function, the sum of the squares of numbers summing to n is minimized when they all equal n/r. Hence if G has no $(r+1)$-clique, we have $2e(G) \le (1 - \frac{1}{r})n^2$.

5.2.17. *Lower bounds on $\omega(G)$ and $\alpha(G)$ for n-vertex graphs with m edges.*

a) $\omega(G) \ge \lceil n^2/(n^2 - 2m) \rceil$, and this is sometimes sharp. Let r be the number of vertices in the largest clique in G. By Exercise 5.2.16, $m \le (1 - 1/r)n^2/2$. This is equivalent by algebraic manipulation to $n^2/(n^2 - 2m) \le r$.

This bound is sometimes best possible. Let $r = \lceil n^2/(n^2 - 2m) \rceil$. Since $m \le \binom{n}{2}$, we have $r \le n$. For the bound to be sharp, it suffices to show that $T_{n,r}$ has at least m edges and that $m \ge \binom{r}{2}$. If these two statements are true, then we can discard edges from $T_{n,r}$ to obtain a graph G with n vertices and m edges such that $\chi(G) = \omega(G) = r$.

If r is an integer that divides n, then $e(T_{n,r}) = (1 - 1/r)n^2/2 = m$ and the desired properties hold. However, when $n = 12$ and $m = 63$, there are three edges in $\overline{G}$. We have $\lceil n^2/(n^2-2m)\rceil = 8$, but every 12-vertex simple graph with only three edges in the complement has clique number 9.

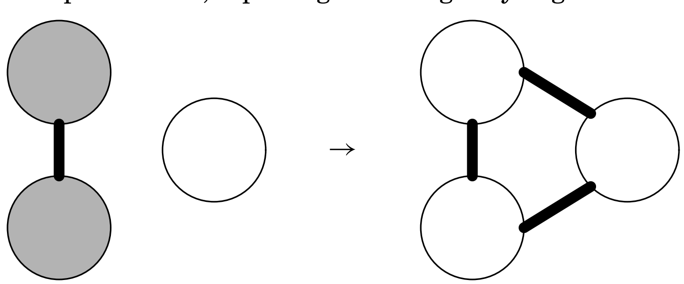
b) $\alpha(G) \ge \lceil n^2/(n+2m)\rceil$ *vertices, and this is sometimes sharp.* We can transform this question into an instance of part (a) by taking complements. Every clique in G becomes an independent set in $\overline{G}$, and vice versa. Let $H = \overline{G}$. Let $m' = \binom{n}{2} - m$ be the number of edges in H. If the largest independent set in G has s vertices, then the largest clique in H has s vertices. From part (b), we have $s \ge \lceil n^2/(n^2-2m')\rceil$. Substituting $m' = \binom{n}{2} - m$ yields $s \ge \lceil n^2/(n+2m)\rceil$. Since this lower bound for s is achieved for some m, n by letting H be the appropriate Turán graph, it is also achieved by letting G be the complement of that graph.

5.2.18. *Counting edges in the Turán graph.* Let $T_{n,r}$ denote the r-partite Turán graph on n vertices, and let $a = \lfloor n/r\rfloor$ and $b = n - ra$.

a) $e(T_{n,r}) = (1-1/r)n^2/2 - b(r-b)/(2r)$. By the degree-sum formula, we need only show that the vertex degrees sum to $(1-1/r)n^2 - b(r-b)/r$. Every vertex has degree $n-a$ or $n-a-1$, with $(r-b)a$ of the former and $b(a+1)$ of the latter. Hence the degree sum is $n(n-a) - b(a+1)$. Substituting $a = (n-b)/r$ yields $n^2 - n(n-b)/r - b(n-b+r)/r$, which equals the desired formula.

b) The least r where $e(T_{n,r})$ can differ from $\lfloor(1-1/r)n^2/2\rfloor$ is $r = 8$, and $e(T_{n,8}) < \lfloor(1-1/r)n^2/2\rfloor$ whenever $n \equiv 4 \pmod 8$. In general, $e(T_{n,r}) < \lfloor(1-1/r)n^2/2\rfloor$ if and only if $b(r-b)/(2r) \ge 1$. Since $e(T_{n,r})$ is an integer, the formula for $e(T_{n,r})$ in part (a) differs from $\lfloor(1-1/r)n^2/2\rfloor$ if and only if the difference between $(1-1/r)n^2/2$ and $e(T_{n,r})$ is at least 1. Hence the condition is $b(r-b)/(2r) \ge 1$. For fixed r, the left side is maximized by $b = r/2$, where it equals $r/8$. Hence the condition occurs if and only if $r \ge 8$, and when $r = 8$ it occurs if and only if $b = 4$.

5.2.19. *Comparison of the Turán graph $T_{n,r}$ with the graph $\overline{K}_a + K_{n-a}$ yields $e(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$.* Here $a = \lfloor n/r\rfloor$. The initial graph $\overline{K}_a + K_{n-a}$ has $\binom{n-a}{2}$ edges. We transform it into $T_{n,r}$ and study the change in the number of edges. Let A be the independent set of size a. We create $T_{n,r}$ by iteratively removing the edges within a set of size a or $a+1$ to make it one of the desired partite sets, replacing these edges by edges to A.



The number of edges from A to a new partite set B is $a\,|B|$. Whether $|B|$ is a or $a+1$, this numerically equals $\binom{|B|}{2} + \binom{a+1}{2}$. Thus replacing the edges of the clique on B with these edges gains $\binom{a+1}{2}$ edges. Repeating this $r-1$ times to create the other partite sets gains $(r-1)\binom{a+1}{2}$ edges, and thus $e(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$.

5.2.20. *For positive integers n and k, if $q = \lfloor n/k\rfloor$, $r = n - qk$, $s = \lfloor n/(k+1)\rfloor$, and $t = n - s(k+1)$, then $\binom{q}{2}k + rq \ge \binom{s}{2}(k+1) + ts$.* The Turán graph $T_{n,k}$ has partite sets of sizes q and $q+1$, with r of the latter. Hence its complement has $\binom{q}{2}k + rq$ edges. Similarly, $\overline{T}_{n,k+1}$ has $\binom{s}{2}(k+1) + ts$ edges. To prove the desired inequality, it thus suffices to show that $e(\overline{T}_{n,k}) \ge e(\overline{T}_{n,k+1})$, or $e(T_{n,k}) \le e(T_{n,k+1})$.

This follows from Turán's Theorem. Since $e(T_{n,k+1})$ is the maximum number of edges in an n-vertex graph not containing K_{k+2}, and $T_{n,k}$ is such a graph, we have $e(T_{n,k}) \le e(T_{n,k+1})$.

5.2.21. *$T_{n,r}$ is the unique n-vertex K_{r+1}-free graph of maximum size.* We use induction on r. The statement is immediate for $r = 1$. For the induction step, suppose $r > 1$. Let G be an n-vertex K_{r+1}-free graph, and let x be a vertex of maximum degree in x. Let $G' = G[N(x)]$. Let $H' = T_{d(x),r-1}$, and let $H = \overline{K}_{n-d(x)} \vee H'$. Since G' has no r-clique, the induction hypothesis yields $e(H') \ge e(G')$, with equality only if $G' = H'$. Let $S = V(G) - N(x)$. Since $e(G) - e(G') \ge \sum_{v\in S} d_G(v)$ and $e(H) - e(H') = (n-d(x))d(x) = |S|\,\Delta(G)$, we have $e(H) - e(H') \ge e(G) - e(G')$. Hence $e(H) \ge e(G)$, with equality only if equality occurs in both transformations.

We have seen (by the induction hypothesis) that equality in the first transformation requires $G' = H'$. Equality in the second transformation requires each edge of $E(G) - E(G')$ to have exactly one endpoint in S and requires each vertex of S to have degree $d(x)$. Thus every vertex of S is adjacent to every vertex of $N(x)$ and to no other vertex of S, which means that G is the join of G' with an independent set. Since G' is a complete $(r-1)$-partite graph, this makes G a complete r-partite graph. Finally, we know by shifting vertices between partite sets that $T_{n,r}$ is the only n-vertex complete r-partite graph that has the maximum number of edges.

5.2.22. *Vertices of high degree.* We have 18 vertices in a region of diameter 4, with $E(G)$ consisting of the pairs at most 3 units apart. Since $3 > 4/\sqrt{2}$, Application 5.2.11 (in particular the absence of independent 4-sets) guarantees that G lacks at most 108 edges of its 153 possible edges and thus has at least 45 edges. If at most one vertex has degree at least five, then the degree-sum is at most $(17)4 + (1)17 = 85$, which only permits 42 edges.

The result can be strengthened by a more detailed argument (communicated by Fred Galvin). Let S be the set of vertices with degree less than 5. Because there cannot be four vertices that are pairwise separated by at least 3 units, the subgraph induced by S has no independent set of size 4. Thus $|S| \le 15$, since the edges incident to the vertices of a maximal independent set in S must cover all the vertices in S. This shows among any 17 vertices there must be two with degree at least 5.

Furthermore, consider a set with 16 vertices. If $|S| < 15$, then again we have two vertices with degree at least 5. If $|S| = 15$, let T be an independent 3-set in S, and let z be the vertex outside S. Since the vertices of $S - T$ have degree at most 4, they have degree-sum at most 48. However, Theorem 1.3.23 or the absence of independent 4-sets guarantees that $S-T$ has at least 18 edges. Adding the 12 edges to S and at least five edges to z yields degree-sum at least 53. The contradiction implies that at least two vertices have degree at least five.

The result for 16 is sharp, because 15 vertices can be placed in three clumps forming cliques, and then all vertices have degree four.

5.2.23. *Turán's proof of Turán's Theorem.*

a) Every maximal simple graph with no $(r+1)$-clique has an r-clique. If making x and y adjacent creates an $(r+1)$-clique, then the graph must already have a clique of $r-1$ vertices all adjacent to both x and y. Thus x or y forms an r-clique with these vertices.

b) $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$. Since n and $n-r$ have the same remainder under division by r, the size of the ith largest partite set of $T_{n-r,r}$ is one less than the size of the ith largest partite set of $T_{n,r}$, for each i. Hence deleting one vertex from each partite set of $T_{n,r}$ leaves a copy of $T_{n-r,r}$ as an induced subgraph. The deleted edges form a complete subgraph on the vertices removed plus an edge from each of the $n-r$ remaining vertices to all but one of the deleted vertices. The terms in the claimed equation directly count these contributions.

c) The Turán graph $T_{n,r}$ is the unique simple graph with the most edges among n-vertex graphs without K_{r+1}. We use induction on n. Basis step: $n \le r$. We can include all edges without forming K_{r+1}. Thus the maximum graph is K_n, and this is $T_{n,r}$.

Induction step: $n > r$. Let G be a largest simple n-vertex graph avoiding K_{r+1}. By part (a), G contains K_r; let S be an r-vertex clique in G. Since G avoids K_{r+1}, every vertex not in S has at most $r-1$ neighbors in S. Therefore, deleting S loses at most $\binom{r}{2} + (n-r)(r-1)$ edges. The remaining graph G' avoids K_{r+1}. By the induction hypothesis, $e(G') \le e(T_{n-r,r})$, with equality only for $T_{n-r,r}$.

Since $e(G) \le \binom{r}{2} + (n-r)(r-1) + e(G')$, part (b) implies that $e(G) \le e(T_{n,r})$. To achieve equality, G' must be $T_{n-r,r}$, and each vertex of G' must have exactly $r-1$ neighbors in S. If some vertex of S has a neighbor in each partite set of G', then G contains K_{r+1}. Hence each vertex of S has neighbors at most $r-1$ partite sets of G'. Since each vertex of G' is adjacent to $r-1$ vertices in S, the vertices of S miss different partite sets in G'. Thus the vertices of S can be added to distinct partite sets in G' to form $T_{n,r}$.

5.2.24. *An n-vertex graph having $t_r(n) - k$ edges and an $(r+1)$-clique has at least $f_r(n) - k + 1$ such cliques, where $f_r(n) = n - r - \lceil n/r \rceil$ and $k \ge 0$.*

Let G be a graph with exactly one $(r+1)$-clique Q; we first use Turán's Theorem to bound $e(G)$. Note that $e(G-Q) \le t_r(n-r-1)$, and furthermore each $v \in V(G) - Q$ has at most $r-1$ neighbors in Q. Thus

$$e(G) \le t_r(n-r-1) + (r-1)(n-r-1) + \binom{r+1}{2}.$$

To express this in terms of $t_r(n)$, we compute $t_r(n) - t_r(n-r-1)$. First, deleting one vertex from each partite set in $T_{n,r}$ loses the edges among them plus an edge from each remaining vertex to $r-1$ deleted vertices. Hence $t_r(n) - t_r(n-r) = \binom{r}{2} + (r-1)(n-r)$. Also, $T_{n-r,r}$ becomes $T_{n-r-1,r}$ when we delete a vertex from a largest partite set, which has degree $n - r - \lceil (n-r)/r \rceil$. Thus $t_r(n-r) - t_r(n-r-1) = (n-r) - \lceil (n-r)/r \rceil$. Hence

$$t_r(n) - t_r(n-r-1) = \binom{r}{2} + r(n-r) - \lceil n/r \rceil + 1.$$

Together,

$$\begin{aligned} e(G) &\le t_r(n) - \binom{r}{2} - r(n-r) + \lceil n/r \rceil - 1 + (r-1)(n-r-1) + \binom{r+1}{2} \\ &= t_r(n) - (n - r - \lceil n/r \rceil) = t_r(n) - f_r(n). \end{aligned}$$

Suppose now that G has $t_r(n) - k$ edges and $s \ge 1$ copies of K_{r+1}. By iteratively deleting an edge that does not belong to every $(r+1)$-clique, we can delete fewer than s edges from G to obtain a graph G' with exactly one $(r+1)$-clique. By the preceding argument, $e(G') \le t_r(n) - f_r(n)$. Since $e(G) - e(G') \le s - 1$, we have $t_r(n) - k = e(G) \le t_r(n) - f_r(n) + s - 1$, or $s \ge f_r(n) - k + 1$.

5.2.25. *Bounds on $\mathrm{ex}(n, K_{2,m})$.*

a) If G is simple and $\sum_{v\in V} \binom{d(v)}{2} > (m-1)\binom{n}{2}$, then G contains $K_{2,m}$. If any pair of vertices has m common neighbors, then G contains $K_{2,m}$. Since there are $\binom{n}{2}$ pairs of vertices $\{x, y\}$, this means by the pigeonhole principle that a graph with no $K_{2,m}$ has at most $(m-1)\binom{n}{2}$ selections $(v, \{x, y\})$ such that v is a common neighbor of x and y. Counting such selections by v shows that there are exactly $\sum_{v\in V} \binom{d(v)}{2}$ of them, which completes the proof.

b) If G has e edges, then $\sum_{v\in V} \binom{d(v)}{2} \ge e(2e/n - 1)$. Because $\binom{x}{2}$ is a convex function of x, $\binom{x}{2} + \binom{y}{2} \ge 2\binom{(x+y)/2}{2}$. Hence $\sum_{v\in V} \binom{d(v)}{2}$ is minimized over fixed degree sum (number of edges) by setting all $d(v) = \sum d(v)/n = 2e/n$, in which case the sum is $e(2e/n - 1)$.

c) A graph with more than $\frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}$ edges contains $K_{2,m}$. Since this edge bound implies $2e/n - 1 > (m-1)^{1/2}n^{1/2} - \frac{1}{2}$, we conclude

$$e\left(\frac{2e}{n} - 1\right) > \frac{1}{2}\left[(m-1)^{1/2}n^{3/2} + \frac{n}{2}\right]\left[(m-1)^{1/2}n^{1/2} - \frac{1}{2}\right]$$
$$= \frac{1}{2}(m-1)n^2 - \frac{n}{8} > (m-1)\binom{n}{2}.$$

By (b), this implies the hypothesis of (a) (if $m \ge 2$), and then (a) implies that G contains $K_{2,m}$.

d) Among n points in the plane, there are at most $\frac{1}{\sqrt{2}}n^{3/2} + \frac{n}{4}$ pairs with distance exactly one. Let $V(G)$ be the n points, with edges corresponding to the pairs at distance 1. If G has more than the specified number of edges, then (c) with $m = 3$ implies that G contains $K_{2,3}$. However, no two points in the plane have three points at distance exactly 1 from each of them.

5.2.26. *Every n-vertex graph G with more than $\frac{1}{2}n\sqrt{n-1}$ edges has girth at most 4.* The sum $\sum_{v\in V(G)}\binom{d(v)}{2}$ counts the triples u, v, w such that v is a common neighbor of u and w. If G has no 3-cycle and no 4-cycle, then we can bound the common neighbors of pairs u, w. If $u \leftrightarrow w$ in G, then they have no common neighbor. If $u \not\leftrightarrow w$ in G, then they have at most one common neighbor. Thus $\sum_{v\in V(G)}\binom{d(v)}{2} \le \binom{n}{2} - e(G)$.

Since the vertex degrees have the fixed sum $2e(G)$, we also have a lower bound on $\sum_{v\in V(G)}\binom{d(v)}{2}$ due to the convexity of $x(x-1)/2$. When $\sum d(v) = 2e(G)$, the sum $\sum\binom{d(v)}{2}$ is numerically minimized when $d(v) = 2e(G)/n$ for each v. Letting $m = e(G)$, we now have $n(m/n)(2m/n - 1) \le n(n-1)/2 - m$. Clearing fractions yields the quadratic inequality $2m(2m - n) \le n^2(n-1) - 2mn$, which simplifies to $m \le \frac{1}{2}n\sqrt{n-1}$.

5.2.27. *For $n \ge 6$, the maximum number of edges in a simple n-vertex graph not having two edge-disjoint cycles is $n+3$.* We argue first that $K_{3,3}$ does not have two edge-disjoint cycles. Deleting the edges of a 6-cycle leaves $3K_2$, and deleting the edges of a 4-cycle leaves a connected spanning subgraph, which must therefore use the remaining five edges. Thus every cycle other than the deleted one shares an edge with the deleted one.

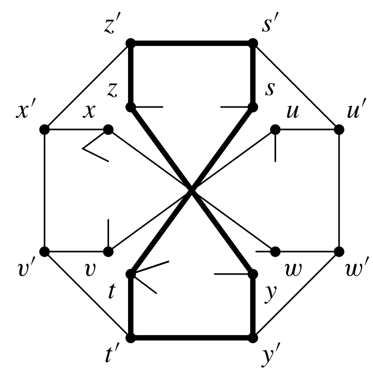
The number of edges in $K_{3,3}$ exceeds the number of vertices by 3. This is preserved by subdividing edges, and the property that every two cycles have a common edge is also preserved by subdividing edges. Hence every n-vertex subdivision of $K_{3,3}$ is a graph of the desired form with $n+3$ edges. This establishes the lower bound.

For the upper bound, consider an n-vertex graph G without two edge-disjoint cycles. We may assume that G is connected, since otherwise we can add an edge joining two components without adding any cycles. To prove that $e(G) \le n+3$, we may assume that G has a cycle C. Let H be a maximal unicyclic subgraph of G containing C. That is, we add edges from G to C without creating another cycle. Since G is connected, H is a spanning subgraph and has n edges. In addition to C, the rest of H forms a spanning forest H', with components rooted at the vertices of C.

Each edge of $G - E(H)$ joins two components of H', since otherwise it creates a cycle edge-disjoint from C using the path joining its endpoints in H'. Furthermore, for any two such edges xy and uv, with endpoints in components of H' having roots x', y', u', v' on C, it must be that C does not contain an x', y'-path and a u', v'-path that are disjoint, since these would combine with xy, uv, and the x, x'-, y, y'-, u, u'-, and v, v'-paths in H' to form edge-disjoint cycles. Therefore, the x', y'- and u', v'-paths on C must alternate endpoints (or share one endpoint).

Suppose that there are four such extra edges, say $\{st, uv, wx, yz\}$, such that the corresponding roots on C in order are $s', u', w', y', t', v', x', z'$ (consecutive vertices in the list may be identical, and any a may equal a'). Suppose that these vertices in order split into distinct pairs, such as $s' \ne u'$, $w' \ne y'$, $t' \ne v'$, and $x' \ne z'$. We now build edge-disjoint cycles by taking the cycle through $s', s, t, t', y', y, z, z'$ indicated in bold below and the analogous cycle through $u', u, v, v', x', x, w, w'$ (note that $t' = y'$, etc., is possible). If for example $s' = u'$, so that these two cycles are not edge-disjoint, then edge-disjoint cycles can be extracted in other ways (we omit the details).

We conclude that only three additional edges are possible, which limits $e(G)$ to $n+3$.



5.2.28. *For $n \ge 6$, the maximum number of edges in a simple n-vertex graph G not having two disjoint cycles is $3n - 6$.* To construct such a graph, form a triangle on a set S of three vertices, and let S be the neighborhood of each remaining vertex. Each cycle uses at least two vertices from S, so there cannot be two disjoint cycles. The graph has $3 + 3(n - 3) = 3n - 6$ edges.

For the upper bound, we use induction on n. Basis step ($n = 6$): G has at most two missing edges. We find one triangle incident to all the missing edges, and then the remaining three vertices also form a triangle.

Induction step ($n > 6$): If G has a vertex v of degree at most 3, then the induction hypothesis applied to $G - v$ yields the claim. Thus we may assume that $\delta(G) \geq 4$. Since $e(G) \geq n$, there is a cycle in G. Let C be a shortest cycle in G, and let $H = G - V(C)$. We may assume that H is a forest, since otherwise we have a cycle disjoint from C.

Since $\delta(G) \geq 4$, every leaf or isolated vertex in H has at least three neighbors on C. This yields a shorter cycle than C unless C is a triangle. Hence we may assume that C is a triangle, and now $\delta(G) \geq 4$ implies that H has no isolated vertices.

Since every leaf of H is adjacent to all of $V(C)$, two leafs in a single component of H plus one additional leaf yield two disjoint cycles. Hence we may assume that H is a single path. Thus every internal vertex of H has at least two neighbors in C, and there is at least one such vertex since $n > 6$. We now have two disjoint triangles: the first two vertices of the path plus one vertex of C, and the last vertex of the path plus the other two vertices of C.

5.2.29. Let G be a claw-free graph (no induced $K_{1,3}$).

a) The subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles. Let H be such a subgraph. Since H is 2-colorable, it is triangle-free. Hence a vertex of degree 3 in H is the center of a claw. Since G is claw-free, every induced subgraph of G is claw-free. Hence $\Delta(H) \leq 2$. Every component of a graph with maximum degree at most 2 is a path or a cycle. Since H is 2-colorable, the cycle components have even order.

b) If G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one. Consider a proper k-coloring of G. If some two color classes differ in size by more than 1, then we alter the coloring to reduce the size of a largest color class A and increase the size of a smallest color class B. Consider the subgraph H induced by $A \cup B$. By part (a), the components of H are paths and even cycles. The even cycles have the same number of vertices from A and B. Since $|A| \geq |B| + 2$, there must be a component of H that is a path P with one more vertex from A than from B. Switching the colors on P brings the two color classes closer together in size. Iterating this procedure leads to all pairs of classes differing in size by at most 1.

5.2.30. *If G has a proper coloring in which each color class has at least two vertices, then G has a $\chi(G)$-coloring in which each color class has at least two vertices.* (Note that C_5 doesn't have either type of coloring.)

Proof 1 (induction on $\chi(G)$; S. Rajagopalan). The statement is immediate if $\chi(G) = 1$. If $\chi(G) > 1$, let f be an optimal coloring of G, and let $g\colon V(G) \to \mathbb{N}$ be a coloring in which each class has at least two vertices. If f has a singleton color set $\{x\}$, let $S = \{v \in V(G)\colon g(v) = g(x)\}$, and let $G' = G - S$. Since f restricts to a $(\chi(G) - 1)$-coloring of G' (because x is omitted) and g restricts to a coloring of G' in which every color is used at least twice (because only vertices with a single color under G were omitted), the induction hypothesis implies that G' has a $(\chi(G) - 1)$-coloring in which every color is used at least twice. Replacing S as a single color class yields such a coloring for G.

Proof 2 (algorithmic version). Define f and g as above. if x is a singleton color in the current $\chi(G)$-coloring f, change all vertices in $\{v\colon g(v) = g(x)\}$ to color $f(x)$. The new coloring is proper, since $f(x)$ appeared only on x and since the set of vertices with color $g(x)$ in g is independent. No new colors are introduced, so the new coloring is optimal. Vertices that have been recolored are never recolored again, so the procedure terminates after at most $\chi(G)$ steps. It can only terminate with an optimal coloring in which each color is used at least twice.

5.2.31. *If G is a connected graph that is not a complete graph or a cycle whose length is an odd multiple of 3, then in every minimum proper coloring of G there are two vertices of the same color with a common neighbor.* For odd cycles, if every two vertices having the same color are at least three apart, then the coloring must be $1, 2, 3, 1, 2, 3, \cdots$, cyclically, so the length is an odd multiple of 3. For other graphs, Brooks' Theorem yields $\chi(G) \leq \Delta(G)$. Since only $\Delta(G) - 1$ colors are available for the neighborhood of a vertex of maximum degree, the pigeonhole principle implies that a vertex of maximum degree has two neighbors of the same color in any optimal coloring.

5.2.32. *The Hajós construction.* Applied to graphs G and H sharing only vertex v, with $vu \in E(G)$ and $vw \in E(H)$, the Hajós construction produces the graph $F = (G - vu) \cup (H - vw) \cup uw$.

a) If G and H are k-critical, then F is k-critical. A proper $(k - 1)$-coloring of F contains proper $(k-1)$-colorings of $G - vu$ and $H - vw$. Since G and H are k-critical, every $(k - 1)$-coloring of F gives the same color to v and u and gives the same color to v and w. Since this gives the same color to u and w, there is no such coloring of F.

Thus $\chi(F) \geq k$, and equality holds because we can combine proper $(k - 1)$-colorings of $G - vu$ and $H - vw$ and change w to a new color.

Finally, for $e \in E(F)$ we show that $F - e$ is $(k-1)$-colorable. For $F - uw$, the coloring described above is proper. Let xy be another edge of F; by symmetry, we may assume that $xy \in E(G)$. Since G is k-critical, we have a proper $(k-1)$-coloring f of $G - xy$. Since uv is an edge in $G - xy$, this coloring gives distinct colors to u and v. In a proper $(k-1)$-coloring of $H - vw$ that gives v and w the same color, we can permute labels so this color is $f(v)$. Combining these colorings now yields a proper $(k-1)$-coloring of $F - xy$.

b) For $k \ge 3$, a k-critical graph other than K_k. Apply the Hajós construction to the graph consisting of two edge-disjoint k-cliques sharing one vertex v. This deletes one edge incident to v from each block and then adds an edge joining the two other vertices that lost an incident edge. The resulting graph is $(k-1)$-regular except that v has degree $2k-4$.

c) Construction of 4-critical graphs with n vertices for all $n \ge 6$. Since the join of color-critical graphs is color-critical, we can use $C_{2k+1} \vee K1$, which yields 4-critical graphs for all even n. In particular, this works for $n \in \{4, 6, 8\}$, which has a member of each congruence class modulo 3.

If we apply the Hajós construction to a 4-critical graph G with $2l$ vertices and the 4-critical graph $H = K_4$, we obtain a 4-critical graph F with $2l + 3$ vertices. Thus we obtain a 4-critical n-vertex graph whenever n exceeds one of $\{4, 6, 8\}$ by a multiple of 3. This yields all $n \ge 4$ except $n = 5$.

5.2.33. *a) If a k-critical graph G has a 2-cut $S = \{x, y\}$, then 1) $x \not\leftrightarrow y$, 2) G has exactly two S-lobes, and 3) we may index them as G_1 and G_2 such that $G_1 + xy$ and $G_2 \cdot xy$ are k-critical.* Since no vertex cut of a k-critical graph induces a clique, we have $x \not\leftrightarrow y$. By k-criticality, every S-lobe of G is $(k-1)$-colorable. If each S-lobe has a proper $(k-1)$-coloring where x, y have the same color, then colors can be permuted within S-lobes so they agree on $\{x, y\}$, so G is $(k-1)$-colorable.

The same can be done if each S-lobe has a proper $(k-1)$-coloring where x, y have different colors. Hence there must be an S-lobe G_1 such that u, v receive the same color in every proper $(k-1)$-coloring and an S-lobe G_2 such that u, v receive the different colors in every proper $(k-1)$-coloring. Deletion of any other S-lobe would therefore leave a graph that is not $(k-1)$-colorable, so criticality implies that there is no other S-lobe.

Since every proper $(k-1)$-coloring of G_1 gives x and y the same color, $G_1 + xy$ is not $(k-1)$-colorable. Since every proper $(k-1)$-coloring of G_2 gives x and y different colors, $G_2 \cdot xy$ is not $(k-1)$-colorable. To see that $G_1 + xy$ is k-critical, let $G' = G_1 + xy$ and consider edge deletions. First $G' - xy = G_1$, which is $(k-1)$-colorable. For any other edge e of G', $G - e$ has a proper $(k-1)$-coloring that contains a proper $(k-1)$-coloring of G_2, hence it gives distinct colors to x and y. Therefore the colors it uses on the vertices of G_1 form a proper $(k-1)$-coloring of $G' - e$. The analogous argument holds for $G_2 \cdot xy$.

b) Every 4-chromatic graph contains a K_4-subdivision. Part (a) can be used to shorten the proof of this. We use induction on $n(G)$, with the basis $n(G) = 4$ and K_4 itself. Given $n(G) > 4$, let G' be a 4-critical subgraph of G. We know G' has no cutvertex. If G' is not 3-connected, then we have a 2-cut $S\{x, y\}$. Part (a) guarantees an S-lobe G_1 such that $G_1 + xy$ is 4-critical. By the induction hypothesis, $G_1 + uv$ contains a subdivision of K_4; if this subdivision uses the edge uv, then this edge can be replaced by a path through G_2 to obtain a subdivision of K_4 in G. If G' is 3-connected, the proof is as in the text.

5.2.34. *In a 4-critical graph G with a separating set S of size 4, $e(G[S]) \le 4$.* If $e(G[S]) = 6$, then S is a 4-clique, and G is not 4-critical. If $e(G[S]) = 5$, then $G[S]$ is a kite. Every proper 3-coloring of the S-lobes of G assigns one color to the vertices of degree 2 in the kite and two other colors to the vertices of degree 3 in the kite. Hence the names of colors in the proper 3-colorings of the S-lobes can be permuted so that the coloring agree on S. This yields a proper 3-coloring of G. The contradiction implies that $G[S]$ cannot have five edges.

5.2.35. *Alternative proof that k-critical graphs are $(k-1)$-edge-connected.*

a) If G is k-critical, with $k \ge 3$, then for any $e, f \in E(G)$ there is a $(k-1)$-critical subgraph of G containing e but not f. Any $(k-1)$-coloring ϕ of $G - e$ assigns the same color to both endpoints of e. The endpoints of f get distinct colors under ϕ; by renumbering colors, we may assume one of them gets color $k-1$. Let $S = \{v : \phi(v) = k-1\}$; note that $G - e - S$ is $(k-2)$-colored by ϕ. However, $G - S$ is $(k-1)$-chromatic, since S is an independent set, so any $(k-1)$-critical subgraph of $G - S$ must contain e and be the desired graph. (Toft [1974])

b) If G is k-critical, with $k \ge 3$, then G is $(k-1)$-edge-connected. Since the 3-critical graphs are the odd cycles, this is true for $k = 3$, and we proceed by induction. For $k > 3$, consider an edge cut with edge set F. If $|F| = 1$, we permute colors in one component of $G - F$ to obtain a $(k-1)$-coloring of G from a $(k-1)$-coloring of $G - F$, so we may assume $|F| \ge 2$. Choose $e, f \in F$. By part (a), there is a $(k-1)$-critical subgraph G' containing e but not f. Deleting $F - f$ from G' separates it, since it separates the endpoints of e. By the induction hypothesis, $|F - f| \ge k-2$, and thus $|F| \ge k-1$.

5.2.36. *If G is k-critical and every $(k-1)$-critical subgraph of G is isomorphic to K_{k-1}, then $G = K_k$ (if $k \ge 4$).* Since K_k is k-critical, a k-critical graph cannot properly contain K_k, so if we can find K_k in G, then $G = K_k$. Let G have the specified properties; since $k \ge 4$, G has a triangle x, y, z. Toft's critical graph lemma says that for any edges e, f, G contains a $(k-1)$-critical subgraph that contains e and avoids f.

Let G_1 be such a graph that contains xy but omits yz. Since every $(k-1)$-critical subgraph is a clique, by hypothesis, G_1 cannot contain z at all. Similarly, let G_2 be a $(k-1)$-critical graph that contains yz but omits x. Both G_1 and G_2 are $(k-1)$-cliques, so a proper $(k-2)$-coloring of $G_1 - xy$ must give x and y the same color, and a proper $(k-2)$-coloring of $G_2 - yz$ must give y and z the same color. This means that the graph $H = (G_1 - xy) \cup (G_2 - yz) \cup xz$ is not $(k-2)$-colorable, so it contains some $(k-1)$-critical subgraph H', which by hypothesis is a $(k-1)$-clique.

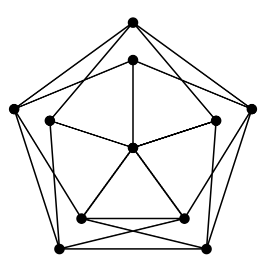
Furthermore, the set of vertices common to G_1 and G_2 induce a clique, which means that the $(k-2)$-colorings of G_1 and G_2 can be made to agree on their intersection. This means that $H - xz$ is $(k-2)$-colorable, which implies that $xz \in H'$. By construction, $N_H(x) = V(G_1) - y$ and $N_H(z) = V(G_2) - y$. Since H' is a clique containing x, z, this forces G_1, G_2 to have $k-3$ common vertices other than y. We add x, y, z to these to obtain a k-clique in G, which as noted earlier implies that $G = K_k$.

5.2.37. *Vertex-color-critical graphs.*

a) Every color-critical graph is vertex-color-critical. Every proper subgraph of a color-critical graph has smaller chromatic number, including those obtained by deleting a vertex, which is all that is needed for vertex-color-critical graphs.

b) Every 3-chromatic vertex-color-critical graph G is color-critical. Since it needs 3 colors, G is not bipartite, but $G - v$ is bipartite for every $v \in V(G)$. Hence every vertex of G belongs to every odd cycle of G; let C be a spanning cycle of G. If G has any edge e not on C, then e creates a shorter odd cycle with a portion of C, leaving out some vertices. Since G is vertex-color-critical, this cannot happen, and G is precisely an odd cycle.

c) the graph below is vertex-color-critical but not color-critical. This graph G is obtained from the Grötzsch graph by adding an edge, so $\chi(G) \geq 4$. An explicit coloring shows that $\chi(G) = 4$. Hence G is not color-critical. Explicit 3-colorings of the graphs obtained by deleting one vertex show that G is vertex-color-critical.



5.2.38. *Every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision.* Call a vertex with degree less than 3 a *deficient* vertex. By considering the larger class of graphs that may have one deficient vertex, we obtain a stronger result than $\delta(G) \geq 3$ forcing a K_4-subdivision, but one that is easier to prove inductively.

We use induction on $n(G)$; the only graph with at most four vertices that satisfies the hypothesis is K_4 itself. For the induction step, we seek a graph G' having at most one deficient vertex and having $n(G') < n(G)$. If G contains G' or a G'-subdivision, we obtain a K_4-subdivision in G, because the K_4-subdivision in G' guaranteed by the induction hypthesis is a subgraph of G or yields a subgraph of G by subdividing additional edges.

If G is disconnected, let G' be a component of G. If G has a cut-vertex x, then some $\{x\}$-lobe of G has at most one deficient vertex; let this be G'. Hence we may assume G is 2-connected. If G is 3-connected, then as in the proof of Theorem 5.2.20 we find a cycle C in $G - x$ and an $x, V(C)$-fan in G to complete a subdivision of K_4.

Hence we may assume that $\kappa(G) = 2$, with S a separating 2-set. Only one S-lobe of G can have a vertex outside S that is deficient in G. Let H be an S-lobe of G containing no vertex outside S that is deficient in G.

Note that x and y each have degree at least 1 in H, since $\kappa(G) = 2$, and in fact they must have distinct neighbors in $V(H) - S$. If x or y has degree at least 3 in H, then let $G' = H$.

If $d_H(x) = d_H(y) = 1$, then x and y cannot be adjacent. Merge x and y to form G' from H; degrees of other vertices don't change, since x and y have no common neighbors in H. Also G contains a subdivision of G' by undoing the merging and adding an x, y-path through another S-lobe.

Hence we may assume that $d_H(x) = 2$. If $xy \notin E(G)$, then add the edge xy to H to form G'; only y can now be deficient. Also G contains a subdivision of G' by replacing the added edge with an x, y-path through another S-lobe. If $xy \in E(G)$, then y has a neighbor in H other that the neighbor of x (and it is the only neighbor of y other than x). Now we contract xy to obtain G', with the new vertex having degree 2. Now H is a subdivision of G' that is a subgraph of G.

5.2.39. *For $n \geq 3$, the maximum number of edges in a simple n-vertex graph G having no K_4-subdivision is $2n-3$.* If G has at least $2n-2$ edges, then $n \geq 4$; we prove by induction on n that G has a K_4-subdivision. For $n = 4$, G has (at least) 6 edges and must be K_4. For $n > 4$, if $\delta(G) \geq 3$, then Dirac's Theorem guarantees that G has a K_4-subdivision.

When $\delta(G) < 3$, let x be vertex of minimum degree. The graph $G - x$ has at least $2(n-1) - 2$ edges; by the induction hypothesis, $G - x$ has a K_4-subdivision, and this subgraph appears also in G.

To show this is the best bound, we observe that $K_2 \vee (n-2)K_1$ has $2n-3$ edges but no K_4-subdivision. It cannot have a K_4-subdivision because it has only two vertices with degree at least 3. Another example is $K_1 \vee P_{n-1}$, but it requires induction to show that this example with $2n-3$ edges has no K_4-subdivision.

5.2.40. *For $G_7 = C_5[K_3, K_2, K_3, K_2, K_3]$ and $G_8 = C_5[K_3, K_3, K_3, K_3, K_3]$, the graph G_k is k-chromatic but contains no K_k-subdivision.* In these constructions, the vertices substituted for two successive vertices of C_5 (call these *groups*) induce a clique. For G_7, we use colors 123, 45, 671, 23, 456 in the successive cliques. For G_8, we use 123, 456, 781, 234, 567.

In these graphs, one cannot take two vertices from the same group or adjacent groups in an independent set. Thus each graph has independence number 2. Thus $\chi(G_7) \ge n(G_7)/2 = 6.5$ and $\chi(G_8) \ge n(G_8)/2 = 7.5$. Since $\chi(G)$ is always an integer, we have $\chi(G_k) \ge k$.

If G_k has a K_k-subdivision H, then H must have two vertices u, v of degree $k-1$ in nonadjacent groups, since adjacent groups together have size at most $k-2$. Since there must be $k-1$ pairwise internally disjoint u, v-paths in H, this is impossible when G_k has a u, v-separating set of size $k-2$. In all cases except one, G_k has such a u, v-separating set consisting of two groups. The exception is u, v chosen from the groups of size 2 in G_7.

In this exceptional case, we have forbidden the high-degree vertices of H from the consecutive groups of size 3, since that would yield the case already discussed. Thus the seven high-degree vertices must consist of the two groups of size 2 and the triangle between them. Now the four needed paths connecting the two groups of size 2 must use the two consecutive groups of size 3, but only three paths can do this.

5.2.41. *If $m = k(k+1)/2$, then $K_{m,m-1}$ contains no subdivision of K_{2k}.* In $K_{m,m}$ there is such a subdivision: place k branch vertices in each partite set, and then there remain $\binom{k}{2}$ unused vertices in each partite set to subdivide edges joining the branch vertices in the other partite set. We prove that if an X, Y-bigraph G contains a subdivision of K_{2k}, then $n(G) \ge 2m$.

Proof 1 (counting argument). The paths representing edges of K_{2k} are pairwise internally-disjoint. When a partite set has a "branch vertices" (degree more than two in the subdivision), the other partite set has at least $\binom{a}{2}$ vertices that are not branch vertices. If the subdivision of K_{2k} has a branch vertices in X, we thus need at least $\binom{a}{2} + 2k - a + \binom{2k-a}{2} + a$ vertices. Using the identity $\binom{a}{2} + a(n-a) + \binom{n-a}{2} = \binom{n}{2}$, the formula becomes $\binom{2k}{2} + 2k - a(2k-a)$. Since $a(2k-a) \le k^2$, the number of required vertices is at least $\binom{2k+1}{2} - k^2$. This quantity is $k(k+1) = 2m$.

Proof 2 (extremal bipartite subgraphs). In a subdivision of K_{2k} within a graph G, there are $2k$ branch vertices. The maximum number of edges in a bipartite graph with $2k$ vertices is k^2. Hence if more than k^2 edges joining branch vertices are left unsubdivided, then the subgraph of G induced by these vertices will not be bipartite. Since we require the host graph ($G = K_{m,m-1}$) to be bipartite, at least $\binom{2k}{2} - k^2$ edges must be subdivided. This requires $k^2 - k$ additional vertices. Together with the branch vertices, a bipartite graph containing a subdivision of K_{2k} must have at least $k^2 + k$ vertices. (Comment: The uniqueness of the $2k$-vertex bipartite graph with k^2 edges leads to the uniqueness of $K_{m,m}$ as a graph with $k^2 + k$ vertices having a subdivision of K_{2k}.)

5.2.42. *If F is a forest with m edges, and G is a simple graph such that $\delta(G) \ge m$ and $n(G) \ge n(F)$, then $F \subseteq G$.* We may assume that F has no isolated vertices, since those could be added at the end.

Let F' be a subgraph of F obtained by deleting one leaf from each nontrivial component of F. Let R be the set of neighbors of the deleted vertices. Map R onto an m-set $X \subseteq V(G)$ that minimizes $e(G[X])$. Since $\delta(G) \ge m$ and $n(F') = m$, we can extend X to a copy of F' in G (each vertex has at least m neighbors, but fewer than m of its neighbors are used already in F' when we need to add a neighbor to it).

To extend this copy of F' to become a copy of F, we show that G contains a matching from X into the set Y of vertices not in this copy of F'. Let H be the maximal bipartite subgraph of G with bipartition X, Y. By Hall's Theorem, the desired matching exists unless there is a set $S \subseteq X$ such that $|N_H(S)| < |S|$. Consider $t \in S$ and $u \in Y - N_H(S)$. Outside S, t has at most $(n(F') - |X|) + |N_H(S)|$ neighbors in G. Since $\delta(G) \ge m$, we have $|N_G(t) \cap S| \ge |X| - |N_H(S)|$. On the other hand, since $u \notin N_H(S)$, we have $|N_G(u) \cap X| \le |X| - |S|$. Since $|N_H(S)| < |S|$, replacing t with u in X reduces the size of the subgraph induced by X. This contradicts the choice of X, and hence Hall's condition holds.

5.2.43. *Every proper k-coloring of a k-chromatic graph contains each labeled k-vertex tree as a labeled subgraph.* We use induction on k, with trivial basis $k = 1$. For $k > 1$, let f be the coloring, and let $C_i = \{v \in V(G) \colon f(v) = i\}$. Suppose that x is a leaf of T with neighbor y and that we seek label p for x and q for y. Let $S \subset C_q$ be the vertices in C_q adjacent to no vertex of C_p. We have $S \ne C_q$, else we can combine color classes in f.

The vertices of S cannot be used in the desired embedding of T, so we will discard them. Let $G' = G - (S \cup C_p)$. We have $\chi(G') \le k-1$ because we have discarded all vertices of color p in f, and we have $\chi(G') \ge k-1$ because $S \cup C_p$ is an independent set in G. By the induction hypothesis, G' has $T - x$ as a labeled subgraph H, and the image of y in H belongs to C_q. We have retained in $C_q - S$ only vertices having a neighbor with color p in

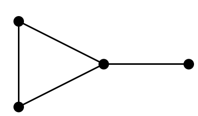
f (by part (a), this set is non-empty). Hence G has a vertex in C_p that we can use as the image of x to obtain T as a labeled subgraph.

5.2.44. *Every k-chromatic graph with girth at least 5 contains every k-vertex tree as an induced subgraph.* If $\chi(G) = k$ and $d(x) < k-1$ for some $x \in V(G)$, then $\chi(G-x) = k$, so it suffices to prove the claim for graphs in which the minimum degree is at least $k-1$. In fact, with this condition, we do not need the condition on the chromatic number. For $k \le 2$, the result is obvious; we proceed by induction for $k > 2$.

Suppose T is a k-vertex tree, x is a leaf of T with neighbor y, and $T' = T - x$. By the induction hypothesis, G has T as an induced subgraph $f(T')$; let $u = f(y)$. It suffices to show that $S = N(u) - f(T')$ contains a vertex adjacent to no vertex of $f(T')$ except u. Each vertex in $f(N(y))$ has no neighbor in S, because G has no triangles. Each vertex in $f(T - N[y])$ has at most one neighbor in S, else it would complete a 4-cycle in G with two such vertices and u. Hence S has at most $n(T') - 1 - d(y)$ unavailable vertices. Since $|S| \ge k - 1 - d(y)$, there remains an available vertex in S to assign to x.

5.3. ENUMERATIVE ASPECTS

5.3.1. *The chromatic polynomial of the graph below is $k(k-1)^2(k-2)$.* The graph is chordal, and the polynomial follows immediately from a simplicial elimination ordering. It can also be obtained from the recurrence, from the inclusion-exclusion formula, etc.



5.3.2. *The chromatic polynomial of an n-vertex tree is $k(k-1)^{n-1}$, by the chromatic recurrence.* We use induction on n. For $n=1$, the polynomial is k, as desired. Contracting an edge of an n-vertex tree leaves a tree with $n-1$ vertices. Deleting the edge leaves a forest of two trees, with orders m and $n-m$ for some m between 1 and $n-1$. The polynomial for a disconnected graph is the porduct of the polynomials for the components. We use the induction hypothesis and the chromatic recurrence and extract the factors k and $(k-1)^{n-2}$ to obtain the polynomial

$$k(k-1)^{m-1}k(k-1)^{n-m-1} - k(k-1)^{n-2} = k(k-1)^{n-2}(k-1) = k(k-1)^{n-1}.$$

5.3.3. *$k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial.* In $\chi(G;k)$, the degree is $n(G)$, and the second coefficient is $-e(G)$. Hence we need a 4-vertex graph with four edges. The only such graphs are C_4 and the paw, which have chromatic polynomials $k(k-1)(k^2-3k+3)$ and $k(k-1)(k-2)(k-1)$, each with nonzero linear term. (Note: The linear term of the chromatic polynomial of a connected graph is nonzero; see Exercise 5.3.12.)

Alternatively, observe that the value at 2 is negative, so it cannot count the proper 2-colorings in any graph.

5.3.4. *a) The chromatic polynomial of the n-cycle is $(k-1)^n + (-1)^n(k-1)$.* **Proof 1** (induction on n). The chromatic polynomial of the loop (C_1) is 0, which the formula reduces to when $n = 1$. Those considering only simple graphs can start with $\chi(C_3;k) = k(k-1)(k-2) = (k-1)^3 - (k-1)$. For larger n, the chromatic recurrence yields $\chi(C_n;k) = \chi(P_n;k) - \chi(C_{n-1};k)$. By the induction hypothesis and the formula for trees, this equals $k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) = (k-1)^n + (-1)^n(k-1)$.

Proof 2 (Whitney's formula). We use $\chi(G;k) = \sum_{S\subseteq E(G)}(-1)^{|S|}k^{c(G(S))}$. For every set S of size j, the number of components of $G(S)$ is $n-j$, except that for $S = E(G)$ the number of components is 1, not 0. Since there are $\binom{n}{j}$ sets with j edges, we obtain $\chi(C_n;k) = (\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}k^{n-j}) + (-1)^n k$. By the binomial theorem, $(k-1)^n = (\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}k^{n-j}) + (-1)^n$. Thus we obtain $\chi(C_n;k)$ from $(k-1)^n$ by adding $(-1)^n k$ and subtracting $(-1)^n$.

b) If $H = G \vee K_1$, then $\chi(H;k) = k\chi(G;k-1)$. Let x be the vertex added to G to obtain H. In every proper coloring, the color used on x is forbidden from the rest of H. Each of the k ways to color x combines with each of the $\chi(G;k-1)$ ways to properly color the rest of H to form a proper coloring of H. Hence $\chi(H;k) = k\chi(G;k-1)$;in particular, $\chi(C_n \vee K_1;k) = k(k-2)^n + (-1)^n k(k-1)$.

5.3.5. *If $G_n = K_2 \square P_n$, then $\chi(G_n;k) = (k^2-3k+3)^{n-1}k(k-1)$.*

Proof 1 (induction on n). Since G_1 is a 2-vertex tree, $\chi(G_1;k) = k(k-1)$. For $n > 1$, let u_n, v_n be the two rightmost vertices of G_n. The proper colorings of G_n are obtained from proper colorings of G_{n-1} by assigning colors also to u_n and v_n. Each proper coloring f of G_{n-1} satisfies $f(u_{n-1}) \ne f(v_{n-1})$. Thus each such f extends to the same number of colorings of G_n.

There are $(k-1)^2$ ways to specify $f(u_n)$ and $f(v_n)$ so that $f(u_n) \ne f(u_{n-1})$ and $f(v_n) \ne f(v_{n-1})$. Of these extensions, $k-2$ give u_n and v_n the same color, and we delete them. Since $(k-1)^2 - (k-2) = k^2 - 3k + 3$, the induction hypothesis yields

$$\chi(G_n;k) = (k^2-3k+3)\chi(G_{n-1};k) = (k^2-3k+3)^{n-1}k(k-1).$$

Proof 2 (induction plus chromatic recurrence). Again $\chi(G_1;k) = k(k-1)$. Let $e = u_n v_n$. For $n > 1$, observe that $\chi(G_n - e;k) = \chi(G_{n-1};k)(k-1)^2$ and $\chi(G_n \cdot e;k) = \chi(G_{n-1};k)(k-2)$, by counting the ways to extend each coloring of G_{n-1} to the last column. Thus

$$\begin{aligned}\chi(G_n;k) &= \chi(G_n - u_n v_n;k) - \chi(G_n \cdot u_n v_n;k)\\ &= \chi(G_{n-1};k)[(k-1)^2 - (k-2)] = (k^2-3k+3)^{n-1}k(k-2).\end{aligned}$$

5.3.6. *Non-inductive proof that the coefficient of $k^{n(G)-1}$ in $\chi(G;k)$ is $-e(G)$.* Let n be the number of vertices in G. By Proposition 5.3.4, $\chi(G;k) = \sum_{r=1}^n p_r k_{(r)}$, where p_r is the number of partitions of G into exactly r nonempty independent sets. Since $k_{(r)}$ is a polynomial in k of degree r, contributiong to the coefficient of k^{n-1} in $\chi(G;k)$ can arise only from the terms for $r = n$ and $r = n-1$.

The only partition of $V(G)$ into n independent sets is the one with each vertex in a set by itself, so $p_n = 1$. When partitioning $V(G)$ into $n-1$ independent sets, there must be one set of size 2 and $n-2$ sets of size 1. Thus each such partition is determined by choosing two nonadjacent vertices. There are $\binom{n}{2} - e(G)$ such pairs (G is simple), so $p_{n-1} = \binom{n}{2} - e(G)$.

The term involving k^{n-1} in $k_{(n-1)}$ arises only by choosing the term k from each factor when expanding the product. Thus the coefficient of k^{n-1} in $k_{(n-1)}$ is 1. Contributions to the coefficient of k^{n-1} in $k_{(n)}$ arise by choosing the term k from $n-1$ factors and the constant from the remaining term. Thus the contributions are $-1, -2, \ldots, -(n-1)$, and the coefficient is $-\sum_{i=0}^{n-1} i$, which equals $-\binom{n}{2}$.

Combining these computations yields the coefficient of k^{n-1} in $\chi(G;k)$ as $1 \cdot [-\binom{n}{2}] + [\binom{n}{2} - e(G)] \cdot 1$.

5.3.7. *Roots of chromatic polynomials.*

a) The chromatic polynomial $\chi(G;k)$ of an arbitrary graph G is a nonnegative linear combination of chromatic polynomials of cliques with at most $n(G)$ vertices. This holds trivially when G itself is a clique, which is the situation where $e(\overline{G}) = 0$. This is the basis step for a proof by induction on $e(\overline{G})$. For $e(\overline{G}) > 0$, let G' be the graph obtained by adding the edge $e = uv$ and contracting it; we have $\chi(G;k) = \chi(G+uv;k) + \chi(G';k)$ by the chromatic recurrence. To apply the induction hypothesis, note that $e(\overline{G+uv}) = e(\overline{G}) - 1$ and $e(\overline{G'}) = e(\overline{G}) - 1 - |\overline{N}(u) \cap \overline{N}(v)|$, where $e = uv$. Hence we can express $\chi(G';k)$ and $\chi(G' \cdot e;k)$ as nonnegative linear combinations of the polynomials $\chi(K_j;k)$ for $j \le n$.

b) The chromatic polynomial of a graph on n vertices has no real root larger than $n-1$. The combinatorial definition of the chromatic polynomial as the function of k that counts the proper colorings of G using at most k colors guarantees that the value cannot be 0 for $k \ge n$, because we can arbitrarily assign the vertices distinct colors to obtain at least $k(k-1)\cdots(k-n+1) > 0$ proper colorings. However, this argument applies only to integers. To forbid all real roots exceeding n, we use part (a). Observe that $\chi(K_j;x)$ is the product of positive real numbers whenever $x > j-1$; hence these polynomials have no real roots larger than $j-1$. Since any chromatic polynomial is a nonnegative linear combination of these for $j \le n$, its value at any $x > n-1$ is the sum of at most n positive numbers and therefore is also positive.

5.3.8. *The number of proper k-colorings of a connected graph G is less than $k(k-1)^{n-1}$ if $k \ge 3$ and G is not a tree.* If G is connected but not a tree, let T be a spanning tree contained in G, and choose $e \in E(G) - E(T)$. Every proper coloring of G must be a proper coloring of the subgraph T, and there are exactly $k(k-1)^{n-1}$ proper k-colorings of T. It suffices to show that at least one of these is not a proper k-coloring of G. Since T is bipartite and $k \ge 3$, we can construct such a coloring by using a 2-coloring of T and then changing the endpoints of e to a third color. This is still a proper k-coloring of T, but it is not a proper k-coloring of G.

If $k = 2$, then T has exactly two proper k-colorings, and these are both proper colorings of G if G is bipartite. Thus the statement fails when $k = 2$ if G is bipartite (if G is not bipartite, then it still holds when $k = 2$).

5.3.9. $\chi(G;x+y) = \sum_{U \subseteq V(G)} \chi(G[U];x)\chi(G[\overline{U}];y)$. Polynomials of degree n that agree at $n+1$ points are equal everywhere. Hence it suffices to prove the claim when x and y are nonnegative integers. We show that then each side counts the proper $(x+y)$-colorings of G.

In each proper $(x+y)$-coloring, the first x colors are used on some subset $U \subseteq V(G)$, and $\overline{U}$ receives colors among the remaining y colors. Since there is no interference between the colors, we can put an arbitrary x-coloring on $G[U]$ and an arbitrary y-coloring on $G[\overline{U}]$ and form such a coloring in $\sum_{U \subseteq V(G)} \chi(G[U];x)\chi(G[\overline{U}];y)$ ways. Furthermore, the set U that receives colors among the first x colors is uniquely determined by the coloring. Hence summing over U counts each coloring exactly once. The left side by definition is the total number of colorings.

5.3.10. *If G is a connected n-vertex graph with $\chi(G;k) = \sum_{i=0}^{n-1} (-1)^i a_{n-i} k^{n-i}$, then $a_i \ge \binom{n-1}{i-1}$ for $1 \le i \le n$.* In order to prove this inductively using the chromatic recurrence, we must guarantee that the graphs in the recurrence are connected and appear "earlier". We use induction on n, and to prove the induction step we use induction on $e(G) - n + 1$.

The statement holds for the only 1-vertex graph, so consider $n > 1$. If $e(G) = n-1$ and G is connected, then G is a tree and has chromatic polynomial $k(k-1)^{n-1}$. The term involving k^i is $k\binom{n-1}{i-1}k^{i-1}(-1)^{n-i}$, so the magnitude of the coefficient is $\binom{n-1}{i-1}$, as desired.

Now consider $e(G) > n-1$. If G is connected and has more than $n-1$ edges, then G has a cycle, and deleting any edge of the cycle leaves a connected graph. Let e be such an edge, and define $\{b_i\}$, $\{c_i\}$ in the chromatic polynomials by $\chi(G-e;k) = \sum_{i=0}^{n-1}(-1)^i b_{n-i}k^{n-i}$ and $\chi(G\cdot e;k) = \sum_{i=0}^{n-1}(-1)^i c_{n-1-i}k^{n-1-i}$.

The recurrence $\chi(G;k) = \chi(G-e;k) - \chi(G\cdot e;k)$ implies that $(-1)^{n-i}a_i$ is the sum of the coefficients of k^i in the other two polynomials. Since $G-e$ and $G\cdot e$ are connected, the induction hypothesis implies $a_i = b_i - (-1)c_i = b_i + c_i \ge \binom{n-1}{i-1} + \binom{n-2}{i-1} > \binom{n-1}{i-1}$ for $1 \le i \le n-1$. Indeed, equality holds in the bound for any of these coefficients only if G is a tree.

5.3.11. *The coefficients of $\chi(G;k)$ sum to 0 unless G has no edges.* The sum of the coefficients of a polynomial in k is its value at $k=1$. The value of $\chi(G;1)$ is the number of proper 1-colorings of G. This is 0 unless G has no edges. (The inductive proof from the chromatic recurrence is longer.)

5.3.12. *The exponent in the last nonzero term in the chromatic polynomial of G is the number of components of G.* We use induction on $e(G)$. When $e(G)=0$, we have $\chi(G;k) = k^{n(G)}$, and G has $n(G)$ components. Let $c(G)$ count the components in G. Both $G-e$ and $G\cdot e$ have fewer edges than G. Also $G\cdot e$ has the same number of components as G, and $G-e$ has the same number or perhaps one more. Since $n(G\cdot e) = n(G-e)-1$ and coefficients alternate signs, the coefficients of $k^{c(G)}$ have opposite signs in $\chi(G\cdot e;k)$ and $\chi(G-e;k)$. Thus we have positive α and nonnegative α' such that

$$
\begin{array}{rlllll}
\chi(G-e;k): & k^n & -[e(G)-1]k^{n-1}+ & \cdots & +(-1)^{n-c(G)}\alpha k^{c(G)} & \\
-\chi(G\cdot e;k): & -(& k^{n-1}- & \cdots & +(-1)^{n-c(G)-1}\alpha' k^{c(G)} &) \\
\hline
=\chi(G;k): & k^n & -e(G)k^{n-1}+ & \cdots & +(-1)^{n-c(G)}(\alpha+\alpha')k^{c(G)} &
\end{array}
$$

Since $\alpha+\alpha' > 0$, the last coefficient of $\chi(G;k)$ is as claimed.

Alternatively, one can reduce to the case of connected graphs by observing that the chromatic polynomial of a graph is the product of the chromatic polynomials of its components. Since an n-vertex tree has chromatic polynomial $k(k-1)^{n-1}$, its last nonzero term is the linear term. For a connected graph that is not a tree, the chromatic recurrence can be applied as above to obtain the result inductively.

If $p(k) = k^n - ak^{n-1} + \cdots \pm ck^r$ with $a > \binom{n-r+1}{2}$, then p is not a chromatic polynomial. If p is a chromatic polynomial of a (simple) graph G, then G has n vertices, a edges, and r components. The maximum number of edges in a simple graph with n vertices and r components is achieved by $r-1$ isolated vertices and one clique of order $n-r+1$. This has $\binom{n-r+1}{2}$ edges (Exercise 1.3.40), which is less than a.

5.3.13. *Chromatic polynomials and clique cutsets.* Let $F = G\cup H$, with $S = V(G)\cap V(H)$ being a clique. Every proper k-coloring of F yields proper k-colorings of G and H, and proper k-colorings of G and H together yield a proper k-coloring of F if they agree on S. Since S induces a clique, in every proper k-coloring of G or H the vertices of S have distinct colors. Therefore, given a proper k-coloring of $G\cap H$, the number of ways to extend it to a proper k-coloring of H [or G, or F] is independent of which proper k-coloring of $G\cap H$ is used.

For each $k \ge 0$, the value of the chromatic polynomial simply counts proper colorings. We have partitioned the proper k-colorings of these graphs into equal-sized classes that agree on S. For a fixed coloring f of $G\cap H$, the number of ways to extend it to a coloring of G, H, or F is thus $\chi(G;k)/\chi(G\cap H;k)$, $\chi(H;k)/\chi(G\cap H;k)$, or $\chi(F;k)/\chi(G\cap H;k)$, respectively. Since every extension of f to G is compatible with every extension of f to H to yield an extension of f to F, the product of the first two of these equals the third, and $\chi(G\cup H;k) = \chi(G;k)\chi(H;k)/\chi(G\cap H;k)$. (*Comment*: 1) When G and H intersect in a clique, it need not be true that $\chi(G;k) = \chi(G-G\cap H;k)\chi(G\cap H;k)$; for example, let G and H be 4-cycles sharing a single vertex.)

When $G\cap H$ is not a clique, this argument breaks down. For example, consider $G = H = P_3$, $F = G\cup H = C_4$, $G\cap H = 2K_1$. We have

$$\chi(F;k)\chi(G\cap H;k) = k^3(k-1)(k^2-3k+3) \ne k^2(k-1)^4 = \chi(G;k)\chi(H;k)$$

5.3.14. *Minimum vertex partitions of the Petersen graph into independent sets.* Let P be the Petersen graph. The Petersen graph P has odd cycles, so it requires 3 colors, and it is easy to partition the vertices into 3 independent sets using color classes of size 4,3,3, as described below.

a) If S is an independent 4-set, then $P-S = 3K_2$. The three neighbors of a vertex have among them an edge to every other vertex, so S cannot contain all the neighbors of a vertex. Hence $P-S$ has no isolated vertex. Deleting S deletes 12 edges, so $P-S$ has 3 edges and 6 vertices. With no isolated vertices, this yields $P-S = 3K_2$.

b) P has 20 partitions into three independent sets. Since P has 10 vertices, every such partition has an independent set of size at least four. There is no independent 5-set, because we have seen that every independent 4-set has two edges to each remaining vertex. For each independent 4-set S, there are 4 ways to partition the vertices of the remaining $3K_2$ into two independent 3-sets. Hence it suffices to count the independent 4-sets and multiply by 4. The number of independent 4-sets containing a specified vertex is 2, since deleting that vertex and its neighbors leaves C_6, which has two independent 3-sets. Summing this over all vertices counts each independent 4-set four times. Hence there are $2 \cdot 10/4 = 5$ independent 4-sets and 20 partitions of the vertices.

c) If $r = \chi(G)$, then $V(G)$ has $\chi(G; r)/r!$ partitions into r independent sets. Each such partition can be converted into a coloring in exactly $r!$ ways.

5.3.15. *A graph with chromatic number k has at most k^{n-k} vertex partitions into k independent sets, with equality achieved only by $K_k + (n-k)K_1$ (complete graph plus isolated vertices.* For $K_k + (n-k)K_1$, the sets of the partition are identified by the vertex of the clique that they contain, and the isolated vertices can be assigned to these sets arbitrarily, so this is the correct number of vertex partitions for this graph.

If G has only k vertices, then G has be a k-clique, and there is only one partition. If $n > k$, choose a vertex $v \in V(G)$. We consider two cases; $\chi(G - v) = k$ and $\chi(G - v) = k - 1$.

If $\chi(G - v) = k$, then partitions of $G - v$ can be extended to partitions of G by putting v in any part to which it has no edges. Thus it extends in at most k ways, with equality only if v is an isolated vertex.

If $\chi(G - v) = k - 1$, then G has a k-partition in which v is by itself and is adjacent to vertices $X = \{x_1, \ldots, x_{k-1}\}$ of the other parts. Let R be the independent set containing v in an arbitrary k-partition, and suppose $|R| = 1 + r$. Note that $\chi(G - R) = k - 1$. By the induction hypothesis, $G - R$ has at most $(k-1)^{n-r-k}$ partitions into $k - 1$ independent sets. Allowing R to be an arbitary subset of $G - (X \cup \{v\})$, we obtain at most $\sum_{r=0}^{n-k} \binom{n-k}{r}(k-1)^{n-k-r}$ partitions of G into k independent sets, which equals k^{n-k} by the binomial theorem. For equality, we must have $N(v) = X$ and $G - (X \cup \{v\}) = (n-k)K_1$ for each such choice of v, which again yields $G = K_k + (n-k)K_1$.

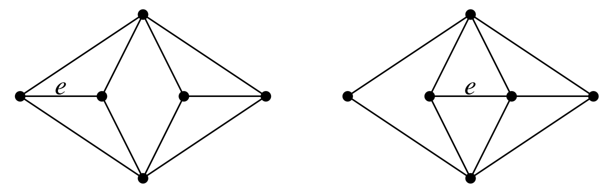
5.3.16. *If G is a simple graph with n vertices and m edges, then G has at most $\frac{1}{3}\binom{m}{2}$ triangles.* Each triangle has three pairs of incident edges, and each edge pair of incident edges appears in at most one triangle. Hence the number of triangles is at most $1/3$ of the number of pairs of edges.

The coefficient of k^{n-2} in $\chi(G; k)$ is positive unless G has at most one edge. In the expression for the chromatic polynomial in Theorem 5.3.10, contributions to the coefficient of k^{n-2} arise from spanning subgraphs with $n - 2$ components. These include all ways to choose two edges (weighted positively) and all ways to choose three edges forming a triangle (weighted negatively). With m edges and t triangles, the coefficient is $\binom{m}{2} - t$. Since $t \le \frac{1}{3}\binom{m}{2}$, the coefficient is positive unless G has at most one edge.

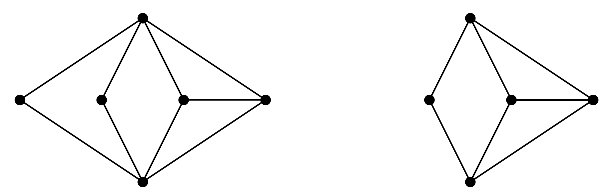
5.3.17. *Chromatic polynomial via the inclusion-exclusion principle.* In the universe of all k-colorings of G, let A_i be the set of colorings that assign the same color to the endpoints of edge e_i. The proper k-colorings of G are the k-colorings outside all the sets A_i. By the inclusion-exclusion formula,

the number of these is $\sum_{S \subseteq E(G)} (-1)^{|S|} g(S)$, where $g(S)$ is the number of k-colorings in $\bigcap_{e_i \in S} A_i$. These are the colorings in which every edge in S has its endpoints given the same color. To count these, we can choose a color independently for each component of the spanning subgraph of G with edge set S. Hence $g(S) = k^{c(G(S))}$, where $c(G(S))$ is the number of these components. We have obtained the formula of Theorem 5.3.10.

5.3.18. *Two chromatic polynomials.*

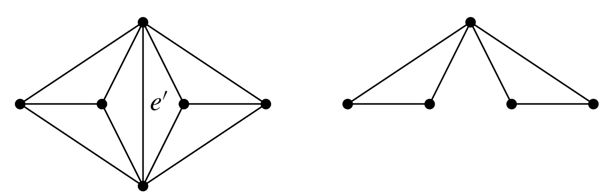


a) The graphs G, H above have the same chromatic polynomial. Applying the chromatic recurrence using the edge labeled e shows that each of these graphs has a chromatic polynomial that is the difference of the chromatic polynomials of the two graphs below.



b) The chromatic polynomial of G. The first graph G is $G' - e'$, where G' is the graph on the left below and e' is the indicated edge. The graph $G' \cdot e'$ appears on the right. Each of these graphs is chordal, as shown by exhibiting a simplicial eliminationordering. For each, the chromatic polynomial is a product of linear factors arising from the reverse of a simplicial elimination ordering. Thus

$$\begin{aligned}\chi(G; k) &= \chi(G' - e'; k) = \chi(G'; k) + \chi(G' \cdot e; k) \\ &= k(k-1)(k-2)^2(k-3)^2 + k(k-1)^2(k-2)^2 \\ &= k(k-1)(k-2)^2(k^2 - 5k + 8)\end{aligned}$$



5.3.19. *The chromatic polynomial of the graph G obtained from K_6 by subdividing one edge is a product of linear factors, although G is not a chordal graph.* Let v be the vertex of degree 2 in G, and let e be an edges incident to e. The cycle consisting of v, its incident edges, and the edges from its neighbors to one other vertex form a chordless 4-cycle, so G is not chordal.

To compute $\chi(G;k)$, observe that $G-e$ consists of a 5-clique Q, an additional vertex w adjacent to four vertices of Q, and v adjacent to w. Hence $G-e$ is a chordal graph, with $\chi(G-e;k) = k(k-1)(k-2)(k-3)(k-4)(k-4)(k-1)$. Let $f(k) = \prod_{i=0}^{4}(k-i)$. The graph $G\cdot e$ is K_6, with $\chi((;K)_6) = f(k)(k-5)$. Thus

$$\begin{aligned}\chi(G;k) &= \chi(G-e;k) - \chi(G\cdot e;k) = f(k)[(k-4)(k-1)-(k-5)] \\ &= f(k)[k^2-6k+9] = k(k-1)(k-2)(k-3)(k-4)(k-3)^2\end{aligned}$$

5.3.20. *Properties of a chordal graph G with n vertices.*

a) G has at most n maximal cliques, with equality if and only if G has no edges. As each vertex v is added in the reverse of a simplicial elimination ordering, it creates one new maximal clique (containing v) if $N(v)$ is not already a maximal clique. If $N(v)$ is already a maximal clique, then the clique grows. No other maximal clique appears or changes. Thus there is at most one new maximal clique for each vertex. The first time an edge is added, a maximal clique is enlarged, not created, so there is a new clique at most $n-1$ times if G has an edge. (Comment: A more formal version of this argument uses the language of induction on n.)

b) Every maximal clique of G that contains no simplicial vertex of G is a separating set of G.

b) Every maximal clique of G that contains no simplicial vertex of G is a separating set of G.

Proof 1 (construction ordering, following part (a).) When a maximal clique Q of G acquires its last vertex v in the construction ordering, v is then simplicial. If all vertices of Q that are simplicial when Q is created are not simplicial in G, then the rest of the construction gives them additional neighbors that are separated by Q from each other and from the vertices of $G-Q$ that are earlier than v. If there are no such earlier vertices, then Q has at least two simplicial vertices at the time it is formed; each of these acquires a later neighbor, so Q separates those later neighbors.

Proof 2 (induction on n.) When $G=K_n$, there is no separating set, but all the vertices are simplicial, so the statement holds. When $G\neq K_n$, let Q be a maximal clique containing no simplicial vertex of G. Every chordal graph that is not a complete graph has two nonadjacent simplicial vertices (this follows, for example, from Lemma 5.3.16). Let u and v be such vertices. Note that Q cannot contain both u and v; we may assume that $v\notin Q$. Hence Q is a maximal clique in $G-v$.

If Q contains no simplicial vertex of $G-v$, then the induction hypothesis implies that Q separates $G-v$. All neighbors of v in G lie in one component of $G-v-Q$, since $N(v)$ is a clique in $G-v$. Hence Q is also a separating set in G.

If Q contains at least one simplicial vertex of $G-v$, then all such vertices lie in $N(v)$, since they are not simplicial in G. Therefore $u\notin Q$, and Q separates v from u.

5.3.21. *A graph G is chordal if and only if $s(H)=\omega(H)$ for every induced subgraph H of G*, where $s(H)$ is the *Szekeres–Wilf number* of H, defined to be $1+\max_{H\subseteq G}\delta(H)$.

Sufficiency. We prove the contrapositive. If G is not chordal, then G has a chordless cycle with length at least 4. Such a cycle is an induced subgraph. Its clique number is 2, and its Szekeres–Wilf number is 3.

Necessity. Since every induced subgraph of a chordal graph is chordal, it suffices to show that $s(G)=\omega(G)$ (the argument for G also applies to each induced subgraph). Since always $s(G)\geq\omega(G)$, it suffices to show that $s(G)\leq\omega(G)$.

Let H be an induced subgraph of G such that $\delta(H)=\max_{G'\subseteq G}\delta(G')$, so $s(G)=1+\delta(H)$. Let x be the first vertex of H that is deleted in some simplicial elimination ordering of G. Since the neighbors of x in H complete a clique with x, we have $\omega(H)\geq 1+d_H(x)\geq 1+\delta(H)=s(G)$.

5.3.22. *If $k_r(G)$ is the number of r-cliques in a connected chordal graph G, then $\sum_{r\geq 1}(-1)^{r-1}k_r(G)=1$.* We use induction on $n(G)$. When $n(G)=1$, the only graph is K_1, which has one 1-clique and no larger clique; this satisfies the formula.

For $n(G)>1$, we know that G has a simplicial elimination ordering. Let v be a simplicial vertex in G. By the induction hypothesis, $\sum_{n\geq 1}(-1)^{r-1}k_r(G-v)=1$. All cliques in $G-v$ appear also in G, so the contribution to the sum from these cliques is the same in G. Thus it suffices to show that the net contribution from cliques containing v is 0.

Each clique of size r containing v consists of v and $r-1$ vertices from $N(v)$. Since v is simplicial, $N(v)$ is a clique, and thus every selection of $r-1$ vertices from $N(v)$ forms an r-clique with v. Therefore, the contribution from these cliques is $\sum_{r\geq 1}(-1)^{r-1}\binom{d(v)}{r-1}$.

The binomial theorem states that $(1+x)^m=\sum_{s=1}^{m}x^s\binom{m}{s}$. Setting $m=d(v)$ and $x=-1$ yields our sum on the right; on the left it yields 0 (since $m>0$). Thus the contribution from cliques containing v is 0, as desired.

5.3.23. *If C is a cycle of length at least 4 in a chordal graph G, then G has a cycle whose vertex set is $V(C)$ minus one vertex.* Given a simplicial elimination ordering of G, let v be the first vertex of C that is deleted. Since the remaining neighbors of v at the time of deletion form a clique, the neighbors of v on C are adjacent. Hence deleting v from the cyclic order of vertices on C yields a shorter cycle.

5.3.24. *If e is an edge of a cycle C in a chordal graph, then e forms a triangle with a third vertex of C.* We use induction on the length of C. If C is a triangle, then we have nothing to do. If C is longer, then because the graph is chordal there is a chord f of C. This splits C into two paths, one of which contains e. Combining this path with f yields a shorter cycle containing e, with all its vertices still in C. Applying the induction hypothesis to this shorter cycle yields the desired vertex of C.

5.3.25. *If Q is a maximal clique in a chordal graph G and $G-Q$ is connected, then Q contains a simplicial vertex.* (Equivalently, a maximal clique containing no simplicial vertex is a separating set.) We use induction on $n(G)$. When $n(G)\le 2$, G is a union of disjoint cliques, and the claim holds. For $n(G)\ge 3$, let Q be a maximal clique of G containing no simplicial vertex. Let v be a simplical vertex of G, and consider $G-v$. Still Q is a maximal clique in $G-v$.

If Q contains no simplicial vertex of $G-v$, then by the induction hypothesis Q is a separating set of $G-v$. If Q is not a separating set of G, then v has a neighbor in each component of $G-v-Q$, which contradicts v being simplicial in G.

Hence we may assume that Q contains a simplicial vertex u of $G-v$ that is not simplicial in G. This requires $v\leftrightarrow u$. If Q is not a separating set, then also v has a neighbor x outside Q. Since $u\leftrightarrow v$ and v is simplicial in G, also $x\leftrightarrow u$. Now since $x,u\in V(G-v)$ and u is simplicial in $G-v$, all of Q must also be adjacent to x. This contradicts the maximality of Q. Hence Q must indeed be a separating set in G.

5.3.26. *Chromatic polynomials of chordal graphs.*

a) If $G\cup H$ is a chordal graph, then $\chi(G\cup H;k)=\frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}$, regardless of whether $G\cap H$ is a complete graph. We use induction on $n(G\cup H)$; the claim is immediate when there is one vertex. When $G\cup H$ is larger, let v be a simplicial vertex in $G\cup H$. By symmetry, we may assume that $v\in V(G)$. Since $N_{G\cup H}(v)$ is a clique, it cannot intersect both $V(G)-V(H)$ and $V(H)-V(G)$, since $G\cup H)$ has no edges joining these two sets. Hence we may assume that $N_{G\cup H}(v)\subseteq V(G)$.

Since v is simplicial, we have $\chi(G\cup H;k)=(k-d(v))\chi((G\cup H)-v;k)$. Note that $(G\cup H)-v=(G-v)\cup(H-v)$ and $(G\cap H)-v=(G-v)\cap(H-v)$. Since $(G\cup H)-v$ is chordal, the induction hypothesis yields $\chi((G\cup H)-v;k)=\frac{\chi(G-v;k)\chi(H-v;k)}{\chi((G-v)\cap(H-v);k)}$. Since $N_{G\cup H}(v)\subseteq V(G)$, we have $\chi(G-v;k)=\chi(G;k)/(k-d(v))$.

If $v\in V(G)\cap V(H)$, then $d_H(v)=d_{G\cap H}(v)$, and v is simplicial in every induced subgraph of $G\cup H$ containing it, so $\chi(H;k)/\chi(G\cap H;k)=\chi(H-v;k)/\chi((G\cap H)-v;k)$. If $v\in V(G)-V(H)$, then this ratio also holds, because in this case $H-v=H$ and $(G\cap H)-v=G\cap H$.

Hence we have

$$\begin{aligned}\chi(G\cup H;k)&=(k-d(v))\chi((G\cup H)-v;k)\\&=(k-d(v))\frac{\chi(G-v;k)\chi(H-v;k)}{\chi((G\cap H)-v;k)}=\frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}\end{aligned}$$

b) If x is a vertex in a chordal graph G, then

$$\chi(G;k)=\chi(G-x;k)k\frac{\chi(G[N(x)];k-1)}{\chi(G[N(x)];k)}.$$

We apply part (a) with $G=F\cup H$, where $H=G[N(x)\cup x]$ and $F=G-x$. Observe that $F\cap H=G[N(x)]$. Also, since x is adjacent to all other vertices in H, we form all proper colorings of H by choosing a color for x and then forming a proper coloring of H from the remaining $k-1$ colors. Hence $\chi(H;k)=k\chi(H-x;k-1)=k\chi(G[N(x)];k)$. Now we simply substitute these expressions into the formula from part (a).

5.3.27. *Characterization of chordal graphs by minimal vertex separators,* where a *minimal vertex separator* in a graph G is a set $S\subseteq V(G)$ that for some pair x,y is a minimal set whose deletion separates x and y.

a) If every minimal vertex separator in G is a clique, then the same property holds in every induced subgraph of G. Let H be an induced subgraph of G. If S is a minimal x,y-separator in H, then $S\cup(V(G)-V(H))$ separates x and y in G. Hence $S\cup(V(G)-V(H))$ contains a minimal x,y-separator of G. Such a set T must contain S, since otherwise $G-T$ contains an x,y-path within H. By hypothesis, T is a clique in G, and hence S is a clique in H.

b) A graph G is chordal if and only if every minimal vertex separator is a clique. Necessity. For two vertices u,v in a minimal x,y-separator S, find shortest u,v-paths through the components of $G-S$ containing x and y. The union of these paths is a cycle of length at least 4, and its only possible chord is uv. Hence the vertices in S are pairwise adjacent.

Sufficiency. By part (a) and induction on $n(G)$, it suffices to show that G has a simplicial vertex if every minimal vertex separator of G is a clique. By induction on $n(G)$, we prove the stronger statement that if every minimal vertex separator of G is a clique and G is not a clique, then G has two nonadjacent simplicial vertices. The basis is vacuous (small cliques).

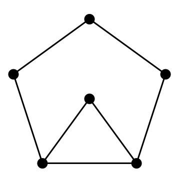
For larger G, let x_1,x_2 be a nonadjacent pair of vertices in G, let S be a minimal x_1,x_2-separator, and let G_i be the S-lobe of G (Definition 5.2.17) containing x_i. Since C holds for induced subgraphs, it holds for G_i. By the induction hypothesis, G_i has a simplicial vertex $u_i \notin S$ (whether or not G_i is a clique). Since no edge connect $V(G_1)$ to $V(G_2)$, the vertices u_1, u_2 are also simplicial in G, and they are nonadjacent in G.

5.3.28. *Every interval graph is a chordal graph and is the complement of a comparability graph.* Consider an interval representation of G, with each v represented by the interval $I(v) = [a(v), b(v)]$. Let v be the vertex with largest left endpoint $a(v)$. The intervals for all neighbors of v contain $a(v)$, so in the intersection graph the neighbors of v form a clique. Hence v is simplicial. If we delete v and procede with the remainder of the representation, which is an interval representation of $G - v$, we inductively complete a perfect elimination ordering.

Alternatively, let C be a cycle in G. Let u be the vertex in C whose right endpoint is smallest, and let v be the vertex whose left endpoint is largest. If u, v are nonadjacent, then the intervals for the two u, v-paths in C must cover $[b(u), a(v)]$. Hence the intersection graph has a chord of C between them. We conclude that an interval graph has no chordless cycle.

If $uv \in E(\overline{G})$, then $I(u)$ and $I(v)$ are disjoint. Orient the edge uv toward the vertex whose interval is to the left. This yields a transitive orientation of $\overline{G}$; if $I(u)$ is to the left of $I(v)$, and $I(v)$ is to the left of $I(w)$, then $I(u)$ is to the left of $I(w)$.

5.3.29. *The smallest imperfect graph G such that $\chi(G) = \omega(G)$.* The only imperfect graph with at most five vertices is C_5. Thus the graph below is the smallest imperfect graph with $\chi(G) = \omega(G)$.



5.3.30. An edge in an acyclic orientation of G is *dependent* if reversing it yields a cycle.

a) Every acyclic orientation of a connected n-vertex graph G has at least $n-1$ independent edges. We use induction on n. When $n = 1$, we have no edges and need none. Consider $n > 1$. Since the orientation has no cycles, every maximal path starts with a source (indegree 0). Hence G has a source v. Define a digraph H with vertex set $N^+(v)$ by putting $x \leftrightarrow y$ in H if G has an x, y-path. Since a closed walk in the digraph G would contain a cycle, H must be acyclic. Let x be a source in H. The edge vx is independent; reversing it cannot create a cycle, since no path in G from v reaches x except the edge vx itself.

Let $G' = G - v$. Edges of G' are independent if and only if they are also independent in G, because there is no path in G through v from one vertex of G' to another. Also, G' is acyclic. Hence we can apply the induction hypothesis to G' to obtain another $n-2$ independent edges.

b) If $\chi(G)$ is less than the girth of G, then G has an orientation with no dependent edges. Given an optimal coloring f, orient edge xy from x to y if and only if $f(y) > f(x)$. The maximum path length in this orientation is less than $\chi(G)$, and hence it is smaller by at least two than the length of any cycle.

An edge in an acyclic orientation is dependent if and only if there is another path from its tail to its head. The length of such a path would be one less than the length of the resulting cycle, but we have shown that our orientation has no paths this long.

5.3.31. *Comparison between acyclic orientations and spanning trees.* The number $\tau(G)$ satisfies the recurrence $\tau(G) = \tau(G - e) + \tau(G \cdot e)$. This is the recurrence satisfied by $a(G)$, but the initial conditions are different. A graph with no edges has one acyclic orientation, but it has no spanning tree unless it has only one vertex. A connected graph containing a loop has spanning trees but no acyclic orientation. A tree of order n has one spanning tree and 2^{n-1} acyclic orientations. A clique of order n has n^{n-2} spanning trees and $n!$ acyclic orientations; $n^{n-2} > n!$ if $n \geq 6$.

5.3.32. *Compatible pairs:* $\eta(G; k) = (-1)^n \chi(G; -k)$. Suppose D is an acyclic orientation of G and f is a coloring of $V(G)$ from the set $[k]$. We say that (D, f) is a *compatible pair* if $u \to v$ in D implies $f(u) \leq f(v)$. Let $\eta(G; k)$ be the number of compatible pairs. If $f(u) \neq f(v)$ for every adjacent pair u, v, then only one orientation is compatible with f. Therefore, $\chi(G; k)$ counts the pairs (D, f) under a slightly different condition: D is acyclic and $u \to v$ in D implies $f(u) < f(v)$ (*equality forbidden*). We know that $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$ for any edge of G; we claim that $\eta(G; k) = \eta(G - e; k) + \eta(G \cdot e; k)$.

The two conditions on pairs being counted are the same when there are no edges, so the two recurrences have the same boundary conditions: $k^n = \eta(\overline{K}_n; k) = \chi(\overline{K}_n; k)$. From this and the recurrence, we obtain $\eta(G; k) = (-1)^n \chi(G; k)$ by induction on $e(G)$. We compute $\eta(G; k) = \eta(G - e; k) + \eta(G \cdot e; k) = (-1)^{n(G)} \chi(G - e; k) + (-1)^{n(G)-1} \chi(G \cdot e; k) = (-1)^{n(G)} \chi(G; k)$. Evaluating η at 1 or χ at -1 yields $(-1)^n \chi(G; -1)$. Because there is only one labeling in which all vertices get label 1, and this is compatible with every acyclic orientation, $\eta(G; 1)$ is the number of acyclic orientations.

It remains only to prove the recurrence for η. Let $e = uv$. As in the recurrence for the chromatic polynomial, we begin with the compatible pairs for $G - e$ and consider the effect of adding e. If (D, f) is a compatible pair for $G - e$ such that $f(u) \neq f(v)$, say $f(u) < f(v)$, then e must be oriented from u to v to obtain an orientation of G compatible with f. The result is indeed acyclic, else it has a directed v, u-path along which the value f must step downward at some point. Conversely, we can delete e from a compatible pair for G with $f(u) \neq f(v)$ to obtain a compatible pair for $G - e$. Hence the compatible pairs with differing labels for u and v are in 1-1 correspondence in G and $G - e$.

Now consider pairs with $f(u) = f(v)$. It suffices to show that each such pair for $G - e$ becomes a compatible pair for G by adding e oriented in at least one way, and that for $\eta(G \cdot e, k)$ of these, *both* orientations of e yield compatible pairs for G. For the first statement, consider an arbitrary compatible pair (D', f) with $f(u) = f(v)$ for $G - e$, and suppose neither orientation for e yields a compatible pair for G. This requires D' to have both a u, v-path and a v, u-path, which cannot happen since D' is acyclic. For the second statement, suppose that (D, f) is a compatible pair for G with $f(u) = f(v)$ and that the orientation obtained by reversing e is also compatible with f. Then $D - e$ has neither a u, v-path nor a v, u-path, and contracting e yields a compatible pair for $G \cdot e$. Conversely, given a compatible pair for $G \cdot e$, we can split the contracted vertex to obtain a compatible pair for $G - e$ with $f(u) = f(v)$ so that orienting e in either direction yields a compatible pair for G.

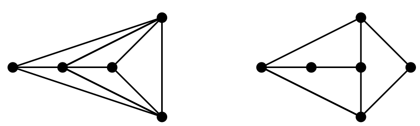
6.PLANAR GRAPHS

6.1. EMBEDDINGS & EULER'S FORMULA

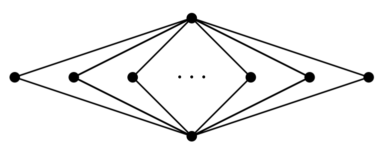
6.1.1. *a) Every subgraph of a planar graph is planar—TRUE.* Given a planar embedding of G, deleting edges or vertices does not introduce crossings, so an embedding of any subgraph of G can be obtained.

b) Every subgraph of a nonplanar graph is nonplanar—FALSE. $K_{3,3}$ is nonplanar, but every proper subgraph of $K_{3,3}$ is planar.

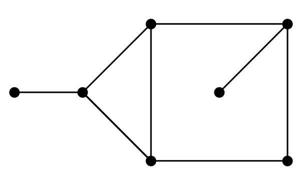
6.1.2. *The graphs formed by deleting one edge from K_5 and $K_{3,3}$ are planar.*



6.1.3. *$K_{r,s}$ is planar if and only if* $\min\{r, s\} \le 2$. If G contains the nonplanar graph $K_{3,3}$, then G is nonplanar; hence $K_{r,s}$ is nonplanar when $\min\{r, s\} \ge 3$. When $\min\{r, s\} = 2$, the drawing below suggests the planar embedding, and $K_{1,s}$ is a subgraph of this.



6.1.4. *The number of isomorphism classes of planar graphs that can be obtained as planar duals of the graph below is 4.*



The 4-cycle C can be embedded in only one way. Let e be the pendant edge incident to it, and let f be the pendant edge incident to the triangle D. We may assume that e immediately follows the edges D when we traverse C clockwise, because the other choice corresponds to reflecting the plane, and the resulting duals will be isomorphic to these.

We may embed e inside or outside C, we may embed D inside or outside C, and we may embed f inside or outside D. This yields eight possible embeddings, all with three faces. These come in pairs yielding the same dual, because flipping the choices involving C (while maintaining the same choice of whether f is inside D) has the effect of exchanging the inside and outside of C without affecting the dual.

In the four pairs, the resulting degree lists for the dual are $(9, 4, 3)$, $(7, 6, 3)$, $(7, 5, 4)$, and $(6, 5, 5)$. These are distinct, so there are four isomorphism classes of duals.

6.1.5. *A plane graph has a cut-vertex if and only if its dual has a cut-vertex—FALSE.* There are many counterexamples. The duals of trees and unicyclic graphs have at most two vertices and hence no cut-vertices. The duals of disconnected graphs without cut-vertices have cut-vertices.

6.1.6. *A plane graph with at least three vertices is 2-connected if and only if for every face, the bounding walk is a cycle.* If multiple edges are being allowed, the restriction to at least three vertices eliminates the cycle of length 2.

A disconnected plane graph has a face whose boundary consists of more than one closed walk, so we restrict our attention to a connected plane graph G. If G has a cut-vertex x, then considering the edges incident to x in clockwise order, there must be two consecutive edges in different $\{x\}$-lobes. For the face incident to these two edges, the boundary intersects more than one $\{x\}$-lobe and hence cannot be a cycle.

Now suppose that G is 2-connected. For a vertex x on the boundary of a face F, there are points inside F near x. By the definition of "face", all the nearby points between two rotationally consecutive incident edges at x are in F. Let e and e' be two such edges. Since G is 2-connected e and e' lie on a common cycle C.

By the Jordan Curve Theorem and the definition of "face", all points interior to F are inside C, or they are all outside C. In either case, as we follow the boundary of F after e and e', C prevents the boundary from visiting x again. Thus every vertex on the boundary of F is incident to exactly two edges of the boundary, and the boundary is a cycle.

6.1.7. *Every maximal outerplanar graph with at least three vertices is 2-connected.* Let G be a planar graph embedded with every vertex on the unbounded face. If G is not connected, then adding an edge joining vertices of distinct components still leaves every vertex on the unbounded face. If G has a cut-vertex v, and u and w are the vertices before and after v in a walk around the unbounded face, then adding the edge uw still leaves every vertex on the unbounded face, since v is visited at another point in the walk. We have shown that an outerplanar graph that is not 2-connected is not a maximal outerplanar graph.

6.1.8. *Every simple planar graph has a vertex of degree at most 5.* Every simple planar graph with n vertices has at most $3n-6$ edges (Theorem 6.1.23). Hence the degree sum is at most $6n-12$, and by the pigeonhole principle there is a vertex with degree less than 6.

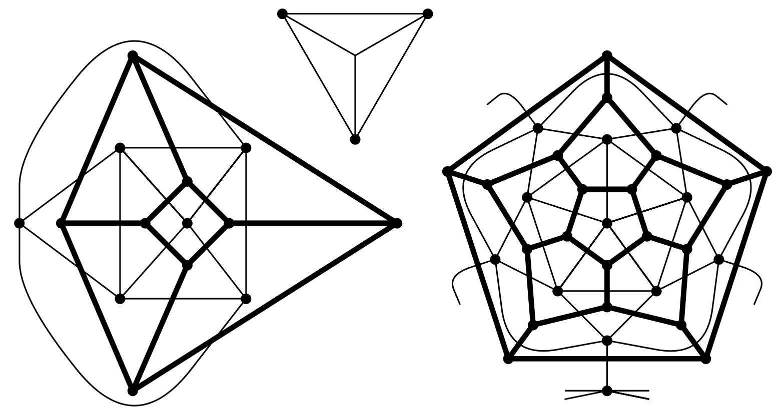
6.1.9. *Every simple planar graph with fewer than 12 vertices has a vertex of degree at most 4.* By Theorem 6.1.23, every simple planar graph with n vertices has at most $3n-6$ edges and degree-sum at most $6n-12$. If $12 > n$, then this degree sum is less than $5n$, and the pigeonhole principle implies that some vertex has degree at most 4.

6.1.10. *There is no simple bipartite planar graph with minimum degree at least 4—TRUE.* Since every face of a simple bipartite planar graph has length at least 4, it has at most $2n-4$ edges and degree sum at most $4n-8$. Hence the average degree of a simple bipartite planar graph is less than 4, and its minimum degree is less than 4.

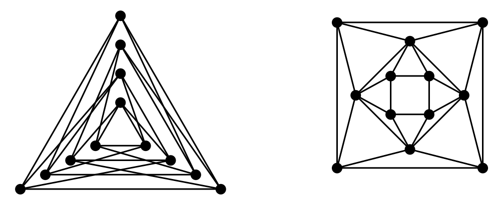
6.1.11. *The dual of a maximal planar graph is 2-edge-connected and 3-regular.* By definition, a maximal planar graph G is a simple planar graph to which no edge can be added without violating planarity. Consider an embedding of G. Every dual G^* is connected. If G^* has a cut-edge, then the edge of G corresponding to this edge is a loop in G, which cannot occur in a simple graph. Thus G^* is 2-connected.

Since G is simple, every face has length at least 3. If some face has length exceeding 3. Let w, x, y, z be four vertices in order on this face. If wy is an edge (outside this face), then xz cannot be an edge. Thus we can add wy or xz, contradicting maximality. This implies that every face has length 3, which is the statement that the dual is 3-regular.

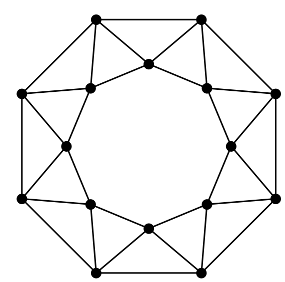
6.1.12. *Drawings of the five regular polyhedra as planar graphs, with the octahedron as the dual of the cube and the icosahedron as the dual of the dodecahedron.* The edges incident to the vertex of the icosahedron corresponding to the unbounded face of the dodecahedron are not fully drawn.



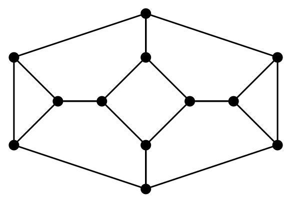
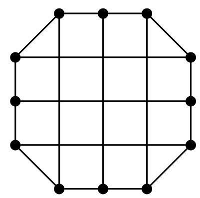
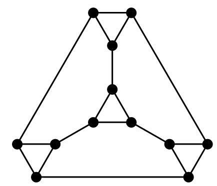
6.1.13. *Planar embedding of a graph.* The drawing on the right is a planar embedding of the graph on the left. (Of course, the isomorphism should be given explicitly.)



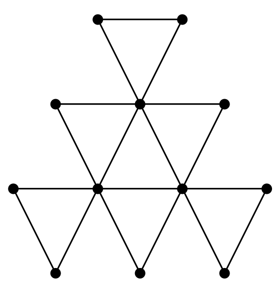
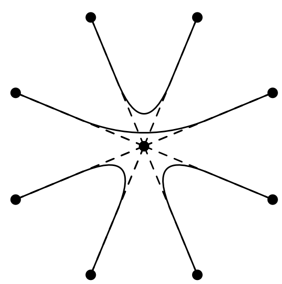
6.1.14. *For each $n \in \mathbb{N}$, there is a simple connected 4-regular planar graph with more than n vertices—TRUE.* When $n \geq 3$, we can form a simple connected 4-regular plane graph with $2n$ vertices by using an inner n-cycle, an outer n-cycle, and a cycle in the region between them that uses all $2n$ vertices. Below we show this for $n = 8$.



6.1.15. *A 3-regular planar graph of diameter 3 with 12 vertices.* By inspection, the graphs below are 3-regular and planar. To show that that they have diameter 3, we conduct a breadth-first search (Dijkstra's Algorithm) to compute distances from each vertex. By symmetry, this need only by done for one vertex each of "type" (orbit under automorphisms). In this sense, the rightmost graph is the best answer, since it is vertex transitive, and the distances need only be checked from one vertex. The graph on the left has five types of vertices, and the graph in the middle has two.

6.1.16. *An Eulerian plane graph has an Eulerian circuit with no crossings.* As the graph on the left below illustrates, it is not always possible to do this by splitting the edges at each vertex into pairs that are consecutive around the vertex. The figure on the right illustrates a non-consecutive planar splitting. We give several inductive proofs. The result does not require the graph to be simple.

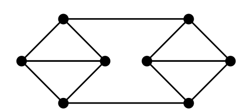



Proof 1 (induction on $e(G)-n(G)$). A connected Eulerian plane graph G has at least as many edges as vertices; if $e(G)=n(G)$, it is a single cycle and has no crossings. If $e(G)>n(G)$, then G has a vertex x on the outer face with degree at least 4. Form G' by splitting x into two vertices: a vertex y incident to two consecutive edges of the outer face that were incident to x, and a vertex z incident to the remaining edges that were incident to x. Locating y in the unbounded face of G yields a planar embedding of G'. Since G' is an even connected plane graph and $e(G')-n(G')=e(G)-n(G)-1$, the induction hypothesis applies to G'. Since the edges incident to y are consecutive at x, the resulting Eulerian circuit of G' translates back into an Eulerian circuit without crossings in G. This proof converts to an algorithm for obtaining the desired circuit.

Proof 2 (induction on $e(G)$). The basis is a single cycle. If G has more than one cycle, find a cycle C with an empty interior. Delete its edges and apply induction to the resulting components $G_1,\ldots,G_k$. For $i=1,\ldots,k$, absorb G_i into C as follows. Find a vertex x where G_i intersects C; all edges incident to x but not in C belong to G_i. Consider a visit $e1, x, e_2$ that the tour on G_i makes to x using an edge e_1 next to an edge f_1 of C in the embedding. Let f_2 be the other edge of C at x. Then replace the visits e_1, e_2 and f_1, f_2 by e_1, f_1 and e_2, f_2. This absorbs G_i into C while maintaining planarity of the circuit. Each component of $G-C$ is inserted at a different vertex, so no conflicts arise.

Proof 3 (local change). Given an Eulerian circuit that has a crossing in a plane graph G, we modify it to reduce the number of crossings formed by pairs of visits to vertices. By symmetry, it suffices to consider four edges a, b, c, d incident to v in counterclockwise order in the embedding such that one visit to v enters on a and leaves on c, and a later visit enters on b and leaves on d. We eliminate this crossing by traversing the portion $c,\ldots,b$ of the circuit in reverse order, as $b,\ldots,c$. Crossings at other vertices are unchanged by this operation. At v itself, if a passage e, f through v now crosses a, b, then e, f crossed a, c or b, d before, and if it now crosses a, b and c, d, then it crossed both a, c and b, d before. Thus there is no increase in other crossings, and we obtain a net decrease by un-crossing a, b, c, d.

6.1.17. *The dual of a 2-connected simple plane graph with minimum degree 3 need not be simple.* For the 2-connected 3-regular plane graph G below, G^* has a double edge joining the vertices of degree 6.



6.1.18. *Duals of connected plane graphs.*

a) If G is a plane graph, then G^* is connected. One vertex of G^* is placed in each face of G. If $u, v \in V(G^*)$, then any curve in the plane between u and v (avoiding vertices) crosses face boundaries of G in its passage from the face of G containing u to the face of G containing v. This yields a u, v-walk in G^*, which contains a u, v-path in G^*.

b) If G is connected, and G^* is drawn by placing one vertex inside each face of G and placing each dual edge in G^* so that it intersects only the corres ponding edge in G, then each face of G^* contains exactly one vertex of G. The edges incident to a vertex $v \in V(G)$ appear in some order around v. Their duals form a cycle in G^* in this order. This cycle is a face of G^*. If w is another vertex of G, then there is a v, w-path because G is connected, and this path crosses the boundary of this face exactly once. Hence every face of G^* contains at most one vertex of G. Equality holds because the number of faces of G^* equals the number of vertices of G: since both G and G^* are connected, Euler's formula yields $n - e + f = 2$ and $n^* - e^* + f^* = 2$. We have $e = e^*$ and $n^* = f$ by construction, which yields $f^* = n$.

c) For a plane graph G, $G^{**} \cong G$ if and only if G is connected. Since G^{**} is the dual of the plane graph G^*, part (a) implies that G^{**} is connected. Hence if G^{**} is isomorphic to G, then G is connected.

Conversely, suppose that G is connected. By part (b), the usual drawing of G^* over the picture of G has exactly one vertex of G inside each face of G^*. Associate each vertex $x \in V(G)$ with the vertex x' of G^{**} contained in the face of G that contains x; by part (b), this is a bijection.

Consider $xy \in E(G)$. Because the only edge of G^* crossing xy is the edge of G^* dual to it, we conclude that the faces of G^* that contain x and y have this edge as a common boundary edge. When we take the dual of G^*, we thus obtain $x'y'$ as an edge. Hence the vertex bijection from G to G^{**} that takes x to x' preserves edges. Since the number of edges doesn't change when we take the dual, G^{**} has no other edges and thus is isomorphic to G.

6.1.19. *For a plane graph G, a set $D \subseteq E(G)$ forms a cycle in G if and only if the corresponding set $D^* \subseteq E(G^*)$ forms a bond in G^*, by induction on $e(G)$.* We prove also that if D forms a cycle, then the two sides of the edge cut that is the bond in G^* corresponding to D are the sets of dual vertices corresponding to the sets of faces inside and outside D.

Basis step: $e(G) = 1$. When G and G^* have one edge, in one it is a loop (a cycle), and in the other it is a cut-edge (a bond).

Induction step: $e(G) > 1$. If D is a loop or a cut-edge, then the statement holds. Otherwise, D has more than one edge. If D forms a cycle, then let e be an edge of the cycle, and let G' be the graph obtained from G by contracting e. In G', the contracted set D' forms a cycle. Also, the set of faces in G' is the same as the set of faces in G; the only change is that the lengths of the faces bordering e (there are two of them since e is not a cut-edge) have shrunk by 1.

Since $e(G') = e(G) - 1$, the induction hypothesis implies that in the dual $(G')^*$, the edges dual to D' form a bond, and the sets of vertices separated by the bond are those corresponding to the faces inside and outside D. By Remark 6.1.15, the effect of contracting e in G was to delete e^* from G^*. Since e^* joins vertices for faces that are inside and outside D, replacing it would reconnect G^*. Hence D^* forms a bond as claimed, and the sets of vertices on the two sides are as claimed.

Now consider the induction step for the converse. We assume that D^* forms a bond, so $D^* - e^*$ forms a bond in $G^* - e^*$ separating the same two vertex sets that D^* separates in G^*. By Remark 6.1.15, $G^* - e^*$ is the dual of G', and the edges of $D^* - e^*$ are the duals to D'. By the induction hypothesis, D' forms a cycle in G', and the two sides of the bond $D^* - e^*$ in $G^* - e^*$ correspond to the faces inside and outside D'. Since e^* joins vertices from these two sets, e (when we re-expand it in G) must bound faces from these two sets. With D being the boundary between two sets of faces, we can argue that D is a cycle.

6.1.20. *A plane graph is bipartite if and only if every face length is even.* A face of G is a closed walk, and an odd closed walk contains an odd cycle, so a bipartite plane graph has no face of odd length.

Conversely, suppose that every face length is even; we prove by induction on the number of faces that G is bipartite. If G has only one face, then by the Jordan Curve Theorem G is a forest and is bipartite.

If G has more than one face, then G has an edge e on a cycle. This edge belongs to two faces F_1, F_2 of even length; these faces are distinct because the cycle embeds as a closed curve, and by the Jordan Curve Theorem the regions on the inside and outside are distinct. Thus deleting e yields a combined face F whose length is the sum of the lengths of F_1 and F_2, less two for the absence of e from each. Hence F has even length. Lengths of other faces remain the same. Thus every face of $G - e$ has even length, and we apply the induction hypothesis to conclude that $G - e$ is bipartite.

To show that G also is bipartite, we replace e. Since F_1 has even length, there is an odd walk in $G - e$ connecting the endpoints of e, so they lie in opposite parts of the bipartition of $G - e$. Hence when we add e to return to G, the graph is still bipartite.

(Comment: Since we deleted one edge to obtain $G - e$, we could phrase this as induction on $e(G)$. Then we must either put all forests into the basis step or consider the case of a cut-edge in the induction step.)

6.1.21. *A set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*.* Since $(G^*)^* = G$ when G is connected, it suffices to prove one direction of the equivalence; the other direction is the same statement applied to G^*. Let T be a spanning tree of G, where G has n vertices and f faces. Let T^* be the spanning subgraph of G^* consisting of the duals of the remaining edges; T^* has f vertices.

Proofs 1, 2, 3. (Properties of trees). It suffices to prove any two of (1) T^* has $f - 1$ edges, (2) T^* is connected, (3) T^* is acyclic.

(1) By Euler's Formula, $e(G) = n + f - 2$; hence if T has $n - 1$ edges there are $f - 1$ edges remaining.

(2) Since T has no cycles, the edges dual to T contain no bond of G^* (by Theorem 6.1.14). Hence T^* is connected.

(3) Since T is spanning and connected, the remaining edges contain no bond of G. Thus T^* contains no cycle in G^* (by Theorem 6.1.14 for G^*).

Proof 4 (extremality and duality). A spanning tree of a graph is a minimal connected spanning subgraph. "Connected" is equivalent to "omits no bond" (see Exercise 4.1.29). Hence the remaining edges form a maximal subgraph containing no bond. By Theorem 6.1.14, the duals of the remaining edges form a maximal subgraph of G^* containing no cycle. A maximal subgraph of G^* containing no cycle is a spanning tree of G^*.

Proof 5 (induction on the number of faces). If G has one face, then G is a tree, $G^* = K_1$, and T^* is empty and forms a spanning tree of G^*. If G has more than one face, then G is not a tree, and hence G has an edge e not in the given tree T. Since e lies on a cycle (in $T + e$) and is not a cut-edge, $G - e$ is a connected plane graph with one less face. Let $G' = G - e$.

The induction hypothesis implies that the duals of $E(G') - E(T)$ form a spanning tree in $(G')^*$. Note that $(G - e)^* = G^* \cdot e^*$; we obtain the dual of G' by contracting the edge dual to e in G^*. Returning to G keeps e^* in $E(G) - E(T)$, so what happens to the duals of the edges outside T is that the vertex of $(G - e)^*$ representing the two faces that merged when e was deleted splits into two vertices joined by e^*. This operation turns a tree into a tree with one more vertex, and it has all the vertices of G^*, so it is a spanning tree.

6.1.22. *The weak dual of an outerplane graph is a forest.* A cycle in the dual graph G^* passes through faces that surround a vertex of G. When every vertex of G lies on the unbounded face, every cycle of G^* therefore passes through the vertex v^* of G^* that represents the unbounded face in G. Hence $G^* - v^*$ is a forest when G is an outerplane graph.

6.1.23. *Directed plane graphs.* In following an edge from tail to head, the dual edge is oriented so that it crosses the original edge from right to left.

a) If D is strongly connected, then D^* has no directed cycle. Such a cycle C^* encloses some region R of the plane. Let S be the set of vertices of D corresponding to the faces of D^* contained in R. Since C^* has a consistent orientation, the construction implies that all the edges of D corresponding to C^* are oriented in the same direction across C^* (entering R or leaving R). This contradicts the hypothesis that D is strongly connected.

b) If D is strongly connected, then $\delta^-(D^*) = \delta^+(D^*) = 0$. A finite acyclic directed graph has $\delta^- = \delta^+ = 0$, because the initial vertex of a maximal directed path can have no entering edge, and the terminal vertex of such a path can have no exiting edge.

c) If D is strongly connected, then D has a face on which the edges form a clockwise directed cycle and a face on which the edges form a counter-clockwise directed cycle. A vertex of D^* with indegree 0 corresponds to a face of D on which the bounding edges must form a clockwise directed cycle, and a vertex of D^* with outdegree 0 corresponds to a face of D on which the edges must form a counter-clockwise directed cycle.

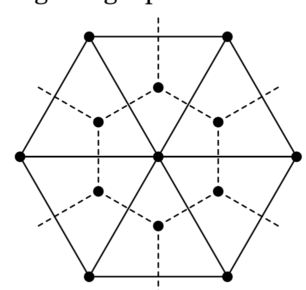
6.1.24. *Alternative proof of Euler's Formula.*

a) Faces of trees. Given a planar embedding of a tree, let x, y be two points of the plane not in the embedding. If the segment between them does not intersect the tree, then x and y are in the same face. If the segment does intersect the tree, then we create a detour for it closely following the embedding. Induction on the number of vertices yields a precise proof that this is possible. Using the detour yields a polygonal x, y-path that does not cross the embedding, so again x and y are in the same face.

b) Euler by edge-deletion. Euler's formula states that for a connected n-vertex plane graph with m edges and f faces, $n - m + f = 2$. If every edge of such a graph is a cut-edge, then the graph is a tree. This implies $m = n - 1$ and $f = 1$, in which case the formula holds. For an induction on e, we need only consider graphs that are not trees in the induction step. Such a graph G has an edge that is not a cut-edge. If e lies on a cycle, then both the interior and the exterior of the cycle have e on their boundary, and hence e is on the boundary of two faces. Therefore, deleting e reduces the number of faces by one but does not disconnect G. By the induction hypothesis, $n - (m - 1) + (f - 1) = 2$, and hence also $n - m + f = 2$.

6.1.25. *Every self-dual plane graph with n vertices has $2n - 2$ edges.* If G is isomorphic to G^*, then G must have the same number of vertices as faces. Euler's formula then gives $n - e + n = 2$ (and hence $e = 2n - 2$) if G is connected. Every self-dual graph is connected, because the dual of any graph contains a path to the vertex for the outside face of the original.

For every $n \geq 4$, the n-vertex "wheel" is self-dual. This is a cycle on $n - 1$ vertices, plus an nth vertex joined to all others. The triangular faces becomes a cycle, and each is adjacent to the remaining face; this is the same description as the original graph.



6.1.26. *The maximum number of edges in a simple outerplanar graph of order n is $2n - 3$.* For the lower bound, we provide a construction. A simple cycle on n vertices together with the chords from one vertex to the $n-3$ vertices not adjacent to it on the cycle forms an outerplanar graph with $2n-3$ edges. For the upper bound, we give three proofs.

a) (induction on n). When $n=2$, such a graph has at most 1 edge, so the bound of $2n-3$ holds. When $n>2$, recall from the text that every simple outerplanar graph G with n vertices has a vertex v of degree at most two. The graph $G'=G-v$ is an outerplanar graph with $n-1$ vertices; by the induction hypothesis, $e(G')\le 2(n-1)-3$. Replacing v restores at most two edges, so $e(G)\le 2n-3$.

b) (using Euler's formula). The outer face in an outerplanar graph has length at least n, since each vertex must be visited in the walk traversing it. The bounded faces have length at least 3, since the graph is simple. With $\{f_i\}$ denoting the face-lengths, we have $2e(G)=\sum f_i = n+3(f-1)$, where f is the number of faces. Substituting $f=e-n+2$ from Euler's formula yields $2e=n+3(e-n+1)$, or $e(G)=2n-3$. (Comment: If one restricts attention to a maximal outerplanar graph, then equality holds in both bounds: the outer face is a spanning cycle, and the bounded faces are triangles.)

c) (graph transformation). Add a new vertex in the outer face and an edge from it to each vertex of G. This produces an $n+1$-vertex planar graph G' with n more edges than G. Since $e(G')\le 3(n+1)-6$ edges, we have $e(G)\le 3(n+1)-6-n=2n-3$.

Comment: If G has no 3-cycles, then the bound becomes $(3n-4)/2$.

6.1.27. *A 3-regular plane graph G with each vertex incident to faces of lengths 4,6, and 8 has 26 faces.* Let n be the number of vertices in the graph. Since each vertex is incident to one face of length 4, one face of length 6, and one face of length 8, there are n incidences of vertices with faces of each length. Since every face of length l is incident with l vertices, there are thus $n/4$, $n/6$, and $n/8$ faces of lengths $4,6,8$, respectively. Hence there are $n(\frac14+\frac16+\frac18)$ faces.

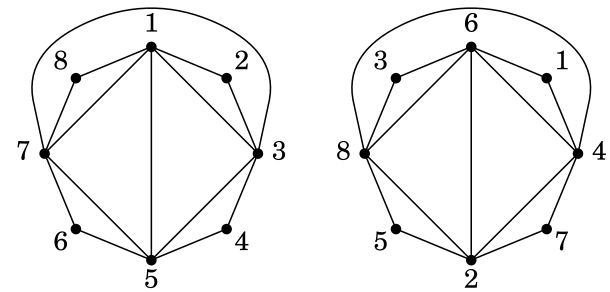
Also, the graph is 3-regular, it has $3n/2$ edges. By Euler's formula, $n-\frac32 n+n\frac{13}{24}=2$. Multiplying by 24 yields $(-12+13)n=48$, so $n=48$. Hence the number of faces is $48\frac{13}{24}$.

6.1.28. *When m chords with distinct endpoints and no triple intersections form p points of intersection inside a convex region, the region is cut into $m+p+1$ smaller regions.* Form a planar graph G by establishing a vertex at each of the p points of intersection and at each endpoint of each chord. The $2m$ endpoints of chords have degree 3 in G, and the p points of intersection have degree 4. By the degree-sum formula, G has $3m+2p$ edges. Since it has $2m+p$ vertices, Euler's Formula yields $m+p+1$ as the number of bounded regions.

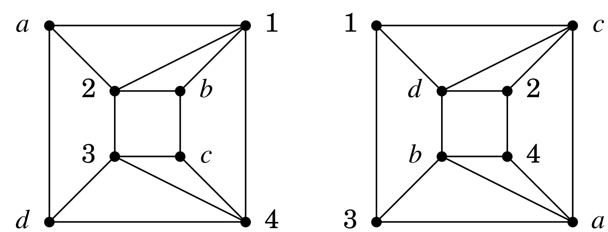
6.1.29. *Complements of planar graphs.*

a) The complement of each simple planar graph with at least 11 vertices is nonplanar. A planar graph with n vertices has at most $3n-6$ edges. Hence each planar graph with 11 vertices has at most 27 edges. Since K_{11} has 55 edges, the complement of each planar subgraph has at least 28 edges and is non-planar. For $n(G)>11$, any induced subgraph with 11 vertices shows that $\overline{G}$ is nonplanar. There is also no planar graph on 9 or 10 vertices having a planar complement, but the easy counting argument here is not strong enough to prove that.

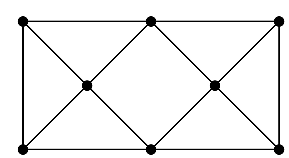
b) A self-complementary planar graph with 8 vertices.



The example below has a different degree sequence.



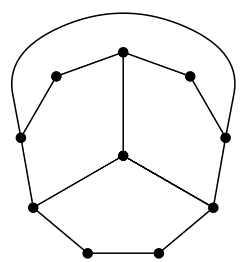
The graph below is planar and has the same degree sequence as that above, but it is not self-complementary.



6.1.30. *A 2-edge-connected n-vertex planar graph G with no cycle of length less than k has at most $(n-2)k/(k-2)$ edges.* Since adding edges will make G connected without reducing face lengths, we may assume that G is connected. Consider an embedding of G in the plane. Each face length is at least k, and each edge contributes twice to boundaries of faces. Therefore, counting the appearances of edges in faces grouped according to the e edges or according to the f faces yields $2e \ge kf$.

Since G is connected, we can apply Euler's formula, $n - e + f = 2$. Substituting for f in the inequality yields $2e \ge k(2 - n + e)$ and thus $e \le k(n-2)/(k-2)$. Note that when $k = 2$, multiple edges are available, and there is no limit on the number of edges.

The Petersen graph has 10 vertices, 15 edges, and girth 5. It has girth 5, so the size of a planar subgraph is at most $\lfloor 5 \cdot 8/3 \rfloor$, which equals 13. Since $15 > 13$, the Petersen graph is not planar, and at least two edges must be deleted to obtain a planar subgraph. The figure below shows that deleting two edges suffices.

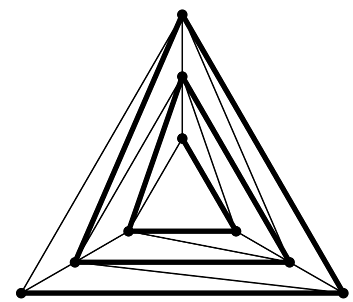


6.1.31. *The simple graph G with vertex set $v_1, \ldots, v_n$ and edge set $\{v_i v_j \colon |i - j| \le 3\}$ is a maximal planar graph.* The maximal planar graphs with n vertices are the simple n-vertex planar graphs with $3n - 6$ edges, so it suffices to prove by induction that G is a planar graph with $3n - 6$ edges. To facilitate the induction, we prove the stronger statement that G has a planar embedding *with all of $\{v_{n-2}, v_{n-1}, v_n\}$ on one face*.

Basis step: $n = 3$. The triangle has $3 \cdot 3 - 6$ edges and has such an embedding.

Induction step: $n > 3$. The graph G' obtained by deleting vertex n from G is the previous graph. By the induction hypothesis, it has $3(n-1) - 6$ edges and has an embedding with $\{v_{n-3}, v_{n-2}, v_{n-1}\}$ on one face. We add edges from v_n to these vertices to obtain G. Thus $e(G) = 3n - 6$. To embed G we place v_n inside the face of the embedding of G' having $\{v_{n-3}, v_{n-2}, v_{n-1}\}$ on its boundary. When we add the edges from n to those vertices to complete the embedding, we form a face with $\{v_{n-2}, v_{n-1}, v_n\}$ on the boundary.

The resulting embedding is illustrated below, with the bold path being $v_1, \ldots, v_n$ in order. The special face remains the outside face as the induction proceeds.



6.1.32. *If G is a maximal plane graph, and S is a separating 3-set of G^*, then $G^* - S$ has two components.* A maximal plane graph is a triangulation and has no loops or multiple edges. Hence its dual is 3-regular and 3-edge-connected. The connectivity of a 3-regular graph equals its edge-connectivity (Theorem 4.1.11). If G^* has a separating 3-set S, then it is a minimal separating set, and each vertex of S has a neighbor in each component of $G^* - S$. Extract a portion of a spanning tree in each component of $G^* - S$ that links the chosen neighbors of S. Combine these with the edges from S to the chosen neighbors. If $G^* - S$ has at least three components, then we obtain a subdivision of $K_{3,3}$. Since G^* is planar, we conclude that $G^* - S$ has at most two components.

6.1.33. *If G is a triangulation, and n_i is the number of vertices of degree i in G, then $\sum (6 - i) n_i = 12$.* A triangulation with n vertices has $3n - 6$ edges and hence degree-sum $6n - 12$. The sum $\sum i n_i$ also equals the degree-sum. Hence $6(\sum n_i) - 12 = \sum i n_i$, as desired.

6.1.34. *An infinite family of planar graphs with exactly twelve vertices of degree 5.* Begin with (at least two) concentric 5-cycles; call these "rungs". For each consecutive pair of rungs, add the edges of a 10-cycle in the region between the two 5-cycles. Inside the innermost rung, place a single vertex adjacent to the 5 vertices of the rung. Outside the outermost rung, place a single vertex adjacent to the 5 vertices of the rung. The vertices of degree 5 are the innermost vertex, the outermost vertex, and the vertices of the innermost and outermost rungs. The other vertices have degree 6. The case with exactly two 5-cycles is the icosahedron.

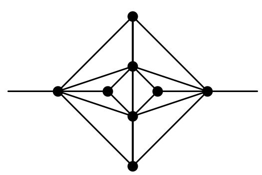
6.1.35. *Every simple planar graph with at least four vertices has at least four vertices with degree less than 6.* It suffices to prove the result for maximal planar graphs, since deleting an edge from a graph cannot make the statement become false. Let G be a maximal planar graph with n vertices.

In a maximal planar graph with at least four vertices, every vertex has degree at least 3.

The sum of the vertex degrees is $6n - 12$. Therefore, the sum of $6 - d(v)$ over the vertices with degree less than 6 is at least 12. Since $\delta(G) \geq 3$, each term contributes at most 3, so we must have at least four such vertices.

For each even value of n with $n \geq 8$, there is an n-vertex simple planar graph G that has exactly four vertices with degree less than 6. By the analysis above, such a graph must be a triangulation with four vertices of degree 3 and the rest of degree 6.

The graph sketched below has eight vertices. If we extend the two half-edges at the left and right to become a single edge, then we have the desired 8-vertex graph. To enlarge the graph, we could instead place vertices at the ends of the two half-edges, make them adjacent also to the top and bottom vertices, and extend half-edges from the top and bottom. If those half-edges become a single edge, then we have the desired 10-vertex graph. Otherwise, we can continue adding pairs of vertices to obtain the sequence of examples.

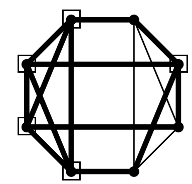


6.1.36. *If S is a set of n points in the plane such that the distance in the plane between any pair of points in S is at least 1, then there are at most $3n - 6$ pairs for which the distance is exactly 1.* If two unit-distances cross, then one of the other distances among these four points is less than 1. Hence the condition implies that the graph of unit distances is a planar graph with n vertices. A planar graph with n vertices has at most $3n - 6$ edges.

6.1.37. *Given integers $k \geq 2$, $l \geq 2$, and kl even, there is a planar graph with exactly k faces in which every face has length l.* (For $l = 1$ and $k > 2$, this does not work.) When $l > 1$ and k is even, use two vertices with degree k joined by k paths of lengths $\lceil l/2 \rceil$ and $\lfloor l/2 \rfloor$ (alternating) through vertices of degree 2. Each face is formed by a path of length $\lceil l/2 \rceil$ and a path of length $\lfloor l/2 \rfloor$. When k is odd, l is even and $\lceil l/2 \rceil = \lfloor l/2 \rfloor$, so k paths of this length suffice.

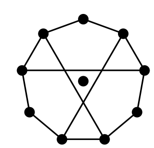
6.2. CHAR'ZN OF PLANAR GRAPHS

6.2.1. *The complement of the 3-dimensional cube Q_3 is nonplanar.* The vertices of Q_3 are the binary triples. Those with an odd number of 1s form an independent set, as do those with an even number of 1s. Each vertex is adjacent to three in the other independent set. Hence $\overline{Q}_3$ consists of two 4-cliques with a matching between them. This graph contains a subdivision of K_5 in which four branch vertices lie in one of the 4-cliques.



6.2.2. *Nonplanarity of the Petersen graph.*

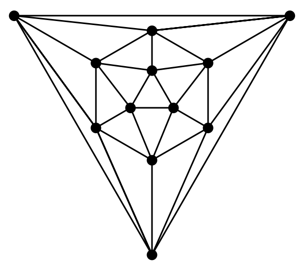
a) via Kuratowski's Theorem. Since the Petersen graph has no vertices of degree at least 4, it contains no K_5-subdivision. Below we show a $K_{3,3}$-subdivision.



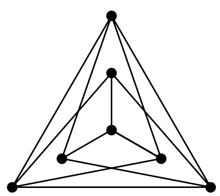
b) via Euler's Formula. To apply Euler's formula, assume a planar embedding. Since the Petersen graph has no cycle of length less than 5, each face has at least 5 edges on its boundary. Each edge contributes twice to boundaries of faces. Counting the appearances of edges in faces grouped by edges or by faces yields $2e \geq 5f$. Since the graph is connected, Euler's formula yields $n - e + f = 2$. Substituting for f in the inequality yields $2e \geq 5(2 - n + e)$, or $e \leq (5/3)(n - 2)$. For the Petersen graph, $15 \leq (5/3)8$ is a contradiction.

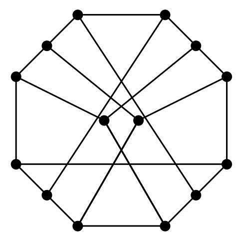
c) via the planarity-testing algorithm. We may start with any cycle. When we start with a 9-cycle C as illustrated, every C-fragment can go inside or outside, so we can pick one of the chords and put it inside. Now the other two chords can only go outside, but after embedding one of them, the remaining chord cannot go on any face. This occurs because this cycle has three pairwise crossing chords.

6.2.3. *A convex embedding.* This is the graph of the icosahedron. It is 3-connected and has a convex embedding in the plane.

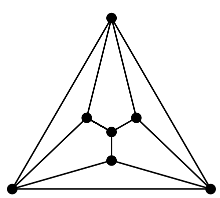


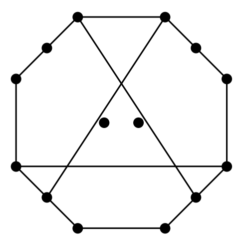
6.2.4. *Planarity of graphs.*





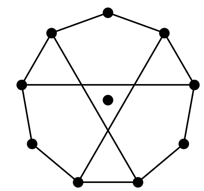
The first graph is planar; a straight-line embedding with every face convex appears below. The second graph is nonplanar. It has many subgraphs that are subdivisions of $K_{3,3}$; one is shown below.

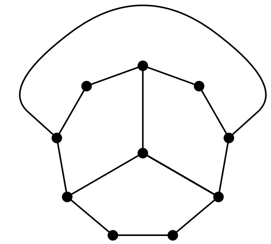




6.2.5. *The minimum number of edges that must be deleted from the Petersen graph to obtain a planar subgraph is 2.* The drawing on the left below illustrates a subdivision of $K_{3,3}$ in the Petersen graph. Since this does not use every edge of the Petersen graph, the graph obtained by deleting one edge from the Petersen graph is still nonplanar (all edges are equivalent, by the disjointness description of the Petersen graph).

Deleting two edges from the Petersen graph yields a planar subgraph as shown on the right below.

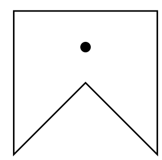


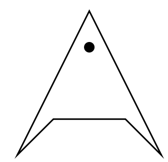


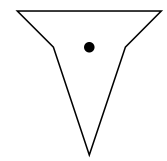
6.2.6. *Fary's Theorem.*

a) Every simple polygon with at most 5 vertices contains a point that sees every point in the polygon. In a convex polygon, by definition, the segment joining any pair of points lies entirely in the polygon. Hence every point in a convex polygon sees the entire polygon.

Proof 1. If a 4-gon is not convex, then the vertex opposite the interior reflex angle (exceeding 180 degrees) sees the entire polygon. A non-convex pentagon has one or two reflex angles, and if two they may be consecutive or not. The cases are illustrated below.







Proof 2. Triangulate the polygon by adding chords between corners that can see each other. This can be done by adding one chord to a 4-gon and by adding two to a 5-gon, with cases as illustrated above. The resulting triangles have one common vertex. Since a corner of a triangle sees the entire triangle, the common corner sees the entire region.

b) Every planar graph has a straight-line embedding. By induction on $n(G)$, we prove the stronger statement that the edges of any plane graph G can be "straightened" to yield a straight-line embedding of G without changing the order of incident edges at any vertex. The statement is true by inspection for $n(G) \le 3$.

For $n(G) \ge 4$, we may assume that G is a triangulation, since any plane graph can be augmented to a maximal plane graph, and deleting extra edges in a straight-line embedding of the maximal planar supergraph yields a straight-line embedding of the original graph. Every planar graph has a vertex of degree at most 5; let x be such a vertex in G.

Since G is a triangulation, the neighborhood of G is a cycle C, and $G - x$ has C as a face boundary. By the induction hypothesis, $G - x$ has a straight-line embedding with C as a polygonal face boundary. By part (a), we can place x at a point inside C and draw straight lines from x to all vertices of C without crossings.

6.2.7. *G is outerplanar if and only if G contains no subdivision of K_4 or $K_{2,3}$.* Let $G' = G \vee K_1$ denote the graph obtained by from G by adding a single vertex x joined to all vertices of G. Then G has an embedding with all vertices on a single face $\Leftrightarrow$ G' is planar $\Leftrightarrow$ G' has no subdivision of K_5 or $K_{3,3}$ $\Leftrightarrow$ G has no subdivision of K_4 or $K_{2,3}$.

Additional details for these statements of equivalence:

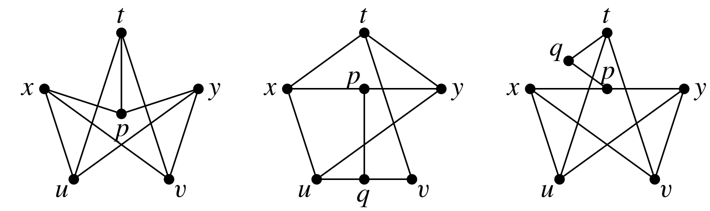
1) If G is outerplanar, then we place x in the unbounded face of an outerplanar embedding of G and join it to all vertices on the face to obtain a planar embedding of G'. Conversely, if G' is planar, then it has a planar embedding in which x lies on the unbounded face. Deleting x from this embedding yields an outerplanar embedding of G, because it has an unobstructed curve from each vertex to the point that had been occupied by x and is now in the unbounded face.

2) Kuratowski's Theorem.

3) If G has a subdivision of K_4 or $K_{2,3}$, then adding x as a additional branch vertex yields a subdivision of K_5 or $K_{3,3}$ in G'. Conversely, if G' has a subdivision F of K_5 or $K_{3,3}$, then deleting x destroys at most one branch vertex or one path of F, which leaves a subdivision of K_4 or $K_{2,3}$ in G.

6.2.8. *Every 3-connected graph with at least 6 vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$.* Let H be a K_5-subdivision in G, with branch vertices x, y, t, u, v. If H itself has only five vertices, then G has another vertex p, and G has a $p, V(H)$-fan of size 3. By symmetry, we may assume that the paths of the fan arrive at x, y, t. Then G has a subdivision of $K_{3,3}$ with branch vertices x, y, t in one partite set and p, u, v in the other partite set.

If H has more than five vertices, then by symmetry we may assume that the x, y-path P in H has length at least two. Since G is 3-connected, $G - \{x, y\}$ has a shortest path Q from $V(P) - \{x, y\}$ to $V(H) - V(P)$. Let the endpoints of Q be p on P and q in $H' = H - V(P)$. If q is on the cycle in H' through t, u, v, then by symmetry we may assume q is on the branch path between u and v and not equal to u. In $H \cup Q$ we now have a subdivision of $K_{3,3}$ with branch vertices x, y, q in one partite set and p, t, u in the other partite set. On the other hand, if q is not on the cycle through t, u, v, then by symmetry we may assume q is on the x, t-path in H' (and not equal to x). Now $H \cup Q$ has a subdivision of $K_{3,3}$ with branch vertices x, y, t in one partite set and p, u, v in the other partite set.



6.2.9. *For $n \geq 5$, the maximum number of edges in a simple planar n-vertex graph not having two disjoint cycles is $2n - 1$.* For the construction, begin with a copy of P_3 and $n - 5$ isolated vertices. Add two vertices x, y adjacent to all of these and each other. In a set of pairwise-disjoint cycles, at most one cycle can avoiding using both x and y, so no two cycles are disjoint. The number of edges is $2 + 2(n - 2) + 1 = 2n - 1$.

For the upper bound, we use induction on n. Basis step ($n = 5$): There is no 5-vertex planar graph with 10 edges, so the bound holds.

Induction step ($n \geq 6$): We need only consider a planar graph G with exactly $2n$ edges and no disjoint cycles. If any vertex has degree at most 2, then we delete it and apply the induction hypothesis to the smaller graph. Hence $\delta(G) \geq 3$. Since G is planar, $e(G) \geq 2n - 4$ forces a triangle on some set $S \subseteq V(G)$. Since G does not have disjoint cycles, $G - S$ is a forest H.

If H has three isolated vertices, then $\delta(G) \geq 3$ yields a copy of $K_{3,3}$ with S as a partite set. Hence H has a nontrivial component.

Main case. If x, y are vertices in a nontrivial component of H, and z is a vertex of H not on the unique x, y-path, and z has two neighbors in S other than a vertex of $N(x) \cap N(y)$, then we form one cycle using the x, y-path in H and a vertex of $N(x) \cap N(y)$, and we form a second cycle using z and the rest of S.

Any two vertices of degree 1 in H have a common neighbor in S. If H has an isolated vertex z, then using two leaves x, y from a nontrivial component of H yields the main case. Hence H has no isolated vertex.

Suppose that H has a component with at least three leaves x, y, z. If x and y both have a neighbor in S outside $N(z)$, then the main case occurs. Otherwise, symmetry yields $N(y) \cap S = N(z) \cap S$, and the main case occurs unless x, y, z all have the same two neighbors in S. Now G contains a subdivision of $K_{3,3}$ with one partite set being $\{x, y, z\}$ and the other consist of their two common neighbors in S and the vertex that is the common vertex of the x, y-, y, z-, and z, x-paths in H. Hence every component of H is a nontrivial path.

If any component of H has endpoints with a common neighbor in S distinct from a common neighbor of the endpoints of another component, then we obtain two disjoint cycles. Hence there is a single vertex $t \in S$ that is adjacent to all endpoints of components in H. In each component the two ends have distinct second neighbors in S; otherwise $n(G) \geq 6$ yields the main case.

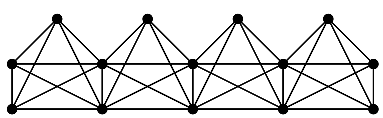
If H has at least two components, we now form one cycle using one component of H plus t and another cycle using another component of H plus the rest of S. Hence H is a single path.

If any internal vertex u of H has a neighbor w in S other than t, then let v be the leaf of H that also neighbors w. We now obtain one cycle using w and the u,v-path in H, and we obtain another cycle using the other endpoint of that component of H plus $S-\{w\}$. Hence every internal vertex in H is adjacent only to t in S.

We now have determined H exactly. Every cycle in H contains t or avoids only t. In fact, G is the wheel $C_{n-1} \vee K_1$, where K_1 is the vertex t. However, this graph has only $2n-2$ edges. This final contradiction completes the proof.

6.2.10. *Simple n-vertex graphs containing no $K_{3,3}$-subdivision.* Let $f(n)$ be the maximum number of edges in such a graph.

a) If $n-2$ is divisible by 3, then $f(n) \geq 3n-5$. We form G by pasting together $(n-2)/3$ copies of K_5 as shown. Since $K_{3,3}$ is 3-connected, a subdivision of $K_{3,3}$ cannot have branch vertices in different S-lobes when $|S|=2$. This confines the branch vertices to a single S-lobe and yields an inductive proof that this graph has no $K_{3,3}$-subdivision.



b) $f(n) = 3n-5$ when $n-2$ is divisible by 3, and otherwise $f(n) = 3n-6$ (for $n \geq 2$). Note that $f(n) \geq 3n-6$ for all n by using maximal planar graphs. For the upper bound, we use induction on n, checking the small cases ($2 \leq n \leq 5$) by inspection.

If $e(G) \geq 3n-5$, then G is nonplanar. By Kuratowski's Theorem, G contains a subdivision of K_5 or $K_{3,3}$. If G is 3-connected, then a subdivision of K_5 also yields a subdivision of $K_{3,3}$, by Exercise 6.2.8. Hence we may assume that G has a separating 2-set S. We avoid a $K_{3,3}$-subdivision in G if and only if each S-lobe *with the addition of the edge joining the vertices of* S has no such subdivision. Since we are maximizing $e(G)$, we may assume that this edge is present in G.

Now the number of edges and existence of $K_{3,3}$-subdivisions is unaffected by how we add the S-lobes. If there are more than two, then we can paste one onto an edge in one of the other S-lobes and maintain the same properties. Hence we may assume that there are only two S-lobes.

Let the two S-lobes have n_1 and n_2 vertices, repectively. The induction hypothesis yields $e(G) \leq f(n_1) + f(n_2) - 1$. Since $n_1 + n_2 = n+2$ and we count the shared edge only once, this total is $3n-c$, where c depends on the congruence classes of n and n_1 modulo 3. If n_1 and n_2 are congruent to 2 modulo 3, then the sum is $3n_1 - 5 + 3n_2 - 5 - 1 = 3n-5$. In other cases, at least one of the contributions is smaller by one. Hence $3n-5$ is achievable only when $n \equiv 2 \pmod 3$, and otherwise $3n-6$ is an upper bound.

Comment. When $n-2$ is divisible by 3, the *only* way to achieve the bound is by pasting together copies of K_5 at edges.

6.2.11. *If $\Delta(H) \leq 3$, then a graph G contains a subdivision of H if and only if G contains a subgraph contractible to H.* A H-subdivision in G is a subgraph of G contractible to H, so the condition is necessary.

For the converse, it suffices to show that if H' is contractible to H then H' contains a subdivision of H.

Proof 1. In contracting H' to H, each vertex of H arises by contracting the edges in some connected subgraph of H'. Let T_v be a spanning tree of the subgraph that is contracted to v. Since $\Delta(H) \geq 3$, at most three edges of H depart from T_v. Let T_v' be the union of the paths in T_v that connect the vertices of T_v from which edges of H depart. In particular, if x, y, z are the vertices of departure for the paths leaving T_v, we can let T_v' be the x,y-path P in T_v plus the path in T_v from z to P. Discard from H' all edges except those of each T_v' and those that in the paths that contract to edges of H. The remaining graph is a subdivision of H in H'.

Proof 2. An alternative proof follows the process from H (that is, $K_{3,3}$) itself back to G, undoing the sequence of deletions and contractions (in the reverse order), keeping only a graph that is a subdivision of H and at the end is H', a subdivision of H contained in G. Deletions are undone by doing nothing (don't add the edge back). Undoing a contraction is splitting a vertex v. At most three edges incident to v have been kept in the current subdivision of H. If u and w are the adjacent vertices resulting from the split, then at least one of them, say w, inherits at most one of these important edges. Keeping that edge and the edge uw allows u to become the vertex playing the role of v in the subdivision, with the same number of paths entering as entered v, going to the same places. If a path went off along an edge now incident to w, then that path is one edge longer.

Comment. The claim fails for graphs with maximum degree 4. Consider the operation of *vertex split*, which replaces a vertex x with two new adjacent vertices x_1 and x_2 such that each former neighbor of x is adjacent to x_1 or x_2. Contracting the edge x_1x_2 in the new graph produces the original graph. In applying a split to a vertex x of degree 4, the two new vertices may each inherit edges to two of the neighbors of x and thus wind up with degree 3. If H has maximum degree 4, then applying vertex splits to vertices of maximum degree can produce a graph G in which each new vertex has degree at most 3. This graph G has H as a minor, but G contain no H-subdivision since G has no vertex of degree 4.

6.2.12. *Wagner's characterization of planar graphs.* The condition is that neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and contractions of edges.

a) Deletion and contraction of edges preserve planarity. Given an embedding of G, deleting an edge cannot introduce a crossing. Also, there is a dual graph G^*. Contracting an edge e in G has the effect of deleting the dual edge e^* in G^*. In other words, $G^* - e^*$ is planar, and $G \cdot e$ is its planar dual, so $G \cdot e$ is also planar. Alternatively, one can follow the transformation that shrinks one endpoint of e continuously into the other and argue that at no point is a crossing introduced.

Since deletion and contraction preserve planarity and K_5 and $K_{3,3}$ are not planar, we cannot obtain these graphs from a planar graph by deletions and contractions. Hence the condition is necessary.

b) Kuratowski's Theorem implies Wagner's Theorem. We prove sufficiency by proving the contrapositive: if G is nonplanar, then K_5 or $K_{3,3}$ can be obtained by deletions and contractions.

If G is nonplanar, then by Kuratowski's Theorem, G contains a subdivision of K_5 or $K_{3,3}$. Every graph containing a subdivision of a graph F can be turned into F by deleting and contracting edges (delete the edges not in the subdivision, then contract edges incident to vertices of degree 2). Hence K_5 or $K_{3,3}$ can be obtained by deleting and contracting edges.

6.2.13. *G is planar if and only if every cycle in G has a bipartite conflict graph.* The condition is necessary because in any planar embedding a cycle C separates the plane into two regions, and the C-bridges embedded in each of the regions must form an independent set in the conflict graph. Conversely, if G is non-planar, then by the preceding theorem it is K_5 (with conflict graph C_5), or it has a cycle C with three crossing chords that produce a triangle in the conflict graph of C.

6.2.14. *If x and y are vertices of a planar graph G, then there is a planar embedding with x and y on the same face if and only if G has no cycle C avoiding $\{x, y\}$ such that x and y belong to conflicting C-fragments.*

If there is a cycle C such that x and y belong to conflicting C-fragments, then in every planar embedding of G, one of these fragments goes inside G and the other goes outside it. Hence in every embedding, x and y are separated by C. (This argument applies when C does not contain x or y, but it suffices to consider such cycles.)

Conversely, suppose there is no such cycle; we show that $G + xy$ is planar. If not, then $G + xy$ contains a Kuratowski subgraph using xy. If this is a $K_{3,3}$-subdivision H with xy on the path between branch vertices u and v, then x and y belong to fragments with alternating vertices of attachment on the cycle in H through the other four branch vertices. If this is a K_5-subdivision H with xy on the path between branch vertices u and v if and only if x and y belong to fragments with three common vertices of attachment on the cycle in H through the other three branch vertices. In either case, x and y belong to conflicting C-fragments for some cycle C.

6.2.15. *A cycle C in a 3-connected plane graph G is the boundary of a face in G if and only if G has exactly one C-fragment.* If G has exactly one C-fragment, then it must be embedded inside C or outside C, and the other of these regions is a face with boundary C.

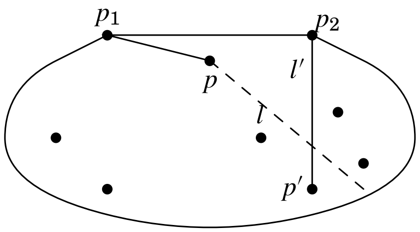
If C is a face boundary, then all C-fragments are embedded on one side of C, say the inside. This prevents two C-fragments H_1, H_2 from having alternating vertices of attachment along C. This means that there is a path P along C that contains all vertices of attachment of H_1 and none of H_2. Now the endpoints of P separate G, which contradicts the assumption of 3-connectedness.

6.2.16. *If G is an n-vertex outerplanar graph and P is a set of n points in the plane with no three on a line, then G has a straight-line embedding with its vertices mapped onto P.* It suffices to consider maximal outerplanar graphs. We prove the stronger statement that if v_1, v_2 are two consecutive vertices of the unbounded face of a maximal outerplanar graph G, and p_1, p_2 are consecutive vertices of the convex hull of P, then G has a straight-line embedding f on P such that $f(v_1) = p_1$ and $f(v_2) = p_2$.)

The statement is trivial for $n = 3$; assume $n > 3$. Let $v_1, v_2, \dots, v_n$ denote the counterclockwise ordering of the vertices of G on the outside face in a particular embedding. Let v_i be the third vertex on the triangle containing v_1, v_2.

Claim: *there is a point $p \in P - \{p_1, p_2\}$ with the two properties (a) no point of P is inside $p_1 p_2 p$, and (b) there is a line l through p that separates p_1 from p_2, meets P only at p, and has exactly $i - 2$ points of P on the side of l containing p_2.* To obtain p, we rotate the line $p_1 p_2$ about p_2 until we reach a line $l' = p_2 p'$ with $p' \in P$ such that exactly $i - 3$ points of P are separated from p_1 by l'. Among the points of $P - \{p_1, p_2\}$ in the closed halfplane determined by l' that contains p_1, let p be the point minimizing the angle $p_2 p_1 p$. By this choice, p satisfies (a), and there are at most $i - 2$ points of P on the side of $p_1 p$ containing p_2. If we rotate this line about p, then before it becomes parallel to l' it reaches a position l satisfying (b).

Let H_1 and H_2 denote the closed halfplanes determined by l containing p_1 and p_2, respectively. By the induction hypothesis, the subgraphs of G induced by $\{v_1, v_2, \ldots, v_i\}$ and $v_i, v_{i+1}, \ldots, v_n, v_1\}$ can be straight-line embedded on $H_1 \cap P$ and $H_2 \cap P$ so that v_1, v_2, v_i are mapped to p_1, p_2, p. Combining these embeddings yields a straight-line embedding of G with the desired properties.



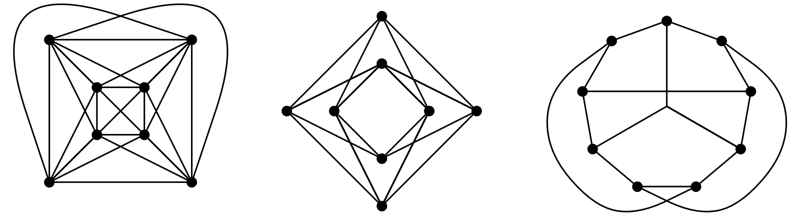
6.3. PARAMETERS OF PLANARITY

6.3.1. *A polynomial-time algorithm to properly color a planar graph G.* First find an embedding in the plane and augment to a maximal plane graph G'. Now delete a vertex v of degree at most 5. Recursively find a proper 5-coloring of $G' - v$. To extend the coloring to v, use Kempe chains if necessary to remove a color from the neighborhood of v.

6.3.2. *If every subgraph of G has a vertex of degree at most k, then G is $k+1$-colorable.* We use induction on $n(G)$. For the basis, K_1 is $k+1$-colorable whenever $k \ge 0$. For the induction step, let v be a vertex of degree at most k in a graph G with at least two vertices. By the definition of k-degenerate, every subgraph of a k-degenerate graph is k-degenerate. He the induction hypothesis yields a proper $k+1$-coloring of $G - v$. Extend the coloring to v by giving v a color that does not appear on its neighbors.

6.3.3. *Every outerplanar graph G is 3-colorable, by the Four Color Theorem.* Adding a vertex v adjacent to all of G yields a planar graph G', which is 4-colorable. A proper 4-coloring of G' restricts to a proper 3-coloring of G, because the colors used on the vertices of G must all be different from the color used on v.

6.3.4. *Crossing number of $K_{2,2,2,2}$, $K_{4,4}$, and the Petersen graph.* Let $k = \lfloor (n-2)g/(g-2) \rfloor$. The maximum number of edges in a planar n-vertex graph with girth g is k, so $\nu(G) \ge e(G) - k$ if G has girth g. This yields $\nu(K_{2,2,2,2}) \ge 6$ and $\nu(K_{4,4}) \ge 4$, and $\nu(G) \ge 2$ when G is the Petersen graph. The drawings below achieve these lower bounds.



6.3.5. *Every planar graph G decomposes into two bipartite graphs.* By the Four Color Theorem, G is 4-colorable. Let the four colors be $0, 1, 2, 3$. Let H consist of all edges of G joining a vertex of odd color with a vertex of even color. Let H' consist of all edges joining two vertices whose color has the same parity. Now H and H' are bipartite and have union G.

6.3.6. *Small planar graphs.* We use induction on $n(G)$; every graph with at most four vertices is planar. A planar graph G with at most 12 vertices has degree-sum at most $6 \cdot 12 - 12$, with equality only for triangulations. The bound is 60. Hence $\delta(G) \le 4$ unless G is a 5-regular triangulation. The only such graph is the icosahedron, which is 4-colorable explicitly.

Hence $\delta(G) \le 4$. The same conclusion holds for graphs with at most 32 edges that have more than 12 vertices: the average vertex degree is at most $64/13$, which is less than 5.

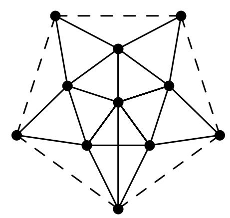
Thus we have $\delta(G) \le 4$. Let v be a vertex of minimum degree. A proper 4-coloring of G can be obtained from a proper 4-coloring of $G - v$. The case of $d(v) = 4$ uses Kempe chain arguments as in the proof in the text that a minimal 5-chromatic planar graph has no vertex of degree at most 4.

6.3.7. *A configuration H in a planar triangulation can be retrieved from the partially labeled subgraph H' obtained by labeling the neighbors of the ring vertices with their degrees and then deleting the ring vertices.* For each vertex v on the external face of H', the data $d_H(v)$ is given. Append $d_H(v) - d_{H'}(v)$ new edges at v, extending to new vertices in the extremal face. When H' is 2-connected, the new neighbors for vertices of H' appear in the same order cyclically in the outside face of H' as the vertices of H' themselves. The requirement that H' is a block is necessary for this reason; when $H' = P_3$, it is easy to construct a counterexample to the desired statement.

For each consecutive pair v, w on the external face of H', the edge vw is on the boundary of a face with a new edge from v and a new edge from w. Since each face of H is a triangle, the endpoints of these edges other than $\{v, w\}$ must merge into a single vertex to complete the face. (When $H' = K_2$, this occurs for both sides of the edge.)

Finally, pass a cycle through the resulting vertices outside H' to form the ring. This is forced, since the bounded faces must all be triangles.

6.3.8. *Configurations with ring size 5 in planar triangulations such that every internal vertex has degree at least five.* The intent was to seek a configuration with more than one internal vertex. For example:



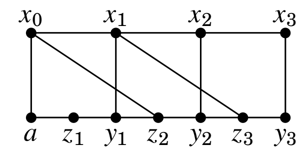
6.3.9. *A minimal non-4-colorable planar graph has no separating cycle of length at most 4.* Let G be a minimal 5-chromatic graph with a vertex cut S that induces a cycle. Let A and B be the S-lobes of G; A is the subgraph induced by S and the vertices inside the cycle, and B is the subgraph induced by S and the vertices outside the cycle. By the minimality of G, A and B are 4-colorable. If $|S| = 3$, then S receives three distinct colors in each of these colorings, and we can permute the names to agree on S.

Hence we may assume that $S = \{v_1, v_2, v_3, v_4\}$, indexed in order around the cycle. Let f and g be proper 4-colorings of $A + v_1v_3$ and $B + v_1v_3$, respectively. We may rename colors in f and g so that $f(v_i) = g(v_i) = i$ for $i \in \{1, 2, 3\}$. If $f(v_4) = g(v_4)$, then we are finished. Otherwise we may assume that $f(v_4) = 4$ and $g(v_4) = 2$. If the subgraph of A induced by vertices with colors 2 and 4 under f does not have a v_2, v_4-path, then we can interchange colors 2 and 4 on the component containing v_4 to obtain a coloring that agrees with g on S.

Otherwise, v_1 and v_3 are in different components of the subgraph of A induced by vertices with colors 1 and 3, and we may interchange colors 1 and 3 on the component containing v_3 to obtain a new proper 4-coloring f' of A that assigns colors 1,2,1,4 to v_1, v_2, v_3, v_4. Now consider $B + v_2v_4$; this planar graph also has a proper 4-coloring g'. Since g' assigns distinct colors to v_1, v_2, v_4, we may assume by renaming colors that these are 1,2,4, respectively. Now g' agrees with f on S if $g'(v_3) = 3$, and g' agrees with f' on S if $g'(v_3) = 1$. In either case, we have 4-colorings of A and B that combine to provide a proper 4-coloring of G.

6.3.10. *Triangle-free planar graphs may have independence number arbitrarily close to $n(G)/3$, so no greater lower bound can be guaranteed.* Consider the sequence of graphs G_k defined as follows: G_1 is

the 5-cycle, with vertices a, x_0, x_1, y_1, z_1 in order. For $k > 1$, G_k is obtained from G_{k-1} by adding the three vertices x_k, y_k, z_k and the five edges $x_{k-1}x_k, x_ky_k, y_kz_k, z_ky_{k-1}, z_kx_{k-2}$. The graph G_3 is shown below. Moving the edges $x_{i-2}z_i$ outside yields a planar embedding.



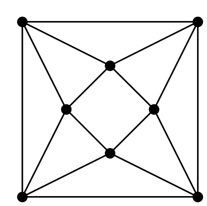
We prove by induction on k that $\alpha(G_k) = k+1 = (n(G)+1)/3$ and that, furthermore, every maximum stable set in G_k contains x_k or y_k or $\{z_k, x_{k-1}\}$. In G_1 the maximum stable sets are the nonadjacent pairs of vertices; the only one not containing x_1 or y_1 is $\{z_1, x_0\}$, so the stronger statement holds.

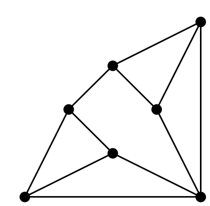
Suppose that the claim holds for G_{k-1}. A maximum stable set S in G_k uses at least one vertex not in G_{k-1}. If it uses two, then they are x_k and z_k. Since $y_{k-1}, x_{k-1}, x_{k-2} \in N(\{x_k, z_k\})$, S cannot contain a maximum stable set from G_{k-1}, and hence $|S| \le k + 1$. Furthermore, when S contains x_k it satisfies the stronger statement.

To complete the proof of the stronger statement, we must show that a stable set S of size $k+1$ in G_k that contains z_k but not x_k also contains x_{k-1}. Since $|S| = k + 1$, S must contain a stable set of size k in G_{k-1}, which contains x_{k-1} or y_{k-1} or $\{z_{k-1}, x_{k-2}\}$, by the induction hypothesis. Since z_k is adjacent to x_{k-2} and y_{k-1}, the only possibility here is $x_{k-1} \in S$, which completes the proof of the statement.

6.3.11. *For the graph G_n defined below, when n is even, every proper 4-coloring of G_n uses each color on exactly n vertices.* Let G_1 be C_4. For $n > 1$, obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to two consecutive vertices of the previous outside face.

Each two consecutive "rungs" of G_n form a subgraph isomorphic to G_2, shown below on the left. This graph is 4-chromatic but not 4-critical, since it contains the 4-chromatic graph shown on the right. Since the remaining graph after deleting any one vertex still needs four colors, every color must appear at least twice (and hence exactly twice) in each copy of G_2.





6.3.12. *Every outerplanar graph is 3-colorable.* The fact that every induced subgraph of an outerplanar graph is outerplanar yields inductive proofs.

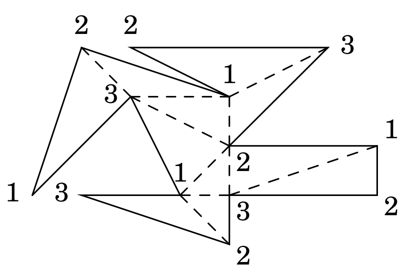
Proof 1 (induction on $n(G)$). If every edge of G is on the outside face, then every block of G is an edge or a cycle, and G is 3-colorable. Otherwise, suppose xy is an internal edge. Then $S = \{x, y\}$ is a separating set. The S-lobes of G are outerplanar; by the induction hypothesis, they are 3-colorable. Since S induces a clique, we can make these colorings agree on S, which yields a 3-coloring of G.

Proof 2 (induction on $n(G)$). Every simple outerplanar graph has a vertex of degree at most 2 (proved in the text); we can delete such a vertex x, 3-color $G - x$ by the induction hypothesis, and extend the coloring to x.

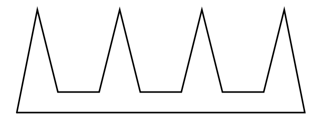
Proof 3 (prior results). Every graph with chromatic number at least 4 contains a subdivision of K_4 (Dirac's Theorem), but a graph containing a subdivision of K_4 cannot be outerplanar.

Every art gallery laid out as a polygon with n segments can be guarded by $\lfloor n/3 \rfloor$ guards so that every point of the interior is visible to some guard. The art gallery is a drawing of an n-cycle in the plane. We add straight-line segments to obtain a maximal outerplanar graph with n vertices. To do this, observe that 3-gons are already triangulated without adding segments. For $n > 3$, some corner can see some other corner across the interior of the polygon. We add this segment and proceed inductively on the two resulting polygons with fewer corners.

Consider a proper 3-coloring of the resulting maximal outerplanar graph (outerplanar graphs are 3-colorable). Since each bounded region is a triangle, its vertices are pairwise adjacent and receive distinct colors. Thus each color class contains a vertex of each triangule. Any point in a triangle, such as a corner, sees all points in the triangle. Thus guards at the vertices of a color class can see the entire gallery. Since the three classes partition the set of vertices, the smallest class has at most $\lfloor n/3 \rfloor$ elements.

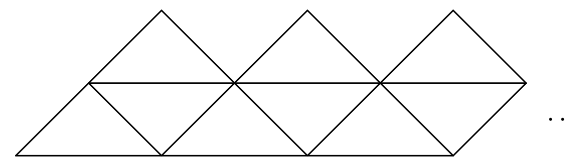


The bound of $\lfloor n/3 \rfloor$ guards is best possible. The alcoves in the polygon below require their own guards; no guard can see into more than one of them. There are $\lfloor n/3 \rfloor$ alcoves. When n is not divisible by 3, we can add the extra vertex (or two) anywhere.



6.3.13. *Every art gallery with walls whose outer boundary is an n-gon can be guarded by $\lfloor (2n-3)/3 \rfloor$ guards, where $n \ge 3$, and this is sharp.* Adding walls cannot make it easier to guard the gallery, so we may assume that the polygon is triangulated by nonintersecting chords. A guard in a doorway can see the two neighboring triangles; we use such guards and guards on the outside walls.

The bound is achieved by an art gallery of the type below.



Proof 1. The embedded n-gon plus the interior walls form a planar embedding of an outerplanar graph whose vertices are the corners; it has $n + (n - 3) = 2n - 3$ edges. Every outerplanar graph is 3-colorable (this can be proved inductively by cutting along chords formed by walls, as in Thomassen's proof of 5-choosability, or by using the existence of a vertex of degree at most 2, which can be proved inductively or by Euler's Formula.)

From a proper 3-coloring of the vertices of the outerplanar graph, 3-color the edges of the graph by assigning to each edge the color *not used* on its endpoints. Now each triangle has each color appearing on its incident edges. If we put guards at the edges occupied by the least frequent color, then each room is guarded, and we have used at most $\lfloor (2n-3)/3 \rfloor$ guards.

Proof 2. Again triangulate the region to obtain an outerplanar graph G. In the dual graph G^*, let v denote the vertex corresponding to the unbounded face of G. The graph $G^* - v$ is a tree with $n - 2$ vertices and maximum degree at most 3. Each edge corresponds to a guard in a doorway, so an edge cover (a set of edges covering the vertices) corresponds to a set of guards in doorways that together can see all the rooms.

It suffices to show that a tree T with $n - 2$ vertices and maximum degree at most 3 has an edge cover with at most $(2n-3)/3$ edges, for $n \ge 4$. We study $n \in \{4, 5, 6\}$ explicitly. For larger n, consider the endpoint x of a longest path in T. By the choice of x, its neighbor y has one non-leaf neighbor and at most two leaf neighbors. We use the pendant edges at y in the edge cover and delete y and its leaf neighbors to obtain a smaller tree

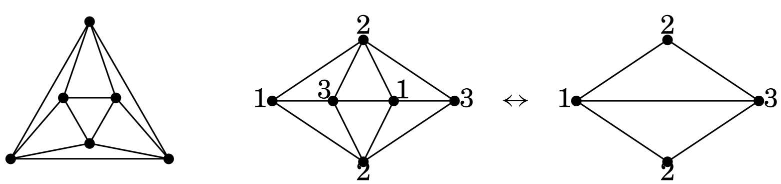
T'. We have placed k edges in the cover and deleted $k+1$ vertices, where $k \in \{1,2\}$. Using the induction hypothesis and $k \le 2$, we obtain an edge cover of size at most

$$\frac{2(n-k-1)-3}{3} + k = \frac{2n-3}{3} - \frac{2(k+1)}{3} + k = \frac{2n-3}{3} + \frac{k-2}{3} \le \frac{2n-3}{3}.$$

6.3.14. *A maximal planar graph is 3-colorable if and only if it is Eulerian.* Let G be a maximal plane graph; G is connected, and every face is a triangle. Suppose that G is 3-colorable. The three colors $\{1,2,3\}$ appear on each face, in order clockwise or counterclockwise. When two faces share an edge, the colors appear clockwise around one face and counterclockwise around the other. This defines a proper 2-coloring of the faces of G, using the colors "clockwise" and "counterclockwise". Hence G^* is bipartite. The degree of a vertex in a connected plane graph G equals the length of the corresponding face in G^*. Since G^* has no odd cycle, G has even degrees and is Eulerian.

Conversely, suppose that G is Eulerian, meaning that each vertex has even degree. **Proof 1**. We use induction on $n(G)$ to obtain a proper 3-coloring. The smallest Eulerian triangulation is K_3, which is 3-colorable. A 2-valent vertex in a larger simple triangulation would belong to two triangles, forcing a double edge. Since G is Eulerian, we may thus assume that $\delta(G) \ge 4$. Since K_5 is nonplanar, the next smallest Eulerian triangulation is the octahedron, with six vertices of degree 4; this is 3-colorable, as illustrated below.

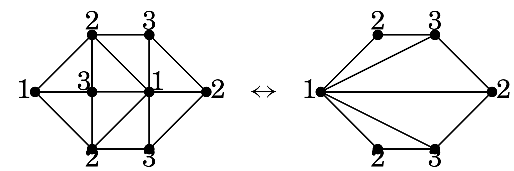
For the induction step, suppose that $n(G) \ge 6$. Since every planar graph has a vertex of degree less than 6, G has a 4-valent vertex. If G has a triangle T of 4-valent vertices, then G the neighbors of T induce a 3-cycle containing T, as in the octahedron. Deleting T reduces the degrees of the neighboring vertices by 2 each, so we can apply the induction hypothesis to the resulting subgraph G'. The coloring assigns distinct colors to the neighbors of T, and this proper coloring extends also to T.



Hence we may assume that when G has two adjacent 4-valent vertices x, y, their two common neighbors a, b have degree greater than 4. Suppose G has adjacent 4-valent vertices $\{x, y\}$, with u and v being the fourth neighbors of x and y, respectively. Form G' by deleting $\{x, y\}$ and adding the edge uv. Because $d(a), d(b) > 4$, u and v are not already adjacent. All vertices

still have even degree; hence G' is an Eulerian triangulation. We apply the induction hypothesis and extend the resulting coloring to a coloring of G, as indicated above.

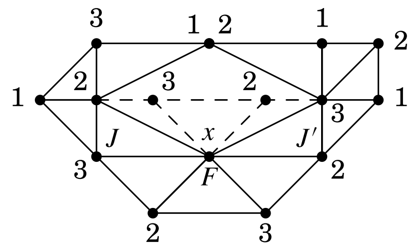
Finally, suppose that G has no adjacent 4-valent vertices. Choose a 4-valent vertex x with neighbor y, and define a, b, u as before. Form G' by deleting $\{x, y\}$ and adding edges from u to all of $N(y) - N(x)$. Because $d(a), d(b) > 4$, z is not already adjacent to any vertex of $N(y) - N(x)$. All vertices still have even degree; hence G' is an Eulerian triangulation. We apply the induction hypothesis and extend the resulting coloring to a coloring of G, as indicated below.



Proof 2. All faces are triangles; we start with an arbitrary 3-coloring on some face F. The color of the remaining vertex on any neighboring face is forced. We claim that iterating this yields a proper 3-coloring f. Otherwise, a contradiction is reached at some vertex v. This means there are two paths of faces from F, distinct after some face F' (we start at F' to obtain disjoint dual paths), that reach v but assigning different colors to v. Let C be the cycle enclosing the faces on these two paths and the regions inside them. Choose an example in which C encloses the smallest possible number of vertices.

The contradiction cannot arise when C encloses only one vertex x. In this case, the faces causing the conflict are only those incident to x, and C is the cycle through the neighbors of x. Since $d(x)$ is even, the colors alternate on C when following the path of faces, and there is no conflict.

We obtain a contradiction by finding such a cycle enclosing fewer vertices. Since the initial face starts two distinct paths of faces, one of its vertices (x below) must be enclosed by C and not on C. Together, the two paths contain a portion of the faces containing x, say from J to J' around x. We replace these by the other faces involving x, but keeping J, J'. Because $d(x)$ is even, the coloring forced on J by $f(V(F))$ forces onto J' the same coloring that $f(V(F))$ forced onto J' directly. From J, J' outward, the paths of faces lead to the same conflict as before. Hence we can start with one of the inner faces involving x and obtain a conflict using paths that enclose fewer vertices than before (x is no longer inside).

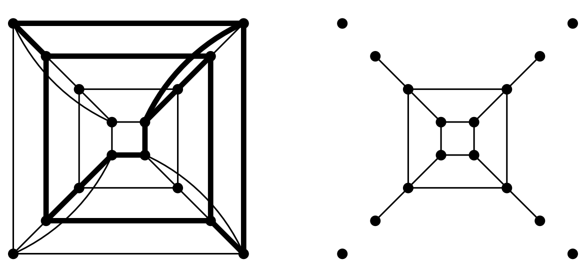


6.3.15. *The vertices of a simple outerplanar graph can be partitioned into two sets so that the subgraph induced by each set is a disjoint union of paths.* Let one set be the set of vertices with even distance from a fixed vertex u, and let the other set be the remainder; call these "color classes". Since no adjacent vertices can have distance from x differing by more than 1, each component of the graph induced by one color class consists of vertices with the same distance from u. Let H be such a component.

To show that H is a path, it suffices to show that H has no cycle and has no vertex of degree at least 3. Given three vertices x_1, x_2, x_3 in H, let P_i be a shortest x_i, u-path in G. Since x_1, x_2, x_3 have the same distance from u, each P_i has only x_i in H. Also, since the paths eventual merge, $P_1 \cup P_2 \cup P_3$ contains a subdivision of a claw; call this F (note that F need not contain u, as the paths may meet before reaching u).

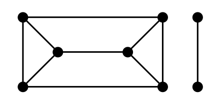
If H contains a cycle C, let x_1, x_2, x_3 be three vertices on C. Now $F \cup C$ is a subdivision of K_4. If H contains a vertex w of degree 3, let x_1, x_2, x_3 be neighbors of w. Now F together with the claw having center w and leaves x_1, x_2, x_3 is a subdivision of $K_{2,3}$. Since an outerplanar graph has no subdivision of K_5 or $K_{2,3}$, H is a path.

6.3.16. *The 4-dimensional cube Q_4 is nonplanar and has thickness 2.* The graph is isomorphic to $C_4 \square C_4$. On the left below, we show a subdivision of $K_{3,3}$ in bold. The graph is also isomorphic to $Q_3 \square K_2$, consisting of two 3-cubes with corresponding vertices adjacent. Taking one of the 3-cubes and the edges to the other 3-cube from one of its 4-cycles yields a planar graph that is isomorphic to the subgraph consisting of the remaining edges.



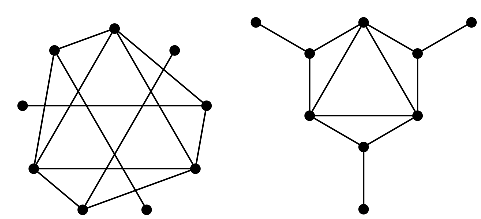
6.3.17. *Thickness. a) The thickness of K_n is at least $\lfloor (n+1)/6 \rfloor$.* Each planar graph used to form G has at most $3n(G)-6$ edges, so the thickness of G is at least $e(G)/[3n(G)-6]$. For $G = K_n$, this yields $\lceil n(n-1)/[6(n-2)] \rceil$, since thickness must be an integer. We compute $n(n-1)/(n-2) = n(1 + 1/(n-2)) = n + n/(n-2) = n+1+2/(n-2)$. Since $\lceil x/r \rceil = \lfloor (x+r-1)/r \rfloor$, the thickness is at least $\lceil [n+1+2/(n-2)]/6 \rceil = \lfloor [n+6+2/(n-2)]/6 \rfloor = \lfloor (n+7)/6 \rfloor$. The last equality holds because there is no integer between these two arguments to the floor function. (Comment: this lower is the exact answer except for $n = 9, 10$.)

b) A self-complementary planar graph with 8 vertices. See the solution to Exercise 6.1.29 for examples of self-complementary planar graphs with 8 vertices. To show that the thickness of K_8 is 2, it suffices to present any 8-vertex planar graph with a planar complement. Many examples are possible. A natural approach is to use a triangulation to eliminate the most possible edges from the complement. An example is $C_6 \vee 2K_1$, putting one vertex inside and one vertex outside a 6-cycle. The complement is $(C_3 \square K_2) + K_2$, which is planar as shown below.



Since K_8 is nonplanar, these examples show that K_8 has thickness 2, and that in fact K_5, K_6, K_7 also have thickness 2. The bound in (a) implies that the thickness of K_n is at least 3 when $n \geq 11$, which is the same as saying there is no planar graph with more than 10 vertices having a planar complement. In fact, there is also no planar graph on 9 or 10 vertices having a planar complement, but the counting argument in (a) is not strong enough to show that.

6.3.18. *Decomposition of K_9 into three pairwise-isomorphic planar graphs.* View the vertices as the congruence classes of integers modulo 9. Group them into triples by their congruence class modulo 3. The graph below consists of a triangle on one triple, a 6-cycle between that and a second triple, and a matching from the second triple to the third. Rotating the picture on the left yields three pairwise isomorphic graphs decomposing K_9. The drawing on the right shows that the graph is planar.



6.3.19. *The chromatic number of the union of two planar graphs is at most 12.* Let G be a graph with n vertices that is the union of two planar graphs H_1 and H_2. For coloring problems, we may assume that G, H_1, H_2 are simple. We claim that G has a vertex of degree at most 11. Since each H_i has at most $3n-6$ edges, G has at most $6n-12$ edges. The degree-sum in G is at most $12n-24$, and by the pigeonhole principle G has a vertex of degree at most 11.

It now follows by induction on $n(G)$ that $\chi(G) \le 12$. This holds trivially for $n(G) \le 12$. For $n(G) > 12$, we delete a vertex x of degree at most 11 to obtain G'. Since $G' = (H_1 - x) \cup (H_2 - x)$, we can apply the induction hypothesis to G' to obtain a proper 12-coloring. Since $d(x) \le 11$, we can replace x and give it one of these 12 colors to obtain $\chi(G) \le 12$.

The chromatic number of the union of two planar graphs may be as large as 9. The graph $C_5 \vee K_6$ has chromatic number 9, since the three colors on the 5-cycle must be distinct from the six colors on the 6-clique, and such a coloring is proper. It thus suffices to show that $C_5 \vee K_6$ is the union of two planar graphs. Since $C_5 \vee K_6$ contains K_8 and $K_9 - e$ (for some edge e) as induced subgraphs, it is reasonable to start with one of these and then try to add the missing vertices with their desired neighbors to the two graphs.

Let the vertices of the C_5 be a, b, c, d, e in order, and let the vertices of the K_6 be $1, 2, 3, 4, 5, 6$. Exercise 6.1.29 requests a self-complementary graph with 8 vertices; in other words, an expression of K_8 as the union of two planar graphs.

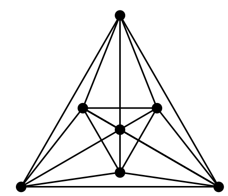
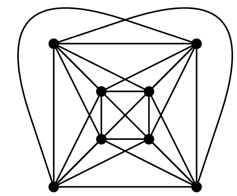
6.3.20. *Thickness of $K_{r,s}$.* Let X, Y be the partite sets of $K_{r,s}$, with $|X| = r$. The graph $K_{2,s}$ is planar. Taking all of Y and two vertices from X yields a copy of $K_{2,s}$. Taking two vertices at a time from X thus yields $r/2$ planar subgraphs decomposing $K_{r,s}$.

Since $K_{r,s}$ is triangle-free, a planar subgraph of $K_{r,s}$ has at most $2(r+s)-4$ edges. Thus the number of planar subgraphs needed in a decomposition is at least $\frac{rs}{2r+2s-4} = \frac{r}{2+(2r-4)/s}$. As s increases, the denominator decreases and the quotient increases. Thus when $s > (r-2)^2/2$, the value of the lower bound is larger than the result of setting $s = (r-2)^2/2$ in the formula. Since $(2r-4)/s = 4/(r-2)$ when s has this value,our lower bound is bigger than $\frac{r}{2+4/(r-2)} = \frac{r(r-2)}{2r-4+4} = r/2 - 1$. Since the crossing number is an integer bigger than $r/2-1$ when r is even and $s > (r-2)^2/2$, it is at least $r/2$. Hence our construction is optimal.

6.3.21. *Crossing number of $K_{1,2,2,2}$.* This simple graph has 7 vertices and 18 edges. The maximum number of edges in a simple planar graph with 7 vertices is $3 \cdot 7 - 6 = 15$. Hence in any drawing of this graph, a maximal plane subgraph has at most 15 edges, and the remaining edges each yield at

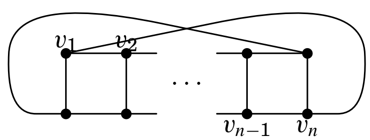
least one crossing with the maximal plane subgraph. Hence $\nu(K_{1,2,2,2}) \ge 3$, and the drawing of this graph on the left below shows that equality holds.

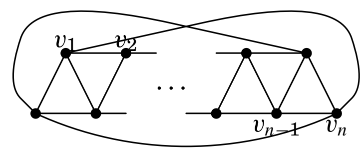
Crossing number of $K_{2,2,2,2}$. Deleting any vertex in a drawing of $K_{2,2,2,2}$ yields a drawing of $K_{1,2,2,2}$, which must have at least 3 crossings. Doing this for each vertex yields a total of at least 24 crossings. Since each crossing is formed by two edges involving 4 vertices, we have counted each crossing $8-4=4$ times. Thus the drawing of $K_{2,2,2,2}$ has at least 6 crossings. We have proved that $\nu(K_{1,2,2,2}) \ge 6$, and the drawing of this graph on the right below shows that equality holds.

6.3.22. *$K_{3,2,2}$ has no planar subgraph with 15 edges, and thus $\nu(K_{3,2,2}) \ge 2$.* The graph has 16 edges, so it suffices to show that deleting one edge leaves a nonplanar graph. Let X be the partite set of size 3. Every 6-vertex induced subgraph containing X contains a copy of $K_{3,3}$, which is nonplanar. Since every edge e is incident to a vertex not in X, the 6-vertex induced subgraph avoiding such an endpoint remains when e is deleted.

6.3.23. *The crossing number of the graph M_n obtained from the cycle C_n by adding chords between vertices that are opposite (if n is even) or nearly opposite (if n is odd) is 0 if $n \le 4$ and 1 otherwise.* For $n \le 4$, all n-vertex graphs are planar. For $n = 5$, $M_n = K_5$. For $n \ge 6$, the cycle with vertices $v_1, \dots, v_n$ plus the chords $v_1v_{1+\lfloor n/2 \rfloor}$, $v_2v_{2+\lfloor n/2 \rfloor}$, $v_3v_{3+\lfloor n/2 \rfloor}$ is a subgraph of M_n that is a subdivision of $K_{3,3}$, so the crossing number is at least 1. The drawings below, by avoiding crossings among the chords and allowing a crossing within the drawing of the cycle, show that one crossing is enough.

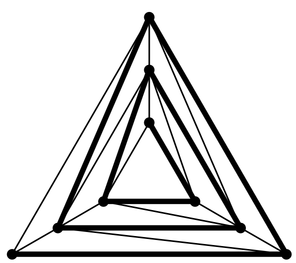




6.3.24. *a) P_n^3 is a maximal planar graph.* The graph P_4^3 is K_4, with six edges. Note that $6 = 3 \cdot 4 - 6$. Each successive vertex in P_n is adjacent to the last three of the earlier vertices, so $e(P_n^3) = 3n-6$. Together with having $3n-6$ edges, showing that P_n^3 is planar implies that it is a maximal planar graph. An embedding is obtained by drawing the path in a spiral as suggested below.

Alternatively, we can prove planarity by proving inductively that there is a planar embedding *with all of* $\{n-2, n-1, n\}$ *on the same face*. This holds for an embedding of P_4^3 (we could also start with $n = 3$ as the basis). For the induction step ($n > 4$), take such an embedding of P_{n-1}^3. Since all of $\{n-3, n-2, n-1\}$ lie on a single face, we can place n in that face and draw edges to all three. This yields a planar embedding of P_n with all of $\{n-2, n-1, n\}$ on a single face.

b) $\nu(P_n^4) = n - 4$. The graph P_5^4 is K_5, with 10 edges. Each additional vertex provides four more edges, so $e(P_n^4) = 4n - 10$. In any drawing, a maximal plane subgraph H has at most $3n - 6$ edges and thus leaves at least $n-4$ edges that each cross an edge of H. That bound is achieved with equality by adding the second diagonals of the trapezoids in the picture below, making each vertex adjacent to the vertex four earlier on the path.



(Alternatively, the earlier induction proof can be strengthened to guarantee an embedding with all of $\{n-2, n-1, n\}$ on a single face and $n-4$ on an adjacent face across the edge joining $n-1$ and $n-3$. That enables the $n-4$ additional edges to be added so that each crosses only the specified edge of P_n^3 and no added edges cross each other.)

6.3.25. *There are toroidal graphs with arbitrarily large crossing number in the plane.* The cartesian product of two cycles, $C_m \square C_n$, embeds naturally on the torus; each face is a 4-cycle. The graph has mn vertices and $2mn$ edges. View the copies of C_m as vertical slices (columns) and the copies of C_n as horizontal slices (rows).

A subgraph of $C_m \square C_n$ consisting of three full columns and three full rows is a subdivision of $C_3 \square C_3$. Since $C_3 \square C_3$ contains a subdivision of $K_{3,3}$, it is nonplanar. Therefore, a planar subgraph of $C_m \square C_n$ cannot contain three full columns and three full rows. This means it must omit at least $\min\{m-2, n-2\}$ edges. By Proposition 6.3.13, $\nu(C_m \square C_n) \ge \min\{m-2, n-2\}$. By making m and n at least $k+2$, we make the crossing number at least k while having a toroidal graph.

6.3.26. *Lower bounds on crossing numbers.* As stated correctly in Example 6.3.15 (not stated correctly in this exercise), the crossing number of $K_{6,n}$ is $6\lfloor n/2\rfloor \lfloor (n-1)/2\rfloor$. *a)* $\nu(K_{m,n}) \ge m\frac{m-1}{5}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. Consider copies of $K_{6,n}$ in a drawing of $K_{m,n}$, with the partite set of size 6 in the subgraph selected from the partite set of size m in the full graph. There are $\binom{m}{6}$ such copies, and each has at least $6\lfloor n/2\rfloor \lfloor (n-1)/2\rfloor$ crossings. Each crossing appears in $\binom{m-2}{4}$ of the subgraphs.

Hence $\nu(K_{m,n}) \ge \binom{m}{6} 6 \left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor / \binom{m-2}{4}$. Cancellation of common factors in the numerator and denominator yields the bound claimed.

b) $\nu(K_p) \ge \frac{1}{80}p^4 + O(p^3)$. Consider copies of $K_{6,p-6}$ in a drawing of K_p. There are $\binom{p}{6}$ of these copies, and each has at least $6\lfloor (p-6)/2\rfloor \lfloor (p-7)/2\rfloor$ crossings. Each crossing appears in $4\binom{p-4}{4}$ of these subgraphs, since the four vertices involved in the crossing can contribute to the smaller partite set in four ways (assuming that $n > 12$), and then four vertices not involved in the crossing must be chosen to fill that partite set.

Hence $\nu(K_p) \ge \binom{p}{6} 6 \left\lfloor\frac{p-6}{2}\right\rfloor\left\lfloor\frac{p-7}{2}\right\rfloor / [4\binom{p-4}{4}]$. The numerator has four more linear factors than the denominator, so the growth is quartic. The leading coefficient is $\frac{6}{6!}\,\frac{1}{2}\,\frac{1}{2}\,\frac{4!}{4}$, which simplifies to $1/80$.

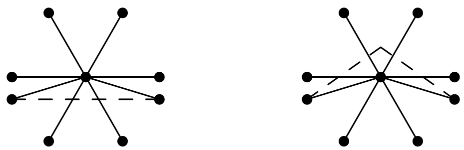
6.3.27. *If the conjecture that* $\nu(K_{m,n}) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. *holds for* $K_{m,n}$ *and* m *is odd, then the conjecture holds also for* $K_{m+1,n}$. In a drawing of $K_{m+1,n}$, there are $m+1$ copies of $K_{m,n}$ obtained by deleting a vertex of the partite set of size m. Each crossing in the drawing of $K_{m+1,n}$ appears in $m-1$ of these copies. Hence $(m-1)\nu(K_{m+1,n}) \ge (m+1)\nu(K_{m,n})$.

Since m is odd, $\lfloor m/2\rfloor \lfloor (m-1)/2\rfloor = (m-1)^2/4$, and $\lfloor (m+1)/2\rfloor \lfloor m/2\rfloor = (m+1)(m-1)/4$. Therefore,

$$\nu(K_{m+1,n}) \ge \frac{m+1}{m-1}\nu(K_{m,n}) = \frac{m+1}{2}\frac{m-1}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor = \left\lfloor\frac{m+1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

6.3.28. *If* m *and* n *are odd, then in all drawings of* $K_{m,n}$*, the parity of the number of pairs of edges that cross is the same.* (We consider only drawings where edges cross at most once and edges sharing an endpoint do not cross.) Any drawing of $K_{m,n}$ can be obtained from any other by moving vertices and edges. The pairs of crossing edges change only when an edge e is moved through a vertex v not incident to it (or vice versa). Let S be the set of edges incident to v other than those also incident to endpoints of e. When e is moved through v (or vice versa), the set of edges incident to v that e crosses is exchanged for its complement in S.

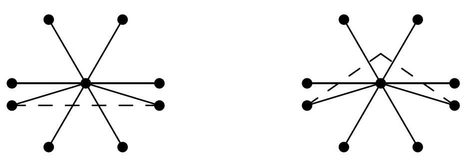
Since the degree of each vertex is odd and v is adjacent to exactly one endpoint of e, the parity of these two sets is the same. Hence the parity of the number of crossings never changes.



If m and n are odd, then $\nu(K_{m,n})$ is odd when $m-3$ and $n-3$ are divisible by 4 and even otherwise. In the naive drawing of $K_{m,n}$ with the vertices on opposite sides of a channel and the edges drawn as segments across the channel, the number of crossings is $\binom{m}{2}\binom{n}{2}$. It suffices to determine the parity of this, since the parity is the same for all other drawings including those with fewest crossings.

The binomial coefficient $r(r-1)/2$ is odd if and only if r is congruent to 2 or 3 modulo 4. Since we require m and n odd, the additional requirement for $\binom{m}{2}\binom{n}{2}$ being odd is thus m and n being congruent to 3 modulo 4.

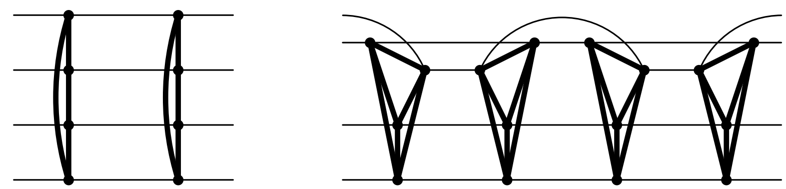
6.3.29. *If n is odd, then in all drawings of K_n, the parity of the number of pairs of edges that cross is the same.* (We consider only drawings where edges cross at most once and edges sharing an endpoint do not cross.) Any drawing of $K_{m,n}$ can be obtained from any other by moving vertices and edges. The crossing pairs change only when an edge e moves through a vertex v not incident to it (or vice versa). Let S be the set of edges incident to v that are not incident to endpoints of e. When e moves through v (or vice versa), the set of edges incident to v that e crosses is exchanged for its complement in S. Since $d(v)$ is even and v is adjacent to both endpoints of e, we have $|S|$ even, so the sizes of complementary subsets of S have the same parity. Hence the parity of the number of crossings does not change.



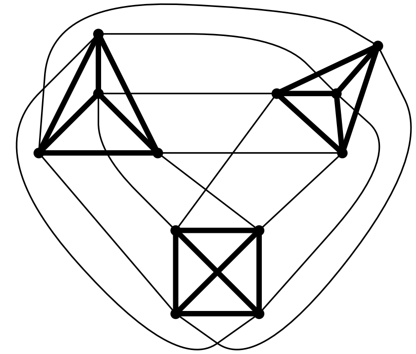
$\nu(K_n)$ is even when n is congruent to 1 or 3 modulo 8 and is odd when n is congruent to 5 or 7 modulo 8. Since the parity is the same in all drawings of K_n, we need only look at one, such as the straight-line drawing with the vertices on a circle. Its number of crossings is $\binom{n}{4}$, which equals $n(n-1)(n-2)(n-3)/24$. When the congruence class is 1 or 3, the numerator has a multiple of 8 and an odd multiple of 2, so it has four factors of 2, and only three are canceled by the denominator. Hence $\binom{n}{4}$ is even. When the class is 5 or 7, the numerator has an odd multiple of 4, an odd multiple of 2, and two odd factors, so the factors of 2 are canceled out, and $\binom{n}{4}$ is odd.

6.3.30. *$\nu(C_m \,\square\, C_n) \le (m-2)n$ and $\nu(K_4 \,\square\, C_n) \le 3n$.* For $C_m \,\square\, C_n$, we draw the copies of C_m along concentric circles. The vertices arising from a single copy of C_n are laid out along a spoke. The "long" edge in each copy of C_n generates $m-2$ crossings, as shown below on the left.

For $K_4 \,\square\, C_n$, we make the cycles concentric again, almost: the two outside cycles weave in and out of each other, as shown on the right below. We draw each copy of K_4 around a spoke, but each copy is the reflection of its neighbors. For each copy of K_n, the two outer cycles cross, and the central cycle crosses two edges in that copy of K_n.



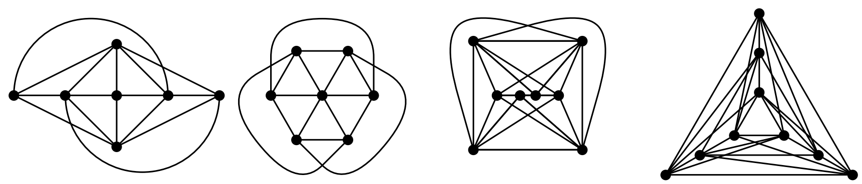
The weaving in and out requires n to be even. When n is odd, a special construction is needed for the case $n=3$ (shown below); there is one crossing in a copy of K_4, and the inner two triangles and outer two triangle each provide four crossings. Copies of K_4 can then be added in pairs by breaking the four edges joining two "neighboring" copies of K_4 and inserting two copies of K_4 with six crossings as in the middle of the figure above.



6.3.31. *Crossing number of complete tripartite graphs.* Let $f(n) = \nu(K_{n,n,n})$. We prove that $n^3(n-1)/6 \le f(n) \le (9/16)n^4 + O(n^3)$.

a) $3\nu(K_{n,n}) \le f(n) \le 3\binom{n}{2}^2$. The lower bound follows from the existence of three pairwise edge-disjoint copies of $K_{n,n}$ in $K_{n,n,n}$. The upper bound follows from a straight-line drawing with the vertices of each part placed on a ray leaving the origin.

b) $\nu(K_{3,2,2}) = 2$. Lower bounds: Since a triangle-free 7-vertex planar graph has at most $2n-4=10$ edges, $\nu(K_{3,4}) \ge 2$, and $K_{3,4}$ is a subgraph of $K_{3,2,2}$. Alternatively, a counting argument for the crossings shows both $\nu(K_{3,2,2}) \ge 2$ and $\nu(K_{3,3,1}) \ge 3$. Consider each vertex-deleted subgraph for some embedding; if it contains $K_{3,3}$, its includes a crossing, and each crossing is counted $n-4=3$ times. Hence $\nu(K_{3,2,2}) \ge \lceil 4/3 \rceil = 2$ and $\nu(K_{3,3,1}) \ge \lceil 7/3 \rceil = 3$. Extending this approach yields $\nu(K_{3,3,2}) \ge \lceil 18/4 \rceil = 5$ and $\nu(K_{3,3,3}) \ge \lceil 45/5 \rceil = 9$. Constructions for $\nu(K_{3,2,2}) \le 2$, $\nu(K_{3,3,1}) \le 3$, $\nu(K_{3,3,2}) \le 7$, and $\nu(K_{3,3,3}) \le 15$ appear below.



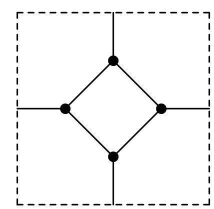
c) Recurrence to improve the lower bound in (a): $K_{n,n,n}$ has n^3 copies of $K_{n-1,n-1,n-1}$. In totalling this, each crossing formed between vertices of two parts counts $n(n-2)^2$ times, but each crossing using vertices of all three parts counts $(n-1)^2(n-2)$ times. We don't know how many crossings of each type there may be, but each crossing is counted at most $(n-1)^2(n-2)$ times, so there are at least $f(n-1)\frac{n^3}{(n-1)^2(n-2)}$ crossings. This recursive bound for $f(n)$ expands to a telescoping product: $f(n) \ge \frac{n^3}{(n-1)^2(n-2)}\frac{(n-1)^3}{(n-2)^2(n-3)}\cdots\frac{5^3}{(5-1)^2(5-2)}\frac{4^3}{(4-1)^2(4-2)}f(3)$. After cancellation, we have $f(n) \ge n^3(n-1)f(3)/54$. Since in (b) we found $f(3) \ge 9$, we at least have $f(n) \ge (1/6)n^4 + O(n^3)$. If in fact $f(3) = 15$, then we have $f(n) \ge (5/18)n^4 + O(n^3)$, which is approximately a factor of 2 from the upper bound below.

d) Improving the upper bound $(3/4)n^4 + O(n^3)$ *of (a) to* $f(n) \le (9/16)n^4 + O(n^3)$. The layout on the tetrahedron splits the n vertices of each part into two sets of size $n/2$ laid out along opposite edges. For the points on a given edge, the four neighboring edges of the tetrahedron contain all points of the other two parts, to which these points have edges laid out directly on the surface of the tetrahedron. Crossings on a face of the tetrahedron are formed by pairs of vertices from two incident edges or by a pair from one edge with one vertex each from the other two edges. If the parts have sizes l, m, n and $l' = \binom{l/2}{2}$, $m' = \binom{m/2}{2}$, $n' = \binom{n/2}{2}$, then the total number of crossings on a single face of the tetrahedron is $[l'm' + l'n' + m'n']$ + $[l'(m/2)(n/2) + m'(l/2)(n/2) + n'(l/2)(m/2)]$. Over the four identical faces,

this sums to $\frac{1}{16}(l^2m^2 + l^2n^2 + m^2n^2 + 2l^2mn + 2m^2ln + 2n^2lm)$, plus lower order terms. When $l = m = n$, this becomes $(9/16)n^4$.

For the other construction, begin with an optimal drawing of K_n. Turn each vertex into an independent set consisting of one vertex from each part. When there are three parts, each edge of the original drawing has now become a bundle consisting of 6 edges. For each crossing in the drawing of K_n, we get 36 crossings between the two bundles. For each edge in the drawing of K_n, we get at most 15 crossings within the bundle. Near a vertex of the original drawing, we get at most 36 crossings (actually slightly less) between the bundles corresponding to incident edges. There are $\binom{n}{2}$ edges in K_n, and $n\binom{n-1}{2}$ pairs of incident edges, but always $\Omega(n^4)$ crossings, so the other contributions are of smaller order. Therefore, we have only $36\nu(K_n) + O(n^3)$ crossings. With the best known bound of $\nu(K_n) \le n^4/64 + O(n^3)$, we get the same constant $9/16$. This generalizes easily to $\binom{r}{2}^2/16n^4$ for the complete multipartite graph with r parts of size n.

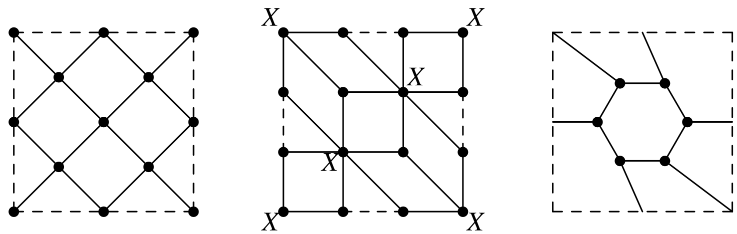
6.3.32. *An embedding of a 3-regular nonbipartite simple graph on the torus such that every face has even length.* It suffices to use K_4 as shown below. Larger examples can be obtained from this.



6.3.33. *If n is at least 9 and n is not a prime or twice a prime, then there is a 6-regular toroidal graph with n vertices.* Given these conditions, express n as rs with r and s both at least 3. Now form $C_r \square C_s$; this 4-regular graph embeds naturally on the torus with each face having length 4. On the combinatorial description of the torus as a rectangle, the embedding looks like the interior of $P_{r+1} \square P_{s+1}$, but the top and bottom rows of vertices are the same, and the left and right columns of vertices are the same.

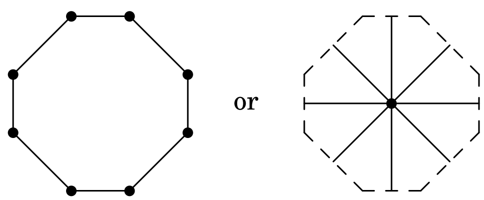
Now add a chord in each face from its lower left corner in this picture to its upper right corner. The resulting graph is 6-regular, toroidal, and has n vertices.

6.3.34. *Regular embeddings of $K_{4,4}$, $K_{3,6}$, and $K_{3,3}$ on the torus.* The number of faces times the face-length is twice the number of edges, and the number of faces is the number of edges minus the number of vertices. For $K_{4,4}$, we need eight 4-faces. For $K_{3,6}$, we need nine 4-faces. For $K_{3,3}$, we need three 6-faces. Such embeddings appear below.



6.3.35. *Euler's Formula for genus γ: For every 2-cell embedding of a graph on a surface with genus γ, the numbers of vertices, edges, and faces satisfy $n - e + f = 2 - 2\gamma$.* We use induction on $e(G)$ via contraction of edges. For the basis step, we need the number of edges required to cut S_γ into 2-cells. Each face in an embedding is a 2-cell; lay it flat. Combining neighboring faces along shared edges yields a large 2-cell R. Identifying shared edges reassembles the surface. The number of edge-pairs needed on the boundary of R is at least the number of cuts required to lay the surface flat, because that is what these boundary edges do.

It takes two cuts to lay a handle flat. If every cut is one edge, then every cut is a loop and there is only one vertex. So, the only 2-cell embeddable graphs on S_γ that have at most 2γ edges are those with 1 vertex and 2γ edges, and the resulting embeddings have 1 face. The polygonal representation of the surface is itself such an embedding, if we view the vertices of the polygonal as copies of the single vertex in the graph, and the edges of the polygon as paired loops. Since $1 - 2\gamma + 1 = 2 - 2\gamma$, all is well.

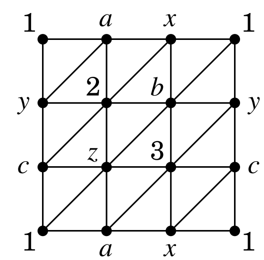


Given a 2-cell embedding with more than 2γ edges, contract an edge e that is not a loop surrounding another loop of the embedding. If e is not a loop, then following the boundaries of the face(s) bounded by e shows they are still 2-cells, and we now have a 2-cell embedding of $G \cdot e$ on the same surface. The induction hypothesis provides Euler's Formula for $G \cdot e$. Since G has one more vertex and edge than $G \cdot e$ but the same number of faces, Euler's Formula holds also for G. On the other hand, if e is a loop, then G has one more edge and face than $G \cdot e$ but the same number of vertices; again the formula holds.

An n-vertex simple graph embeddable on S_γ has at most $3(n - 2 + 2\gamma)$ edges. If G embeds on S_γ, then G has a 2-cell embedding on $S_{\gamma'}$ for some γ' with $\gamma' \le \gamma$, so we may assume that G has a 2-cell embedding on S_γ. Since the sum of the face-lengths is $2e(G)$ and each face in an embedding of a simple graph has length at least three, we have $2e \ge 3f$. Substituting in Euler's Formula $n - e + f = 2 - 2\gamma$ yields $e \le 3(n - 2 + 2\gamma)$.

6.3.36. *Genus of $K_{3,3,n}$.* Since $K_{3,3,n}$ has $n + 6$ vertices and $6n + 9$ edges, Euler's formula yields $\gamma(K_{3,3,n}) \ge 1 + (6n + 9 - 3n - 18)/6 = (n - 1)/2$. This can be improved by apply Euler's Formula to the bipartite subgraph $K_{6,n}$. Here the genus is at least $1 + (6n - 2n - 12)/4$, which simplifies to $n - 2$.

For $0 \le n \le 3$, the genus is 1, since $K_{3,3}$ is nonplanar and $K_{3,3,3}$ embeds in the torus. Such an embedding is obtained by adding vertices for the third part in the faces of a regular embedding of $K_{3,3}$ as found in Exercise 6.3.34 (each of the three faces in the regular embedding is incident to all six vertices of $K_{3,3}$). Alternatively, a pleasing triangular embedding of $K_{3,3,3}$ can be found directly as shown below.

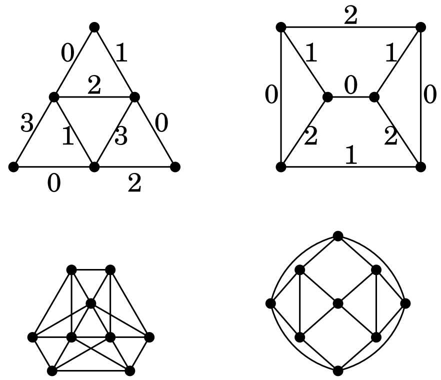


6.3.37. *For every positive integer k, there exists a planar graph G such that $\gamma(G \,\square\, K_2) \ge k$.* Let $H = G \,\square\, K_2$. The definition of cartesian product yields $n(H) = 2n(G)$ and $e(H) = 2e(G) + n(G)$. Since an n-vertex graph embeddable on S_γ has at most $3(n - 2 + 2\gamma)$ edges, we have $\gamma(H) \ge 1 + (e(H) - 3n(H))/6 = 1 + (2e(G) - 5n(G))/6$. If G is a triangulation with n vertices, then $e(G) = 3n - 6$, and we obtain $\gamma(H) \ge -1 + n/12$. It suffices to choose $n \ge 12k + 12$.

7.EDGES AND CYCLES

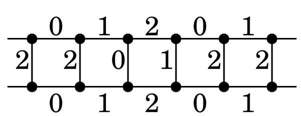
7.1. LINE GRAPHS & EDGE COLORING

7.1.1. *Edge-chromatic number and line graph for the two graphs below.* The labelings are proper edge colorings, the number of colors used is the maximum degree, so the colorings are optimal.



7.1.2. $\chi'(Q_k) = \Delta(Q_k)$, *by explicit coloring.* In the cube Q_k, the edges between vertices differing in coordinate j form a complete matching. Over the k choices of j, these partition the edges.

7.1.3. $\chi'(C_n \square K_2) = 3$. The lower bound is given by the maximum degree. For the upper bound, when n is even colors 0 and 1 can alternate along the two cycles, with color 2 appearing on the edges between the two copies of the factor C_n. When n is odd, colors 0 and 1 can alternate in this way except for the use of one 2. Color 2 appears on all cross edges except those incident to edges on the cycles with color 2, as shown below.

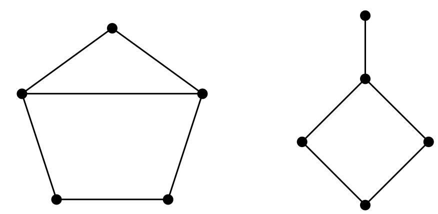


7.1.4. *For every graph G, $\chi'(G) \geq e(G)/\alpha'(G)$.* In a proper edge-coloring, each color class has at most $\alpha'(G)$ edges. The lower bound follows because all $e(G)$ edges must be colored.

7.1.5. *The Petersen graph is the complement of $L(K_5)$.* The vertices of $L(K_5)$ are the edges in K_5, which can be named as the 2-element subsets of [5]. Two such pairs are adjacent in the Petersen graph if and only if they are disjoint, which is the condition for them being nonadjacent in $L(K_5)$.

7.1.6. *The line graph of the Petersen graph has 10 triangles.* For a simple graph G, there is a triangle in $L(G)$ for every set of three edges in G that share one common endpoint and for every set of three edges that form a triangle in G. The Petersen graph has no triangles, so the latter type does not arise. However, the Petersen graph has 10 triples of edges with a common endpoint, one for each of its vertices.

7.1.7. *$\overline{P}_5$ is a line graph.* The complement of P_5 is a 5-cycle with a chord. It is the line graph of a 4-cycle with a pendant edge.



$\overline{P}_5 = L(H)$ $\quad$ H

7.1.8. *The line graph of $K_{m,n}$ is the cartesian product of K_m and K_n.* For each edge $x_i y_j$ in $K_{m,n}$, we have a vertex (i, j) in $L(K_{m,n})$; these are also the vertices of $K_m \square K_n$. Pairs (i, j) and (k, l) in $V(K_m \square K_n)$ are adjacent in $K_m \square K_n$ if and only if $i = k$ or $j = l$. This is the same as the condition for adjacency in $L(K_{m,n})$, because $x_i y_j$ and $x_k y_l$ share an endpoint in $K_{m,n}$ if and only if $i = k$ or $j = l$.

7.1.9. *A set of vertices in the line graph of a simple graph G form a clique if and only if the corresponding edges in G have one common endpoint or form a triangle.* Let S be the corresponding set of edges in G, and choose $e \in S$. If all other elements of S intersect e at the same endpoint of e, we have one common endpoint. Otherwise, we have edges f and g such that f shares endpoint x with e and g shares endpoint y with e. Since f and g must share an endpoint, they share their other endpoint z and complete a triangle. Since no single vertex lies in all of e, f, g, no additional edge of the simple graph G can share a vertex with all of these.

7.1.10. *If $L(G)$ is connected and regular, then either G is regular or G is a bipartite graph in which vertices of the same partite set have the same degree.* If $L(G)$ is connected, then G is connected (except for isolated vertices, which we ignore). For $e = uv \in E(G)$, the degree of e in $L(G)$ is $d(u) + d(v) - 2$. If the edges incident to v in G have the same degree in $L(G)$, then they must join v to vertices of the same degree in G.

If G is not regular, then G has adjacent vertices u, v with different degrees, since G is connected. By the observation above about maintaining constant degree, every walk from v in G must alternate between vertices of degrees $d(v)$ and $d(u)$. Thus G has no odd walk and is bipartite. Furthermore, the vertices of one partite set have degree $d(v)$, and those of the other partite set have degree $d(u)$.

7.1.11. *Line graphs of simple graphs.*

a) $e(L(G)) = \sum_{v\in V(G)} \binom{d(v)}{2}$.

Proof 1 (bijective argument). The edges of $L(G)$ correspond to the incident pairs of edges in G. Such pairs share exactly one vertex, and each vertex $v \in V(G)$ contributes exactly $\binom{d(v)}{2}$ such incident pairs.

Proof 2 (degree-sum formula). The degree in $L(G)$ of the vertex corresponding to $uv \in E(G)$ is $d_G(u) + d_G(v) - 2$, the number of edges of G sharing an endpoint with it. When this is summed over all edges of G, the term $d_G(u)$ appears $d_G(u)$ times. Hence the degree sum in $L(G)$ is $\sum d(v)^2 - 2e(G)$, and $L(G)$ has $\frac{1}{2}\sum d(u)^2 - e(G)$ edges. Replacing $e(G)$ by $\frac{1}{2}\sum d_i$ yields $\sum \binom{d(v)}{2}$.

(Comment: The formula holds also for graphs with multiple edges under the convention that when edges share both endpoints we have two edges between the corresponding vertices of $L(G)$.)

b) G is isomorphic to $L(G)$ if and only if G is 2-regular.

Sufficiency. A 2-regular graph is a disjoint union of cycles. The line graph of any cycle is a cycle of the same length (successive edges on a cycle in G turn into successive vertices on a cycle in $L(G)$).

Necessity.

Proof 1 (numerical argument). If G is isomorphic to $L(G)$, then $L(G)$ has the same number of vertices and edges as G. Thus $n(G) = n(L(G)) = e(G) = e(L(G))$. By (a), this becomes $n(G) = \sum_{v\in V(G)} \binom{d(v)}{2}$. Using the degree-sum formula, $\sum d(v) = 2e(G) = 2n(G)$.

We have shown that the average degree is 2. When the degrees all equal 2, the sum $\sum \binom{d(v)}{2}$ equals $n(G)$, as desired. It suffices to show that when the average degree is 2 but the individual degrees do not all equal 2, $\sum \binom{d(v)}{2}$ is larger than $n(G)$.

In this case, there is at least one number bigger than 2 (the average) and one smaller than 2. Since $\binom{r}{2} + \binom{s}{2} > \binom{r-1}{2} + \binom{s+1}{2}$ when $r > s+1$, we can iteratively bring the values toward the average while always decreasing $\sum \binom{d(v)}{2}$. Hence the equality $n(G) = \sum \binom{d(v)}{2}$ is achieved only when every vertex degree is 2.

(Comment: This is the discrete version of a calculus argument. Because $\binom{x}{2}$ is quadratic in x with positive leading coefficient, it is convex. For a convex function, the sum of values at a set of n arguments with fixed sum s is minimized by setting each argument to s/n.)

Proof 2 (graph structure). As above, $n(G) = \sum \binom{d(v)}{2}$. If all degrees are at least 2, then equality holds only when all equal 2. Hence it suffices to forbid vertices of degree less than 2.

For a graph H, observe that $L(H)$ is a path if and only if H is a nontrivial path. If G has any component that is a path, then let k be the maximum number of vertices in such a component. In $L(G)$ there is no component isomorphic to P_k. Hence G does not have a component that is a path. In particular, G has no isolated vertex.

Suppose that G has a path $(v_0, \ldots, v_l)$ such that $d(v_0) \ge 3$, $d(v_l) = 1$, and internal vertices have degree 2. Let $e_1, \ldots, e_l$ be the edges of P. In $L(G)$, the vertices $e_1, \ldots, e_l$ form a path such that $d(e_1) \ge 3$, $d(e_l) = 1$, and internal vertices have degree 2. This path is shorter than P. Also, a pendant path in $L(G)$ can only arise in this way.

Let m be the maximum length of a path from a vertex of degree at least 3 through vertices of degree 2 to a vertex of degree 1. By the reasoning above, $L(G)$ has no such path of length m. Hence $L(G)$ cannot be isomorphic to G if G has a vertex of degree 1.

7.1.12. *If G is a connected simple n-vertex graph, then $e(L(G)) < e(G)$ if and only if G is a path.* In the preceding problem, it is shown that $e(L(G)) = \sum_{v\in v(G)} \binom{d(v)}{2}$ and that this is numerically minimized when each $d(v)$ is $2e(G)/n$. Hence we require $n\binom{2e(G)/n}{2} < e(G)$, which simplifies to $e(G) < n$. This holds if and only if G is a tree.

Hence it is necessary that G be a tree, but this is not sufficient, because the degrees may be far from equal (consider $L(K_{1,n-1}$, for example). If G has k leaves, then these contribute 0 to $\sum \binom{d(v)}{2}$. The sum of the other vertex degrees is $2n-2-k$. Again the sum is smallest when these degrees are equal. The requirement for the edge inequality becomes $(n-k)\binom{\frac{2n-2-k}{n-k}}{2} < n-1$. Since $\frac{2n-2-k}{n-k} - 1 = \frac{n-2}{n-k}$, the inequality simplifies to $(2n-2-k)\left(\frac{n-2}{n-k}\right) < 2n-2$, which further simplifies to $(2n-2)\left(\frac{n-2}{n-k} - 1\right) < k\left(\frac{n-2}{n-k}\right)$ and eventually to $k < 4(n-1)/n$.

We conclude that $k \le 3$. If a tree has three leaves, then its actual degree list must consist of one 3, three 1s, and the rest 2s, and $L(G)$ has exactly $n-1$ edges. Hence the only graphs with $e(L(G)) < e(G)$ are paths.

7.1.13. *If G is a simple graph such that $\overline{G} \cong L(G)$, then G is C_5 or the graph consisting of a triangle plus a matching from the triangle to an independent 3-set (shown on the right below).* Since $\overline{G}$ and $L(G)$ have the same number of vertices, $e(G) = n(G)$. Also G has only one nontrivial component, since otherwise $\overline{G}$ would be connected while $L(G)$ would not.

If G has an isolated vertex, then $\overline{G}$ and hence $L(G)$ has a dominating vertex. Hence G has an edge xy incident to all other edges. Since there are $n(G)$ edges, the number of common neighbors of x and y is one more than the number of common nonneighbors. Hence G has a triangle, which means that $\overline{G}$ and hence $L(G)$ has an independent set of size 3. Hence G has three disjoint edges, which contradicts that every edge contains x or y.

Thus we may assume that G is connected. Since $e(G) = n(G)$, there is exactly one cycle in G (since deleting an edge of a cycle, which must exist, leaves a tree). Also, $e(G) = n(G)$ implies that the average vertex degree is 2. If G is 2-regular, then $G \cong L(G) \cong \overline{G}$. The only 2-regular graph isomorphic to its complement is C_5.

Otherwise, G has a vertex of degree 1. Thus $\overline{G}$ and hence $L(G)$ has a vertex adjacent to all but one other vertex. Such a vertex in $L(G)$ corresponds to an edge in G that is incident to all but one other edge; we call such an edge *semidominant*. Indeed, we have argued that the number of semidominant edges in G equals the number of vertices of degree 1. Thus G has a semidominant edge xy. Since G is unicyclic, x and y have at most one common neighbor.

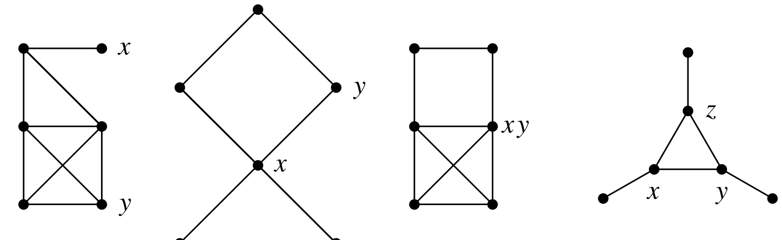
First suppose that x and y have no common neighbor. The one edge not incident to xy now joins neighbors of x and y (creating a 4-cycle) or joins (by symmetry) two neighbors of x.

In the 4-cycle case, semidominant edges other than xy must be on the 4-cycle and incident to xy, and these work only if there are no pendant edges incident to the opposite side of the 4-cycle. Thus there cannot be pendant edges at both x and y, and there must be exactly two pendant edges at one of them, say x. We have now specified G completely, as shown below, but its complement and line graph are not isomorphic.

On the other hand, if an edge zw joins two neighbors of x, all semidominant edges other than xy are incident to x. Since xz and xw are incident to all edges except the pendant edges at y, there must be at least one pendant edge at y. If there is exactly one such edge, then xz and xw are semidominant. If there is more than one, then xy is the only semidominant edge. Both possibilities contradict the equality between the number of semidominant edges and the number of vertices of degree 1.

In the remaining case, x and y have a common neighbor, z. The one edge not incident to xy is incident to z, since G is unicyclic. Since G has only one triangle, $\overline{G}$ and $L(G)$ have only one independent set of size 3.

Hence there is only one way to choose three disjoint edges in G. Hence x and y, like z, have only one neighbor each of degree 1, and the graph is as shown on the right below.



$\overline{G}$ G $L(G)$

7.1.14. *Connectivity and edge-connectivity of line graphs of k-edge-connected graphs.* Suppose $L(G)$ has a separating t-set S. Then S corresponds to a set of t edges whose deletion disconnects G, because the line graph of a connected graph is connected. Therefore $t \geq k$. This can also be proved using edge-disjoint paths in G, but not as cleanly.

Now consider edge-connectivity. Since $\delta(G) \geq k$, we have $\delta(L(G)) \geq 2k - 2$, since each edge is incident to at least $k - 1$ others at each endpoint. Let $[T, T']$ be a minimum edge cut of $L(G)$, with κ' edges. Because a minimum edge cut yields only two components, T, T' corresponds to a partition of $E(G)$ into two connected subgraphs, which we call F, F', respectively. There is an edge of $L(G)$ in $[T, T']$ each time an edge of F is incident to an edge of F'.

These incidences take place at vertices of G. At a vertex $x \in V(G)$, there are $d_F(x)$ edges of F (corresponding to vertices of T) and $d_{F'}(x)$ edges of F (corresponding to vertices of T'). Since each such edge of F is incident to each such vertex of F', this vertex x in G yields $d_F(x)d_{F'}(x)$ edges in $[T, T']$. Since $d_F(x) + d_{F'}(x) = d_G(x) \geq k$, this product is at least $k - 1$ whenever x is incident to edges of both F and F'.

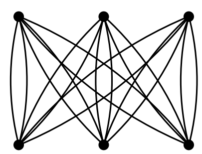
Hence it suffices to show that there are at least two vertices of G that are incident to edges from both F and F'. If F and F' are incident at only one vertex x, then this must be a cut-vertex of G, because any path from F to F' that avoids x would yield another vertex where F and F' are incident. Deleting the edges of F incident to x or the edges of F' incident to x disconnects G. Since G is k-edge-connected, we conclude that $d_F(x), d_{F'}(x)\ H \geq k$ and $\left|[T, T']\right| \geq k^2 > 2k - 2$.

7.1.15. *Every connected line graph of even order has a perfect matching.* Note that a graph without isolated vertices has the same number of components as its line graph. Let S' be the set of edges in G corresponding to a set $S \subseteq V(L(G))$. Deleting S from $L(G)$ corresponds to deleting S' from G, but each edge deletion increases the number of components by at most one. Thus $G - S'$ (and $L(G) - S$) have at most $1 + |S|$ components of any sort, odd or otherwise. For a graph of even order, $o(L(G) - S) \le 1 + |S|$ implies Tutte's condition $o(L(G) - S) \le |S|$, since the order is even.

The edges of a connected simple graph of even size can be partitioned into paths of length two. The paired vertices of a perfect matching in $L(G)$ correspond in G to paired edges forming paths of length 2. Since the matching saturates $V(L(G))$, the corresponding paths partition $E(G)$.

7.1.16. *If G is a simple graph, then $\gamma(L(G)) \ge \gamma(G)$, where $\gamma(G)$ denotes the genus of G (Definition 6.3.20).* Consider an embedding of $L(G)$ on a surface S; it suffices to obtain an embedding of G on the same surface. For each $x \in V(G)$, the edges of G with endpoint x form a clique Q_x in $L(G)$. For the embedding of G, locate x at one vertex xx' of Q_x in the embedding of $L(G)$. For each edge xy, embed it along the path in the embedding of $L(G)$ from xx' to xy to yy'. Since xy is used in only one such path, the edges of the new embedding of G on this surface have no crossings.

7.1.17. *The number of proper 6-edge-colorings of the graph below (from a specified set of six colors) is* $900 \cdot 512$.



It suffices to count the ways to assign pairs of colors to the double edges so that the pairs at two double edges with a common endpoint are disjoint, because we can then multiply by 2^9 to assign the colors within the pairs.

We can view such an assignment as a 3-by-3 matrix in which the entry in position (i, j) is the pair assigned to the two edges joining the ith top vertex and the jth bottom vertex. Each color must appear exactly once in some pair in each row and each column. We can choose entry $(1, 1)$ in $\binom{6}{2}$ ways, and for each such way there are $\binom{4}{2}$ choices for entry $(1, 2)$. Thus we can choose the first row in 90 ways, and for each way the number of completions will be the same. Let the pairs in the first row be $\{a, b\}$, $\{c, d\}$, and $\{e, f\}$, in order.

If entry $(2, 1)$ is one of the pairs in the first row, then we have two such pairs to choose from. By symmetry, let it be $\{c, d\}$. Now entry $(2, 2)$ must be $\{e, f\}$, and entry $(2, 3)$ is $\{a, b\}$, and the bottom row is determined.

If entry $(2, 1)$ is not one of the pairs in the first row, then we fill it using one element from entry $(1, 2)$ and one element from entry $(1, 3)$; these can be chosen in 4 ways. For example, suppose that entry $(2, 1)$ is $\{c, e\}$. Now f must appear in entry $(2, 2)$ and d in entry $(2, 3)$, and the second row is completed by deciding which of $\{a, b\}$ goes into entry $(2, 2)$ and which goes into entry $(2, 3)$. There are two ways to make this choice, and again the bottom row is determined.

Thus after choosing the first row, there are two ways to complete the matrix with entry $(2, 1)$ not being a pair from the first row. Since there are two ways when entry $(2, 1)$ is a pair from the first row, the total number of colorings is $10 \cdot 90 \cdot 2^9$, as claimed.

7.1.18. *$\chi'(K_{m,n}) = \Delta(K_{m,n})$, by explicit coloring.* We may assume that $m \le n$, so the maximum degree is n. If the vertices are $X \cup Y$ with $X = x_1, \ldots, x_m$ and $Y = y_1, \ldots, y_n$, we give the edge $x_i y_j$ the color $i + j \pmod n$. Since incident edges differ in the index of the vertex in X or the vertex in Y, they receive different colors.

7.1.19. *Every simple bipartite graph G has a $\Delta(G)$-regular simple bipartite supergraph.* Let $k = \Delta(G)$, and let X and Y be the partite sets of G.

Construction 1. A huge simple k-regular supergraph of G can be constructed iteratively as follows: If G is not regular, add a vertex to X for each vertex of Y and a vertex to Y for each vertex of X. On the new vertices, construct another copy of G. For each vertex in G with degree less than k, join its two copies in the new graph to get G'. Now k is the same as before, the minimum degree has increased by one, and G' is a supergraph of G. Iterating this $k - \delta(G)$ times yields the desired simple supergraph H. It is connected if G was connected.

Construction 2. We may assume that $|X| = |Y|$ by adding vertices to the smaller side, if necessary. Let $M = nk - \sum_i d(x_i)$; this is the total "missing degree". Add M vertices to both X and Y, and place a $(k-1)$-regular graph H on these, which may be constructed using successively tilted matchings as in the natural 1-factorization of $K_{M,M}$. Now add edges joining deficient vertices of X and Y to vertices of H on the opposite side. Each vertex of H receives one such edge, which remedies the M deficiencies in each of X and Y.

7.1.20. *Edge-coloring of digraphs.* Given a digraph D with indegrees and outdegrees at most d, form a bipartite graph H as follows. The partite sets are $A = \{x^-: x \in V(D)\}$ and $B = \{x^+: x \in V(D)\}$. For each edge xy in D, place an edge $x^- y^+$ in H; the vertex x^- inherits the edges exiting x and the vertex x^+ inherits the edges entering x. The resulting bipartite graph H is the "split" of D (Section 1.4).

Since the maximum number of edges entering or exiting a vertex of D is d, $\Delta(H) = d$. Since H is bipartite, $\chi'(H) = d$. The d-edge-coloring on the edges of H is the desired coloring of the corresponding edges in D.

7.1.21. *Algorithmic proof of $\chi'(G) = \Delta(G)$ for bipartite graphs.* Let G be a bipartite graph with maximum degree k. Let f be a proper k-edge-coloring of a subgraph H of G. Let uv be an edge of G not in H. We produce a proper $\Delta(G)$-edge-coloring of the subgraph consisting of H plus the edge uv.

Since uv is uncolored, among the $\Delta(G)$ available colors there is a color α not used at u. Similarly, some color β is not used at v. If α is missing at v or β at u, then we can extend the coloring to uv using α or β. Otherwise, follow the path P from u that alternates in colors α and β. The path is well-defined, since each color appears at most once at each vertex.

Since α does not appear at u, the path P ends somewhere and does not complete a cycle. The path reaches the partite set of v along edges of color β, and it reaches the partite set of u along edges of color α. Hence P cannot reach v, where β is missing. We can now interchange colors α and β on the edges of P to make β available for the edge uv.

7.1.22. *If G is a simple graph with maximum degree 3, then $\chi'(G) \leq 4$.* Let $H = L(G)$; since $\chi'(G) = \chi(L(G))$, we seek a bound on $\chi(H)$. By making the same argument for each component, we may assume that G and H are connected. Since $\Delta(G) = 3$, an edge of G intersects at most two other edges at each end, and hence $\Delta(H) \leq 4$. If H is 4-regular, then G must be 3-regular. The smallest 3-regular simple graph has 6 edges, so $H \neq K_5$.

By Brooks' Theorem, $\chi(H) \leq \Delta(H)$ if H is not a clique or odd cycle. If H is an odd cycle, then $\chi(H) \leq 3$. If H is a clique, then it has at most 4 vertices. Otherwise, $x'(G) = \chi(H) \leq \Delta(H) \leq 4$. (Note: when $\Delta = 3$, $\Delta + 1 = 2(\Delta - 1)$. For larger Δ, Brooks' Theorem is not strong enough to prove $\chi'(G) \leq \Delta(G) + 1$.)

7.1.23. *1-factorization of $K(p,q)$, the complete p-partite graph with q vertices in each partite set.* With $G[H]$ denoting composition, we have $K(p,q) = K(p,d)[\overline{K}_{q/d}]$ when d divides q.

a) If G decomposes into copies of F, then $G[\overline{K}_m]$ decomposes into copies of $F[\overline{K}_m]$. Expanding a copy of F in the decomposition of G into a copy of $F[\overline{K}_m]$ uses precisely the copies in $G[\overline{K}_m]$ of the edges in that copy of F. Thus these expansions exhaust the copies in $G[\overline{K}_m]$ of edges in G. Since $\overline{K}_m$ is an independent set, there are no other edges to consider.

The relation "G decomposes into spanning copies of F" is transitive. If G decomposes into spanning copies of F and H decomposes into spanning copies of G, then the F-decomposition of G can be used on each graph in a G-decomposition of H to decompose H into spanning copies of F.

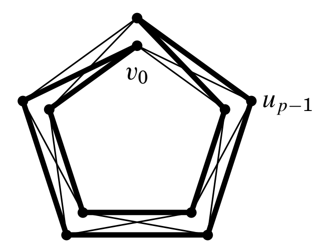
b) $K(p,q)$ decomposes into 1-factors when pq is even. When p is even, K_p has a 1-factorization - a decomposition into copies of $(p/2)K_2$. By part (a), $K_p[\overline{K}_q]$ decomposes into spanning copies of $(p/2)K_2[\overline{K}_q]$, which

equals $(p/2)K_{q,q}$. Since $(p/2)K_{q,q}$ is a regular bipartite graph, it has a 1-factorization. By transitivity, $K(p,q)$ also has a 1-factorization.

When p is odd, we have q even, and thus $K(p,q) = K(p,2)[\overline{K}_{q/2}]$. If $K(p,2)$ has a 1-factorization (into spanning copies of pK_2), then we decompose $K(p,q)$ into spanning copies of $pK_{q/2,q/2}$ and obtain a 1-factorization of $K(p,q)$ by transitivity.

It remains only to decompose $K(p,2)$ into 1-factors when p is odd. Cliques of odd order decompose into spanning cycles; thus it suffices to decompose $C_p[\overline{K}_2]$ into 1-factors. Since this 4-regular graph has an even number of vertices ($2p$), it suffices to decompose it into two spanning cycles. Let the vertices be $v_0, \ldots, v_{p-1}$ and $u_0, \ldots, u_{p-1}$, with $\{u_i, v_i\} \leftrightarrow \{u_{i+1}, v_{i+1}\}$ (indices modulo p). The two desired cycles are

$$(v_0, \ldots, v_{p-1}, u_0, u_{p-1}, \ldots, u_1)$$
$$(u_0, v_1, u_2, \ldots, u_{p-1}, v_0, v_{p-1}, u_{p-2}, \ldots, v_2, u_1).$$



7.1.24. *If $\chi'(H) = \Delta(H)$, then $\chi'(G \square H) = \Delta(G \square H)$.* The graph $G \square H$ consists of a copy of G for each vertex of H and a copy of H for each vertex of G. A vertex $(u,v) \in G \square H$ has neighbors (u, v') for every $v' \in N_H(v)$ and (u', v) for every $u' \in N_G(u)$. Hence $d_{G \square H}(u,v) = d_G(u) + d_H(v)$. With $u \in G$ and $v \in H$ having maximum degree, we obtain $\Delta(G \square H) = \Delta(G) + \Delta(H)$. We construct a proper $(\Delta(G) + \Delta(H))$-edge-coloring.

Use the same proper $(\Delta(G)+1)$-edge-coloring (guaranteed by Vizing's Theorem) on each copy of G. With $\Delta(G) + 1$ colors allowed, some single color i is missing from all edges incident to all copies of the vertex $u \in V(G)$. To color the copy of H on the vertices with first coordinate u, we need only $\Delta(H)$ colors. We use color i and $\Delta(H) - 1$ additional colors. Doing this for each $u \in V(G)$ constructs a proper edge-coloring of $G \square H$ with $\Delta(G) + 1 + \Delta(H) - 1 = \Delta(G \square H)$ colors.

7.1.25. *Kotzig's Theorem on Cartesian products.*

a) $\chi'(G \square K_2) = \Delta(G \square K_2)$. By Vizing's Theorem, we can properly color $E(G)$ with $\Delta(G) + 1$ colors. Use a single such coloring on both copies of G. The two copies of a vertex of G are joined by an edge, but both are missing the same color i in the coloring of G, so color i can be assigned to the edge between them. Hence $G \times K_2$ is $(\Delta(G)+1)$-edge-colorable. We cannot properly color $E(G \times K_2)$ with $\Delta(G)$ colors, because $\chi'(G \,\square\, K_2) \geq \Delta(G \,\square\, K_2) = \Delta(G)+1$.

b) If G_1, G_2 are edge-disjoint graphs with the same vertex set and H_1, H_2 are edge-disjoint graphs with the same vertex set, then $(G_1 \cup G_2) \square (H_1 \cup H_2) = (G_1 \,\square\, H_2) \cup (G_2 \,\square\, H_1)$. We view G_1 and G_2 as a red/blue edge-coloring of $G_1 \cup G_2$, and we view H_1 and H_2 as a yellow/green edge-coloring of H. Since every edge of $G \,\square\, H$ is a copy of an edge of G or H, this induces a red/blue/yellow/green edge-coloring of the product. The spanning subgraph containing the red and green edges is $G_1 \,\square\, H_2$, and the spanning subgraph containing the blue and yellow edges is $G_2 \,\square\, H_1$.

c) $G \,\square\, H$ is 1-factorable if G and H each have a 1-factor. Let G_1 be a 1-factor of G, $G_2 = G - E(G_1)$, H_1 a 1-factor of H, and $H_2 = H - E(H_1)$. Since $H_1 = mK_2$, we have $G_2 \,\square\, H_1 = G_2 \,\square\, mK_2 = m(G_2 \,\square\, K_2)$. By part (a), there is a proper edge-coloring of $G_2 \,\square\, H_1$ with $\Delta(G_2)+1 = \Delta(G)$ colors. Similarly, there is a proper edge-coloring of $G_1 \,\square\, H_2$ with $\Delta(H)$ colors. By part (b), these together yield a proper edge-coloring of $G \,\square\, H$ with $\Delta(G)+\Delta(H) = \Delta(G \square H)$ colors. (This result is Kotzig's Theorem, usually stated for regular graphs; the proof is from the thesis of J. George.)

7.1.26. *If G is a regular graph with a cut-vertex x, then $\chi'(G) > \Delta(G)$.*

Proof 1. Because G is regular, $\chi'(G) = \Delta(G)$ requires that each color class be a 1-factor. Hence $n(G)$ is even. Since $n(G)-1$ is odd, $G-x$ has a component H of odd order. Let y be a neighbor of x not in H. A 1-factor of G that contains xy must contain a 1-factor of H, which is impossible since H has odd order.

Proof 2. Again each color class must be a 1-factor. Let M_1 and M_2 be color classes containing edges incident to x whose other endpoints are in different components of $G-x$. Since these are perfect matchings, their symmetric difference consists of isolated vertices and even cycles. In particular, it contains a cycle through x that visits different components of $G-x$, but there is no such cycle.

7.1.27. *Density conditions for $\chi'(G) > \Delta(G)$.*

a) If G is regular and has $2m+1$ vertices, then $\chi'(G) > \Delta(G)$. For a regular graph, being $\Delta(G)$-edge-colorable means being 1-factorable, which is impossible with odd order since such graphs have no 1-factor.

b) If G has $2m+1$ vertices and more than $m \cdot \Delta(G)$ edges, then $\chi'(G) > \Delta(G)$. Each color class is a matching, and each matching has size at most m, so Δ matchings cover at most $m\Delta$ edges. Since G has more edges than that, every proper edge-coloring of G requires more than Δ colors.

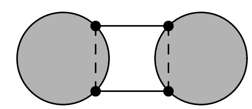
c) If G arises from a k-regular graph with $2m+1$ vertices by deleting fewer than $k/2$ edges, then $\chi'(G) > \Delta(G)$. Since fewer than k vertices have

lost an edge and $k \leq 2m$, some vertex of degree k remains; hence $\Delta(G) = k$. Also $e(G) > (2m+1)k/2 - k/2 = m\Delta(G)$, so (b) implies $\chi'(G) > \Delta(G)$.

7.1.28. *The Petersen graph has no overfull subgraph.* A subgraph H is overfull if and only if it has an odd number of vertices and has more than $(n(H)-1)\Delta(G)/2$ edges. Subgraphs of order $3, 5, 7, 9$ would need more than $3, 6, 9, 12$ edges, respectively. Since the Petersen graph has no cycle of length less than 5, the smaller cases are excluded. For the last case, deleting a single vertex leaves a subgraph with 9 vertices and 12 edges, but 12 is not more than 12.

7.1.29. *A non-1-factorable regular graph with high degree.* Let G be the $(m-1)$-regular connected graph formed from $2K_m$ by deleting an edge from each component and adding two edges between the components to restore regularity. If m is odd and greater than 3, then G is not 1-factorable.

To see this, observe that the central edge cut of size 2 leaves an odd number of vertices on both sides. Hence every 1-factor in G includes an edge of this cut. If G is 1-factorable, this forces the degree to be at most 2, and hence $m \leq 3$.



$K_m - e$ $\quad$ $K_m - e$

7.1.30. *Overfull Conjecture $\Rightarrow$ 1-factorization Conjecture.* Let G be a k-regular simple graph of order $2m$.

An induced subgraph of G is overfull if and only if the subgraph induced by the remaining vertices is overfull. Let H be the subgraph induced by vertex set S. We have $2e(H) = kn(H) - \left|[S,\overline{S}]\right|$ (Proposition 4.1.12). Overfullness for H is thus the inequality $kn(H) - \left|[S,\overline{S}]\right| > k(n(H)-1)$ (and $n(H)$ odd), since $\Delta(G) = k$. This inequality simplifies to $\left|[S,\overline{S}]\right| < k$, and it is satisfied for S if and only if it is satisfied for $\overline{S}$.

If G has an overfull subgraph, then $k \leq 2\lfloor (m-1)/2 \rfloor$. Again, we have $2e(H) = kn(H) - \left|[S,\overline{S}]\right|$. If H is overfull, then $\left|[S,\overline{S}]\right| < k$, by the computation in part (a). Also, $2e(H) \leq n(H)[n(H)-1]$, since G is simple. Together, these inequalities yield $n(H)[n(H)-1] < k[n(H)-1]$, or $k < n(H)$. By part (a) we may assume that $n(H) \leq m$, since both $V(H)$ and $V(G)-V(H)$ induce overfull subgraphs. Hence we may conclude that $k < m$. Furthermore, since $n(H)$ is odd when H is overfull, we strengthen this to $k < m-1$ when m is even.

If the constraint on k fails, then there is no overfull subgraph, so if the Overfull Conjecture holds, then the 1-factorization Conjecture also holds.

7.1.31. *Optimal edge-colorings.* A k-edge-coloring of a multigraph G is *optimal* if it has the maximum possible value of $\sum_{v\in V(G)} c(v)$, where $c(v)$ is the number of distinct colors appearing on edges incident to v.

a) If G does not have a component that is an odd cycle, then G has a 2-edge-coloring that uses both colors at each vertex of degree at least 2. If G is Eulerian, we follow an Eulerian circuit, alternating between the colors; each visit to a vertex enters and leaves on different colors. If $e(G)$ is even, then the first and last edge also contribute both colors to their common vertex. If $e(G)$ is odd and the starting vertex has degree at least 4, then it receives both colors from another visit. If G has no vertex of degree at least 4 at which the odd circuit can be started, then G is an odd cycle, which is the exceptional case and has no such 2-edge-coloring.

If G is not Eulerian, then we add a new vertex x having an edge to each vertex of G with odd degree. Let C be an Eulerian circuit starting at x in the new graph G'; alternate the two colors along C. The problem of first and last edge having the same color is irrelevant, because we discard the edges incident to x. For vertices other than x, the degree in G' is at least 2, and there is at most one edge to x. Hence each vertex v of degree at least 2 in G has a visit to it in C using only edges of G, and this visit contributes edges of both colors at v.

b) If f is an optimal k-edge-coloring of G, having color a at least twice at $u \in V(G)$ and color b not at u, then in the subgraph of G consisting of edges colored a or b, the component containing u is an odd cycle. Let H be the specified component, consisting of edges reachable from u using paths of colors a and b. If H is not an odd cycle, then part (a) allows us to recolor $E(H)$ with colors a and b so that both colors appear at every vertex of degree at least 2. Now the number of colors appearing at each vertex of H is at least as large as before, and at u the number has increased. This new coloring of G has a larger value of $\sum c(v)$, which contradicts the optimality of f. Hence H must be an odd cycle.

c) If G is bipartite, then G is $\Delta(G)$-edge-colorable, and G has a $\delta(G)$-edge-coloring in which each color appears at every vertex. Consider an optimal $\Delta(G)$-edge-coloring f of G. If f is not a proper edge-coloring, then some color appears at least twice at some vertex. Since the degree of that vertex is at most $\Delta(G)$, some other color must be missing at that vertex. Since f is optimal, part (b) implies that G has an odd cycle, which cannot occur in a bipartite graph. Hence f is a proper edge-coloring.

For the second claim, consider an optimal δ-edge-coloring of G. If some color i is missing at u, then some color j must appear twice, because the number of edges at u is at least δ. By part (b), this requires an odd cycle in G. Thus an optimal δ-coloring must have δ different colors appearing at each vertex of G.

7.1.32. *Every bipartite graph G with minimum degree k has a k-edge-coloring in which at each vertex v, each color appears $\lceil d(v)/k\rceil$ or $\lfloor d(v)/k\rfloor$ times.* Modify G to obtain another bipartite graph H by iteratively splitting each vertex v of G into $\lceil d_G(v)/k\rceil$ vertices, each inheriting k of the edges incident to v, except for one vertex that may receive fewer. Let the resulting graph be H; note that $\Delta(H) = k$. Since a bipartite graph H has a proper $\Delta(H)$-edge-coloring, we can properly color H with the desired number of colors. Each color is used at each vertex that was split from v except possibly the one that received fewer incident edges. Hence we recombine the split vertices to return to G, we have each of the k colors appearing $\lceil d(v)/k\rceil$ or $\lfloor d(v)/k\rfloor$ times at each vertex v. (Comment: The same argument holds for every k, not only the minimum degree.)

7.1.33. *Every simple graph with maximum degree Δ has a proper $(\Delta+1)$-edge-coloring in which each color is used $\lceil e(G)/(\Delta+1)\rceil$ or $\lfloor e(G)/(\Delta+1)\rfloor$ times.* By Vizing's Theorem, there is a proper coloring with $\Delta+1$ colors. If the total usage of some two colors differs by more than one edge, consider the subgraph formed by the edges with these two colors. Since the coloring is proper, this consists of components that are paths and/or cycles alternating between the two colors. The color appearing more often must occur on the end edges of a path of odd length. Switching colors on such a path yields a new proper coloring that is less out of balance. Such improvements can be made until the frequencies differ by at most one, at which point they must all be $\lceil e(G)/(\Delta+1)\rceil$ or $\lfloor e(G)/(\Delta+1)\rfloor$.

7.1.34. *Shannon's bound on $\chi'(G)$, almost.*

a) Every loopless graph G has a $\Delta(G)$-regular loopless supergraph. Given G with vertex set $x_1,\ldots,x_n$, add another copy of G, disjoint from it, with vertex set $y_1,\ldots,y_n$. Add $\Delta(G)-d_G(v_i)$ copies of the edge x_iy_i to complete the construction. (Comment: If G is simple and a simple supergraph is desired, modify the construction by taking $2(\Delta(G)-\delta(G))$ copies of G. Since a complete graph with an even number of vertices is 1-factorable, on the copies of $x \in V(G)$ we can add $\Delta(G)-d_G(x)$ edge-disjoint matchings to raise the degree of these vertices to $\Delta(G)$).

b) If G is a loopless graph with even maximum degree, then $\chi'(G) \le 3\Delta(G)/2$. By part (a), we can find a $\Delta(G)$-regular supergraph H of G; by Petersen's Theorem, we can partition H into $\Delta(G)/2$ 2-factors. Since each 2-factor is a disjoint union of cycles, each 2-factor is 3-edge-colorable. Hence we can color $E(H)$ with $3\Delta(G)/2$ colors, and we can delete the edges of $H-G$ to obtain a proper edge-coloring of G with $3\Delta(G)/2$ colors.

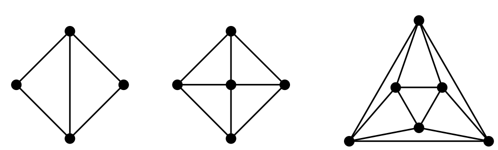
7.1.35. *Bounds on $\chi'(G)$.* Let P denote the set of 3-vertex paths in G, expressed as edges xy and yz, and let $\mu(e)$ denote the multiplicity of edge

e. the last bound below (Anderson–Goldberg) implies the earlier bounds.
Shannon: $\chi'(G) \le \lfloor 3\Delta(G)/2 \rfloor$.
Vizing, Gupta: $\chi'(G) \le \Delta(G) + \mu(G)$.
Ore: $\chi'(G) \le \max\{\Delta(G), \max_P \left\lfloor \frac{1}{2} d(x) + d(y) + d(z) \right\rfloor\}$.
$\chi'(G) \le \max\{\Delta(G), \max_P \left\lfloor \frac{1}{2}(d(x) + \mu(xy) + \mu(yz) + d(z)) \right\rfloor\}$.
The last implies Ore because $\mu(xy)+\mu(yz) \le d(y)$. It implies Vizing–Gupta because $[\mu(xy) + mu(yz)]/2 \le \mu(G)$ and $[d(x) + d(z)]/2 \le \Delta(G)$. It implies Shannon because it implies Ore and $[d(x) + d(y) + d(z)]/2 \le 3\Delta(G)/2$.

7.1.36. (+) *Line graphs of complete graphs.* If $n \ne 8$, prove that $G = L(K_n)$ if and only if G is a $(2n-4)$-regular simple graph with $\binom{n}{2}$ vertices in which nonadjacent vertices have four common neighbors and adjacent vertices have $n-2$ common neighbors. (When $n = 8$, there are three exceptional graphs satisfying the conditions.) (Chang [1959], Hoffman [1960])

7.1.37. (+) *Line graphs of complete bipartite graphs.* Unless $n = m = 4$, prove that $G = L(K_{m,n})$ if and only if G is an $(n+m-2)$-regular simple graph of order mn in which nonadjacent vertices have two common neighbors, $n\binom{m}{2}$ pairs of adjacent vertices have $m-2$ common neighbors, and $m\binom{n}{2}$ pairs of adjacent vertices have $n-2$ common neighbors. (Moon [1963], Hoffman [1964]) (Comment: for $n = m = 4$, there is one exceptional graph - Shrikande [1959].)

7.1.38. *Sufficiency of van Rooij-Wilf condition for connected graphs containing a double triangle with two even triangles.* We claim that the only possibilities for G in this case are the three graphs appearing below. By inspection, these graphs are line graphs, being $L(K_{1,3}+e)$, $L(K_4-e)$, and $L(K_4)$, respectively. Let F be a double triangle of G with two even triangles axy and xyb. If G has another vertex, then G has another edge to one of $\{x, y\}$, say an edge xz, else any edge joining F to $G-F$ creates an odd triangle in F. Now $N(z) \cap \{a, b, y\}$ is $\{y\}$ or $\{a, b\}$, but in the former case $\{x, a, b, z\}$ induces $K_{1,3}$. Hence $z \leftrightarrow \{a, b\}$, and the graph induced by $S = \{x, y, z, a, b\}$ is the wheel $L(K_r - e)$ in the middle below.



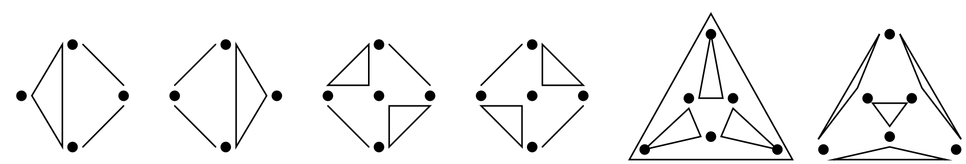
If x or y has another neighbor w, then by the same argument the other neighbors of w in F are $\{a, b\}$. If the edge is yw, then we must have $z \leftrightarrow w$ to avoid making both zxa and zxb odd; this graph is now $L(K_4)$ on the right below. If the edge is xw, then we must have $z \leftrightarrow w$ to avoid inducing $K_{1,3}$ on $\{x, y, z, w\}$. Now $\{y, z, w, a, b\}$ induce the expanded 4-cycle, which we

saw earlier is not a line graph; it violates the hypothesis because y makes both azw and bzw odd. This argument shows that y also has at most one neighbor not in F. The only remaining way to attach additional vertices is $z \leftrightarrow v$ (or, equivalently, $w \leftrightarrow v$ if y does have a neighbor w outside F), but then $\{z, v, a, b\}$ induces $K_{1,3}$.

7.1.39. *Characterization of graphs with the same line graph.* A **Krausz decomposition** of a simple graph H is a partition of $E(H)$ into complete graphs such that each vertex of H is used at most twice.

a) For a connected simple graph H, two Krausz decompositions of H that have a common complete graph are identical. Beginning with the common complete graph Q_1, we iteratively find common complete graphs in the decomposition until no more edges remain. While an edge remains, it has a path to the subgraph that has been decomposed, since G is connected. Thus there is an unabsorbed edge incident to a clique that has been absorbed; call the common vertex v. Since each vertex is used at most twice, all the unabsorbed edges incident to v must lie in the same complete graph in each decomposition. Its vertex set must be v together with the neighbors of v along the remaining incident edges, so no other neighbors of v are available. Hence this complete graph must also be in each decomposition.

b) Distinct Krausz decompositions for the graphs in Exercise 7.1.38.



c) No connected simple graph except K_3 and those in part (b) has two distinct Krausz decompositions. By part (a), it suffices to show that in any other graph G, there is some complete graph that appears in every Krausz decomposition. Call the complete graphs in some Krausz decomposition *K-graphs*.

Suppose first that G has a clique Q of size at least 4. We may assume that three vertices of Q appear together in a some K-graph, since otherwise each vertex of Q is in at least three K-graphs. If three vertices of Q appear together but not with all of Q, then an omitted vertex of Q, since it is used only twice, appears in another K-graph with at least two vertices of Q, and now some edge is covered twice. Therefore, if G has a maximal clique of size at least 4, it appears in every Krausz decomposition.

If G has an edge in no triangle, then it appears in every Krausz decomposition. Hence we may assume that every edge of G is in a triangle and G has no 4-clique. Since there is no 4-clique and every vertex is used at most twice, no edge appears in three triangles.

Suppose that every edge of G appears in exactly one triangle. If there are two triangles sharing a vertex, then there are four edges at that vertex and both triangles appear in every decomposition, since the vertex can only be used twice. Hence $G = K_3$, which has two decompositions.

Therefore, we may assume that some edge e appears in two triangles and there are two decompositions. Now e is the common edge of a double triangle (no K_4), and each triangle is used in one decomposition (since the endpoints are used only once). By symmetry, we may let $e = xy$, let x, y, z be the triangle used, and let w be the other vertex of the double triangle.

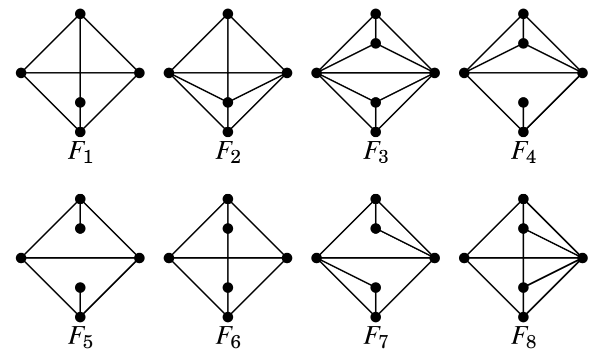
If w has two other neighbors u and v, then each is adjacent to exactly one of $\{x, y\}$, since w has been used twice and there is no K_4. By symmetry, let these edges be ux and vy. Since we have assumed another decomposition using x, y, w in a triangle, the edges uw and vw must appear together in a triangle in that decomposition. But now also yv and yz must lie in a triangle, and similarly xu and xz. Hence u, v, z form another triangle in that decomposition. Now every vertex is in two triangles in both decompositions, there is no room for additional incident edges, and the graph is the last graph in part (a).

If w has exactly one other neighbor, by symmetry we may assume that it is u, adjacent to x. Now since x, y, w form a triangle in the other decomposition, xu and xz must lie in a triangle, so uz is an edge and x, u, z form a triangle in the other decomposition. There are no other neighbors of w or y. Another neighbor of u or z would have to form a triangle with uz, but in the other decomposition these edges could not be absorbed. Hence G is the 5-vertex wheel (the middle graph in part (a)).

Hence w has no other neighbor. This implies that x and y have no other neighbor, since they are already used twice. If z has another neighbor, then we are in one of the cases described above with respect to the other decomposition where x, y, w is the triangle used. Hence z also has no other neighbor, and our graph is the kite (the left graph in part (a)).

d) $K_{1,3}$ and K_3 are the only two nonisomorphic simple graphs with isomorphic line graphs. When G is the line graph of a graph H, the vertices of H correspond to complete subgraphs in a Krausz decomposition of G. Furthermore, given a Krausz decomposition, there is one way to retrieve H satisfying this correspondence, as in Theorem 7.1.16. Thus if G is the line graph of two graphs H_1 and H_2, then G must have distinct Krausz decompositions. For the graphs in part (b), the Krausz decompositions are "isomorphic", retrieving the same graph as H_1 and H_2. For K_3, the decomposition using one triangle yields $L(K_{1,3}) = K_3$, and the decomposition into three edges yields $L(K_3) = K_3$. For every other line graph G, there is only one Krausz decomposition and hence only one solution to $L(H) = G$.

7.1.40. *A simple claw-free graph with has a double triangle with both triangles odd if and only if it some graph below is an induced subgraph.*



In each graph shown, for each triangle of the double triangle T there is a vertex with an odd number of neighbors on that triangle. If such a graph is an induced subgraph of G, then T also has both triangles odd in G.

Conversely, suppose that G has a double triangle T with triangles X and Y both odd. Let $\{u, w, z\}$ and $\{v, w, z\}$ be the vertex sets of X and Y, respectively. A vertex outside T is adjacent to w or z and to nothing else in T would yield an induced claw, so G has no such vertex. However, a vertex can have one neighbor in X or Y and two in the other by being adjacent to w or z and to u or v. A vertex with exactly one neighbor in X or Y and none in the other is adjacent only to u or v in T.

A single vertex outside G cannot be adjacent to three vertices in X or Y and one in the other, but it can be adjacent to one in each or to three in each, which yield F_1 and F_2 above.

Otherwise, we use two vertices x and y, respectively, to make X and Y odd. Suppose first that neither x nor y has one neighbor on one triangle and two on the other. Let $a = |N(x) \cap X|$ and $b = |N(y) \cap Y|$. If $x \not\leftrightarrow y$, then we obtain F_3, F_4, or F_5 when (a, b) is $(1, 1)$, ($(1, 3)$ or $(3, 1)$), or $(3, 3)$, respectively. If $x \leftrightarrow y$ and $(a, b) = (1, 1)$, then we obtain F_6. If $x \leftrightarrow y$ and $b = 3$, then deleting v yields F_1 or F_2, depending on whether a is 1 or 3.

In the remaining case, we may assume by symmetry that y has one neighbor in Y and two neighbors in X, with $N(y) \cap (X \cup Y) = \{z, u\}$. If x has one neighbor in X and none in Y, then $N(x) \cap (X \cup Y) = \{u\}$, and $\{u, x, y, w\}$ induces a claw. If x has three neighbors in X, then $\{z, x, y, v\}$ induces a claw if $x \not\leftrightarrow y$, and deleting v leaves F_2 if $x \leftrightarrow y$.

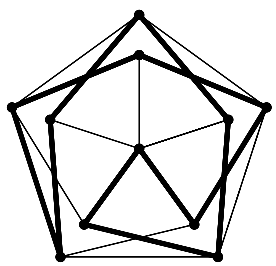
Hence we may assume that x has one neighbor in X and two neighbors in Y. Depending on $N(x) \cap \{y, z\}$, we have these outcomes:

$N(x) \cap \{y, z\}$	$\varnothing$	$\{y\}$	$\{z\}$	$\{y, z\}$
outcome	F_7	$G[\{v, w, x, y, z\}] \cong F_1$	$G[\{z, w, x, y\}] \cong K_{1,3}$	F_8

7.2. HAMILTONIAN CYCLES

7.2.1. *The complete bipartite graph $K_{r,r}$ is Hamiltonian if and only if $r \geq 2$.* Since $K_{1,1}$ has no cycle, we exclude it. For $r \geq 2$, we list the vertices alternately from the two partite sets. Consecutive vertices are adjacent, and the last is adjacent to the first, so we obtain a spanning cycle.

7.2.2. *The Grötzsch graph is Hamiltonian.*



7.2.3. *$K_{n,n}$ has $n!(n-1)!/2$ Hamiltonian cycles.* Specifying the order in which the vertices of each partite set will be visited determines a cycle starting at a given vertex x. Since there are n vertices in the other partite set and $n-1$ remaining to be visited in the same partite set as x, there are $n!(n-1)!$ ways to specify these orderings. This counts each cycle twice, since each cycle can be followed in either direction from x.

7.2.4. *If a graph G has a Hamiltonian path, then for every vertex set S, the number of components in $G-S$ is at most $|S|+1$.* Let $c(H)$ denote the number of components of a graph H, let P be a Hamiltonian path in G, and consider $S \subseteq V(G)$.

Proof 1 (counting components). Successive deletion of vertices from a path increases the number of components of the path by at most one each time, so $c(P-S) \leq 1+|S|$. Since P is a spanning subgraph of $G-S$ and adding edges cannot increase the number of components, we have $c(G-S) \leq c(P-S) \leq 1+|S|$.

Proof $1'$ (following the path). P starts somewhere and visits each component of $G-S$. It must exit all but one of these before it first enters the last such component, and these exits must go to distinct vertices of S. Hence $|S| \geq c(G-S)-1$.

Proof 2 (graph transformation). Let u, v be the endpoints of P. If $u \leftrightarrow v$, then G is Hamiltonian, and then $c(G-S) \leq |S| < |S|+1$. If uv is not an edge, then $G' = G+uv$ is Hamiltonian, which implies $c(G'-S) \leq |S|$.

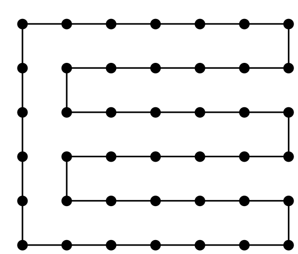
However, $c(G-S) \leq c(G'-S)+1$, since adding an edge to a graph reduces the number of components by at most 1. Hence again $c(G-S) \leq |S|+1$.

7.2.5. *Every 5-vertex path on the dodecahedron extends to a Hamiltonian cycle.* The dodecahedron has an automorphism taking a given face to any other face, with any rotation. Thus it suffices to shown that the spanning cycle shown at the beginning of this section in the text contains all types of 5-vertex paths relative to a given face. These are 1) four edges on one face, 2) three edges on one face and an edge extending off it, 3) at most two edges on every face. Since each pair of successive edges lie on a common face, there is essentially only one path of type 3 relative to a given central vertex. Every such path can be mapped to any other because the central vertex can be mapped to any other with an arbitrary rotation of its three incidence edges (mapped by an automorphsim, that is).

7.2.6. *Matchings in Hamiltonian bipartite graphs.*

a) If G is a Hamiltonian bipartite graph, then $G-x-y$ has a complete matching if and only if x and y are on opposite sides of the bipartition of G. Let G be an X,Y-bigraph with a Hamiltonian cycle C. Since C alternates between X and Y, we have $|X| = |Y|$. If two vertices are deleted from one partite set, then the other cannot be saturated by a matching. If we delete $x \in X$ and $y \in Y$, then each of the two paths forming $C - \{x, y\}$ must alternate between colors and have endpoints of opposite colors, since the endpoints are neighbors of x and y. If the vertices on these two paths are $u_1, \ldots, u_{2r}$ and $v_1, \ldots, v_{2s}$ in order, then the edges $u_{2i-1}u_{2i}$ and $v_{2j-1}v_{2j}$ for all i and j together form the desired matching in $G - \{x, y\}$.

b) Defective chessboards (missing two squares) can be covered by dominoes (1 by 2 rectangles) if and only if the two missing squares have opposite colors. By part (a), it suffices to show that the graph G corresponding to the chessboard is Hamiltonian. This is true for every grid ($P_m \square P_n$) with an even number of rows, as illustrated below. Follow the rows back and forth, but reserve one end column to tie together the first and last row and complete the cycle. If the number of rows is even, then this path ends on the same side of the grid in the first and last rows.



Comment: Without using the result on Hamiltonian bipartite graphs, there are several other ways to prove that a chessboard missing two squares of opposite has a tiling by dominoes. Proof 2: prove by induction on n that the property holds for all n by n chessboards with n even. Proof 3: Explicitly construct a matching, given that (i, j) and (k, l) are the two missing squares. Proof 4: Establish the existence of an alternating path joining two unsaturated squares whenever a set of dominoes does not fully cover the defective chessboard. These proofs involve a fair amount of detail and are not as general as the method above.

7.2.7. *A mouse eating cheese.* Model this with a graph G on 27 vertices in which vertices are adjacent if they correspond to adjacent subcubes. We ask whether G has a Hamiltonian path between the vertex corresponding to the center cube and a vertex corresponding to a corner cube. The vertices correspond to the 3-digit vectors with entries 0,1,2. The edges join vectors that differ by 1 in one position. Since they join vertices with opposite parity of coordinate-sum, G is bipartite.

If G has the desired Hamiltonian path, then the graph G' obtained by adding an edge between the corner cube (0,0,0) and the center cube (1,1,1) has a Hamiltonian cycle. These vectors lie in opposite partite sets of G, so G' is also bipartite. Hence the desired path yields a Hamiltonian cycle in a bipartite graph with an odd number of vertices, which is impossible.

7.2.8. *The $4 \times n$ chessboard has no knight's tour.* Let G be the graph having a vertex for each square and an edge for each pair of squares whose positions differ by a knight's move. Every neighbor of a square in the top or bottom row is in the middle two rows, so the top and bottom squares form an independent set. Deleting the $2n$ squares in the middle rows leaves $2n$ components remaining; that is not enough to prohibit the tour.

Instead, note that every neighbor of a white square in the top and bottom rows is a black square in the middle two rows. Therefore, if we delete the n black squares in the middle two rows, the white squares in the top and bottom rows become n isolated vertices, and there remain $2n$ other vertices in the graph, which must form at least one more component. Hence we have found a set of n vertices whose deletion leaves at least $n + 1$ components, which means that G cannot be Hamiltonian. (For most n, the graph has a Hamiltonian path.)

7.2.9. *An infinite family of non-Hamiltonian graphs satisfying the necessary condition of Proposition 7.2.3 for Hamiltonian cycles.* It is easy to generalize the first example of a non-Hamiltonian graph satisfying the condition. Begin with a complete graph H with n vertices. Let x, y, z, w be vertices in H. Add vertices a, b, c and edges xa, yb, zc, wa, wb, wc to form G. Every separating set includes w and another vertex for each small component cut off from the clique, so the condition holds. However, visiting a, b, c requires three edges incident to w.

7.2.10. *Spanning cycles in line graphs.*

a) A 2-connected non-Eulerian graph whose line graph is Hamiltonian. The kite has two vertices of odd degree and hence is not Eulerian. Its line graph is $K_1 \vee C_4$, which is Hamiltonian.

b) $L(G)$ is Hamiltonian if and only if G has a closed trail that includes a vertex of every edge. If G is a star, then $L(G)$ is Hamiltonian and G has such a closed trail of length 0. Otherwise, there is no vertex cover of size 1, so a closed trail with a vertex of each edge must be nontrivial.

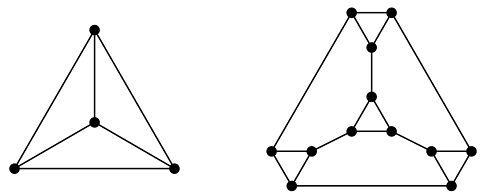
Sufficiency. Let T be such a trail in G, with vertices $v_1, \ldots, v_t$ in order. Consecutive edges on T are incident in G, so $E(T)$ in order becomes a cycle C in $L(G)$. For each edge $e \in E(G) - E(T)$, select an endpoint v of e that occurs in $V(T)$. Although v may occur repeatedly on T, select one particular occurrence of v in T as v_i. Between the vertices of C corresponding to the edges $v_{i-1}v_i$ and v_iv_{i+1} in T, insert the vertices of $L(G)$ for all edges of $E(G) - E(T)$ whose selected vertex occurrence is v_i. Since these edges all share endpoint v_i, the corresponding vertices replace an edge in $L(G)$ with a path. Every vertex of $L(G)$ is in the original cycle C or in exactly one of the paths used to enlarge it, so the result is a spanning cycle of $L(G)$.

Necessity. Given a spanning cycle in $L(G)$, we obtain such a closed trail in G. First we shorten the cycle. If there are three successive vertices e_{i-1}, e_i, e_{i+1} on the remaining cycle in $L(G)$ that correspond to edges in G with a common endpoint, we delete e_i from the cycle. Since e_{i-1} and e_{i+1} have a common endpoint, what remains is still a cycle in $L(G)$. Each deletion preserves the property that the remaining edges include an endpoint of every edge in G.

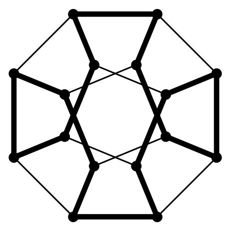
When no more deletions are possible, every three successive vertices in the resulting cycle C in $L(G)$ correspond to edges in G with no common endpoint, but two successive vertices on C are incident edges in G. Orient each such edge in G by letting the tail be the endpoint it shares withits predecessor on C in $L(G)$; the head is the vertex it shares with its successor on C. This expresses the edge set in C as the vertex set of a closed trail in G (and it contains a vertex of every edge in G).

7.2.11. *A 3-regular 3-connected graph whose line graph is not Hamiltonian.* Let G be a non-Hamiltonian 3-regular 3-connected graph, such as the Petersen graph. Form G' by replacing each vertex v of G with a triangle T_v. Each original edge uw becomes an edge joining a vertex of T_u with a vertex of T_w. Observe that G' is 3-regular. Also, if G' has a 2-vertex cut, then deleting the corresponding two or one vertices in G also cuts G. Below we illustrate the application of the transformation to K_4.

Suppose that C is a closed trail in G' that touches every edge. Since edges of T_v are incident only to vertices of T_v, the trail C must enter each T_v. Since only three edges enter T_v, the trail C can enter and leave T_v only once. Hence contracting C back to G by contracting the triangles yields a cycle that visits each vertex once. Since G has no spanning cycle, G' has no such trail.



7.2.12. *The graph below is Hamiltonian.*



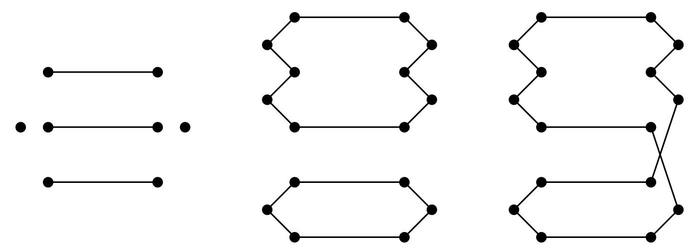
7.2.13. *The 3-regular graph obtained from the Petersen graph by expanding one vertex into a triangle matched to the former neighbors of the deleted vertex is not Hamiltonian.* Since there are only three edges incident to the triangle, it can only be entered once on a cycle. It must visit all vertices of the triangle during one visit to the triangle. Therefore, shrinking the triangle to a single vertex shortens the cycle by two steps and yields a Hamiltonian cycle in the Petersen graph. There is no such cycle, so the original cycle also cannot exist.

7.2.14. *Every uniquely 3-edge-colorable 3-regular graph is Hamiltonian.* Each color class induces a perfect matching. Consider the subgraph H formed by the edges in two of these matchings. It has degree 2 at every vertex, and thus H a 2-factor, i.e. a union of disjoint cycles. The cycles have even length, since the two colors alternate on its edges. If H is not a single (i.e. Hamiltonian) cycle, then we can switch these two colors on one of the cycles to obtain a 3-edge-coloring with a different partition of the edges. Thus **unique** 3-edge-colorability requires that the union of any two color classes is a Hamiltonian cycle.

7.2.15. *C_n^2 is the union of two disjoint Hamiltonian cycles.* This graph consists of n vertices in cycle order, with each adjacent to the nearest two in each direction. Let the points be $v_1, \ldots, v_n$ in order. If we use $(v_1, \ldots, v_n)$ as one cycle, then the remaining edges form a cycle $(v_1, v_3, \ldots., v_{n-2})$ (traveling around twice) if n is odd. If n is even, then the remaining edges form two disjoint cycles of length $n/2$, and we must make a switch. In this case replace the three edges v_{n-1}, v_n, v_1, v_2 in the original cycle by the edges v_{n-1}, v_1, v_n, v_2; the result is still a cycle. Now the remaining edges form the cycle $(v_1, v_3, \ldots, v_{n-1}, v_n, v_{n-2}, \ldots, v_4, v_2)$. All indices change by two on each edge of this cycle except the two edges $v_{n-1}v_n$ and v_2v_1 that were switched out of the original cycle.

7.2.16. *The graph G_k obtained from two disjoint copies of $K_{k,k-2}$ by adding a matching between the two "partite sets" of size k is Hamiltonian if and only if $k \geq 4$.* If $k = 2$, then G_k is disconnected. If $k = 3$, then deleting the two centers of claws leaves three components (on the left below).

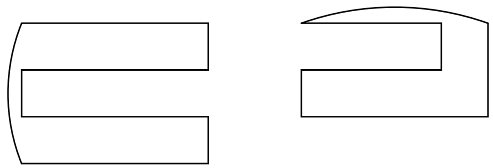
When $k \geq 4$, first take two cycles to cover $V(G_k)$: use two edges of the middle matching in each cycle, and use one fewer vertex from the outside parts than from the inside parts (as in the middle below). Then switch a pair of edges on one side to link the two cycles (as on the right).



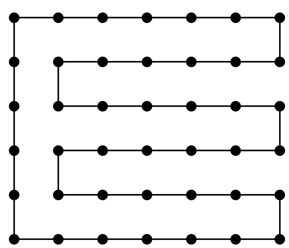
7.2.17. *The Cartesian product of two Hamiltonian graphs is Hamiltonian.* It suffices to show that the product of two cycles is Hamiltonian, because the product of two Hamiltonian graphs has a spanning subgraph of this form. Index the vertices of the first cycle as $1, \ldots, m$ and those of the second cycle as $1, \ldots, n$; the vertices of the product are then $\{(i, j)\colon 1 \leq i \leq m,\ 1 \leq j \leq n\}$. The product is a grid with m rows and n columns plus a wraparound edge in each row and column.

If m is even, then we start in the upper left corner (1,1) and follow rows alternately to the right and left, finishing in the lower left corner $(m, 1)$ after visiting all vertices; the wraparound edge in the first column completes the cycle. If m is odd, then we follow the same zigzag from $(1, 1)$ to traverse the first $n - 1$ columns, ending at $(m, n - 1)$. We then traverse the last column from (m, n) to $(1, n)$ and take the wraparound edge from $(1, n)$ to $(1, 1)$ to complete the cycle.

To show that the hypercube Q_k is Hamiltonian, we use induction on k. Note that Q_1 is K_2, which is not Hamiltonian. Basis step: For $k = 2$ and $k = 3$, we have explicit constructions. Induction step: For $k \geq 4$, we observe that $Q_k \cong Q_2 \,\square\, Q_{k-2}$. Since each factor is Q_l for some l with $2 \leq l \leq k-2$, the induction hypothesis tells us that both factors are Hamiltonian, and then the first part of the problem yields this for the product.



7.2.18. *The product of graphs with Hamiltonian paths has a Hamiltonian cycle unless both factors have odd order.* Since deleting edges never introduces a Hamiltonian path or cycle, it suffices to prove the claim when the two graphs are paths. In this case the product is the grid $P_m \,\square\, P_n$. If the factors do not both have odd order, then we may assume that the grid has an even number of rows. Follow the rows back and forth, but reserve one end column to tie together the first and last row and complete the cycle. Since the number of rows is even, the zigzag path ends on the same side of the grid in the first and last rows.



The product of two graphs with Hamiltonian paths fails to have a Hamiltonian cycle if and only if both graphs are bipartite and have odd order, in which case the product has a Hamiltonian path. If both graphs are bipartite and have odd order, then the product is bipartite and has odd order, so it cannot be Hamiltonian. The discussion above handles the case where at least one factor has even order. Hence we may assume that both factors have odd order and at least one is not bipartite. Since paths are bipartite, the Hamiltonian path in one factor must have an chord that completes an odd cycle. It thus suffices to construct a Hamiltonian path when one factor is a path of odd order and the other (the "horizontal" factor in the grid) is a path of odd order plus a single edge that forms such a chord.

Let $P = P_m$ with vertices $v_1, \ldots, v_m$ in order, and let $Q = P_n + e$ with vertices $u_1, \ldots, u_n$ in order on the path, plus the edge $e = u_r u_s$ where $s - r$ is even and positive. Let $P_{i,j}$ denote the v_i, v_j-path in P, and let $Q_{i,j}$ denote the u_i, u_j-path in Q (along the path P_n in Q). If P, Q are disjoint paths such that the last vertex of P is adjacent to the first vertex of Q, then $P : Q$ denotes the path consisting of P followed by Q. Let $s_1(P_{i,j}, Q_{k,l})$ be the "back-and-forth" Hamiltonian path of $P_{i,j} \,\square\, Q_{k,l}$ that follows the rows, switching from one row to the next in the end columns corresponding to u_k and u_l. The path starts at (v_i, u_k). It ends at (v_j, u_k) if $j - i$ is odd and at (v_j, u_l) if $j - i$ is even. Similarly, $s_2(P_{i,j}, Q_{k,l})$ is the Hamiltonian path following using all the column edges, starting at (v_i, u_k) and ending at (v_i, u_l) if $l - k$ is odd and at (v_j, u_l) if $l - k$ is even.

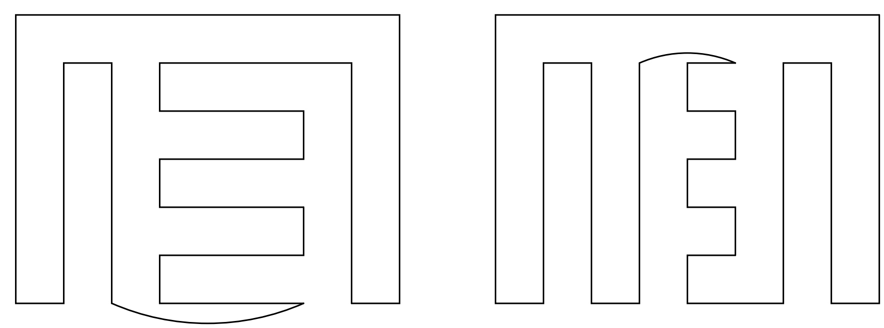
Recall that $u_r u_s$ is the extra edge in the factor H. If r, s are odd (such as when H is an odd cycle), then

$$s_2(P_{2,m}, Q_{1,r}) : s_1(P_{m,2}, Q_{s,r+1}) : s_2(P_{2,m}, Q_{s+1,n}) : (u_1 \,\square\, Q_{n,1})$$

is a Hamiltonian cycle of $G \,\square\, H$. If r, s are even, then

$$s_2(P_{2,m}, Q_{1,r}) : s_1(P_{2,m}, Q_{s,r+1}) : s_2(P_{m,2}, Q_{s+1,n}) : (u_1 \,\square\, Q_{n,1})$$

is a Hamiltonian cycle of $G \,\square\, H$. In each case all vertices are listed, and the last vertex of each segment is adjacent to the first vertex of the next.



7.2.19. *Construction of a $(k-1)$-connected k-regular non-Hamiltonian bipartite graph for odd k.* Let H be the graph with vertex set $W \cup X \cup Y \cup Z$, where W, Z have size $(k-1)/2$ and X, Y have size $(k+1)/2$. Add the edges $W \times X$, $X \times Y$, and $Y \times Z$; now the vertices X, Y have degree k. Take k copies of H. Add special vertex sets A and B, each of size $(k-1)/2$. Add an edge from a_i to each of the k copies of w_i, for each i; this gives a_i degree k and increases the degree of w_i to k. Similarly add an edge from b_i to each of the k copies of z_i, for each i. This completes the desired graph G_k. The set $A \cup B$ has size $k-1$, and $G_k - A - B$ has k components, all isomorphic to H. We omit the verification that G_k is $(k-1)$-connected. It is conjectured that every k-connected k-regular bipartite graph is Hamiltonian.

7.2.20. *Hamiltonian cycles in powers of graphs.*

a) If $G - x$ has at least three nontrivial components in which x has exactly one neighbor, then G^2 is not Hamiltonian. Let v_1, v_2, v_3 be the unique neighbor of x in three such components H_1, H_2, H_3 of $G - x$. Let $S = \{x, v_1, v_2, v_3\}$. Since each H_i is non-trivial, $G^2 - S$ has at least three components. Within S, only x and v_i have neighbors in $H_i - v_i$. A spanning cycle of G^2 must enter and leave $H_i - v_i$ via distinct vertices of S; these can only be x and v_i. This forces at least three edges incident to x, one to each H_i, which is impossible in a Hamiltonian cycle.

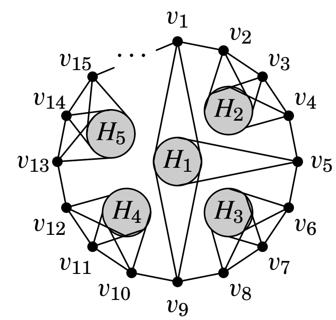
b) The cube of each connected graph (with $n \geq 3$) is Hamiltonian. The cube of a connected graph contains the cube of each of its spanning trees, so it suffices to prove the claim for trees. We use induction on $n(T)$ to prove the stronger result that T^3 has a Hamiltonian cycle such that a specified pair x, y of adjacent vertices in T are consecutive on the cycle. For $n(T) = 3$, this is trivial since T^3 is a clique; suppose $n(T) \geq 4$. The graph $T - xy$ consists of two disjoint trees R and S containing x and y, respectively. By symmetry, we may assume $n(R) \leq n(S)$. Choose $z \in N_S(y)$ and, if $n(R) > 1$, choose $w \in N_R(x)$. If each subtree has at least three vertices, then the induction hypothesis provides Hamiltonian cycles of R^3 and S^3 containing the edges xw and yz, respectively. Since T^3 contains both R^3 and S^3, we obtain the desired Hamiltonian cycle of T^3 by replacing the edges xw and yz with xy and wz, which exist because $d_T(w, z) = 3$. If $n(R) = 2$, we replace yz in the cycle through S^3 by y, x, w, z. If $n(R) = 1$, we replace yz by y, x, z.

7.2.21. *Non-Hamiltonian complete k-partite graphs.* If $m < n/2$, then there is a non-Hamiltonian complete k-partite graph with minimum degree m and all partite sets nonempty as long as $m \geq k-1$. Simply make the largest partite set X have size $n - m$, and partition the remaining set S into $k-1$ parts. Now $G - S$ has more than $|S|$ components, which violates the necessary condition for Hamiltonicity.

For the values stated in the text, $m = \frac{n}{2}\frac{k-1}{k}\frac{2l}{2l+1} = (k-1)l$, and thus $m < n/2$ with $n = k(2l+1)$.

7.2.22. *For $k, t \geq 4$, in the class $\mathbf{G}(k, t)$ of connected k-partite graphs in which each partite set has size t and every two parts induce a matching of size t, there is a graph that is not Hamiltonian.* Let $V_1, \ldots, V_k$ be the partite sets. To construct G, start with $C_{3t} \cup tK_{k-3}$ as follows. Form C_{3t} with vertices $v_1, \ldots, v_{3t}$ in order so that $v_i \in V_j$ if and only if $i \equiv j \pmod 3$. For $1 \leq i \leq t$, add edges to make a clique consisting of the ith vertex of each of $V_4, \ldots, V_k$. Now we have $C_{3t} \cup tK_{k-3}$; let $H_1, \ldots, H_t$ be the copies of K_{k-3}. Each vertex in H_i needs a neighbor in each of V_1, V_2, V_3. Add all edges joining $V(H_1)$ and $\{v_1, v_5, v_9\}$; note that $v_1 \in V_1$, $v_5 \in V_2$, $v_9 \in V_3$. Add all edges joining $V(H_2)$ and $\{v_2, v_3, v_4\}$. Add all edges joining $V(H_3)$ and $\{v_6, v_7, v_8\}$. For $4 \leq j \leq t$, add all edges joining $V(H_j)$ and $\{v_{3j-2}, v_{3j-1}, v_{3j}\}$.

Since each vertex has exactly one neighbor in each other part, and the graph is connected, the graph belongs to $\mathbf{G}(k, t)$. However, deletion of $\{v_1, v_5, v_8\}$ leaves a graph with four components, and hence the graph is not Hamiltonian.



7.2.23. *The Petersen graph G has toughness* 4/3. We seek the smallest value of $|S|/c(G-S)$ achieveable by a separating set S, where $c(H)$ counts the components of H. To separate into two components we need $|S| \geq 3$, since G is 3-connected.

To separate G into three components, we claim that $|S| \geq 4$. Deleting any three vertices deletes at most 9 edges, which leaves at least six edges among the remaining seven vertices. If $c(G-S) = 3$, then $G - S$ is a forest with at most four edges unless one component is a 5-cycle, which allows only five edges. Hence deleting three vertices cannot create more than two components. We can separate the graph into three components by deleting an independent set of size 4, so $t \leq 4/3$.

To separate G into more than three components, we must leave an independent set T of size 4 with one vertex in each component. Since a vertex neighborhood in G is a dominating set, T has no three vertices with a common neighbor. Hence there are six different vertices that are common neighbors of two vertices in T, and these must all be deleted to leave 4 components. This ratio is 6/4.

After consider all separating sets, we conclude that 4/3 is the smallest ratio and hence is the toughness.

7.2.24. *The toughness $t(G)$ of a $K_{1,3}$-free graph is half its connectivity.* We may assume that G is connected, since otherwise the toughness and connectivity are 0. Let $c(H)$ denote the number of components of H.

For every connected graph, the toughness is at most half the connectivity, since a minimum vertex cut separates the graph into at least two components and has size $\kappa(G)$. The inequality $|S| \ge t(G)c(G-S)$ thus yields $t(G) \le |S|/c(G-S) \le |S|/2 = \kappa(G)/2$ when S is a minimum cut.

Now let S be a vertex cut achieving the minimum ratio of $|S|/c(G-S)$; in other words, $t(G) = |S|/c(G-S)$. Let u be a vertex in a component C of $G-S$, and let v be a vertex of S. By Menger's Theorem, there exist $\kappa(G)$ pairwise internally disjoint u,v-paths in G. These paths enter S at distinct vertices, establishing edges to C from $\kappa(G)$ distinct vertices of S. This holds for each component of $G-S$. Since G is $K_{1,3}$-free, each vertex of S has neighbors in at most two components of $G-S$ and hence is incident to at most 2 of the edges we have generated. This yields the inequality $\kappa(G)c(G-S) \le 2|S|$, and hence $t(G) \ge \kappa(G)/2$.

7.2.25. *If G is a simple graph that is not a forest and has girth at least 5, then $\overline{G}$ is Hamiltonian.* Let $H = \overline{G}$. If H satisfies Ore's Condition, then H is Hamiltonian. Otherwise, H has nonadjacent vertices x and y such that $d_H(x)+d_H(y) \le n-1$. Thus $xy \in E(G)$ and $d_G(x)+d_G(y) \ge n-1$. Avoiding cycles of length less than 5 in G yields $N_G(x) \cap N_G(y) = \varnothing$, and also there is no edge from $N(x)$ to $N(y)$.

We have argued that $N_G(x) \cup N_G(y)$ induces a tree with at least $n-1$ vertices. Since G is not a forest, exactly one vertex, z, remains outside this set. Furthermore, girth at least 5 implies that z has exactly one neighbor a in $N_H(x)$ and one neighbor b in $N_G(y)$. No other edge can appear in G. Now H has a spanning cycle that visits the vertices in the order $z, x, b, N_H(y) - \{b\}, N_H(x) - \{a\}, a, y$.

7.2.26. *The maximum number of edges in a non-Hamiltonian n-vertex simple graph is $\binom{n-1}{2}+1$.* The graph consisting of an $(n-1)$-clique plus a single pendant edge has $\binom{n-1}{2}+1$ edges and is not Hamiltonian. To show that this is the maximum size, suppose that G is not Hamiltonian, and let $d_1,\ldots,d_n$ be the vertex degrees of G, indexed in nondecreasing order.

Since G must fail Chvátal's Condition, there is some $i < n/2$ such that $d_i \le i$ and $d_{n-i} < n-i$. Let u be the vertex with the ith smallest degree, and let v be the vertex with the $n-i$th smallest degree. Thus $d_G(u) + d_G(v) \le i + (n-i-1) = n-1$. In the complement, we have $d_{\overline{G}}(u) + d_{\overline{G}}(v) = (n-1-d_G(u)) + (n-1-d_G(v) \ge 2(n-1)-(n-1) = n-1$.

Since u and v have degree sum at least $n-1$ in $\overline{G}$, and since a simple graph has at most one edge joining them (counted twice in the degree sum), there must be at least $n-2$ edges in $\overline{G}$ incident to $\{u,v\}$. Hence $e(G) \le \binom{n}{2} - (n-2) = \binom{n-1}{2}+1$.

7.2.27. *By induction on n, the maximum number of edges in a non-Hamiltonian n-vertex simple graph is $\binom{n-1}{2}+1$.* The graph consisting of an $(n-1)$-clique plus a single pendant edge has $\binom{n-1}{2}+1$ edges and is not Hamiltonian. For $n=2$, this graph is K_2 and is trivially the largest. For $n=3$, exceeding the bound requires three edges, and the resulting simple graph can only be K_3.

For $n > 3$, suppose that $e(G) > \binom{n-1}{2}+1$. Thus $e(\overline{G}) < n-2$, and $\overline{G}$ has a vertex v of degree at most 1. In G, we have $d(v) \ge n-2$. Since $\binom{n-1}{2} - (n-2) = \binom{n-2}{2}$, the induction hypothesis provides a Hamiltonian cycle C in $G-v$. Since v has at most one nonneighbor in $V(G)-\{v\}$ and $n-1 \ge 3$, vertex v has two consecutive neighbors on C. Hence we can enlarge C to include v and obtain a spanning cycle in G.

7.2.28. *Generalization of the edge bound.*

a) If $f(i) = 2i^2 - i + (n-i)(n-i-1)$ and $n \ge 6k$, then on the interval $k \le i \le n/2$, the maximum value of $f(i)$ is $f(k)$. The derivative is $6i-2n$, and the second derivative is 6. Since the second derivative is always positive, the maximum occurs only at the endpoints. The minimum is at $i = n/3$ (where the derivative is 0), and the parabola is symmetric around $i = n/3$. Hence to show that $f(k) \ge f(n/2)$ and complete the proof, it suffices to show that k is farther from the axis $n/3$ than $n/2$ is. This is the inequality $n/3 - k \ge n/2 - n/3$, which is equivalent to the hypothesis $n \ge 6k$.

b) If $\delta(G) = k$ and G has at least $6k$ vertices and has more than $\binom{n-k}{2}+k^2$ edges, then G is Hamiltonian. By Chvátal's Condition, it suffices to show that $d_i > i$ or $d_{n-i} \ge n-i$ for every $i < n/2$, where $d_1 \le \cdots \le d_n$ are the vertex degrees of G. If this condition fails for some i, we have $d_i \le i$; this requires $i \ge k$, since every vertex has degree at least k. Hence we may assume $k \le i < n/2$.

The number of edges is half the degree-sum; hence the hypothesis guarantees a degree-sum greater than $(n-k)(n-k-1)+2k^2$. If Chvátal's Condition fails for some i, we know that $d_1 = k$, the $i-1$ next smallest degrees are at most i, the degrees $d_{i+1},\ldots,d_{n-i}$ are at most $n-i-1$, and the i largest degrees are at most $n-1$. This places an upper bound on the degree-sum of $k+i(i-1)+(n-2i)(n-i-1)+i(n-1) = k+2i^2-i+(n-i)(n-i-1)$. We now have $2k^2+(n-k)(n-k-1) < \sum d_i \le k+2i^2-i+(n-i)(n-i-1)$, but this contradicts the conclusion $f(k) \ge f(i)$ from part (a).

7.2.29. *If G is simple with vertex degrees $d_1 \le d_2 \le \cdots \le d_n$ and $\overline{G}$ has vertex degrees $d_1' \le d_2' \le \cdots \le d_n'$, then $d_m \ge d_m'$ for all $m \le n/2$ guarantees that G has a Hamiltonian path.* Chvátal proved (Theorem 7.2.17) that if $d_m \ge m$ or $d_{n-m+1} \ge n-m$ for all $m < (n+1)/2$, then G has a Hamiltonian path.

Consider $m < (n+1)/2$, which is equivalent to $m \le n/2$ when m,n are integers. We have $d_m' = n-1-d_{n+1-m}$. Hence $d_m \ge d_m'$ implies that $d_m \ge n-1-d_{n+1-m}$, or $d_m + d_{n+1-m} \ge n-1$. If $d_m \le m-1$, then $d_{n+1-m} \ge$

$n - m$. Thus $d_m \ge m$ or $d_{n+1-m} \ge n - m$. Since this holds for all m in the desired range, we have proved that Chvátal's Condition for spanning paths is satisfied by G, and we conclude that G has a Hamiltonian path.

If G is isomorphic to $\overline{G}$, then $d_m = d'_m$ for all m, and the condition holds. Thus every self-complementary graph has a Hamiltonian path.

7.2.30. *Chvátal's Theorem implies Ore's Theorem.* It suffices to show that Ore's Condition implies Chvátal's Condition, because then Chvátal's Theorem implies that the graph is Hamiltonian.

Consider $i < n/2$. If $d_i \le i$, then a vertex v with degree d_i has at least $n - 1 - i$ nonneighbors. By Ore's Condition, each nonneighbor has degree at least $n - i$. Hence at least $n - 1 - i$ vertices have degree at least $n - i$. Thus $d_{i+2} \ge n - i$. Since $i + 2 \le n/2 + 1 \le n - i$, we have $d_{n-i} \ge n - i$. Thus $d_i > i$ or $d_{n-i} \ge n - i$, and Chvátal's Condition holds.

7.2.31. *If G has at least $\alpha(G)$ vertices of degree $n(G) - 1$, then G is Hamiltonian.* Any set whose deletion separates G must include all vertices of degree $n(G) - 1$. Hence $\kappa(G)$ is at least the number of vertices of degree $n(G) - 1$, and the specified condition implies $\kappa(G) \ge \alpha(G)$. This implies that G is Hamiltonian, by the Chvátal-Erdős Theorem.

7.2.32. Let $d_1 \le \cdots \le d_n$ be the degree sequence of an X, Y-bigraph G with equal-size partite sets. Let G' be the supergraph of G obtained by adding edges so that $G[Y] = K_{n/2}$.

a) G is Hamiltonian if and only if G' is Hamiltonian, and the degree sequence of G' is formed by adding $n/2 - 1$ to the degrees of vertices in Y and moving them (in order) to the back. Let X, Y be the partite sets. Because $|X| = |Y|$, we can add arbitrary edges within Y without affecting whether G is Hamiltonian; the independence of X forces a Hamiltonian cycle to alternate between the sets anyway. Hence we add a clique on Y to obtain a graph G' that is Hamiltonian if and only if G is Hamiltonian. This raises the degree of each vertex in Y by $n/2 - 1$.

b) If $d_k > k$ or $d_{n/2} > n/2 - k$ whenever $k \le n/4$, then G is Hamiltonian. By part (a), it suffices to show that this condition on G implies that G' satisfies Chvátal's Condition. In G', the vertices of Y are the $n/2$ vertices of largest degree (otherwise, G has a vertex in Y with degree 0 and a vertex in X with degree $n/2$, which is impossible). If there is a value $k < n/2$ such that G' has k vertices of degree at most k and $n - k$ vertices of degree less than $n - k$, then G has k vertices in X with degree at most k and $n/2 - k$ vertices in Y with degree less than $n/2 - k + 1$ (at most $n/2 - k$). If $i = \min\{k, n/2 - k\}$, then G has i vertices of degree at most i and $i + n/2 - i = n/2$ vertices of degree at most $n/2 - i$, contradicting the given condition. Thus there is no such k, G' satisfies Chvátal's condition, and G' and G are both Hamiltonian.

7.2.33. *If G has n vertices and $e(G) \ge \binom{n-1}{2} + 2$, then G is Hamiltonian; if $e(G) \ge \binom{n-1}{2} + 3$, then G is Hamiltonian-connected.* We prove the two statements simultaneously by induction on n. The statements are vacuous for very small graphs. For $n = 4$, both conditions can hold; $K_4 - e$ is Hamiltonian and K_4 is Hamiltonian-connected. For the induction step, suppose that $n > 4$. For clarity, we write the conditions as $e(\overline{G}) \le n - 4$ for a Hamiltonian-connected graph and $e(\overline{G}) \le n - 3$ for a Hamiltonian graph.

If $e(\overline{G}) \le n - 4$, then we seek a Hamiltonian x, y-path, where x, y are arbitrary vertices of G. If x is not isolated in $\overline{G}$, then $e(\overline{G - x}) \le n - 5$, and the induction hypothesis guarantees that $G - x$ is Hamiltonian-connected. Since at most $n - 4$ edges are missing, we can choose $z \in N(x) - \{y\}$ and add xz to a Hamiltonian z, y-path in $G - x$ to obtain a Hamiltonian x, y-path in G. If x is isolated in $\overline{G}$, then $e(\overline{G - x}) \le n - 4$, and the induction hypothesis guarantees that $G - x$ is Hamiltonian. We break an edge involving y (say yw) on an arbitrary Hamiltonian cycle in $G - x$ and add the edge wx to obtain the desired Hamiltonian x, y-path in G.

Since a Hamiltonian-connected graph is Hamiltonian (using a Hamiltonian x, y-path when $x \leftrightarrow y$), we may assume for the second statement that $e(\overline{G}) = n - 3$. Hence $2 \le \delta(G)$. Since the complement of a matching is Hamiltonian, we may assume that some vertex x has degree at least 2 in $\overline{G}$. Now $e(\overline{G - x}) \le n - 5$, and the induction hypothesis guarantees that $G - x$ is Hamiltonian-connected. Since $d_G(x) \ge 2$, we can select $y, z \in N(x)$ and add the path z, x, y to a Hamiltonian y, z-path in $G - x$ to complete a Hamiltonian cycle in G.

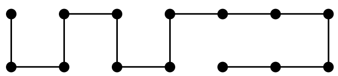
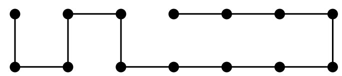
7.2.34. *Hamiltonian-connected graphs - necessary condition.*

a) A Hamiltonian-connected graph G with $n \ge 4$ vertices has at least $\lceil 3n/2 \rceil$ edges. It suffices to show that $\delta(G) \ge 3$, because then $e(G) = \sum d(v)/2 \ge 3n/2$; since the number of edges is an integer, this means $e(G) \ge \lceil 3n/2 \rceil$. If a vertex has degree 0 or 1, there is no Hamiltonian path or no Hamiltonian path without it as an endpoint. If x has degree 2, then since there is no Hamiltonian path that has the neighbors of x as the endpoints (when $n \ge 4$), since the two neighbors of x appear immediately next to x in any Hamiltonian path where x is not the endpoint.

b) If m is odd, then $G = C_m \square K_2$ is Hamiltonian-connected. We phrase the cases for general odd m but illustrate with $C_7 \square K_2$. Express $V(G)$ as $U \cup W$, where $U = \{u_0, \ldots, u_{m-1}\}$ and $W = \{w_0, \ldots, w_{m-1}\}$; thus $G[U] = G[W] = C_m$, and the remaining edges are $\{u_i w_i \colon 0 \le i \le m - 1\}$. We construct a Hamiltonian y, z-path for each pair $y, z \in V(G)$. Since G is vertex-transitive, we may assume that $y = u_0$. By up/down symmetry in the indices, we may assume that $z = w_{2j}$ when $z \in W$ and $z = u_{2j+1}$ when $z \in U$. (Note: There are many other ways to describe the cases.)

Case 1: *A u_0, w_{2j}-path.* Begin the path by zig-zagging: $u_0, w_0, w_1, u_1, \ldots$. The step is from U to W on even indices and from W to U on odd indices, thus finishing at u_{2j-1} after w_{2j-1}. Now finish the path by traversing U from u_{2j-1} to u_{m-1} and W from w_{m-1} to w_{2j}.

Case 2: *A u_0, u_{2j+1}-path.* Begin in the same way, stopping the zig-zag at w_{2j}. Now finish the path by traversing W from w_{2j} to w_{m-1} and U from u_{m-1} to u_{2j+1}.

7.2.35. *Hamiltonian-connected graphs - sufficient condition.*

a) A simple n-vertex graph G is Hamiltonian-connected if $\delta(G) > n/2$. We must guarantee a Hamiltonian path from each vertex to every other; let u, v be an arbitrary pair of vertices in G. Let G' be the graph obtained from G by adding a vertex w and adding the edges wu, wv. Then G has a Hamiltonian u, v-path if and only if G' has a Hamiltonian cycle. We prove that G' has a Hamiltonian cycle.

A graph is Hamiltonian if and only if its closure is Hamiltonian. The closure of G' contains a clique induced by the vertices of G, because $d_G(x) + d_G(y) \geq n(G) + 1 = n(G')$ when x and y are nonadjacent vertices of G. After adding all the edges on $V(G)$, the degrees are high enough that the edges to w will also be added. Thus the closure of G' is a clique and G' is Hamiltonian, which yields the spanning u, v-path in G.

b) An n-vertex graph with minimum degree $n/2$ that is not Hamiltonian-connected. Let G_n consist of two cliques of order $n/2 + 1$ sharing an edge xy. The minimum degree is $n/2$, and because $\{x, y\}$ is a separating 2-set, there is no Hamiltonian path with endpoints x, y.

Another example is $K_{n/2,n/2}$. Since a spanning path must alternate between the partite sets and the total number of vertices is even, there is no spanning x, y-path when x and y lie in the same partite set.

7.2.36. *Las Vergnas' Condition.* The condition, which implies that the n-closure is complete, is the existence of a vertex ordering $v_1, \ldots, v_n$ for which there is no nonadjacent pair v_i, v_j such that $i < j$, $d(v_i) \leq i$, $d(v_j) < j$, $d(v_i) + d(v_j) < n$, and $i + j \geq n$.

a) Chvátal's Condition implies Las Vergnas' Condition. Consider a vertex ordering with $d(v_i) = d_i$ and $d_1 \leq \cdots \leq d_n$. If Las Vergnas' condition fails, then every ordering (including this one) has a bad pair (i, j) of indices. Badness requires $i < j$, $i + j \geq n$, $d_i \leq i$, $d_j < j$, and $d_i + d_j < n$. Given such a j, choose a minimal i satisfying these properties.

Since $i < j$ and $i + j \geq n$, we have $j > n/2$. If $i + j = n$, then Chvátal's Condition yields $d_i > i$ or $d_j \geq j$, a contradiction. If $i + j > n$, then $d_i = i$,

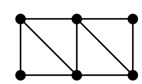
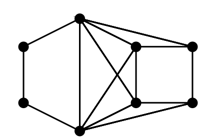
since otherwise $d_{i-1} \leq d_i \leq i - 1$, and the properties would hold also for the pair $(i - 1, j)$. If $i \geq n/2$, we now have $d_i + d_j \geq n$, again a contradiction. Thus Chvátal's Condition yields $d_{n-i} \geq n - i$. Now the final contradiction:

$$n = i + (n - i) \leq d_i + d_{n-i} \leq d_i + d_j < n.$$

b) Las Vergnas' Condition on small graphs. The smaller graph below has degree sequence 223344. Chvátal's Condition fails at $i = n/2 - 1 = 2$, since $d_2 = 2$ and $d_4 < 4$. The Hamiltonian closure is complete, because each 2-valent vertex receives an edge to a non-neighbor of degree 4, and then minimum degree 3 allows every edge to be added.

To verify Las Vergnas' Condition, place the vertices in increasing order of degree, but choose v_2 and v_4 to be adjacent vertices of degrees 2 and 3. A violation the condition requires a pair of nonadjacent vertices with degree sum less than 6 and index sum at least 6. Such degrees must be 2,3 or 2,2. The index sum for the latter pair is 3. The index sum for the former pair is at least 6 only for v_2 and v_4, but these vertices are adjacent.

With degrees 22334455, the larger graph fails Chvátal's Condition for $i = n/2 - 1 = 3$. The closure raises the degrees to 22444466 and then to 33555577, and then all remaining edges can be added. Suppose that Las Vergnas' condition holds, with $v_1, \ldots, v_8$ a suitable ordering. The 2-valent vertices are independent of the 3-valent vertices and 4-valent vertices. Since some 2-vertex has index at least 2, no 3-vertex or 4-vertex has index at least 6. With both 3-vertices and both 4-vertices among the first 5, some 2-vertex has index at least 6. Now no 3-vertex has index at least 3, and no 4-vertex has index at least 4. This forces four vertices into the first three positions, which is impossible.

7.2.37. *Lu's Theorem implies the Chvátal–Erdős Theorem.* Lu proved that if $t(S) \geq \alpha(G)/n(G)$ whenever $\varnothing \neq S \subset V(G)$, where $t(S) = \frac{|\overline{S} \cap N(S)|}{|\overline{S}|}$, then G is Hamiltonian. To show that this implies the Chvátal–Erdős Theorem, it suffices to show that the condition $\kappa(G) \geq \alpha(G)$ implies Lu's Condition.

Let $k = \kappa(G)$. If $|S| \geq n(G) - k$, then $t(S) = 1$. If $|S| < n(G) - k$, then $t(S) \geq k/(n(G) - |S|)$. Since $n(G) > n(G) - |S|$, this yields $t(S)n(G) > k = \kappa(G)$. Hence $\kappa(G) \geq \alpha(G)$ implies $\theta(G)n(G) \geq \alpha(G)$.

7.2.38. *A connected graph G with $\delta(G) = k \ge 2$ and $n(G) > 2k$ has a path of length at least $2k$.*

a) The vertices of a maximal path P in G form a cycle in some order if the path has at most $2k$ vertices. Let u, v be the endpoints of P, and let $H = G[V(P)]$. Since P is maximal, $N(u)$ and $N(v)$ are contained in $V(H)$. Hence $d_H(u) + d_H(v) \ge 2k \ge n(H)$, and $H + uv$ is Hamiltonian. By Ore's Theorem, H is Hamiltonian; that is, the vertices of P form a cycle.

b) G has a path with at least $2k + 1$ vertices. Choose a longest path P in G. If P has at most $2k$ vertices, then part (a) guarantees a cycle through $V(P)$. Since G is connected, there is an edge from $V(P)$ to $V(G) - V(P)$. Together with the vertices of P in the order of the cycle, this gives a longer path, contradicting the choice of P.

c) Quadratic algorithm for finding a Hamiltonian cycle if $d(u) + d(v) \ge n(G)$ whenever $u \nleftrightarrow v$. Find a maximal path P (greedily, in linear time). If the endpoints u, v are adjacent, then there is a cycle C through $V(P)$. Otherwise, the condition $d(u) + d(v) \ge n(G) \ge n(P)$ forces a neighbor of u following a neighbor of v by the usual switch argument; again we have a cycle C through $V(P)$. By following P, we find C in linear time.

The condition $d(u) + d(v) \ge n(G)$ also forces diameter at most 2. If $V(C) \ne V(G)$, we select a vertex not on C. Either it has a neighbor on C, or it has a neighbor with a neighbor on C. Thus we find an edge from $V(C)$ to $V(G) - V(C)$ in linear time. This gives us a path longer than P, which we extend greedily through the new vertex. We repeat the process.

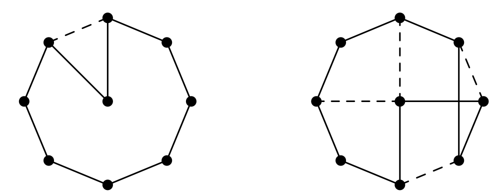
Each iteration takes only linear time, and the length of P increases fewer than n times, so in quadratic time we find a spanning cycle of G.

7.2.39. ($\bullet$) Prove that if a simple graph G has degree sequence $d_1 \le \cdots \le d_n$ and $d_1 + d_2 < n$, then G has a path of length at least $d_1 + d_2 + 1$ unless G is the join of $n - (d_1 + 1)$ isolated vertices with a graph on $d_1 + 1$ vertices or $G = pK_{d_1} \vee K_1$ for some $p \ge 3$. (Ore [1967b])

7.2.40. *Every $2k$-regular simple graph G on $4k + 1$ vertices is Hamiltonian (using Dirac's theorem that a 2-connected simple graph has a cycle of length at least 2δ).* To apply Dirac's theorem, we first must show that G is 2-connected. Suppose G has a vertex x whose removal leaves a disconnected graph (this includes the case where G is not connected). Let H_1 be the smallest component of $G - x$, and let H_2 be another component. H_1 has at most $2k$ vertices. If any vertex in H_1 was not joined to x in G, then it still has degree $2k$ in $G - x$. This is impossible, since H_1 has at most $2k$ vertices and G is simple. So, H_1 must have exactly $2k$ vertices; all joined to x. This means that H_2 also has at most $2k$ vertices. The same argument requires that every vertex of H_2 have x as a neighbor in G, but this assigns $4k$ neighbors to x. This contradiction means there could not have been such

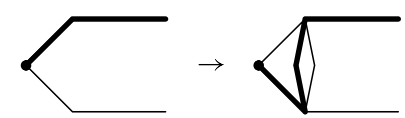
an x, and G is 2-connected.

Dirac's theorem now implies that G has a cycle C of length at least $4k$. If G is not Hamiltonian, let x be the vertex not included in C, and let X and Y denote the neighbors and non-neighbors of x. If X has two adjacent vertices on C, then we can visit x between them and augment C to a Hamiltonian cycle (see first figure below). Since X has half the vertices on C, C therefore alternates between X and Y. Now, if any two vertices of Y are adjacent, then it is possible to form a Hamiltonian cycle as indicated in the second figure below. On the other hand, if the only neighbors of vertices in $Y \cup \{x\}$ are the vertices in X, they must each neighbor every vertex in X (since there are only $2k$ of them), and thus every vertex in Y has $2k + 1$ neighbors. Since this contradicts $2k$-regularity, one of the possibilities mentioned above, in which G is Hamiltonian, must occur. (Note: the fact that an $(n-1)/2$-regular graph is Hamiltonian if $n \equiv 1 \pmod 4$ is just a slight improvement over minimum degree $n/2$. It has in fact been proved that an $n/3$-regular graph is Hamiltonian).



7.2.41. *Scott Smith's Conjecture (for $k \le 4$ only).*

a) Let G be a 4-regular graph with l vertices that is the union of two cycles, and suppose that $l \le 3$. If G' is a 4-regular graph with $l + 2$ vertices obtained from G by subdividing one edge from each of the cycles forming G and adding a double edge between the two new vertices, then G' is also the union of two spanning cycles. Since $l \le 3$, the two subdivided edges in G share an endpoint. Hence the new edges can be traversed as detours when following the subdivided edges on the old cycles, as shown below.



b) For $2 \le k \le 4$, any two longest cycles C and D in a k-connected graph H have at least k common vertices. Suppose that C and D have l common vertices, where $l < k$. Let $S = V(C) \cap V(D)$. Since $k > l$, there is a path P from $V(C) - S$ to $V(D) - S$ in H. Discard all edges not in $C \cup D \cup P$. In the remaining graph, replace threads (maximal paths whose internal vertices have degree 2) with single edges. Now $C \cup D \cup P$ is a subdivision of the resulting graph G'. Also, let G be the graph obtained from $C \cup D$ by the same replacement operation, so G is a subdivision of $C \cup D$. Note that G has l vertices. Now G' is obtained from G by the operation in part (a). By part (a), G' is the union of two cycles. These cycles correspond to cycles in H whose union is $C \cup D \cup P$. The total lengths of these two cycle exceeds that of C and D together. This contradicts the hypothesis that C and D were longest cycles, so C and D must have k common vertices.

7.2.42. *The Eulerian circuit graph is Hamiltonian when* $\Delta(G) = 4$. For convenience, we use *tour* here to mean Eulerian circuit. Let G be a loopless Eulerian multigraph, and let V' be the set of tours of G. We treat tours as equivalent if they have the same pairs of consecutive edges (hence a tour and its reversal are equivalent). Two tours are adjacent in G' if and only if one can be obtained from the other by reversing the direction of a proper closed subtour, which is the portion between some two visits to one vertex. Since every tour passes through a vertex v exactly $d(v)/2$ times, each vertex of V' thus has $\sum_{v\in V(G)} \binom{d(v)/2}{2}$ neighbors in G', and the Eulerian circuit graph is regular. (Its degree is generally too small to apply general results about spanning cycles.)

Using induction on the number of 4-valent vertices, we prove that G' is Hamiltonian when $\Delta(G) = 4$ and G' has at least three vertices. The graph G' is l-regular, where l is the number of 4-valent vertices in G (there is one switch available at each 4-valent vertex). If $l = 0$, then $G' = K_1$; if $l = 1$, then $G' = K_2$. These graphs have spanning paths. If G has two 4-valent vertices x and y, then G' is a 3-cycle when G has four x,y-paths, and G' is a 4-cycle when G has two x,y-paths.

To facilitate the induction step, we prove the stronger statement that if $l \ge 2$, then G' has a spanning cycle through any specified edge t_1t_2, where t_1 and t_2 are adjacent tours. We have verified this for $l = 2$. For the induction step, consider $l \ge 3$. Let v be the vertex where the reversal occurs to obtain t_2 from t_1. Since $\Delta(G) = 4$, we have $d(v) = 4$, with incident edges $e_0, e_1, e_2, e_3 \in E(G)$.

Let V_i' be the subset of $V(G')$ consisting of tours in which the visit through v that uses e_0 also traverses e_i, for $i \in \{1,2,3\}$. Each vertex of G' lies in exactly one of these sets; call this the *v-partition* of G'. Let $G_i' = G'[V_i']$. The induced subgraph G_i' is isomorphic to the Eulerian circuit graph of the graph G_i obtained from G by splitting v into two 2-valent vertices x, x', where the edges incident to x are $\{e_0, e_i\}$, and those incident to x' are the other two edges at v. For any tour, the tour adjacent to it by the reversal at v lies in a different set in the v-partition. Reversal at any other vertex does not change the pairing at v and thus reaches another tour in the same block of the v-partition. Therefore, the edges of G' that join two sets in $\{V_1', V_2', V_3'\}$ form a perfect matching of G' and correspond to reversals at v. Call these the *cross-edges*.

If v is a cut-vertex, then because all vertex degrees are even, v has two edges to each component of $G-v$, say e_0 and e_3 to one component and e_1 and e_2 to the other. In this case, G_3 is empty and $G_1 \cong G_2$, with corresponding vertices joined by an edge. That is $G' = G_1 \square K_2$. Since G_1 is Hamiltonian or is a single edge, G' has a spanning cycle through any cross-edge (see Exercise 7.2.17).

If v is not a cut-vertex, then each set in the v-partition is nonempty. The reason is that $G - v$ is connected, and hence the graph obtained from $G - v$ by adding vertices x and x' whose neighbors are the endpoints of $\{e_0, e_i\}$ and the other neighbors of v, respectively, is connected. This is precisely the graph G_i; being even and connected, it is Eulerian. (The sets V_i need not have the same size, as shown by letting G be the 4-regular graph consisting of K_4 with an extra copy of two disjoint edges, where the sizes of the V_i are 16,16,6.)

By the induction hypothesis, each G_i has a Hamiltonian cycle through any specified edge (or a path through the single edge, if it has two vertices). Thus it suffices to find a cycle C in G' that contains t_1t_2 and alternates between cross-edges and non-cross-edges, using exactly one edge within each G_i (consecutive cross-edges are acceptable if $G_i = K_1$). Using the cross-edges on C plus a Hamiltonian path of each G_i joining its vertices on C yields a Hamiltonian cycle of G containing t_1t_2

Let t_1t_2 be the specified edge, using a reversal at v. We may assume that $t_1 \in G_1$ and $t_2 \in G_2$. The vertex v cuts t_1 into two segments. Since v is not a cut-vertex, the two segments share another vertex u, which therefore has degree (at least) 4. The desired cycle C is now obtained by alternating reversals at v and u.

To list the tours of C explicitly, break t_1 into four successive trails with endpoints v and u; that is, express t_1 as $[v,Q,u,R,v,S,u,T]$, in the sense that A starts at v and ends at u, etc. We may further assume that Q starts with e_1, R ends with e_2, and T ends with e_0, so that $t_1 \in G_1$ and $t_2 \in G_2$. Let $\overline{Q}, \overline{R}, \overline{S}, \overline{T}$ denote the reversals of these trails. For the six successive tours on C, we have

$$\begin{array}{ll} t_1 = [v,Q,u,R,v,S,u,T] \in G_1 & t_4 = [v,S,u,R,v,\overline{Q},u,T] \in G_3 \\ t_2 = [v,\overline{R},u,\overline{Q},v,S,u,T] \in G_2 & t_5 = [v,S,u,\overline{Q},v,\overline{R},u,T] \in G_3 \\ t_3 = [v,\overline{R},u,\overline{S},v,Q,u,T] \in G_2 & t_6 = [v,Q,u,\overline{S},v,\overline{R},u,T] \in G_1 \end{array}$$

With this approach, the construction of the desired Hamiltonian cycle is easy. The approach also works for the general case without limits on $\Delta(G)$. For the general problem, Zhang and Guo [1986] use three cases like this when $d(v) = 6$ and two cases when $d(v) = 2t > 6$.

7.2.43. *For a graph G, the Eulerian circuit graph G' of Exercise 7.2.42 is $(\sum_{v\in V(G)}\binom{d(v)/2}{2})$-regular, which is not enough to apply general results on Hamiltonicity of regular graphs.* The formula for the degree is obtained in the first paragraph of the solution to Exercise 7.2.42. For a given Eulerian orientation, Theorem 2.2.28 computes the number of Eulerian circuits as $c\prod_v(d(v)/2-1)!$, where c is the number of in-trees or out-trees from any vertex. Already this number is very much bigger than the degree, and in addition there are many Eulerian orientations. Summing over all the orientations and dividing by 2 counts the vertices in G'. Hence $n(G')$ is hugely bigger than the degree, not bounded by a factor of 2 or 3 times the degree, which would be needed to apply general sufficiency conditions for Hamiltonian cycle. This explains why a specialized structural argument is needed in Exercise 7.2.42.

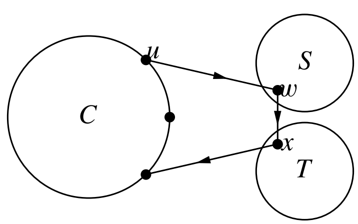
7.2.44. *Every tournament has a Hamiltonian path.*

Proof 1. If a directed path P of maximum length omits x, then $u\to x\to v$, where u and v are the origin and terminus of P. Considering the vertices of P in order, there must therefore be a consecutive pair y,z on P such that $y\to x\to z$. This detour absorbs x to form a longer path. Hence a path of maximum length in a tournament omits no vertex.

Proof 2. The result follows immediately from the Gallai-Roy Theorem, since $\chi(K_n)=n$ and every tournament is an orientation of K_n.

7.2.45. *Strong tournaments are Hamiltonian.* We prove first that a vertex on a k-cycle is also on a $(k+1)$-cycle, if $k<n$. suppose C is a k-cycle containing u. If some vertex w not on C has both a predecessor and a successor on C, then there is a successive pair v_i,v_{i+1} on C such that $v_i\to w$ and $w\to v_{i+1}$, and we can detour between them to pick up w and obtain a longer cycle through u.

Hence we may assume that every vertex off C has no successors on C or no predecessors on C; let these sets of vertices be S and T, respectively. Since there is a vertex not on C and the tournament is strong, there must be an edge wx from S to T. We can leave C at u and detour through wx, skipping the successor of u on C, to obtain a cycle of length $k+1$ through u.



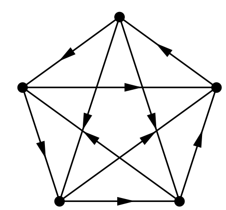
If a tournament is strong, then for every edge uv there is also a v,u-path, which together with uv completes a cycle through u. Successive

application of the statement above turns this into a spanning cycle. (In fact, by considering chords we can first get down to a 3-cycle, and then we obtain a cycle of every length through u).

7.2.46. *If G is a 7-vertex tournament in which every vertex has outdegree 3, then G has two disjoint cycles.* If G is not strong, then G has a cut $[S,\overline{S}]$ with every vertex of S pointing to every vertex of $\overline{S}$. Since outdegrees in S are 3, $|\overline{S}|\le 3$, but now vertices of $\overline{S}$ don't have enough successors.

Hence G is strong. By Exercise 7.2.45, G has a 3-cycle C. Let $H=G-V(C)$. If H has a cycle, we are done. Otherwise, H is a 4-vertex transitive tournament: vertices $v_0,\dots,v_3$ with $v_i\to v_j$ when $i<j$. Outdegree 3 implies that v_i has i successors in $V(C)$, for each i. Let u be the successor of v_1 in $V(C)$; we have a 3-cycle with vertices (v_0,v_1,u). Since v_2 has two successors in C, we can choose w as the predecessor of v_2 in $V(C)$. Now we obtain a second 3-cycle with vertices (v_2,v_3,w).

7.2.47. (+) Prove that every tournament has a Hamiltonian path with the edge between beginning and end directed from beginning to end, except the cyclic tournament on three vertices and the tournament T_5 on five vertices drawn below. (Grünbaum, in Harary [1969, p211])



(Hint: this can be proved by induction, which requires a bit of care for invoking the induction hypothesis to prove the claim for six vertices. In all cases, find the desired configuration or $G=T_5$.)

7.2.48. *Sharpness of Ghouila-Houri's Theorem.* We construct for each even n a n-vertex digraph D that is not Hamiltonian even though it satisfies "at most one copy of each ordered pair is an edge" and $\min\{\delta^-(D),\delta^+(D)\}\ge n/2$. Take two sets A and B of size $n/2$. Add edges $A\square A$ and $B\square B$ (hence there is a loop at each vertex and opposed edges joining each pair in one set), and add a matching from A to B. Each vertex has indegree and outdegree $n/2$ within its own set, but the full digraph is not strongly connected.

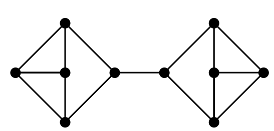
7.2.49. *Ghouilà-Houri's Theorem implies Dirac's Theorem for Hamiltonian cycles.* Suppose that a simple graph G satisfies Dirac's Condition $\delta(G)\ge n(G)/2$. From G we form a digraph D be replacing each edge with a pair of oppositely directed edges having the same endpoints. Thus

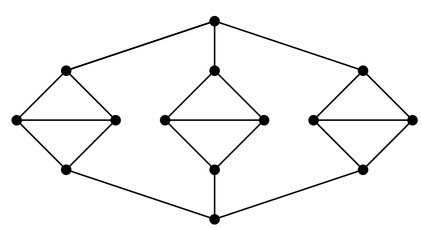
$d_D^+(x) = d_D^-(x) = d_G(x)$ for all $x \in V(G)$. Since $n(D) = n(G)$, we obtain $\min\{\delta^+(D), \delta^-(D)\} = \delta(G) \geq n(G)/2 = n(D)/2$. Hence Ghouilà-Houri's Theorem implies that D is Hamiltonian. Since a Hamiltonian cycle C in D does not use two oppositely directed edges from G, the edges of G giving rise to the edges in C also form a Hamiltonian cycle in G.

7.3. PLANARITY, COLORING, & CYCLES

7.3.1. *Every Hamiltonian 3-regular graph has a Tait coloring.* A 3-regular graph has even order, so two colors can alternate along a Hamiltonian cycle C. Deleting $E(C)$ leaves a 1-factor to receive the third color.

7.3.2. *Examples of 3-regular simple graphs:* a) planar but not 3-edge-colorable. b) 2-connected but not 3-edge-colorable. c) planar with connectivity 2, but not Hamiltonian. For part (b), the Petersen graph is an example. For (a) and (c), suitable graphs appear below. Regular graphs with cut-vertices are not 1-factorable, and graphs having 2-cuts that leave 3 components are not Hamiltonian.





7.3.3. *Every maximal plane graph other than K_4 is 3-face-colorable.* With fewer than four vertices, the maximal plane graphs have fewer than three faces. For larger graphs, every face is a triangle, so the dual is 3-regular. Since the dual is planar, it does not contain K_5. Hence the dual is not a complete graph, and by Brooks' Theorem it is 3-colorable. This becomes a proper 3-coloring of the original graph.

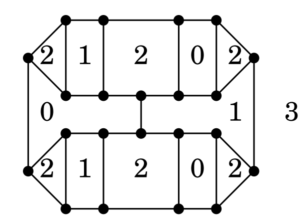
7.3.4. *Every Hamiltonian plane graph G is 4-face-colorable.* It suffices to show that the faces inside C can be properly 2-colored, since the same argument applies to the faces outside C using two other colors. View the union of C and the edges embedded inside C as an outerplane graph H; all the vertices are on the outer face.

Proof 1. In the dual H^*, the bounded faces in H become vertices. We claim that the subgraph of H^* induced by these vertices is a tree T^*. If they induce a cycle, then that cycle lies inside C in the embedding of G and encloses a face of H^*, which in turn contains a vertex of H. This is a vertex of G that does not lie on the outer face of H, which contradicts C being a spanning cycle.

Proof 2. We properly 2-color the faces inside C using induction on the number of edges inside. With no such edges, H has one bounded face and is 1-colorable. Otherwise, let e be an inside edge whose endpoints are as close together as possible on C. By the choice of e, there is a face whose boundary consists of e and edges of C. This face F is adjacent to only one other, F'. Deleting e merges F into F' in a smaller graph H'. By the induction hypothesis, H' has a proper 2-face-coloring f'. To obtain the proper 2-face-coloring f of H, let f give the same color as f' for each face other than F and give F the opposite color from $f(F')$.

7.3.5. *A 2-edge-connected plane graph is 2-face-colorable if and only if it is Eulerian.* Let G be a 2-edge-connected plane graph; note that $(G^*)^* = G$. We have G 2-face-colorable if and only if G^* is bipartite, which by $(G^*)^* = G$ and Theorem 6.1.16 is equivalent to G being Eulerian.

7.3.6. *The graph below is 3-edge-colorable.* By Tait's Theorem, it suffices to show that the graph is 4-face-colorable.



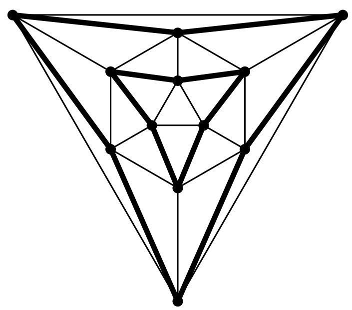
7.3.7. Let G be a plane triangulation.

a) The dual G^* has a 2-factor. The dual of a plane triangulation is 3-regular and has no cut-edge (since G has no loop). Hence the dual has a 1-factor, by Petersen's Theorem (Corollary 3.3.8). Deleting the 1-factor leaves a 2-factor.

b) The vertices of G can be 2-colored so that every face has vertices of both colors. Given the 2-factor F of G^* resulting from part (a), we can 2-color the faces of the dual by given each face the parity of the number of cycles in F that contain it. This assigns colors to the vertices of G, which correspond to the faces of G^*.

Each face of G correspond to a vertex v of G^*, with degree 3. The 2-factor F uses two edges at v, lying on one cycle of F. Hence each face of G is entered by one cycle of F. This cycle cuts one of the vertices of F from the other two, and hence the face has vertices of both colors.

7.3.8. *The icosahedron is Class 1.* The graph is 5-regular; we describe a proper 5-edge-coloring. Show in bold is a 2-factor consisting of even cycles; on this we use two colors. For the remaining three colors, we color by the angle in the picture. Color 0 goes on the six edges that are vertical or horizontal. Colors 1 and 2 go on the edges obtained by rotating this 1-factor by 120 or 240 degrees in the picture.



7.3.9. *Every proper 4-coloring of the icosahedron uses each color exactly 3 times.* The icosahedron has 12 vertices; it suffices to show that it has no independent set of size 4. In the figure above, an independent set takes at most one vertex from the inner triangle, one vertex from the outer triangle, and at most three from the 6-cycle C between them. If it takes three vertices from C, then they alternate on C and include neighbors of all other vertices Two opposite vertices on C also kill off the rest. Two vertices at distance 2 along C kill off one triangle but leave one vertex on the other triangle that can be added.

7.3.10. *By Whitney's result that every 4-connected planar triangulation is Hamiltonian, the Four Color Problem reduces to showing that every Hamiltonian planar graph is 4-colorable.* The Four Color Problem reduces to showing that triangulations are 4-colorable. Let S be a minimal separating set in a triangulation G; we show first that $|S| \geq 3$. Each vertex x of S has a neighbor in each component of $G - S$. Since there is no edge joining two components of $G - S$ and every face is a triangle, in the embedding of G edges must emerge from x between edges to different components of $G - S$. These edges go to other vertices of S. Hence $G[S]$ has minimum degree at least two, and $|S| \geq 3$.

If $|S| = 3$, then $\delta(G[S]) \geq 2$ implies that S is a clique. Hence a proper 4-coloring of each S-lobe of G uses distinct colors on S, and we can permute the names of the colors to agree on S. This yields a proper 4-coloring of G. Hence a minimal planar triangulation that is not 4-colorable must be 4-connected. Since every such graph is Hamiltonian, it suffices to show that Hamiltonian planar graphs are 4-colorable.

7.3.11. *Highly connected planar graphs.* The icosahedron is 5-connected. The symmetry of the solid icosahedron is such that two vertices at distance d in the graph can be mapped into any other pair at distance d by rotating the solid. Hence it suffices to consider one pair at distance d, for each d, and show that they are connected by five pairwise internally disjoint paths. This can be done on any drawing of the graph.

Since every planar graph has a vertex of degree at most 5, there is no 6-connected planar graph.

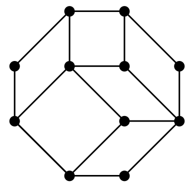
7.3.12. *A plane triangulation has a vertex partition into two sets inducing forests if and only if the dual is Hamiltonian.* Every plane triangulation F is connected, so $(F^*)^* = F$. Let $G = F^*$.

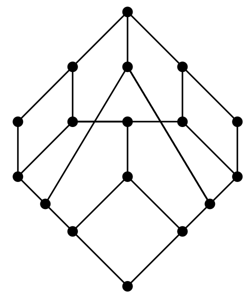
Let G be a Hamiltonian plane graph. A spanning cycle C is embedded as a closed curve, and the subgraph H of G consisting of C and all edges drawn inside C is outerplanar. In the dual of an outerplane graph H, every cycle contains the vertex for the outer face, since every cycle in H^* encloses a vertex, and thus a cycle in H^* not including the vertex for the outer face in H would yield a vertex of H not on the outer face. We conclude that in G^*, the vertices for faces of H induce a forest. The same argument applies to the graph consisting of C and the edges of G drawn outside G.

Conversely, let F be a plane triangulation with such a vertex partition. Since F is connected, there exist edges joining components in the union of these two forests. We add edges joining components, possibly changing the vertex partition while doing this, until we obtain a vertex partition into two sets $S, \overline{S}$ inducing trees.

Adding any edge from S to $\overline{S}$ yields a spanning tree of G, so $[S, \overline{S}]$ is a bond. Hence the duals of the edges in $[S, \overline{S}]$ form a cycle. We claim that this is a spanning cycle in the dual. It suffices to show that $\left|[S, \overline{S}]\right| = f$. Since F has $3n - 6$ edges and we use $n - 2$ edges in the two trees, we have $2n - 4$ edges from S to $\overline{S}$. By Euler's Formula, this is indeed the number of faces in a triangulation.

7.3.13. *Grinberg's Theorem.* Neither of the graphs below is Hamiltonian. Grinberg's Theorem requires $\sum(i-2)(\phi_i - \phi_i') = 0$, where ϕ_i and ϕ_i' are the number of i-faces inside and outside the Hamiltonian cycle. The plane graph on the left has six 4-faces and one 8-face. Since $2(\phi_4 - \phi_4')$ must be a multiple of 4 and $6(\pm 1)$ cannot be a multiple of 4, there is no way these can sum to 0. Similarly, redrawing the graph on the right yields a plane graph with three 4-faces and six 6-faces. This time $2(\phi_4 - \phi_4')$ cannot be a multiple of 4, but $4(\phi_6 - \phi_6')$ must be; again they cannot sum to 0.

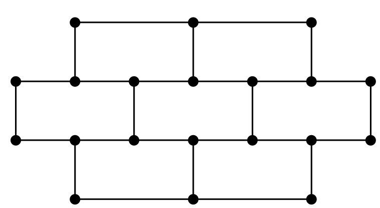




7.3.14. *A non-Hamiltonian graph.* In any spanning cycle of the graph below, both edges incident to a vertex of degree 2 must appear. Applying this to the vertices of degree 2 on the outside face generates a non-spanning cycle that must appear.

Irrelevance and relevance of Grinberg's Theorem. This plane graph has four 5-faces, three 6-faces, and one 14-face. It is possible to choose nonnegative integers f_i' and f_i'' such that $f_5' + f_5'' = 4$, $f_6' + f_6'' = 3$, $f_{14}' + f_{14}'' = 1$, and $\sum(i-2)(f_i' - f_i'') = 0$. This is achieved by $f_5' = f_5'' = 2$, $f_6' = 3$, $f_{14}'' = 1$, and $f_6'' = f_{14}' = 0$. Hence the graph does not violate the numerical conditions of Grinberg's Theorem.

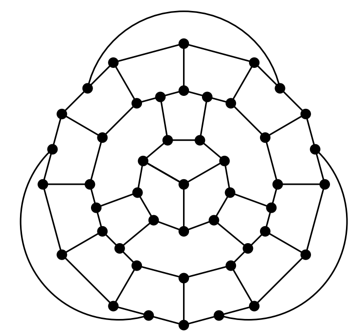
On the other hand, since the four long horizontal edges in the drawing are incident to vertices of degree 2 and therefore must appear in any Hamiltonian cycle, subdividing them once each does not affect whether the graph is Hamiltonian. The new plane graph has seven 6-faces and one 18-face. Since the difference of two numbers summing to 7 is odd, Grinberg's Condition now requires an odd multiple of 4 to equal an even multiple of 4, which is impossible. Hence the graph is not Hamiltonian.



7.3.15. *Proof of Grinberg's Theorem from Euler's Formula.* Let C be a Hamiltonian cycle in a plane graph G, and let f_i' be the number of faces of length i inside C. It suffices to prove that $\sum_i (i-2)f_i' = n-2$, since the same argument applies to the regions outside the cycle. We apply Euler's Formula to the outerplanar graph G' formed by C and the chords inside it.

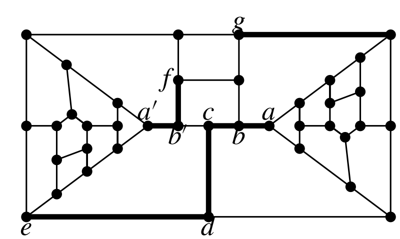
We can rewrite the desired formula as $2 = n - \sum_i if_i' + 2\sum_i f_i'$. Note that $\sum_i if_i'$ counts every internal edge of G' twice and every edge on the cycle once. Thus $\sum_i if_i' = 2e - n$. Also, $\sum_i f_i' = f - 1$, the total number of bounded faces in G'. Thus we want to prove that $4 = 2n - 2e + 2f$, which follows immediately from Euler's Formula.

7.3.16. *The Grinberg graph is not Hamiltonian.* In the plane graph below, all faces have length 5, except for three of length 8 and the one unbounded face of length 9. If it is Hamiltonian and f_i', f_i'' denote the number of faces of length i inside and outside the cycle, respectively, then Grinberg's Condition requires that $3(f_5' - f_5'') + 6(f_8' - f_8'') + 7(f_9' - f_9'') = 0$. This can happen only when $7(f_9' - f_9'')$ is divisible by 3, which is impossible since there is exactly one face of length 9.

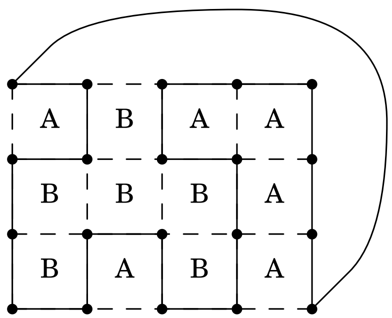


7.3.17. *The smallest known 3-regular planar non-Hamiltonian graph.* The triangular portion on both ends is the subgraph of the Tutte graph called H. Since it has three entrance points here, it must be traversed by a spanning path connecting the entrance points. Example 7.3.6 in the text shows that no such path exists joining the top and bottom entrances.

Hence edges $a'b'$ and ab must be used. By symmetry, we may assume that bc is used. If cb' is used, then completion of a cycle will miss d or the portion on the top. Hence cd is used. Since each copy of H can be visited only once, de must be used. Now the cycle must traverse the left copy of H, emerge at b', and turn up to f. On the other end, the cycle exits the right copy of H at g. Now the cycle cannot be completed without missing one of the common neighbors of f and g.

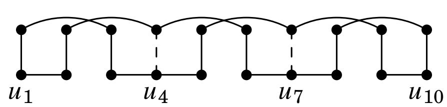


7.3.18. *A Hamiltonian path between opposite corners of a grid splits the squares of the grid into two sets of equal size.* Suppose Q is a Hamiltonian path from the upper-leftmost vertex to the lower-rightmost vertex of $P_m \square P_n$. Adding an edge through the unbounded face from the upper-leftmost vertex to the lower-rightmost vertex completes a Hamiltonian cycle. Each face containing the added edge has length $m+n-1$, and they are on opposite sides of the cycle. By Grinberg's Theorem, then, the number of 4-faces inside the cycle must equal the number of 4-faces outside the cycle. One of these measures the area of the regions escaping to the top and right, and the other measures the area of the regions escaping to the bottom and left.



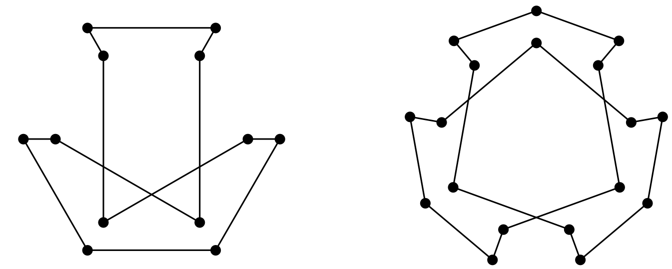
7.3.19. The *generalized Petersen graph* $P(n,k)$ is the graph with vertices $\{u_1,\ldots,u_n\}$ and $\{v_1,\ldots,v_n\}$ and edges $\{u_iu_{i+1}\}$, $\{u_iv_i\}$, and $\{v_iv_{i+k}\}$, where addition is modulo n. The usual Petersen graph is $P(5,2)$ with $\chi'=4$.

a) If $k\equiv 1 \pmod 3$ with $r\ge 1$, then we can cover $\bigcup_{i=j+1}^{j+k}\{u_i,v_i\}$ (k consecutive pairs) using a single cycle involving the edge $v_{j+k}u_{j+k}$. For $r=1$, the remainder of the cycle is the path u_{j+4}, u_{j+3}, v_{j+3}, v_{j+1}, u_{j+1}, u_{j+2}, v_{j+2}, v_{j+4}. To obtain the cycle for $r+1$ from the cycle for r, we replace the edge $v_{j+k}u_{j+k}$ with a path through the six new vertices: v_{j+k}, v_{j+k+2}, u_{j+k+2}, u_{j+k+3}, v_{j+k+3}, v_{j+k+1}, u_{j+k+1}, u_{j+k}. This proves the claim by induction on r.



b) $\chi'(P(n,2))=3$ if $n\ge 6$. The construction of part (a) produces a spanning cycle if $k\equiv 1 \pmod 3$ and $k\ge 4$; since there are $2k$ vertices, this is a 2-factor using even cycles. If $k\equiv 2 \pmod 3$ and $k\ge 8$, then we can combine a cycle on 4 pairs with a cycle on the remaining $k-4$ pairs, since $k-4\equiv 1 \pmod 3$ and is at least 4. If $k\equiv 3 \pmod 3$ and $k\ge 12$, then we can combine two cycles on 4 pairs each with a cycle on the remaining $k-8$ pairs, since $k-8\equiv 1 \pmod 3$ and is at least 4. This resolves all

cases except $k=6$ and $k=9$, for which we present explicit spanning cycles below. (The Petersen graph $P(n,2)$ is not 3-edge-colorable.)



Alternatively, there is an explicit coloring when $k\equiv 0 \pmod 3$. Let $c(e)$ denote the color on edge e. Treating the vertex indices modulo n and the colors as $\mathbb{Z}_3$, we let $c(u_iv_i)\equiv i \pmod 3$, $c(u_iu_{i+1})\equiv (i-1) \pmod 3$, and $c(v_iv_{i+2})\equiv (i+1) \pmod 3$.

7.3.20. *If a 3-regular graph is the union of two cycles, then it is 3-edge-colorable.* (Note: The statement is not true with "three" in place of "two", since the Petersen graph can be expressed as the union of three cycles.) At each vertex, each incident edge is in one of the cycles, so one of the edges must be in both. If C and C' are the two cycles, then the desired 1-factors are $E(0)-E(C')$, $E(C')-E(C)$, and $E(C)\cap E(C')$.

7.3.21. *The flower snarks: G_k* consists of three "parallel" k-cycles with vertex sets $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ and vertices $w_1,\ldots,w_k$ such that $N(w_i)=\{x_i,y_i,z_i\}$ for each i. H_k is obtained from G_k by replacing $\{x_kx_1, y_ky_1\}$ with $\{x_ky_1, y_kx_1\}$.

a) G_k is Type 1. A proper 3-edge-coloring has distinct color pairs on the cycle edges at x_i, at y_i, and at z_i, because the three edges to w_i have distinct colors. Thus, if the x,y,z-edges from $i-1$ to i are colored a,b,c, then those from i to $i+1$ can be colored b,c,a or c,a,b, respectively. We travel around the k-cycles, always stepping the cyclic permutation forward. Upon reaching the last set of edges, the forward rotation or the backward rotation is compatible with both the previous triple and the first triple.

b) H_k is Type 2 when k is odd. Again we must have distinct color pairs on the cycle edges at x_i,y_i,z_i. The x,y,z-edges from $i-1$ to i cannot have the same color, because the color pairs at i could not then be distinct. Hence they are a permutation or one color is omitted. If one color is omitted, then the fact that each color appears twice among the three pairs implies that the omitted color appears twice between i and $i+1$ (and the color that was twice is now omitted). Since this argument applies in both directions (and between k and 1), all triples are permutations or all triples omit one color.

If all triples are permutations, then the permutation must maintain the same parity. In particular, placing a cyclic permutation of c,b,a next to a, b, c produces an incident pair with the same color. When the edges $x_k y_1$, $y_k x_1$, $z_k z_1$ are reached, the x, y-switch in edges switches the parity of the permutation. Thus edge-coloring using permutations on these triples cannot be compatible all the way around.

If all triples omit one color, then by the remarks above one color appears in every triple, and the other two colors alternate between appearing twice and not appearing. The alternation of appearances of these two colors implies that k must be even. When k is odd, H_k is not 3-edge-colorable. (When k is even, this discussion leads to a 3-edge-coloring of H_k.)

7.3.22. *Every edge cut of $K_k \,\square\, C_t$ that does not isolate a vertex has at least $2k$ edges, unless $k = 2$ and $t = 3$.* Edge cuts that isolate vertices have only $k+1$ edges, since $K_k \,\square\, C_t$ is $(k+1)$-regular. The graph consists of t copies of K_k (the "cliques") arranged in a ring, with corresponding vertices from the cliques forming a cycle. Two successive cliques are joined by a matching.

When a clique is split into sets of size l and $k-l$ by the edge cut, with $l \le k-l$, we call it an l-split. Such a clique induces $l(k-l)$ edges of the cut, which increases with l. Suppose that there is an l-split clique Q and another clique Q' that is unsplit. On the paths from Q to Q' in both directions around the cycle, we cut at least $2l$ edges. Hence in this case we cut at least $l(k+2-l)$ edges. For $l \ge 2$ this is at least $2k$.

Hence every clique is split or all splits are 1-splits. Since splitting a clique cuts at least $k-1$ edges and $t \ge 3$, the former case cuts at least $3k-3$ edges and suffices unless $t = 3$ and $k = 2$. This is the exceptional case, and indeed $K_2 \,\square\, C_3$ has a nontrivial cut of size 3.

Therefore, all split cliques are 1-splits and some clique is unsplit. If two cliques are 1-split, then we can find an unsplit clique preceding a split clique and a second split clique followed by an unsplit clique. In addition to the $k-1$ edges from each split clique, we have at least one edge in the cut to the neighboring unsplit clique, for a total of at least $2k$ edges.

Hence at most one clique Q is split, and if so it is a 1-split. Because the cut does not isolate a vertex, some other clique Q' is entirely on the same side with the singleton from Q. We cut $k-1$ edges within Q and some edge along each path in each direction from the large part of Q to Q'. Hence we cut at least $3k-3$ edges, which suffices unless $k = 2$ and no other edges are cut. In this case, the "large" part of Q is isolated by the cut.

Finally, if no clique is split, then having a nonempty cut requires some clique on one side and another on the other side, and then we cut at least $2k$ edges of the paths along the cycles as we go back and forth from one side to the other. This case achieves equality for all k and t.

7.3.23. *Applying Isaacs' dot product operation (Definition 7.3.12) to two snarks yields a third snark.* Let G_1 and G_2 be snarks, with disjoint edges

uv and wx from G_1 and adjacent vertices y and z from G_2 deleted to perform the dot product operation. Let G be the resulting graph, adding the edges ua, vb, wc, and xd.

Since G_1 and G_2 are 3-regular, by construction G is 3-regular; the four vertices left with degree 2 in each subgraph receive new neighbors.

Since G_2 has girth at least 5, the vertices in G_2 receiving neighbors in G_1 form an independent set. Hence any cycle in G involving the new edges must use at least two of them plus at least one edge of G_1 and at least two of G_2. Other cycles lie in G_1 or G_2. Hence G has girth at least 5.

Any edge cut of G that separates $V(G_1)$ or separates $V(G_2)$ has as many edges as a corresponding edge cut of G_1 or G_2, and the only edge cut that cuts neither of those sets has size 4. Hence G is 3-edge-connected and cyclically 4-edge-connected.

Finally, if G has a proper 3-edge-coloring f, then G_1 or G_2 is 3-edge-colorable. Being 3-regular, G_1 and G_2 have even order. Since each color class is a perfect matching in G, it appears an even number of times in $\{ua, vb, wc, xd\}$. Call this property "parity". If $f(ua) = f(vb)$, then $f(wc) = f(xd)$, by parity, and assigning $f(ua)$ to uv and $f(wc)$ to wx yields a proper 3-edge-coloring of G_1. If $f(ua) \ne f(vb)$, then by parity and symmetry we may assume that $f(wc) = f(ua)$, and hence parity yields $f(xd) = f(vb)$. Now assign the color $f(ua)$ to ay and cz, the color $f(vb)$ to by and dz, and the third color to yz; this completes a proper 3-edge-coloring of G_2.

7.3.24. *If G_1 has a nowhere-zero k_1-flow and G_2 has a nowhere-zero k_2-flow, then $G_1 \cup G_2$ has a nowhere-zero k_1k_2-flow.* Let D be an orientation of $G = G_1 \cup G_2$, extend the flows on G_i by giving weight 0 to edges of $E(G) - E(G_i)$ them, and change signs of weights as needed so that both extended flows have orientation D. Thus G has a k_1-flow (D, f_1) and k_2-flow (D, f_2) such that (D, f_i) is nonzero on the edges of G_i, for each i

Let $f = f_1 + k_1 f_2$. By Proposition 7.3.16, (D, f) is a flow. For $e \in E(G_1)$, $f(e)$ is nonzero because $|f_1(e)| < k_1$. For $e \in E(G_2) - E(G_1)$, $f(e)$ is nonzero because $f_2(e)$ is nonzero. Furthermore, $|f(e)| \le (k_1 - 1) + k_1(k_2 - 1) = k_1k_2 - 1$. Thus (D, f) is a nowhere-zero k_1k_2-flow on G.

7.3.25. *Every spanning tree of a connected graph G contains a parity subgraph of G.* (A **parity subgraph** of G is a spanning subgraph H such that $d_H(v) \equiv d_G(v) \pmod 2$ for all $v \in V(G)$.) Let T be a spanning tree of G.

Proof 1 (induction on $k = e(G) - n(G) + 1$): We have $k = 0$ if and only if $G = T$, in which case G itself is the desired subgraph H. For the induction step, consider $k > 0$, and let $e = xy$ be an edge outside T. Since T is a spanning subgraph of $G' = G - e$, the induction hypothesis yields a parity subgraph H' of G' contained in T. Vertex degrees are the same in G and G' except for x and y. Form $E(H)$ by taking the symmetric difference of H' with the unique x, y-path in T. This changes the parity of the degree only at x and y, as desired.

Proof 2 (construction): Let $U = \{v_1, \ldots, v_{2l}\}$ be the set of vertices in G with odd degree. Let P_i be the unique v_{2i-1}, v_{2i}-path in T. Let H_0 be the spanning subgraph of G with no edges, and let $H_i = H_{i-1} \triangle P_i$ for $1 \le i \le l$. (Equivalently, $H = H_l$ has precisely those edges appearing in an odd number of $P_1, \ldots, P_l$.) The processing of P_i changes the degree parity only at v_{2i-1} and v_{2i}. Thus H_l has odd degree at precisely the vertices of U.

Proof 3 (induction on $n(G)$): We prove the statement more generally for multigraphs. When $n(G) = 1$, the vertex has even degree and the 1-vertex spanning tree is a parity subgraph. For $n(G) > 1$, select a leaf x of T, and let y be its neighbor in T. Form G' from $G - x$ by adding a matching of size $\lfloor d_G(x)/2 \rfloor$ on $N_G(x)$; this may introduce multiple edges. When $d_G(x)$ is odd, let y be the vertex omitted from the matching. By construction, $d_{G'}(v) \equiv d_G(v) \pmod 2$ for all $v \in V(G')$, except for y when $d_G(x)$ is odd. Also $T - x$ is a spanning tree of G'. By the induction hypothesis, G' has a parity subgraph H' contained in $T - x$. If $d_G(x)$ is even, then we add x to H' as an isolated vertex to obtain the desired parity sugraph H. If $d_G(x)$ is odd, then we also add the edge yx.

7.3.26. *For $k \ge 3$, a smallest nontrivial 2-edge-connected graph G having no nowhere-zero k-flow must be simple, 2-connected, and 3-edge-connected.* We may assume that G is connected. Loops never contribute to net flow out of a vertex, so their presence does not affect the existence of nowhere-zero k-flows. If G has a vertex v of degree 2, then G has a nowhere-zero k-flow if and only if the graph obtained by contracting an edge incident to v has a nowhere-zero k-flow. Thus we may assume that $\delta(G) \ge 3$.

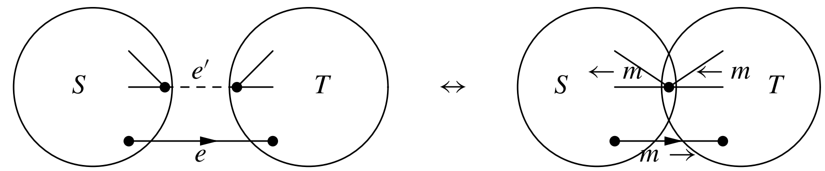
Since G has no cut-edge, each block of G is 2-edge-connected. If G has a cut-vertex, G has no nowhere-zero k-flow only if some block of G has no nowhere-zero k-flow. Thus We may assume that G is 2-connected.

Suppose that $e, e' \in E(G)$ have the same endpoints. If $G - e'$ is not 2-edge-connected, then $\{e, e'\}$ is a block in G, and we can apply the preceding paragraph. Thus we may assume that $G - e'$ has a nowhere-zero k-flow (D, f). We obtain such a flow for G by shifting $f(e)$ up or down by 1 and letting $f(e') = 1$, oriented with or against e depending on whether we shifted $f(e)$ down or up. The shift is possible because $k \ge 3$. Thus we may assume that G is simple.

It remains only to consider a nontrivial 2-edge cut $\{e, e'\}$ (we have eliminated the case where e, e' share a vertex of degree 2). The bridgeless graphs are those where every two vertices lie in a common circuit, and contracting an edge of such a graph with at least three vertices does not destroy this property. Thus we may assume that $G \cdot e'$ has a nowhere-zero k-flow

(D, f). Let S, T be the vertex sets of the components of $G - \{e, e'\}$, and let w be the vertex of $G \cdot e'$ obtained by contracting e'. We may assume that e is oriented from S to T in D.

Let $m = f(e)$. Because $f^*(S \cup w) = 0$, the edges between w and T contribute $-m$ to $f^*(w)$. Similarly, the edges between S and w contribute m to $f^*(w)$. Thus we let $f(e') = m$, oriented from T to S, to obtain a nowhere-zero k-flow on G.



7.3.27. *Every Hamiltonian graph G has a nowhere-zero 4-flow.* Since G has a nowhere-zero 4-flow if and only if it is the union of 2 even subgraphs (Theorem 7.3.25), we express G in this way. The Hamiltonian cycle C is one such subgraph. Let P be a spanning path obtained by omitting one edge of C. For each $e \in G - E(C)$, let $C(e)$ be the cycle created by adding e to P. Each edge outside C appears in exactly one of these cycles. Let C' be the spanning subgraph whose edge set consists of all edges appearing in an odd number of the cycles $\{C(e) \colon e \in E(G) - E(C)\}$. Since C' is a binary sum of even graphs, it is an even graph. It also contains $E(G) - E(C)$.

7.3.28. *Every bridgeless graph G with a Hamiltonian path has a nowhere-zero 5-flow.* If G is Hamiltonian, then G is 4-flowable (Exercise 7.3.27). Otherwise, let G' be the graph obtained from G by adding the edge e joining the endpoints of a spanning path in G.

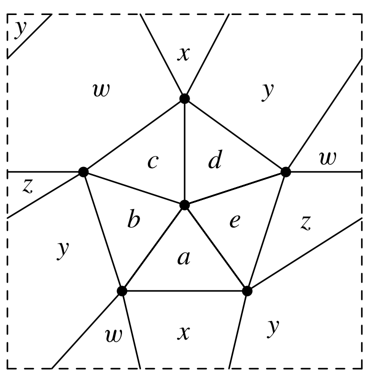
We claim that G' has a nowhere-zero 4-flow with weight 1 on e. Let C be a spanning cycle in G' through e, with vertices $v_1, \ldots, v_n$ in order starting and ending at the endpoints of e. The remaining edges are chords of C; let there be m of them. Let $u_1, \ldots, u_{2m}$ be the endpoings of chords of C, in order on C, listed with multiplicity (a vertex may be an endpoint of many chords. Let C' be the subgraph of G consisting of the chords of C together with the u_{2i-1}, u_{2i}-path on C not containing e, for $1 \le i \le m$. The parity of the number of chords incident to v_j is the same as the parity of the number of edges of C incident to v_j in C'.

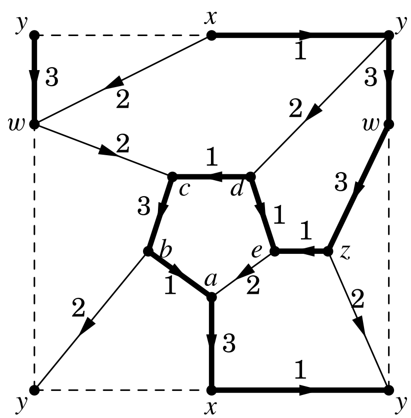
Hence C' and C are both even subgraphs and have positive 2-flows. Let (D, f') and (D, f) be the extension of these to G that are 0 outside C' and C, respectively. Since $C' \cup C = G'$, $(D, f + 2f')$ is a positive 4-flow on G' with weight 1 on e.

Let x be the tail and y the head of e under D. Let S be the set of vertices reachable from x under D without using e. If $S \ne V(D)$, then $[S, \overline{S}] = \{e\}$, with total flow 1. Since the net flow out of S is 0, exactly one unit of flow returns. Since flow comes in integer units, there is only one edge e' in $[\overline{S}, S]$. This makes e' a cut-edge of $G - e$, which has been forbidden.

We conclude that $S = V(D)$, and hence there is a x, y-path P not using e. Increasing the weights by 1 on P and decreasing the weight to 0 on e yields a nowhere-zero 5-flow on G.

7.3.29. *The dual of K_6 on the torus.* Below we show an embedding of K_6 and its surface dual in separate pictures for clarity. The vertices on the right correspond to the faces on the left; note that vertex degrees on the right correspond to face lengths on the left. The heavy edges on the right show that the dual is Hamiltonian. As in Exercise 7.3.27, we combine a constant flow of 1 along the spanning cycle with a constant flow of 2 on an even subgraph containing the remaining edges to obtain a nowhere-zero 4-flow; the resulting flow is shown below. Every nowhere-zero 4-flow is also a nowhere-zero 5-flow.





7.3.30. *A graph G is the union of r even subgraphs if and only if G has a nowhere-zero 2^r-flow. Necessity*. Let $G_1, \ldots, G_r$ be even subgraphs with union G. Given an orientation D of G that restricts to D_i on G_i, for each i, let (D_i, f_i) be a nowhere-zero 2-flows on G_i. Extend f_i to $E(G)$ by letting $f_i(e) = 0$ for $e \in E(G) - E(G_i)$. By Proposition 7.3.16, linear combinations of flows with the same orientation are flows. Hence (D, f) is a flow on G, where $f = \sum_{i=1}^{r} 2^{i-1} f_i$. Since $0 < \sum_{i=1}^{j} 2^{i-1} < 2^j$, always $0 < |f(e)| < 2^r$.

Sufficiency. Proposition 7.3.19 observes the case $r = 1$. We proceed by induction on r, modeling the induction step on the proof in Theorem 7.3.25.

Let (D, f) be a positive 2^r-flow on G. Let $E_1 = \{e \in E(G): f(e)$ is odd$\}$. By Lemma 7.3.23, E_1 forms an even subgraph of G. Thus there is a nowhere-zero 2-flow (D_1, f_1) on E_1, where D_1 agrees with D. Extend f_1 to $E(G)$ by letting $f_1(e) = 0$ for $e \in E(G) - E_1$; now (D, f_1) is a 2-flow on G.

Define f_2 on $E(G)$ by $f_2 = (f - f_1)/2$. By Proposition 7.3.16, (D, f_2) is a flow on G. It is an integer flow, since $f(e) - f_1(e)$ is always even. Since $1 \le f(e) \le 2^r - 1$ and $-1 \le f_1(e) \le 1$, we have $0 \le f_2(e) \le 2^{r-1}$. Equality may hold in either bound. Let $E_2 = \{e \in E(G): 1 \le f_2(e) \le 2^{r-1} - 1\}$. Let G_2 be the spanning subgraph of G with edge set E_2.

If $f_2(e) \ne 2^{r-1}$ for all e, then (D, f_2) restricts to a nowhere-zero 2^{r-1}-flow on G_2. Otherwise, we lose inflow or outflow that is a multiple of 2^{r-1} at each vertex. Since the total of net outflows is 0, at least one has positive outflow and at least one has negative outflow. Furthermore, the set of vertices reachable in D from vertices with positive outflow contains a vertex with negative outflow, because otherwise the set of reachable vertices induces a subgraph with positive total of net outflows.

Let P be a x, y-path in D, where x has with positive net outflow and y has negative net outflow under (D, f); both outflows are multiples of 2^{r-1}. Obtain D' from D by flipping the orientation of each edge in P, and obtain f_2' from f_2 by letting $f_2'(e) = 2^{r-1} - f_2(e)$ for each $e \in E(P)$. Now f_2' is a positive weight on D', and net outflow is unchanged at all vertices except x and y. Furthermore, the total of the absolute values of the net outflows declines by $2 \cdot 2^{r-1}$.

Repeating this process yields a positive 2^{r-1}-flow on G_2. Since $f_1(e) \ne 0$ if $e \in E(G) - E(G_2)$, we have expressed G as the union of subgraphs G_1 and G_2, where G_1 with edge set E_1 is an even subgraph and G_2, by the induction hypothesis, is a union of $r - 1$ even subgraphs.

7.3.31. $(*)$ Let G be a graph having a cycle double cover forming 2^r even subgraphs. Prove that G has a nowhere-zero 2^r-flow. (Jaeger [1988]) *Comment: The short proof of this exercise uses group-valued flows and the flow polynomial of a graph. The arguments are not long, but the concepts are beyond the scope of the brief treatment here, so this exercise will be deleted.*

7.3.32. *A graph has a nowhere-zero 3-flow if and only if it has a modular 3-orientation.*

Necessity. Given that G is 3-flowable, let (D', f') be a positive 3-flow. Let $E_2 = \{e \in E(G): f(e) = 2\}$. Let (D, f) be defined by switching the orientation on edges of E_2 to obtain D from D' and changing the weights on those edges from 2 to 1 to obtain f from f'. The change at edge of E_2 changes the net outflow at each endpoint by 3. Since all weights under f equal 1, we have net outflow at v equal to $d_D^+(v) - d_D - (v)$. Since the net outflow is a multiple of 3, D is a modular 3-orientation.

Sufficiency. Given a modular 3-orientation D, let $f(e) = 1$ for all $e \in E(G)$. Under (D, f), the net outflow at each vertex is a multiple of 3. An argument as in Exercise 7.3.30 now converts this to a nowhere-zero 3-flow on G. If each vertex has net outflow 0, then (D, f) is such a flow.

Otherwise, since the total of the net outflows is 0, the net outflow is a positive multiple of 3 at some vertex and a negative multiple of 3 at some other vertex, with at least one of each type. Furthermore, the set of vertices reachable in D from vertices with positive outflow contains a vertex with negative outflow, because otherwise the set of reachable vertices induces a subgraph with positive total of net outflows.

Let P be a x,y-path in D, where x has with positive net outflow and y has negative net outflow under (D,f); both outflows are multiples of 3. Obtain D' from D by flipping the orientation of each edge in P, and obtain f' from f by letting $f'(e) = 3 - f(e)$ for each $e \in E(P)$. Now f' is a positive weight on D', and net outflow is unchanged at all vertices except x and y. Furthermore, the total of the absolute values of the net outflows declines by $2 \cdot 3$. Repeating this process produces a positive 3-flow on G.

7.3.33. *If G is a bridgeless graph, D is an orientation of G, and $a,b \in \mathbb{N}$, then the following statements are equivalent.*

A) $\frac{a}{b} \le \frac{|[S,\overline{S}]|}{|[\overline{S},S]|} \le \frac{b}{a}$ for every nonempty proper vertex subset S.

B) G has an integer flow using D and weights in the interval $[a,b]$.

C) G has a real-valued flow using D and weights in the interval $[a,b]$.

B $\Rightarrow$ C. Every integer flow is a real-valued flow.

C $\Rightarrow$ A. Let (D,f) be such a flow. Weights in the interval $[a,b]$ are positive. Comparing total inflow and outflow, we have $|[\overline{S},S]|\,a \le f^-(S) = f^+(S) \le |[S,\overline{S}]|\,b$, which yields the first inequality. Similarly, $|[S,\overline{S}]|\,a \le f^+(S) = f^-(S) \le |[\overline{S},S]|\,b$, which yields the second.

A $\Rightarrow$ B. The all-0 flow is a nonnegative integer b-flow on G. Let (D,f) be a nonnegative integer b-flow on G that maximizes m, the minimum weight used, and within that minizimes the number of edges with weight m. Let e^* be an edge with weight m, directed as uv.

If there is a v,u-path P in G that travels forward along edges in D with weight less than b or backward along edges in D with weight more than $m+1$, then we can increase $f(e^*)$ by 1, increase weight by 1 on forward edges of P, and decrease weight by 1 on backward edges of P to obtain another flow that contradicts the choice of (D,f).

Let S be the set of all vertices reachable from v by paths in G whose weights satisfy these constraints. We may assume that $u \notin S$, so $[S,\overline{S}]$ is a nontrivial edge cut. We have $f(e) = b$ for $e \in [S,\overline{S}]$ and $f(e) \in \{m, m+1\}$ for $e \in [\overline{S},S]$. Since the net flow across any cut is 0, we have

$$b\,|[S,\overline{S}]| = \sum_{e\in[S,\overline{S}]} f(e) = \sum_{e\in[\overline{S},S]} f(e) \le (m+2)\,|[\overline{S},S]| < a\,|[\overline{S},S]|\,.$$

Thus $|[S,\overline{S}]| / |[\overline{S},S]| < a/b$, which contradicts the hypothesis. We conclude that such switches can be made until the desired positive b-flow is obtained.

7.3.34. *Cycle double covers for special graphs.* Let C_m have vertices $v_1,\dots,v_m$ in order.

$C_m \vee K_1$. Use each triangle of the form xv_iv_{i+1}, where x is the central vertex and indices are taken modulo m, and use the cycle through $v_1,\dots,v_m$.

$C_m \vee 2K_1$. Use all $2m$ triangles.

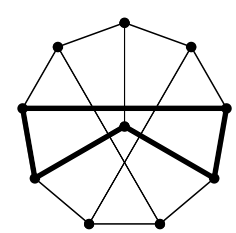
$C_m \vee K_2$. Let x and y be the two vertices outside the m-cycle. Use all triangles of the form xv_iv_{i+1} and yv_iv_{i+1}. We have not yet touched the edges xy and v_mv_1, and the edges xv_1, yv_1, xv_m and yv_m have been used only once. To finish the job, add the cycle through x,y,v_m,v_1 and the cycle through x,y,v_1,v_m.

7.3.35. *For every 3-regular simple graph with 6 vertices, the cycle double covers with fewest cycles consist of three 6-cycles.* There are only two such graphs, $K_{3,3}$ and $C_3 \square K_2$, which follows readily from the two cases of whether the graph contains a triangle or not. The total length of the cycles in the cover is 18, which can be achieved with as few as three cycles only if three 6-cycles are used.

In $K_{3,3}$, a 6-cycle C leaves 3 disjoint edges uncovered. There are two ways to pass a 6-cycle through these three edges, each using alternate edges on C. Hence every 6-cycle in $K_{3,3}$ lies in exactly one smallest cycle double cover.

If $C_3 \square K_2$, a 6-cycle C must visit both triangles, so it uses exactly two of the three edges joining the triangles. Hence it also uses exactly two edges on each triangle. Again, the remaining three edges are disjoint and must appear in both of the other 6-cycles. Again the two ways to complete a 6-cycle through these edges use alternate edges on C and complete a cycle double cover, so again each 6-cycle appears in one such cover.

7.3.36. *Cycle double cover of the Petersen graph using not all 5-cycles.* Consider the drawing of the Petersen graph with a 9-cycle on the "outside". Use the 9-cycle and the 6-cycle formed from the three pairwise-crossing edges and three edges on the outside 9-cycle. Finally, add three 5-cycles; each consists of one of the three crossing edges, two edges incident to the central vertex, and two edges on the outside 9-cycle, as shown below.



7.3.37. *Cycle covers in the Petersen graph.* Let G be the Peterson graph.

Every two 6-cycles in G share at least two edges. Since there are 10 vertices, two 6-cycles have at least two common vertices. In a 3-regular graph, two cycles have a common edge at each common vertex. Hence we have two common edges unless we have exactly two common vertices as the endpoints of one shared edge. In this case, the symmetric difference of the two 6-cycles is a 10-cycle, which does not exist in G.

G has no CDC consisting of five 6-cycles. One 6-cycle in such a CDC would share two edges with each of the other four 6-cycles. Since it has only 6 edges, this produces an edge covered three times.

G has no CDC consisting of even cycles. There is no 10-cycle or 4-cycle, and the total length is 30. Hence the possibilities for cycle lengths are five 6-cycles or $(8, 8, 8, 6)$. We have forbidden the former.

The latter would be a CDC using four even subgraphs. The symmetric difference of one with the other three (see Exercise 7.3.39) yields a CDC consisting of three even subgraphs. Since the graph is 3-regular, the full graph is the union of any two of the even subgraphs in a CDC by three even subgraphs. By Theorem 7.3.25, the graph then has a nowhere-zero 4-flow and hence a proper 3-edge-coloring. Since this does not exist for the Petersen graph, it has no CDC consisting of four cycles.

7.3.38. *Orientable CDC and even subgraphs.*

a) If a graph G has a nonnegative k-flow (D, f), then f can be expressed as $\sum_{i=1}^{k-1} f_i$, where each (D, f_i) is a nonnegative 2-flow on G.

Proof 1 (networks, Menger's Theorem, and induction on k). The case $k = 2$ is Proposition 7.3.19; consider $k > 2$. Let $E_0 = \{e \in E(D)\colon f(e) = 0\}$ and $E' = \{e \in E(D)\colon f(e) = k-1\}$, and let $r = |E'|$. Construct a network N from $D - E' - E_0$ by adding vertices s and t with edges sy and xt of weight $k-1$ for each edge $xy \in E'$. View f as both flow and capacity. Adding an edge ts of weight $r(k-1)$ would turn f into a circulation, so f on N is a flow of value $r(k-1)$.

Hence every s,t-cut in N has capacity at least $r(k-1)$. Since every edge has weight at most $k-1$, every s,t-cut has at least r distinct edges. By Menger's Theorem, N has r pairwise edge-disjoint s,t-paths. These combine with E' to form an even subgraph E_{k-1} of D, containing E' and contained in $D - E_0$. Let (D, f_{k-1}) be the nowhere-zero 2-flow that is nonzero on E_{k-1}.

Reducing weights by 1 on E_{k-1} yields a nonnegative $(k-1)$-flow (D, f') on G. By the induction hypothesis, there are 2-flows (D, f_i) for $1 \le i \le k-1$ such that $f' = \sum_{i=1}^{k-2} f_i$. Now $f = \sum_{i=1}^{k-1} f_i$, as desired.

Proof 2 (manipulation of flows and induction on k). Define E_0 and E' as above. When (D, f) is restricted to $D - E'$, the net outflow at each vertex is a multiple of $k-1$. The argument in Exercise 7.3.30 and Exercise 7.3.32 that switches orientation along paths from vertices with positive outflow to vertices with negative outflow produces a positive $(k-1)$-flow (D', f') on $G - E' - E_0$. If D' and D are opposite on an edge e, then $f'(e) = k-1-f(e)$. Switching these edges back yields a nowhere-zero $(k-1)$-flow (D, f_1) on $G - E' - E_0$ with $f_1(e) \equiv f(e) \pmod{k-1}$ for all e.

Since f is nonnegative, $(D, f - f_1)$ is an integer k-flow on G in which every edge has weight $k-1$ or 0. Let $f_2 = (f - f_1)/(k-1)$. Now $(D, f - f_2)$ is a nonnegative $(k-1)$-flow on G, and (D, f_2) is a nonnegative 2-flow on G. By the induction hypothesis, there are $k-2$ 2-flows with orientation D that sum to $f - f_2$, and adding f_2 to them yields f.

b) A graph G has a positive k-flow (D, f) if and only if D is the union of $k-1$ even digraphs such that each edge e in D appears in exactly $f(e)$ of them. Necessity follows from part (a). Since the $k-1$ guaranteed 2-flows on G have the same orientation D and are all nonnegative, they yield the desired even digraphs $D_1, \ldots, D_{k-1}$ by letting $E(D_i) = \{e \in E(D)\colon f_i(e) = 1\}$. For sufficiency, the even digraphs convert to 2-flows that sum to (D, f).

c) A graph G has a nowhere-zero 3-flow if and only if it has an orientable cycle double cover forming three even subgraphs.

Necessity. If G has a nowhere-zero 3-flow, then G has a positive 3-flow (D, f), and part (b) yields two even digraphs such that each edge e appears in $f(e)$ of them. Reversing the orientation on one of them yields two even digraphs D_1 and D_2 that are oppositely oriented on the edges with $f(e) = 2$. At a given vertex v, those edges contribute the same indegree and outdegree. Therefore, $D_1 \cup D_2$ has the same number of edges entering and leaving v among the edges with $f(e) = 1$. Let D_3 be the reverse of this subgraph of $D_1 \cup D_2$. Now D_3 is a third even subgraph completing an oriented cycle double cover with D_1 and D_2.

Sufficiency. Let D_1, D_2, D_3 be the subgraphs in the given oriented CDC. Let D_1' be the reversal of D_1. Now $D_1' \cup D_3$ is an orientation of G; call it D. Let $f(e) = 2$ if $e \in D_1' \cap D_3$; otherwise, $f(e) = 1$. By part (b), (D, f) is a positive 3-flow on G.

Direct proof that (D, f) is a positive 3-flow without using part (b). In D, each edge is oriented the same way as the higher-indexed of the two among $\{D_1, D_2, D_3\}$ that contains it, and $f(e)$ is the difference of those indices. For $1 \le i < j \le 3$, let $a_{i,j}$ be the number of edges in G that enter v in D_j and leave it in D_i, and let $b_{i,j}$ be the number that leave it in D_j and enter it in D_i. The net outflow at v is $b_{1,2} + 2b_{1,3} + b_{2,3} - a_{1,2} - 2a_{1,3} - a_{2,3}$. The computations below show that this quantity is 0, since D_1, D_2, D_3 are even digraphs. Thus (D, f) is a positive 3-flow on G.

$$b_{1,3} + b_{2,3} = d^+_{D_3}(v) = d^-_{D_3}(v) = a_{1,3} + a_{2,3}$$
$$b_{1,2} + b_{1,3} = d^-_{D_1}(v) = d^+_{D_1}(v) = a_{1,2} + a_{1,3}$$

7.3.39. *If a graph G has a CDC formed from four even subgraphs, then G also has a CDC formed from three even subgraphs.* Let E_0, E_1, E_2, E_3 be the edge sets of the four even subgraphs. Form instead the sets $E_0 \triangle E_1$, $E_0 \triangle E_2$, and $E_0 \triangle E_3$. These form even subgraphs, since the symmetric difference of two even subgraphs is an even subgraph.

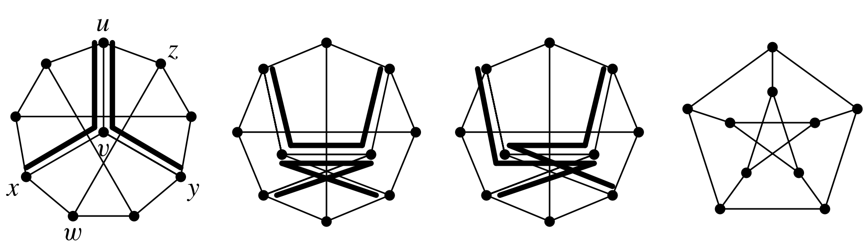
If previously an edge belonged to E_i and E_j but not E_0, then now in belongs to $E_0 \triangle E_i$ and $E_0 \triangle E_j$ but not to $E_0 \triangle E_k$. If it belonged to E_0 and E_k, then the same statement holds. Hence the new family is a CDC by three even subgraphs.

7.3.40. *The solution to the Chinese Postman Problem in the Petersen graph has length 20, but the least total length of cycles covering the Petersen graph is 21.* Since the Petersen graph is regular of odd degree, at least one edge must be added incident to each vertex to obtain an Eulerian spanning supergraph. Hence at least 5 edges must be added, and the total length of an Eulerian circuit in a supergraph is at least 20. Since the graph has a 1-factor, a solution of length 20 exists.

Cycles covering the graph together solve the Chinese Postman Problem, so their total length must be at least 20. If equality holds, then the lengths must be one of $(9,6,5)$, $(8,6,6)$, $(5,5,5,5)$, since the cycle lengths are in $\{5,6,8,9\}$.

If a 9-cycle is used, then one vertex is missed, and the other two cycles must visit it. Let v be the vertex missed by the 9-cycle, let u and x be the neighbors of v on the 6-cycle, and let u and y be the neighbors of v on the 5-cycle. Now the 6-cycle must be completed using a u,x-path of length 4, and the 5-cycle must be completed using a u,y-path of length 3. There are two choices for each of these paths, but all choices miss one particular edge, which we have labeled zw in the drawing on the left below.

If an 8-cycle is used, then two adjacent vertices are missed. The two 6-cycles must both visit these vertices, and girth 5 requires that all 6-cycles through two adjacent vertices use the edge joining them. The remaining edges in the 6-cycles can be distributed in several ways, but in each way some edge remains uncovered.



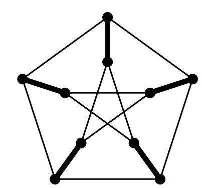
If four 5-cycles are used, then draw the graph so that one of them is the outer 5-cycle. A 5-cycle uses 0 or 2 of the edges crossing from the outer

5-cycle to the inner one. In order to cover these cross edges, the remaining three 5-cycles must use two cross edges each. Such a 5-cycle uses one or two edges on the central cycle; two if the cross edges reach consecutive vertices on the outer cycle, one if they do not.

In order to cover the inner 5-cycle, at least two of the three crossing 5-cycles must have two edges on it. If they share a crossing edge, then the remaining 5-cycle covers the two remaining cross edges, cover no new edges on the inner 5-cycle, and an edge of the inner cycle is left uncovered. If they do not share a crossing edge, then their union avoids a vertex of the inner 5-cycle. Hence the remaining 5-cycle is left to cover three edges at one vertex, which it cannot do.

7.3.41. *Given a perfect matching in the Petersen graph, there is no list of cycles that together cover every edge of M exactly twice and all other edges exactly once.* Suppose that such a list of cycles exists, where M is the bold matching in the drawing below. If a cycle in the cover uses two edges of M, then it uses at least three edges not in M. If a cycle uses four edges of M, then it uses at least four edges not in M. Thus every cycle uses at least as many edges not in M as edges in M.

Together, the cycles must cover edges of M 10 times and edges not in M 10 times. Thus each cycle used must cover the same number of edges of each type. Such cycles have four edges of M and four edges not in M. Thus the total length is a multiple of 8, which is impossible since it is 20.



7.3.42. *If an optimal solution to the Chinese Postman Problem on a graph G decomposes into cycles, then G has a cycle cover of total length at most $e(G)+n(G)-1$.* By Exercise 7.3.25, every spanning tree of G contains a parity subgraph, and this has at most $n(G)-1$ edges. Taking all edges of G and adding one extra copy of each edge in a parity subgraph yields an Eulerian supergraph with at most $e(G)+n(G)-1$ edges. Hence the optimal solution to the Chinese Postman Problem has at most this size, and the hypothesis guarantees a cycle cover of at most this size.

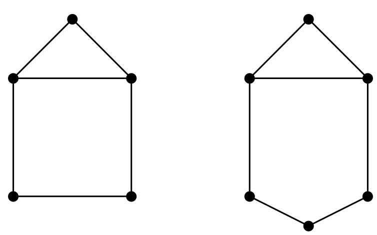
The minimum length of a cycle cover of $K_{3,t}$ is $4t$. Two cycles must visit each vertex of the larger part, and each such visit requires two edges, so the total length of the cycles is at least $4t$. This total length is achievable for $t \ge 2$ using cycles of lengths 4 and 6.

8.ADDITIONAL TOPICS

8.1. PERFECT GRAPHS

8.1.1. *Clique number and chromatic number of $\overline{C}_{2k+1}$ equal k and $k+1$ when $k \ge 2$.* A clique in $\overline{G}$ is an independent set in G. The largest independent set in C_{2k+1} has size k (using alternate vertices along the cycle). A proper coloring of $\overline{G}$ is a covering of $V(G)$ by cliques in G. The only cliques on a chordless cycle are two consecutive vertices, so we need at least $k+1$ of them to cover $V(C_{2k+1})$. When $k=1$, we can use a single triangle, and $\chi(\overline{C}_3) = 1$.

8.1.2. *The smallest imperfect graph G such that $\chi(G) = \omega(G)$.* The only graph with at most five vertices that is not chordal or bipartite is the "house" H on the left below. This is perfect, since its proper subgraphs are chordal or bipartite and for the full graph $\chi(H) = 3 = \omega(H)$. Hence we need at least 6 vertices. The graph H' on the right below is imperfect, since it has a chordless cycle, but $\chi(H') = 3 = \omega(H')$.



8.1.3. *Cographs.*

a) G is P_4-free if and only if G can be reduced to the empty graph by iteratively taking complements within components. If G is not P_4-free, then a 4-set inducing P_4 also induces P_4 after complementation of its component, so these vertices can never be separated.

For the converse, we use induction on $n(G)$. By the induction hypothesis, it suffices to show that if G is P_4-free and connected, then $\overline{G}$ is

disconnected. Since G is P_4-free, diam$(G) = 2$. If G has a cut-vertex x, then diam$(G) = 2$ implies that every pair of vertices from distinct components of $G-x$ have x as a common neighbor. In this case $x \leftrightarrow V(G)-x$ and x is isolated in $\overline{G}$. Hence we may assume that $G-x$ is connected and P_4-free for all $x \in V(G)$. The induction hypothesis implies that $\overline{G}-x$, which is the same as $\overline{G-x}$, is disconnected. If deletion of an arbitrary vertex of $\overline{G}$ leaves a disconnected subgraph, then $\overline{G}$ has no spanning tree and is itself disconnected.

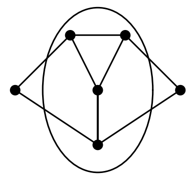
b) Every P_4-free graph is perfect.

Proof 1. If the graphs in a hereditary family **G** are not all perfect, then **G** contains a p-critical graph. By the Perfect Graph Theorem, a minimal imperfect graph and its complement are connected. Hence part (a) implies that there is no p-critical P_4-free graph, and all such graphs are perfect.

Proof 2. Since the class is hereditary, it suffices to prove that G has a clique consisting of a vertex of each color in the greedy coloring with respect to an arbitrary vertex ordering. If k colors are used, let Q be the largest clique of the form $\{u_{i+1}, \ldots, u_k\}$ such that u_j has color j. If $i=0$, we are done. Otherwise, there is no vertex of color i adjacent to all of Q, but by the greedy coloring algorithm every vertex of Q is adjacent to some (earlier) vertex of color i. Let v be a vertex of color i adjacent to the maximum number of vertices in Q, and choose $x \in Q - N(v)$. Let y be a vertex of color i adjacent to x. By the choice of v, there is a vertex $w \in Q$ such that $w \nleftrightarrow y$. Since also $y \nleftrightarrow v$, we have P_4 induced by v, w, x, y. This idea appears in Chvátal's proof that obstruction-free orderings are perfect orderings; this more specialized situation allows a simpler discussion.

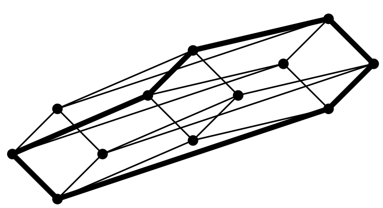
8.1.4. *Clique identification preserves perfection.* Let G_1 and G_2 be two induced subgraphs of G that are perfect and share only a clique S. If a proper induced subgraph of G is not an induced subgraph of G_1 or G_2, then it is formed by pasting together induced subgraphs of G_1 and G_2 at a subset of S. Therefore, we need only verify $\chi(G) = \omega(G)$ to complete an inductive proof that G is perfect. Since there are no edges between $G_1 - S$ and $G_2 - S$, we have $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$. For $\chi(G)$, consider optimal colorings of G_1 and G_2. Since S is a clique, each uses different colors on the vertices of S. By permuting the labels of the colors, we can thus make the colorings agree on S to get a coloring of G with $\max\{\chi(G_1), \chi(G_2)\}$ colors. Since $\chi(G_1) = \omega(G_1)$ and $\chi(G_2) = \omega(G_2)$ by hypothesis, we have $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} = \max\{\omega(G_1), \omega(G_2)\} = \omega(G)$.

8.1.5. *Identification at star-cutsets does not preserve perfection.* The union of two houses sharing a paw has an induced 5-cycle, as shown below. The star-cutset C is circled.



8.1.6. *If G is a cartesian product of complete graphs, then $\alpha(G) = \theta(G)$.* We have $G = K_{n_1} \square \cdots \square K_{n_r}$, indexed so that $n_1 \le \cdots \le n_t$. Always $\theta(G) \ge \alpha(G)$, so it suffices to exhibit a stable set and a clique covering of the same size. Using copies of the largest clique, we have $\theta(G) \le \prod_{i=1}^{r-1} n_i$. To obtain a stable set of this size, recall that H is m-colorable if and only if $\alpha(H \square K_m) = n(H)$ (Exercise 5.1.31). We apply this with $H = \prod_{i=1}^{r-1} K_{n_i}$; it suffices to show that H is n_r-colorable. Here we use another coloring result: $\chi(F_1 \square F_2) = \max\{\chi(F_1), \chi(F_2)\}$ (Proposition 5.1.11). Iterated, this yields $\chi(H) = n_{r-1} \le n_r$, and the proof is complete.

$K_2 \square K_2 \square K_3$ is not perfect. This graph has chromatic number and clique number both equal to 3 but has an induced 7-cycle, as shown below.



8.1.7. *The only color-critical 4-chromatic graph with six vertices is $C_5 \vee K_1$.* Let G be a 4-chromatic graph with six vertices. If G is perfect, then G contains K_4. If it is also 4-critical, then it cannot contain anything but K_4. Therefore, we may assume that G is imperfect. A 6-vertex graph is imperfect if and only if it has an induced 5-cycle. The 5-cycle is 3-colorable, with the third color used only once, anywhere. Therefore, the only way for G to be 4-chromatic it for it to be $C_5 \vee K_1$, which is in fact 4-critical.

8.1.8. (+) Prove that G is an odd cycle if and only if $\alpha(G) = (n(G) - 1)/2$ and $\alpha(G - u - v) = \alpha(G)$ for all $u, v \in V(G)$. (Melnikov–Vizing [1971], Greenwell [1978])

8.1.9. Let $v_1, \ldots, v_n$ be a simplicial elimination ordering of G.

a) Applying the greedy coloring algorithm to the construction ordering $v_n, \ldots, v_1$ yields an optimal coloring, and $\omega(G) = 1 + \max \sum_{x \in V(G)} |Q(x)|$, where $Q(v_i) = \{v_j \in N(v_i)\colon j > i\}$. Since $\{x\} \cup Q(x)$ is a clique, we have $\chi(G) \ge \omega(G) \ge 1 + \max|Q(x)|$. The greedy algorithm with this ordering yields $\chi(G) \le 1 + \max|Q(x)|$, since x has $\{Q(x)\}$ earlier neighbors, so equality holds throughout.

Then the stable set $\{y_1, \ldots, y_k\}$ obtained greedily from the elimination ordering is a maximum stable set, and the sets $\{y_i\} \cup Q(y_i)$ form a minimum clique covering. "Obtained greedily" means set $y_1 = v_1$, discard what remains of $Q(y_1)$ from the remainder of the ordering, and iterate. Let $S = \{y_1, \ldots, y_k\}$. When a vertex is included, it has no edge to an earlier chosen vertex, so S is stable. Furthermore, the vertices discarded when y_i is chosen form a subset of $Q(y_i)$, and $\{x\} \cup Q(x)$ is always a clique. Since every vertex in the list is chosen or discarded, $\{\{y_i\} \cup Q(y_i)\}$ forms a clique cover with the same size as S, so this is a minimum clique cover and S is a maximum stable set.

8.1.10. (•) Add a test to the MCS algorithm to check whether the resulting ordering is a simplicial elimination ordering. (Tarjan–Yannakakis [1984])

8.1.11. *The intersection graph G of a family of subtrees of a tree has no chordless cycle.* Suppose that G has a chordless cycle $[v_1, \ldots, v_k]$, with $T_1, \ldots, T_k$ being the corresponding subtrees of the host tree T. For each i, the edge $v_i v_{i+1}$ (indices modulo k) yields a vertex $w_i \in V(T_i) \cap V(T_{i+1})$. The subtree T_i has a unique w_{i-1}, w_i-path in T. Let x_i be the last common vertex of the w_i, w_{i+1}-path and the w_i, w_{i-1}-path. The x_{i-1}, x_i-path in T is contained in T_i, but its internal vertices belong to no other trees in the list. Therefore, the union of the x_{i-1}, x_i-paths in T, over all i, is a closed walk with no repeated vertices, which contradicts T having no cycles.

8.1.12. *Every graph is the intersection graph of a family of subtrees of some graph.* Given a graph G, let G' be the graph obtained by subdividing each edge of G. Associate with $v \in V(G)$ the star in G' formed by the edges incident to v. The stars for u and v intersect in G' if and only if u and v are adjacent in G.

8.1.13. *Every chordal graph has an intersection representation by subtrees of a host tree with maximum degree 3.* A chordal graph has an intersection representation by subtrees of a host tree T. If T has a vertex x of degree exceeding 3, then form T' as follows. In T, subdivide each edge incident to x (introducing a set S of new vertices), delete x, and add edges forming a path with vertex set S; this yields T'. Modify the subtrees that contained x by replacing x with S. The pairs of subtrees that intersect remain the same, and T' has one less vertex of degree exceeding 3 than T does. Iterating this process produces the desired representation.

8.1.14. *If Q is a maximal clique in a connected chordal graph G and $x \in V(G)$, then Q has two vertices whose distances from x are different.*

Proof 1 (chordless cycles). If $x \in Q$, then we take x and some other vertex of Q. Otherwise, suppose that all vertices of Q have the same distance from x. For $v \in Q$, let $R(v)$ be the set of vertices just before v on shortest x, v-paths. For $v, v' \in Q$, we claim that $R(v)$ and $R(v')$ are ordered by inclusion. Otherwise, there exist $u \in R(v)$ and $R(v')$ such that $u \not\leftrightarrow v'$ and $u' \not\leftrightarrow v$. Now let P and P' be a shortest x, u-path and a shortest x, u'-path. The subgraph induced by $V(P) \cup V(P')$ is connected; let D be a shortest u, u' path in this subgraph. The length of D is at least 1. No internal vertices of D are adjacent to v or v', since this would yield shorter paths to Q. Therefore, adding the edges $uv, vv', v'u'$ to D completes a chordless cycle C of length at least 4. This is a contradiction.

We conclude that $R(v)$ and $R(v')$ are ordered by inclusion. Since this is true for all pairs of vertices in Q, there is a vertex $w \in Q$ such that $R(w)$ is contained in $R(v)$ for all $v \in Q - \{w\}$. Thus there is a vertex, in $R(w)$, that is adjacent to all of Q. This contradicts Q being a maximal clique. We conclude that Q has two vertices with different distance from x.

Proof 2 (clique trees). Consider a clique tree representation of G in a smallest host tree T. By Lemma 8.1.16, the vertices of T correspond to the maximal cliques in G. Let q be the vertex corresponding to Q. Since $x \notin Q$, the tree T_x assigned to x cannot contain q. Let P be the path in T from q to T_x, with q' being the neighbor of q on P. Let Q' be the maximal clique of G corresponding to q', and choose $v \in Q - Q'$. Since G is connected, some shortest path (of subtrees) links T_x to q. Hence some vertex w belongs to $Q \cap Q'$. Thus P encounters T_w before T_v when followed from T_x. This implies that $d_G(x, w) < d_G(x, v)$.

8.1.15. *Intersection graphs of subtrees of graphs.* A *fraternal orientation* of a graph is an orientation such that any pair of vertices with a common successor are adjacent.

a) A simple graph G is chordal if and only if it has an acyclic fraternal orientation. If G is chordal, then G has a simplicial elimination ordering $v_1, \ldots, v_n$. With respect to this ordering, orient the edge v_iv_j from v_j to v_i if $i < j$. Then $v_j \to v_i$ and $v_k \to v_i$, implies that v_j, v_k are remaining neighbors of v_i when v_i is deleted, so the simplicial property of the ordering guarantees $v_j \leftrightarrow v_k$ in G. If G is not chordal, let C be a chordless cycle in G. Let F be an arbitrary acyclic orientation of G. Along C there must be a successive triple u, v, w such that $u \to v$ and $w \to v$ in F, which means F is not fraternal.

b) Example of a graph with no fraternal orientation. Let G be the graph consisting of two 4-cycles sharing a vertex v, and suppose G has a fraternal orientation. Since $N(v)$ is independent, v has at most one edge oriented inward, so we may choose one of the 4-cycles to have both edges incident to v oriented out from v. If u is the remaining vertex of that 4-cycle, we cannot have both edges involving u oriented into u, but $u \not\leftrightarrow v$ forbids either edge to be oriented out from u.

c) G is the intersection graph of a rootable family of trees if and only if G has a fraternal orientation (a family of subtrees is *rootable* if the trees can be assigned roots so that a pair of them intersects if and only if at least one of the two roots belongs to both subtrees). Suppose G is the intersection graph of a rooted family of trees in a graph, with $f(v)$ the tree assigned to $v \in V(G)$. If $xy \in E(G)$, orient the edge xy toward the vertex in $\{x, y\}$ whose tree has root lying in both of $\{f(x), f(y)\}$ (choose the orientation arbitrarily if both roots satisfy this). If $u \to v$ and $w \to v$, then the root of $f(v)$ lies in both $f(u)$ and $f(w)$; hence $f(u)$ and $f(w)$ intersect and u, w are adjacent.

Conversely, let G be a fraternally oriented graph. For each vertex v, let $f(v)$ be the substar of G consisting of all edges of G oriented out from v, and root it at v. For any edge xy oriented as $x \to y$, we have the root of $f(y)$ in $f(x)$. To complete the proof that G is the intersection graph of $\{f(v)\}$, we show that $xy \notin E(G)$ implies $f(x) \cap f(y) = \varnothing$. Nonadjacency of x, y implies that neither of $f(x), f(y)$ contains the root of the other, and hence $f(x) \cap f(y) \neq \varnothing$ requires that x, y have a common successor. This contradicts the assumption that the orientation is fraternal.

8.1.16. *A simple graph G is a forest if and only if every pairwise intersecting family of paths in G has a common vertex.* If G contains a cycle, then the three paths on the cycle joining any two of three specified vertices on the cycle form a pairwise intersecting family of paths with no common vertex.

Conversely, if G is a forest, then a pairwise intersecting family of paths lies in a single component of G and hence is a pairwise intersecting family of subtrees of a tree. By Lemma 8.1.10, the paths have a common vertex.

8.1.17. *For a graph G, the following are equivalent (and define split graphs).*

A) $V(G) = S \cup Q$, where S and Q induce a stable set and clique in G.

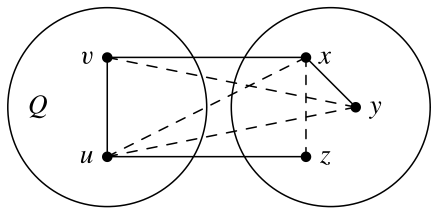
B) G and $\overline{G}$ are chordal.

C) G has no induced C_4, $2K_2$, or C_5.

A $\Rightarrow$ B. A cycle in C cannot visit vertices of S in succession, so a cycle of length at least 4 has two nonconsecutive vertices in Q; they are adjacent.

B $\Rightarrow$ C. C_4 and C_5 are chordless cycles. The vertices of a $2K_2$ in G would induce a chordless 4-cycle in $\overline{G}$.

C $\Rightarrow$ A. Let Q be a maximum clique minimizing $e(G - Q)$. Suppose that $G - Q$ has an edge xy. Since G is C_4-free, $N(x) \cap Q$ and $N(y) \cap Q$ are ordered by inclusion; we may assume that $N(y) \cap Q \subseteq N(x) \cap Q$. Since G is $2K_2$-free, x cannot have two nonneighbors in Q. Since Q is a maximum clique, x has a nonneighbor u in Q and y another nonneighbor v in Q. The edges (solid) and non-edges (dashed) within $\{x, y, u, v\}$ are shown below.

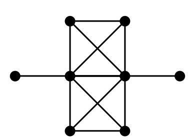


Since $Q - u + x$ is a clique, the choice of Q implies that u as many neighbors as x outside Q. Since $y \in N(x) - N(u)$, there exists $z \notin Q$ such that $z \in N(u) - N(x)$. Now $G[x, y, z, u] \neq 2K_2$ requires $y \leftrightarrow z$, after which $G[x, y, z, v] \neq C_4$ requires $v \not\leftrightarrow z$ and $G[u, v, x, y, z] \neq C_5$ requires $v \leftrightarrow z$. The contradiction implies that $G - Q$ has no edges.

8.1.18. *If $d_1 \geq \cdots \geq d_n$ is the degree sequence of a simple graph G, and m is the largest value of k such that $d_k \geq k - 1$, then G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$.* The Erdős-Gallai condition says that $d_1 \geq \cdots \geq d_n$ are the vertex degrees of such a graph if and only if $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i r$ for $1 \leq k \leq n$; call this the *k-th E-G condition*. *Split graphs* are those with a vertex partition $V(G) = Q \cup S$ such that Q is a clique and S is a stable set.

Necessity. If G is a split graph with maximum clique Q, then the $|Q|$ vertices of Q have degree at least $|Q| - 1$, and other vertices have degree at most $|Q| - 1$. Hence $m = |Q|$, and the vertices of Q are $|Q|$ vertices of highest degree. Counting their degree by the edges inside Q and out shows that the mth E-G condition holds with equality.

Sufficiency. Conversely, assume equality in the mth E-G condition. Whenever the kth E-G condition holds with equality, the k vertices of highest degree must form a clique. If in addition the contribution in $\sum_{i=k+1}^{n} \min\{k, d_i\}$ is always d_i, then all edges from the remaining vertices go to the k highest-degree vertices, and G is a split graph. When $k = m$, $d_i < i - 1 \leq k$ for all $i > k$, so this does indeed hold. (Note: The graph below is not a split graph, but satisfies the E-G condition with equality at $k = 2 \neq 4 = m$.)



8.1.19. *The trees that are split graphs are stars or double-stars. No nonisomorphic trees that are split graphs have the same degree list. Nonisomorphic split graphs with the same degree list arise by finding nonisomorphic bipartite graphs with the same degrees within partite sets and placing a complete graph on one partite set.* The only cliques in a tree T are single vertices or two adjacent vertices. If T is a split graph, then the remaining vertices are adjacent only to these. Hence T is a star or double-star. Such a tree is determined by the degrees of the vertices with degree exceeding 1.

For explicit nonisomorphic split graphs, note that C_8 and $2C_4$ have the same degree list, and adding edges to make one of the partite sets into a clique maintains that property. The graphs are not isomorphic.

8.1.20. *G is a k-tree (obtained from a k-clique by successively adding simplicial vertices of degree k) if and only if G 1) is connected, 2) has a k-clique but no $k+2$-clique, and 3) has only k-cliques as minimal vertex separators.*

Necessity. By induction on $n(G)$. The conditions hold for the unique k-tree obtained by adding two vertices to the initial k-clique. Each subsequent vertex addition maintains connectedness. Let v be the last vertex added. Any new clique must contain new edges and thus must contain v, but v belongs only to a $k+1$-clique. For (3), suppose S is a minimal vertex separator of G. If S is a minimal vertex separator of $G - v$, we apply the induction hypothesis. A vertex of a minimal vertex separator S has neighbors in two components of $G - S$; hence no minimal vertex separator of a graph contains a simplicial vertex. Hence $v \notin S$. If v is not isolated in $G - S$, then S is a minimal vertex separator in $G - v$, because $N(v)$ is a clique. We conclude that v is isolated in $G - S$, this requires $S = N(v)$, which induces a k-clique.

Alternative proof of (3), without induction. For property (3), the construction procedure implies that G is chordal, which implies that every minimal vertex separator induces a clique. If S is a minimal x, y-separator, let x be the first of $\{x, y\}$ in the construction ordering. Among the component of $G - S$ containing y, let z be the first vertex in the construction ordering. When z is added, the only vertices that z can have as neighbors lie in S. Hence $k \leq |S| \leq k + 1$. Suppose $|S| = k + 1$ and H is a component of $G - S$. Every vertex of S has a neighbor in H, but no vertex of H is adjacent to all of S. Hence we can choose $u, v \in S$ and $x, y \in V(H)$ such that $u \leftrightarrow x$, $v \leftrightarrow y$, $u \not\leftrightarrow x$, $v \not\leftrightarrow y$. Adding a shortest x, y-path in H yields a chordless cycle; hence $|S| = k$.

Sufficiency. (3) implies that G is a chordal graph. Let v be the first vertex in a simplicial elimination ordering. Since v is isolated in $G - N(v)$, $N(v)$ contains a minimal vertex separator. Hence (3) implies $|N(v)| \geq k$. Since by (2) G has no $k+2$-clique, we have $|N(v)| \leq k$. Hence $d(v) = k$. To complete the proof, we must show that deleting a simplicial vertex of degree k does not destroy the conditions, so we can complete a "k-valent" simplicial elimination sequence by applying induction. Deleting a simplicial vertex does not disconnect a graph or create a $k+2$-clique, and if G is not a clique, then (3) implies that $G-v$ retains a k-clique. To prove that (3) is preserved, if a minimal vertex separator of $G-v$ is a minimal vertex separator of G, then it induces a k-clique.

We claim that every minimal x,y-separator of an induced subgraph of a graph is contained in a minimal x,y-separator of the full graph. If so, then a minimal x,y-separator of $G-v$ that is not a k-clique must be part of a minimal x,y-separator of G that contains v, which is impossible since no simplicial vertex belongs to a minimal vertex separator. To prove the claim, suppose S is a minimal x,y-separator in an induced subgraph H of G, so $S\cup(V(G)-V(H))$ separates x and y in G. Hence this set contains a minimal x,y-separator of G, but such a separator must include all of S, else we retain an x,y-path from H.

8.1.21. *An n-vertex chordal graph with no $(k+2)$-clique has at most $kn-\binom{k+1}{2}$ edges, with equality if and only if it is a k-tree.* This is the special case of Exercise 8.1.23 obtained by setting $r=k+1$.

8.1.22. *The number of k-trees with vertex set $[n]$ is $\binom{n}{k}[k(n-k)+1]^{n-k-2}$.* We show that the number of rooted k-trees with vertex set $[n]$ that have a fixed set of k vertices as a root clique is $[k(n-k)+1]^{n-k-1}$. To obtain the formula from this, note that every k tree has $1+k(n-k)$ k-cliques, beginning with a root and adding k each time a new vertex is grown from an old k-clique. On the other hand, there are $\binom{n}{k}$ ways to pick a set of k vertices to form a root clique; hence we multiply by $\binom{n}{k}$ and divide by $[k(n-k)+1]$ to obtain the final formula. Note that when $n=k$ there is only one k-tree, which agrees with the formula, so henceforth we may assume $n>k$.

To count the k-trees with label set $[n]$ and a fixed root $R\subseteq[n]$, we put them in 1-1 correspondence with lists of length $n-k-1$ chosen from a fixed alphabet of size $1+k(n-k)$. The alphabet consists of 0, which refers to the root, together with pairs (v,i) such that $v\in([n]-R)$ and $i\in[k]$. Since $n>k$, every vertex belongs to a $k+1$-clique; when we deal with *rooted* k-trees, the *leaves* are the vertices not in the root that belong to only one $k+1$-clique. Given a k-tree with root R, we form a list by iteratively deleting the leaf u with the least label and recording an appropriate member of the alphabet. If $N(u)=R$, we record 0. If $N(u)\neq R$, we want to record some other code in the alphabet that will enable us to recover the k-clique to which u is joined in growing the current tree from the root.

In growing the current tree from the root, there is a unique list of vertex additions that leads from the root to u (ignoring other additions not needed to reach u). When $N(u)\neq R$, there is a last non-root vertex v before u on this list; let this be the vertex part of the code recorded. When we add u, $N(u)$ consists of v together with all but one vertex of the k-clique to which v was connected when added. Let the index part of the code recorded be the *position* among this list of k of the vertex not in $N(u)$. After $n-k-1$ iterations, there remains one non-root vertex joined to the root.

This defines a unique list from each k-tree. To reconstruct from any list on these labels the unique k-tree that generates it, and thereby show that the map is a bijection, there are two phases. In the first phase, at each iteration select the least non-root label u that has not yet been marked finished. If the current code is 0, create edges from u to R. If the code is a vertex-index pair, create an edge from u to the vertex v that is the vertex part of the code. Mark u finished. After $n-k-1$ iterations, there remains one unfinished non-root vertex; join it to R.

The first phase produces a "skeleton" describing possible ways to grow the k-tree from the root. If we shrink the root to a single node, this is in fact a rooted tree that, for each non-root vertex, describes by its path to the root the list of vertices that must be added before it is added. The second phase fleshes out this skeleton. Moving outward from the root as the construction procedure would, we iteratively "expand" a non-root vertex u such that every previous vertex on the path to the root has already been expanded; this expansion creates the other edges formed when the k-tree is grown to u. Let u be a vertex whose deletion generated a non-root code (v,i). When we expand u, the vertex v is the last vertex on the path to it from the root and has already been expanded, which means that we know the set of vertices S to which v was joined when the tree grew to it. The code i tells us which element of S should not be joined to u. This two-phase procedure generates a unique k-tree from every list, and the k-tree generated from a list τ yields τ under the first procedure, so this is a bijection.

8.1.23. *An n-vertex chordal graph G with clique number r has at most $\binom{r}{j}+\binom{r-1}{j-1}(n-r)$ cliques of order j, with equality (for all j simultaneously) if and only if G is an $r-1$-tree.* We use induction on n. The formula holds for $n=r$. For $n>r$, let v be the first vertex to be deleted in a simplicial elimination order. Since v has at most $r-1$ neighbors, it is involved in at most $\binom{r-1}{j-1}$ cliques of order j. The j-cliques not containing v are bounded by the induction hypothesis. Furthermore, equality holds if and only if it holds for $G-v$ and adding v adds $\binom{r-1}{j-1}$ cliques of order j, which by the inductive hypothesis implies that G is an $r-1$-tree.

8.1.24. *Pairwise intersecting real intervals have a common point.* Let a be the rightmost left endpoint among these intervals, and let b be the leftmost right endpoint. If some right endpoint occurs before some left endpoint, then those two intervals do not intersect. Hence $a\le b$. For every interval, its left endpoint is at most a, and its right endpoint is at least b. Hence every interval in the family contains the interval $[a, b]$, which we have shown is nonempty.

8.1.25. *A tree is an interval graph if and only if it is a caterpillar.* We prove the following equivalent for a tree G.
A) G is an interval graph.
B) G is a caterpillar.
C) G does not contain the tree Y formed by subdividing each edge of a claw.

B $\Rightarrow$ A. Create an interval for each vertex on the spine of the caterpillar, such that each interval intersects its the intervals for its neighbors on the spine and no others. This leaves part of each interval intersecting no other. Place small intervals for the leaf neighbors of each vertex x of the spine in the "displayed" area of the interval for x.

C $\Rightarrow$ B. A longest path P contains an endpoint of every edge. If some edge is missed, then there is an edge with neither endpoint on P but having a neighbor x on P (since the tree is connected). Since P is a longest path, P continues at least two edges in each direction from x. Now the tree contains Y, consisting of these six edges within distince 2 of x.

A $\Rightarrow$ C. If G contains Y but is an interval graph, then in an interval representation of G the intervals for the leaves of Y are pairwise disjoint. Name the leaves x, y, z in the order of the corresponding intervals, from left to right. The union of the intervals for the x, z-path in G must cover the gap between the intervals for x and z in the representation. Since this gap contains the interval for y, we obtain a contradiction, because y has no neighbor on this path.

8.1.26. *Every interval graph is a chordal graph and is the complement of a comparability graph.* If it is not a chordal graph, then it has a chordless cycle. A chordless cycle has no interval representation, because the two paths along the cycle between the vertices corresponding to the leftmost and rightmost intervals among these vertices must occupy all the space between them on the line, which produces chords between the two paths when the intersections are taken. Hence the full graph has no interval representation.

Given an interval representation of a graph G, orienting $\overline{G}$ by $x \to y$ if the interval for x is completely to the right of the interval for y expresses $\overline{G}$ as a comparability graph.

8.1.27. *A graph G has an interval representation if and only if the clique-vertex incidence matrix of G has the consecutive 1s property.*

Necessity. From an interval representation, we obtain a natural ordering of the maximal cliques. By the Helly property (Exercise 8.1.24) the intervals corresponding to the vertices of a maximal clique have a common point. These points are different for distinct maximal cliques, because the

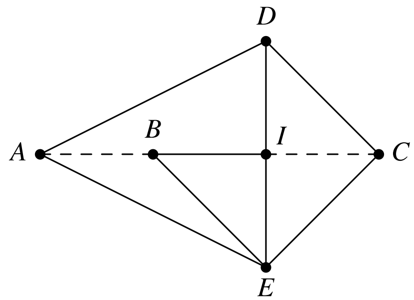
interval for a vertex nonadjacent to some vertex of a maximal clique must be disjoint from the interval for that vertex. Therefore, we can place the cliques in a linear order by the order of the chosen points. Using this ordering on the clique-vertex incidence matrix exhibits the consecutive 1s property, because the interval for a vertex extends from the first chosen point for a clique containing it to the last. The vertex belongs to all maximal cliques whose chosen cliques are between these, and it belongs to no other maximal cliques, since intervals have no gaps.

Sufficiency. Let M be the clique-vertex incidence matrix of G, and suppose that M has the consecutive 1s property. We construct an interval representation. Permute the rows of M so the 1s are consecutive in the columns. On a line, select points in order left to right corresponding to the rows of M. For each column of M (vertex of G), specify an interval from the point for the first 1 in it to the point for the last 1 in it. This defines one interval for each vertex because the 1s are consecutive. It yields an interval representation of G because vertices are adjacent if and only if there is a maximal clique that contains both of them.

8.1.28. *A graph is an interval graph if and only if it has a vertex ordering $v_1, \dots, v_n$ such that the neighborhood of each v_k among the lower-indexed vertices is a terminal segment $v_i, \dots, v_{k-1}$.* Given an interval representation f, index the vertices in order of the right endpoints of the corresponding intervals. If $v_k \leftrightarrow v_i$ with $i < k$, then $f(v_k)$ extends as far to the left and the right endpoint of $f(v_i)$, so it contains the right endpoints of $f(v_i), \dots, f(v_{k-1})$ and is adjacent to all those vertices.

8.1.29. *Interval graphs have no asteroidal triples.* Consider an interval representation for a graph G. Suppose that G has an asteroidal triple $\{x, y, z\}$; that is, three vertices such that connecting any two of them there is a path avoiding the neighborhood of the third. Rename these vertices x, y, z in the order of the corresponding intervals in the representation, from left to right. The union of the intervals for the x, z-path in G must cover the gap between the intervals for x and z in the representation. Since this gap contains the interval for y, we obtain a contradiction, because y has no neighbor on this path. (Comment: Interval graphs are precisely the chordal graphs that have no asteroidal triples.)

8.1.30. *The lying professor.* The intersection graph of the professor's presences in the library is an interval graph. The claimed sightings yield the graph below, where dotted edges are those confirmed from both sides and therefore presumed true. The graph contains two chordless 4-cycles, $DABI$ and $DAEC$. It is not possible to turn this into an interval graph by adding a single edge, and there is no reason to think a suspect would lie by leaving out other possible suspects. Therefore the most reasonable conclusion is that someone lied by trying to cast suspicion on someone else. The only single edge that can be removed to turn this into an interval graph (by destroying both chordless 4-cycles) is the edge due to Desmond's claim of seeing ("Honest") Abe. Hence we conclude that Desmond is the probable thief.



8.1.31. *G is a unit interval graph if and only if the matrix $A(G) + I$ has the consecutive ones property.* For necessity, take a unit interval representation and number the vertices in increasing order of left endpoint. Think of the interval for v_i as representing a loop at v_i. Then the fact that all intervals have the same length puts the right endpoints in the same order, and makes the vertices adjacent to v_i a consecutively-numbered sequence including v_i. In other words, with this ordering, the ones in each column of $A(G) + I$ appear consecutively.

If $A(G) + I$ has the consecutive ones property, then the 1's in each column of $A(G)$ are consecutive and include the diagonal. Let m be the number of distinct rows (or columns, since $A(G) + I$ is symmetric) of G. Construct a unit interval representation f by induction on m, using copies of m distinct intervals. If k columns are the same as the first, delete vertices $v_1, \ldots, v_k$. The remaining graph has the consecutivity property and $m - 1$ distinct rows, so by induction it has a unit representation. If the highest-indexed vertex adjacent to $v_1, \ldots, v_k$ is the jth type of row in $A(G) + I$, assign an interval to $v_1, \ldots, v_k$ that meets the first $j-1$ classes of intervals in f. Note that v_{k+1}, etc., also are adjacent to that high-indexed vertex, by the consecutivity property in $A(G) + I$.

8.1.32. (+) Prove that G is a proper interval graph (representable by intervals such that none properly contains another) if and only if the clique-vertex incidence matrix of G has the consecutive 1s property for both rows and columns. (Fishburn [1985]

8.1.33. *Every P_4-free graph is a Meyniel graph.* Let C be an odd cycle of length at least 5 in a P_4-free graph. Deleting one endpoint of the chord if C has one chord (or deleting an arbitrary vertex if C has no chord) leaves an induced path with at least four vertices. This cannot occur in a P_4-free graph, so C has at least two chords.

8.1.34. *Every odd cycle of length at least 5 in a chordal graph has two noncrossing chords.* Since the graph is chordal, such a cycle C has a chord xy. This chord forms a cycle C' of length at least 4 with one of the x, y-paths along C. A chord of C' is also a chord of C, and its endpoints are on one of the x, y-paths in C, so the two chords are noncrossing.

8.1.35. *If C is an odd cycle in a graph with no induced odd cycle, then $V(C)$ has three pairwise-adjacent vertices such that paths joining them in C all have odd length.* We use induction on the length of C; the statement is trivial when C is a triangle. When C is longer, we know that it has a chord xy. One of the x, y-paths along C has even length; with xy it forms an odd cycle. By the induction hypothesis, this cycle C' has three has three pairwise-adjacent vertices such that paths joining them in C' all have odd length. Two of these paths are along C, and one uses the edge xy. Replacing xy with the other x, y-path along C in this path yields a path of odd length, since the lengths of xy and the path replacing it are both odd. Therefore, the triple provided by the induction hypothesis for C' has the desired properties for C.

8.1.36. *The conditions below are equivalent.*
A) Every odd cycle of length at least 5 has a crossing pair of chords.
B) For every pair $x, y \in V(G)$, chordless x, y-paths are all even or all odd.

B $\Rightarrow$ A. Two vertices on an odd cycle C are connected by paths of different parity along C, so by the parity condition at least one of the paths has a chord. Applying the same argument to points separated on C by the endpoints of such a chord yields another chord of C.

If the two chords are non-crossing, then one or both combines with part of C to form a smaller odd cycle of length at least 5, unless C itself has length 5. Two crossing chords in the smaller cycle would be crossing chords in C. Hence we may assume that C has length exactly 5 and has two non-crossing chords. Now applying the parity condition to a vertex pair on C that does not induce one of these chords yields a third chord that crosses at least one of the other two.

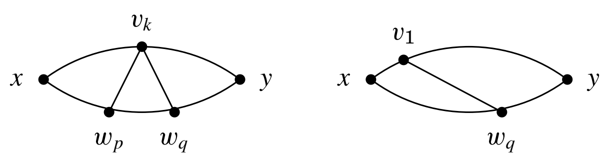
A $\Rightarrow$ B. Suppose that G has a vertex pair connected by chordless paths of opposite parity. Choose the pair $\{x, y\}$ and chordless x, y-paths P_1 and P_2 of even and odd length so that the sum of the lengths of P_1 and P_2 is as small as possible; call this "minimality". If P_1 and P_2 have a common vertex z, then z splits P_1 into two paths whose lengths have the same parity (and length at least two). Also z splits P_2 into two paths of opposite parity (and length at least two). The resulting x, z- or z, y-portions of P_1 and P_2

contradict minimality. Hence we may assume that $P_1 \cup P_2$ is a cycle C. Since P_2 cannot be a chord of P_1, the length of C is odd and at least 5.

We prove that C has no crossing chords. All chords join $P_1 - \{x, y\}$ and $P_2 - \{x, y\}$. Let P_1 and P_2 have vertices $x, v_1, \dots, v_s, y$ and $x, w_1, \dots, w_t, y$ in order, respectively (s is odd and t is even). Let w_p and w_q be the first and last neighbors of v_k in $P_2 - \{x, y\}$, if v_k has any such neighbors.

Suppose first that $2 \le k \le s-1$, so v_k partitions P_1 into chordless paths with the same parity as k. The parity of p is opposite to k, else two x, v_k-paths contradict minimality. Similarly, the parity of $t+1-q$ is opposite to k, which makes it the same as the parity of p. If $q > p$, then $q - p$ is even, else w_p, v_k, w_q and the w_p, w_q-portion of P_2 contradict minimality. We have now partitioned P_2 into three subpaths, of which the middle path has even length and the two extreme paths have the same parity; this is impossible and implies that v_k belongs to no chords.

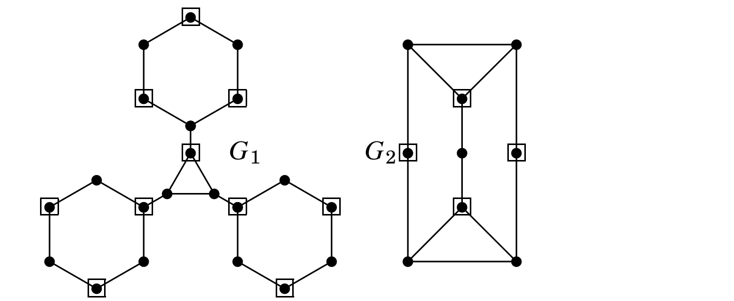
Now consider $k = 1$. As before, the v_1, y-paths yield $t + 1 - q$ even, and when $q > 1$ the x, w_q-paths yield q even. This is impossible, since $t + 1$ is odd. We conclude that $v_1 w_1$ is the only possible chord involving v_1. Similarly, $v_s w_t$ is the only possible chord involving v_s. We have proved that $v_1 w_1$ and $v_s w_t$ are the only possible chords of C, and they do not cross; this contradicts the hypothesis.



8.1.37. *Every perfectly orderable graph is strongly perfect.* Let G be a perfectly orderable graph and L an admissible ordering of G. I.e., G has no induced P_4 such that in L each endpoint appears before its neighbor. Let S be the greedy stable set with respect to L, i.e., place the first vertex of L in S, delete its neighbors, and iterate this step with the remaining vertices. Note that S is the set receiving color 1 under the greedy coloring for L.

We show that S meets every maximal clique. If S misses a maximal clique Q, then each vertex of Q must be deleted from the ordering due to having a prior neighbor that is in S. If all vertices of Q share a prior neighbor in S, then Q is not maximal. Hence we can choose $x, y \in Q$ and $u, v \in S$ such that $u \leftrightarrow x$, $v \leftrightarrow y$, but $u \not\leftrightarrow y$, $v \not\leftrightarrow x$. Since $x \leftrightarrow y$ and $u \not\leftrightarrow v$, these vertices induce P_4; since u comes before x and v before y, they induce an obstruction, contradicting the assumption that L is admissible.

8.1.38. *The graphs below are strongly perfect.* In each case, the marked stable set intersects all maximal cliques, but strong perfection also requires this for all induced subgraphs.



For G_1, an induced subgraph that omits a vertex of the central triangle is bipartite. Every bipartite graph is strongly perfect, because we can form a stable set intersecting all maximal cliques by taking one partite set from each nontrivial component plus all isolated vertices, and the family is hereditary. This takes care of all induced subgraphs of G_1 except those that retain the central triangle. For such a subgraph H, deleting an edge of the triangle yields a bipartite graph H' in which the three central vertices are in the same component. From this component of H', we choose the partite set containing only one vertex of the triangle in H; from others we take either partite set. The resulting set is stable in H' and intersects all maximal cliques in H', and it has the same properties in H.

For G_2, suppose that some induced subgraph H has a maximal clique Q avoiding the marked stable set S. This requires H to omit a vertex of S on a triangle. We may assume that Q is the lower horizontal edge. Now a "rotation" of S around the triangles intersects all maximal cliques in H unless H omits both of the top vertices. Now $H \subseteq P_4 + P_2$, but every disjoint union of paths has the desired property.

The graphs above are not perfectly orderable. A perfectly orderable graph has an orientation (associated with a perfect ordering) such that no induced P_4 has its pendant edges oriented outward. We show that these graphs have no such orientation; suppose that one exists.

For G_1, two of the cut-edges must be oriented in toward the triangle. Let yz be the oriented edge joining them, with xy being the entering cut-edge at its tail. The edges in a matching of size 3 on the 6-cycle containing z must be consistently oriented along the cycle, but one choice of this orientation conflicts with xy, and the other choice conflicts with yz.

For G_2, in the top half of the drawing, two of the three vertical edges must be oriented upward to avoid completing an obstruction with the top triangle. By symmetry, we may assume that these are the left and right vertical edges, but now either orientation of the horizonal edge on the bottom completes an obstruction with one of them.

8.1.39. *The graphs in Exercise 8.1.38 are a Meyniel graph but are not perfectly orderable.* The graphs have no chordless odd cycle (of length at least 5), so they vacuously satisfy the definition of a Meyniel graph. The task of showing they are not perfectly orderable is done in Exercise 8.1.38.

The graph $\overline{P}_5$ is perfectly orderable but is not a Meyniel graph. The graph $\overline{P}_5$ is the "house", a 5-cycle with one chord, so the cycle does not have the requisite two chords. There are two induces 4-vertex paths (each containing one endpoint of the cycle. If the cycle is numbered v_1, v_2, v_3, v_4, v_5 in order so that the chord is v_1v_4, then both copies of P_4 have one endpoint at v_5, so the associated orientation directs that pendant edge in toward the center, and there is no obstruction. Hence this is a perfect ordering.

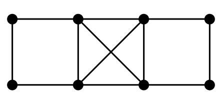
8.1.40. *Every chordal graph is weakly chordal.* If a graph has no chordless cycle, then it has no chordless cycle of length at least five. Suppose $v_1, \ldots, v_k$ in order induce in $\overline{G}$ a chordless cycle, meaning that G contains an antihole on these vertices. If $k = 5$, then $v_1v_3v_5v_2v_4$ is a chordless 5-cycle in G. If $k \geq 6$, then $v_1v_4v_2v_5$ is a chordless 4-cycle in G.

The graph H below is weakly chordal. Any cycle with more than four vertices has at least three in the central clique Q and hence has a chord. In $\overline{H}$, we need only forbid induced C_k for $k \geq 6$, since $\overline{C}_5 = C_5$. Note that $\overline{H}$ has 16 edges (too many for C_8), of which 3 are incident to each vertex of Q and 5 to each of the other vertices. Hence every 7-vertex subgraph has at least 11 edges. The 6-vertex induced subgraphs of $\overline{H}$ with only 6 edges are those where the deleted vertices are neighboring vertices of degree 2 in H (deleting 10 edges from $\overline{H}$), but such a subgraph of $\overline{H}$ is a 4-cycle with two pendant edges.

H is not strongly perfect.

Proof 1. Since $V(H)$ is covered by three disjoint cliques, $\alpha(H) \leq 3$. However, each vertex appears in two maximal cliques, so three vertices cannot meet all 7 maximal cliques.

Proof 2. There are 7 maximal cliques in H: one 4-clique and 6 edges. In a chordless path of three edges, a stable set meeting every maximal clique must contain at least two vertices, including at least one endpoint. Hence if a stable set S meets every maximal clique, the paths on the left and right force S to contain two vertices of the central clique.



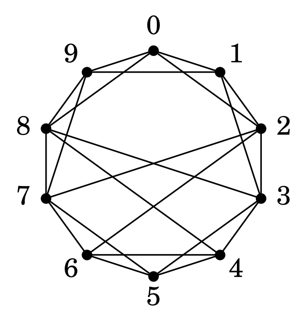
8.1.41. *SPGC $\Rightarrow$ Skew Partition Conjecture $\Rightarrow$ Star-Cutset Lemma.* The *Skew Partition Conjecture* states that no p-critical graph has a skew partition (a *skew partition* of G is a partition of $V(G)$ into nonempty sets X and Y such that $G[X]$ is disconnected and $\overline{G}[Y]$ is disconnected).

The SPGC states that every p-critical graph is an odd cycle or the complement of an odd cycle. Since a skew partition of G is also a skew partition of $\overline{G}$, we obtain the Skew Partition Conjecture from the SPGC by showing that an odd cycle has no skew partition. A skew partition requires X to use more than one segment along the cycle, but then the subgraph of the complement induced by the remaining vertices is connected.

To prove that the Skew Partition Conjecture implies the Star-Cutset Lemma, which states that no p-critical graph has a star-cutset, it suffices to show that a graph with a star-cutset has a skew partition. If C is a star-cutset in G, let $X = V(G) - C$ and $Y = C$. Now $G[X]$ and $\overline{G}[Y]$ are both disconnected, since the dominating vertex in C becomes an isolated] vertex in $\overline{G}[Y]$.

8.1.42. *The graph below is* $3,3$*-partitionable.* Due to the horizontal symmetry through the vertical axis, we need only check six classes of vertices to show that each $V(G-x)$ partitions into three 3-cliques and into three stable 3-sets. This is easy but tedious and seems to require a picture for each vertex.

Alternatively, by Theorem 8.1.39, since $\alpha(G) = \omega(G) = 3$, it suffices to show that (1) each vertex belongs to three 3-cliques and to 3 stable 3-sets, and (2) G has 10 3-cliques and 10 stable 3-sets, paired so that each intersects every set of the other type except its mate. We show this giving a matrix that lists the 3-cliques and stable 3-sets in the rows and columns and has the elements of $Q_i \cap S_j$ as the entries. Each vertex appears in three of the row labels and three of the column labels. However, the matrix does not contain a proof that there are no other cliques or stable sets of size 3. Curiously, the maximum cliques and stable sets are the same as in C_{10}^3 except for a switch of membership in two cliques and two stable sets, underlined below.



$$\begin{array}{cc}
 & \begin{array}{cccccccccc} 369 & 470 & 925 & \underline{8}16 & 703 & 49\underline{2} & 581 & 036 & 147 & 258 \end{array} \\
\begin{array}{c} 012 \\ 123 \\ \underline{8}34 \\ 345 \\ 456 \\ 567 \\ 67\underline{2} \\ 789 \\ 890 \\ 901 \end{array} &
\begin{pmatrix}
\varnothing & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 2 \\
3 & \varnothing & 2 & 1 & 3 & 2 & 1 & 3 & 1 & 2 \\
3 & 4 & \varnothing & 8 & 3 & 4 & 8 & 3 & 4 & 8 \\
3 & 4 & 5 & \varnothing & 3 & 4 & 5 & 3 & 4 & 5 \\
6 & 4 & 5 & 6 & \varnothing & 4 & 5 & 6 & 4 & 5 \\
6 & 7 & 5 & 6 & 7 & \varnothing & 5 & 6 & 7 & 5 \\
6 & 7 & 2 & 6 & 7 & 2 & \varnothing & 6 & 7 & 2 \\
9 & 7 & 9 & 8 & 7 & 9 & 8 & \varnothing & 7 & 8 \\
9 & 0 & 9 & 8 & 0 & 9 & 8 & 0 & \varnothing & 8 \\
9 & 0 & 9 & 1 & 0 & 9 & 1 & 0 & 1 & \varnothing
\end{pmatrix}
\end{array}$$

8.1.43. *If x and v are nonadjacent vertices in a partitionable graph G, then every maximum clique containing x consists of one vertex from each stable set that is the mate of a clique containing v.* (The complementary assertion is that if x and v are adjacent vertices, then every maximum stable set containing x consists of one vertex from each clique that is the mate of a stable set containing v.)

By Theorem 8.1.41, the unique minimum coloring of $G - v$ consists of the $\omega(G)$ stable sets that are mates of the maximum cliques containing v. Since x and v are nonadjacent, a maximum clique containing x omits v and hence must contain exactly one vertex from each stable set in this coloring.

8.1.44. *No p-critical graph has antitwins.* Antitwins are a pair of vertices such that every vertex outside them is adjacent to exactly one of them. Consider a p-critical graph G, and let $\omega = \omega(G)$ and $\alpha = \alpha(G)$.

We first prove that a p-critical graph with antitwins $\{x, y\}$ has a clique of size $\omega - 1$ in $N(x)$ that doesn't extend into $N(y)$. Recall that $\omega(G - S) = \omega(G)$ for any stable set S in a p-critical G (reminder of proof - since $G - S$ is perfect, smaller clique-number would give a smaller coloring, extending to an ω-coloring of G by replacing S). Since G is partitionable, $G - x$ has a unique coloring by ω stable sets of size α; let S be the stable set containing y in this coloring, and let Q be an ω-clique in $G - S$. Since $G - x - S$ is $\omega - 1$-colorable (S is a color in the ω-coloring of $G - x$), Q must contain x. Since $G - x - S$ has no ω-clique, $Q' = Q - x \subset N(x)$ is the desired clique.

Reversing the roles of x and y yields a similar $\omega - 1$-clique in $N(y)$. Since the complement of a p-critical graph is p-critical, we also can apply the same argument to obtain $\alpha - 1$-cliques in $N_{\overline{G}}(x)$ and $N_{\overline{G}}(y)$ that translates into the desired $\alpha - 1$-stable sets in $N(y)$ and $N(x)$. Let S' be the resulting stable set of size $\alpha - 1$ in $N(y)$ that doesn't extend in $N(x) \cup N(y)$.

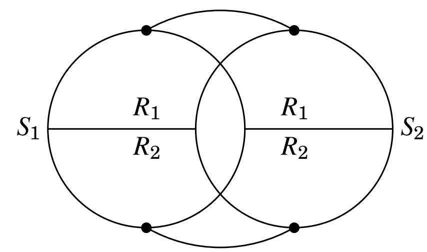
Choose u to be the vertex of Q' with the minimum number of neighbors in S'; u must have at least one neighbor v in S', else S' extends to u. Similarly, v must have a non-neighbor z in Q'. Since $v \in N(u) - N(z)$ and z has at least as many neighbors as u in S', z must have a neighbor w in S' that is not adjacent to u. Now y, v, u, z, w induce a chordless 5-cycle in G. This misses x, so G is not p-critical. (Note: For the special non-circulant partitionable graph pictured in the text, which is not p-critical, the top and bottom vertices are antitwins.)

8.1.45. *Stable sets and even pairs in partitionable graphs.*

a) If S_1, S_2 are maximum stable sets in a partitionable graph G, then $G[S_1 \triangle S_2]$ is connected. Let $S = S_1 \triangle S_2$. Let R_1 be the vertex set of a component of $G[S]$, and let $R_2 = S - R_1$. The sets $T_1 = (S_1 - R_1) \cup (S_2 - R_2)$ and $T_2 = (S_1 - R_2) \cup (S_2 - R_1)$ are stable sets with the same union and intersection as S_1 and S_2 (see figure). Hence $|T_1| + |T_2| = 2\alpha(G)$, which implies

that $|T_1| = |T_2| = \alpha(G)$ since each has size at most $\alpha(G)$.

Since the rows of the incidence matrix between maximum stable sets and vertices are linearly independent, we cannot have two pairs of stable sets with the same union and intersection. Either $T_1 = S_1$ and $T_2 = S_2$, which yields the contradiction $R_1 = \varnothing$, or $T_1 = S_2$ and $T_2 = S_1$, in which case $R_1 = S$ and $G[S]$ is connected.



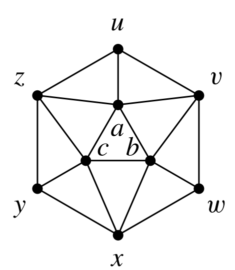
b) No partitionable graph (and hence no p-critical graph) has an even pair. Let x, y be any two vertices in a partitionable graph G. Let S be a maximum stable set containing x in $G - y$, and let T be a maximum stable set containing y in $G - x$. Let $H = G[S \triangle T]$. By part (a), H is connected. Since S and T are stable sets, H is bipartite, with partite sets $S - T$ and $T - S$. By construction, $x \in S - T$ and $y \in T - S$, so every x, y-path in H has odd length. Since H is an induced subgraph of G, a shortest x, y-path in H is a chordless x, y-path in G. Hence x, y is not an even pair in G.

8.1.46. *If G is partitionable, and S_1, S_2 are stable sets in the optimal coloring of $G - x$, then $G[S_1 \cup S_2 \cup \{x\}]$ is 2-connected.* Since S_1, S_2 are maximal, x is adjacent to a vertex of each. Since S_1, S_2 are disjoint, $S_1 \oplus S_2 = S_1 \cup S_2$. Thus part (a) of the preceding problem implies that $H = G_{S_1 \cup S_2 \cup x}$ is connected and that x cannot be a cut-vertex of H. If H has a cut-vertex, we may assume it is $s \in S_1$. Let G_1 be a component of $H - s$ not containing x, and let G_2 be the rest of $H - s$, with $V_i = V(G_i)$.

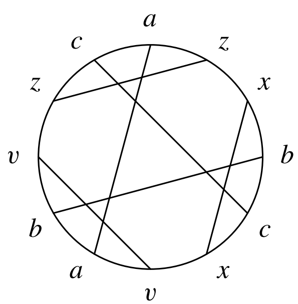
Recall (*): whenever v, x are nonadjacent vertices of a partitionable graph G, any maximum clique containing v omits x and therefore consists of one vertex from each stable set in the unique minimum coloring of $G - x$. Since x has no neighbor in V_1, we can apply (*) to any $v \in V_1$. I.e., each clique in $\Theta(G - x)$ that contains a vertex of G_1 must contain exactly one vertex of each of S_1, S_2. Both these vertices must be in G_1, else we introduce an edge between G_1 and G_2. Thus V_1 has an equal number of vertices from S_1 and S_2, both equal to the number of cliques in $\Theta(G - x)$ that meet G_1.

Choose $u \in S_2 \cap V_1$. To any clique Q of $\Theta(G - u)$, we can apply (*) again to guarantee that Q contains one vertex each of S_1, S_2. In particular, for each $v \in S_1 \cap V_1$, there is a clique of $\Theta(G - u)$ containing it, and this yields a vertex $v' \in S_2 \cap V_1 - u$ adjacent to it. Since S_1 is stable, these cliques of $\Theta(G-u)$ are disjoint, and so the vertices $\{v'\}$ are distinct. This implies $|S_1 \cap V_1| \le |S_2 \cap V_1 - u| < |S_2 \cap V_1|$, contradicting the result of the previous paragraph.

8.1.47. *The graph G below is a circular-arc graph but not a circle graph.* To represent G as a circular-arc graph, we let the arcs for the inner cycle and the outer cycle in the drawing each cover the circle. More precisely, consider a circle of circumference 9, with points on the circle described by numbers modulo 9. Assign arcs as in the middle table below to form a circular-arc representation.



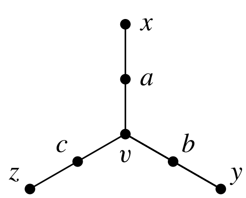
a: [0, 3]	b: [3, 6]	c: [6, 0]
u: [1, 2]	w: [4, 5]	y: [7, 8]
v: [2, 4]	x: [5, 7]	z: [8, 1]

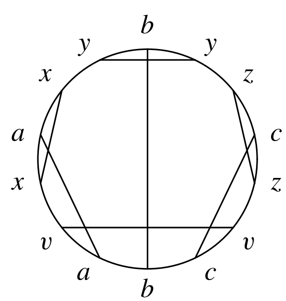


To show that G is not a circle graph, suppose that G has an intersection representation by chords in a circle. The chords for $\{a, b, c\}$ are pairwise intersecting, so their endpoints occur in the order a, b, c, a, b, c on the circle.

The chord for v cannot cross the chord for c, so to intersect the chords for a and b the endpoints for v must precede an a and follow the subsequent b, yielding a, b, c, v, a, b, v, c. We make the analogous argument for x and for z. However, $\{v, x, z\}$ is independent, so the endpoints of chords for any two of them cannot alternate. This means that when we add the endpoints for x and z to satisfy the constraints, we must obtain $a, z, x, b, c, x, v, a, b, v, z, c$, as shown above. Now we cannot add the chord for u to cross the chords for $\{z, a, v\}$ without crossing the chord for b or c.

The graph H below is a circle graph but not a circular-arc graph. A circle representation is shown in the middle below.





To show that H has no circular-arc representation, note that the arcs for $\{a, b, c\}$ must be pairwise disjoint. Since the arc for v must intersect all three, it must contain one of them completely; by symmetry, we may let it be a. Now the arc for x cannot intersect the arc for a without intersecting the arc for v.

8.1.48. *Paw-free graphs satisfy the SPGC.* The "paw" is the graph obtained from the claw $K_{1,3}$ by adding an edge joining two leaves. We must prove that every paw-free graph having no odd hole and no odd antihole is perfect. It suffices to prove that every paw-free graph G having no odd hole is a Meyniel graph, meaning that odd cycles of length at least 5 have at least two chords. Let C be an odd cycle of length at least 5 in G. Since G has no odd hole, C has a chord xy. This forms two cycles with the x, y-paths on C; one is odd. If the odd one has length at least 5, we obtain another chord of C. Otherwise, it has length 3. Since the subgraph induced by these three vertices and the next vertex on C must not be a paw, it contains an additional chord of C.

8.1.49. *Sets S and T of sizes $a+2$ and $w+2$ that intersect every maximum clique and every maximum stable set, respectively, in the cycle-power C_{aw+1}^{w-1}.* (This completes the proof of Theorem 8.1.51.)

Let $S = \{v_{aw}, v_1, v_w, v_{w+2}\} \cup \{v_{iw+1} \colon 2 \le i \le a-1\}$. The maximum cliques in C_{aw+1}^{w-1} are the sets of w vertices with consecutive indices. The first four indices listed for S are separated successively by 2, $w-1$, and 2, respectively. The next step is $w-1$, and the subsequent gaps are w until the final step of $w-1$ that returns to the beginning. Since the set never skips as many as w consecutive indices, it intersects all maximum cliques.

Let $T = \{v_{(a-1)w+1}, v_{aw}, v_1, v_w\} \cup \{v_{w+i} \colon 2 \le i \le w-1\}$. The maximum stable sets in C_{aw+1}^{w-1} are the sets of a vertices whose indices increase successively by w (cyclically) starting from some point. In particular, a set of w successive vertices intersects all but one maximum stable set. The set T has $w-1$ of w successive indices from w through $2w-1$. The stable set skipping this interval starts at v_{2w} and contains v_{aw}, so it intersects T. The remaining stable sets are those containing v_{w+1}. These all contain v_1 except the one that starts at $w+1$, but this stable set intersects T at $(a-1)w+1$.

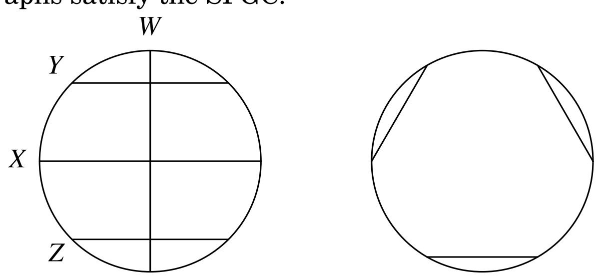
8.1.50. *SPGC for circle graphs.*

a) If x is a vertex in a partitionable graph G, then $G - N[x]$ is connected. If $G - N[x]$ is disconnected, then $N[x]$ is a star-cutset. It thus suffices to show that partitionable graphs have no star-cutsets. Since $\chi(G-x) = \omega$ for each $x \in V(G)$, every proper induced subgraph of G is $\omega(G)$-colorable.

Because $G - x$ has a partition into $\alpha(G)$ disjoint maximum cliques, a stable set intersecting all maximum cliques must be a maximum stable set. However, every maximum stable set misses its mate, so no stable set intersects every maximum clique. These are the hypotheses of the Star-Cutset Lemma Lemma, so G has no star-cutset.

b) Partitionable circle graphs are claw-free. Three pairwise-disjoint chords Y, X, Z of a circle can be intersected by a single chord W only if the endpoints occur as shown below. Suppose that a circle graph G has a claw induced by central vertex w and stable set $\{y, x, z\}$. If x is the vertex corresponding to the middle chord among $\{y, x, z\}$ in the circle representation of G, we have $G - N[x]$ disconnected, since every y, z-path in G must contain a vertex whose chord intersect the chord for x in the representation. By part (a), this cannot occur in a partitionable circle graph.

c) Circle graphs satisfy the SPGC. By part (b), partitionable circle graphs are claw-free. By Corollary 8.1.53, claw-free graphs satisfy the SPGC. Thus every p-critical circle graph is an odd hole or an odd antihole, and circle graphs satisfy the SPGC.



8.2. MATROIDS

8.2.1. *The family of independent vertex sets of a graph need not be the family of independent sets of a matroid.* In the star $K_{1,n}$, let the leaves have weight 1 and the remaining vertex have weight 2. The resulting maximum weighted stable set has weight n, but the greedy algorithm stops with a stable set of weight 2.

8.2.2. *The family of stable sets of a graph G is the family of independent sets of a matroid on its vertex set if and only if every component of G is a complete graph.* If some component of G is not complete, then G has P_3 as an induced subgraph. The stable set of size 1 consisting of the middle of this path cannot be augmented from the stable set of size 2 consisting of its endpoints, so the augmentation inequality fails.

Conversely, if every component is complete, then the hereditary system is a partition matroid, with the stable sets being those sets of vertices having at most one vertex in each component.

8.2.3. *Every partition matroid is a transversal matroid* A partition matroid on E is defined by sets $E_1, \ldots, E_k$ partitioning E such that a subset of E is independent if and only if it contains at most one element of each E_i. This is the same as the transversal matroid on E arising from the $E, [k]$-bigraph whose ith component is the star with center i and leaf set E_i, for $1 \le i \le k$.

8.2.4. *Greedy algorithm with arbitrary real weights.* Since $\varnothing$ is always an independent set and has weight 0, a maximum weighted independent set contains no elements of negative weight. Hence it suffices to run the usual greedy algorithm on the restriction of the matroid obtained by discarding the elements of negative weight. This is accomplished simply by stopping the greedy algorithm when all the elements of nonnegative weight have been considered.

8.2.5. *The family of matchings in a graph G is the family of independent sets of a matroid on $E(G)$ if and only if every component of G is a star or a triangle.* The family of matchings in G is the family of stable sets in $L(G)$. By Exercise 8.2.2, the characterization is that every component of $L(G)$ is a complete graph. A component of $L(G)$ is a complete graph if and only if the corresponding component of G is a star or a triangle.

8.2.6. *The cycle matroid of a multigraph G is a uniform matroid if and only if G is a forest, a cycle, a multiple edge, or a collection of loops (plus possible isolated vertices in each case). The uniform matroid $\mathbf{U}_{k,n}$ is a cycle matroid if and only if $k \in \{0, 1, n-1, n\}$.* With G having n edges, the cycle matroids in the cases listed are the uniform matroids $\mathbf{U}_{n,n}, \mathbf{U}_{n-1,n}, \mathbf{U}_{1,n}, \mathbf{U}_{0,n}$, respectively.

For the converse, note that a matroid is non-uniform if and only if it has a dependent set and an independent set of the same size. Any multigraph G that is not a forest or a cycle (plus isolated vertices) has a cycle that does not contain all the edges. The edge set of a smallest cycle C in G is a circuit in $M(G)$. If its size exceeds 2, then deleting an edge of C and replacing it by any edge not in C cannot yield another cycle; hence it yields an independent set of the same size as C in the cycle matroid.

8.2.7. *The cycle matroid of a multigraph G is a partition matroid if and only if the blocks of G are sets of parallel edges. Every partition matroid is graphic.* A matroid M is a partition matroid with blocks $E_1, \ldots, E_k$ if and only if the circuits of M are all sets of size 2 contained in single blocks.

This is also the cycle matroid of the graph whose blocks are sets of parallel edges of sizes $|E_1|, \ldots, |E_k|$.

In order for $M(G)$ to be a partition matroid, the circuits must have size 2. Hence G has no loops and no cycles of length greater than 2. The latter occurs if and only if the simple graph obtained by discarding extra copies of multiple edges is a forest. Hence G must be as claimed.

8.2.8. *Vectorial matroids satisfy the induced circuit property: adding an element to a linearly independent set of vectors creates at most one minimal dependent set.* Let I be an independent set of vectors in a vector space, and let e be a vector. Let C_1 and C_2 be minimal dependent sets of vectors in $I \cup \{e\}$. The definition of dependence for sets of vectors is the existence of an equation of dependence. The coefficent on e in such an equation of dependence is nonzero, since I is independent; indeed, since these are minimal dependent sets, all the coefficients are nonzero. Hence in each equation we can solve for e expressing e as a linear combination of $C_1 - \{e\}$ and of $C_2 - \{e\}$. Setting these expressions equal yields an equation of dependence for I if C_1 and C_2 are different, since the coefficients in the original equations are nonzero. Hence $C_1 = C_2$, and $I \cup \{e\}$ contains only one minimal dependent set.

8.2.9. *Circuits of a partition matroid.* By definition, sets of elements are independent if they have at most one element from each block of the partition. Hence a set is a circuit if and only if it consists of two elements from one block of the partition. If distinct circuits have a common element e, then they have the form $\{e, x\}$ and $\{e, y\}$, where e, x, and y all lie in a common block. Hence $\{x, y\}$ is also a circuit, and weak elimination holds.

8.2.10. *Direct verification of submodularity for rank functions of cycle matroids.* Given a graph G, let $k(X)$ denote the number of components of the spanning subgraph G_X with edge set X. Let H be the bipartite graph whose partite sets are the sets of components in G_X and G_Y, with vertices adjacent if the corresponding subgraphs share a vertex.

a) H has $k(X) + k(Y)$ vertices and $k(X \cup Y)$ components, and $k(X \cap Y) \ge e(H)$. By construction, the sizes of the partite sets are $k(X)$ and $k(Y)$. Components of G_X and G_Y that share a vertex lie within a single component in $G_{X\cup Y}$. Hence $k(X \cup Y)$ is the number of components of H.

Every edge of H has the form $C_X C_Y$, where C_X and C_Y are components of G_X and G_Y, respectively. Let $S = V(C_X) \cap V(C_Y)$; since $C_X C_Y$ is an edge, $S \ne \varnothing$. Every vertex outside S is outside $V(C_X)$ or outside $V(C_Y)$. Hence $X \cap Y$ has no edge leaving S, and $G_{X\cap Y}[S]$ is a nonempty union of components of $G_{X\cap Y}$. Thus $k(X \cap Y) \ge e(H)$, since we generate at least one component of $G_{X\cap Y}$ for each edge of H (maybe more than one, such as when G is a 4-cycle and X and Y decompose G into two copies of P_3).

b) For the cycle matroid $M(G)$, the submodularity property $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$ holds. In the cycle matroid, $r(X) = n(G) - k(X)$, so it suffices to show that k is supermodular.

A graph with n vertices and c components has at least $n - c$ edges. Since H has $k(X) + k(Y)$ vertices and $k(X \cup Y)$ components, we conclude that $e(H) \ge k(X) + k(Y) - k(X \cup Y)$. By part (a), also $k(X \cap Y) \ge e(H)$. Hence $k(X \cap Y) + k(X \cup Y) \ge k(X) + k(Y)$, as desired.

8.2.11. *Submodularity of rank functions of transversal matroids, using matching theory.* A transversal matroid on a set E is induced by a family $A_1, \ldots, A_m$ of subsets of E by letting the independent sets be the systems of distinct representatives of subfamilies. Equivalently, the independent sets are the subsets of E that can be saturated by matchings in the $E, [m]$-bigraph G. that is the incidence bigraph of the family.

By definition, then, the rank of a set $X \subseteq E$ is the maximum size of a matching in the subgraph $G[X \cup [m]]$, which we denote by G_X. By the König–Egerváry Theorem, $\alpha'(G_X)$ equals the minimum size of a vertex cover in G_X. For $S \subseteq X$, the smallest vertex cover Q such that $S = X - Q$ is $(X - S) \cup N(S)$. Hence the minimum size of a vertex cover of G_X is $|X| - \max_{S\subseteq X}(|S| - |N(S)|)$. The quantity $|S| - |N(S)|$ is the *deficiency* of S, denoted $\mathrm{def}(S)$, and the fact that $\alpha'(G_X) = |X| - \max_{S\subseteq X} \mathrm{def}(S)$ is due to Ore (Exercise 3.1.32).

Now consider subsets $X, Y \subseteq E$. For the submodularity inequality, we must bound $r(X \cup Y) + r(X \cap Y)$ by $r(X) + r(Y)$. For this we begin by studying the neighborhoods of the union and intersection of two sets $S \subseteq X$ and $T \subseteq Y$. The key to the inequality is that for $S, T \subseteq E$, we have $N(S \cap T) \subseteq N(S) \cap N(T)$ (equality need not hold!). Also $N(S \cup T) = N(S) \cup N(T)$. Thus

$$|N(S \cup T)| + |N(S \cap T)| \le |N(S) \cup N(T)| + |N(S) \cap N(T)| = |N(S)| + |N(T)|$$

Since $|S \cup T| + |S \cap T|) = |S| + |T|$, this yields $\mathrm{def}(S \cup T) + \mathrm{def}(S \cap T) \ge \mathrm{def}(S) + \mathrm{def}(T)$. Furthermore, the deficiency of a set S is the same in each G_X such that $X \supseteq S$. Therefore, if we let S and T be subsets of X and Y with maximum deficiency in G_X and G_Y, we obtain

$$\begin{aligned} r(X) + r(Y) = |X| - \mathrm{def}(S) + |Y| - \mathrm{def}(T) &\ge |X| + |Y| - \mathrm{def}(S \cup T) - \mathrm{def}(S \cap T) \\ &\ge |X \cup Y| - \max_{U\subseteq(X\cup Y)} \mathrm{def}(U) + |X \cap Y| - \max_{V\subseteq(X\cap Y)} \mathrm{def}(V), \end{aligned}$$

using in the last step that $S \cup T$ and $S \cap T$ are particular subsets of $X \cup Y$ and $X \cap Y$, respectively. Thus the submodularity inequality holds.

8.2.12. *For a digraph D with distinguished source s and sink t, and $r(X)$ defined for $X \subseteq V(D) - \{s, t\}$ to be the number of edges from $s \cup X$ to $\overline{X} \cup t$,* *the function r is submodular.* When we view D as a network by giving each edge capacity 1, the statement of submodularity for r is precisely the statement of part (a) of Exercise 4.3.12.

8.2.13. *For an element x in a hereditary system, the following properties are equivalent and characterize loops.* The definition of a loop (an element comprising a circuit of size 1) is statement C.

A) $r(x) = 0$. D) x belongs to no base.
B) $x \in \sigma(\varnothing)$. E) Every set containing x is dependent.
C) x is a circuit. F) x belongs to the span of every $X \subseteq E$.

F $\Rightarrow$ B. If $x \in \sigma(X)$ for all $X \subseteq E$, then $x \in \sigma(\varnothing)$.

B $\Rightarrow$ C. If $x \in \sigma(\varnothing)$, then x completes a circuit with $\varnothing$; hence $\{x\}$ is a circuit.

C $\Rightarrow$ A. The rank of a circuit C is $|C| - 1$.

A $\Rightarrow$ D. Every subset of every base is independent. If x belongs to a base, then $r(\{x\}) = 1$.

D $\Rightarrow$ E. If x belongs to an independent set, then it can be augmented to a maximal independent set (a base) containing x.

E $\Rightarrow$ F. Let Y be a maximal independent subset of X. If every set containing x is dependent, then $Y \cup x$ contains a circuit C, which must contain x since Y is independent. Hence x completes a circuit with a subset of X, so $x \in \sigma(X)$.

8.2.14. *The following characterizations of parallel elements in a hereditary system are equivalent, assuming that $x \neq y$ and neither is a loop.* Property B is the definition of parallel elements, given that neither is a loop.

A) $r(\{x, y\}) = 1$.
B) $\{x, y\} \in \mathbf{C}$.
C) $x \in \sigma(y)$, $y \in \sigma(x)$, $r(x) = r(y) = 1$.

B $\Leftrightarrow$ A. The rank of a circuit C is $|C| - 1$, so B $\Rightarrow$ A. Conversely, if $r(\{x, y\}) < 2$ with x and y being non-loops, then $\{x, y\}$ is a minimal dependent set.

B $\Leftrightarrow$ C. Since neither is a loop, $r(x) = r(y) = 1$, and each element by itself forms an independent set. Now $\{x, y\} \in \mathbf{C}$ is equivalent to $x \in \sigma(\{y\})$ and $y \in \sigma(\{x\})$, by the definition of the span function.

If x and y are parallel elements in a matroid and $x \in \sigma(X)$, then $y \in \sigma(X)$. From $x \in \sigma(X)$, we have $x \in X$ or $Y \cup \{x\} \in \mathbf{C}$, where $Y \subseteq X$. If $x \in X$, then $y \in \sigma(X)$, since y completes a circuit with x. If $Y \cup \{x\} \in \mathbf{C}$ and $\{x, y\} \in \mathbf{C}$, then the weak elimination property guarantees a circuit in $Y \cup \{y\}$, and hence $y \in \sigma(X)$.

8.2.15. *If $r(X) = r(X \cap Y)$ in a matroid, then $r(X \cup Y) = r(Y)$.*

Proof 1 (submodularity). Submodularity yields $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$. Cancelling $r(X) = r(X \cap Y)$ leaves $r(X \cup Y) \leq r(Y)$, but $r(X \cup Y) \geq r(Y)$ always, so equality holds.

Proof 2 (span function and absorption). The hypothesis implies $X \subseteq \sigma(X \cap Y)$, which in turn is contained in $\sigma(Y)$ since σ is order-preserving. Now $X \subseteq \sigma(Y)$ and the absorption property yield $r(X \cup Y) = r(Y)$.

8.2.16. *If M is a hereditary system that satisfies the base exchange property (B), then the greedy algorithm generates a maximum-weighted base whenever the elements have nonnegative weights.* This is actually more direct using the dual version of the base exchange property (Lemma 8.2.33): if $B_1, B_2 \in \mathbf{B}$ and $e \in B_1 - B_2$, then there exists $f \in B_2 - B_1$ such that $B_2 + e - f$ is a base. This follows from the induced circuit property in Lemma 8.2.33, and the induced circuit property follows directly from the base exchange property in Exercise 8.2.17.

Since the weights are nonnegative, the greedy algorithm generates a base. Let B be a base generated by the greedy algorithm. Among the bases of maximum weight, let B^* be one having largest intersection with B. If $B^* \neq B$, then there exists an element $e \in B - B^*$, since the bases form an antichain. Let e be a heaviest element of $B - B^*$. By the dual base exchange property, there exists $f \in B^* - B$ such that $B^* + e - f$ is a base. Since B^* is optimal, $w(f) \geq w(e)$. Since the greedy algorithm chose e after choosing the heavier elements of B, even though f was also avaiable, $w(e) \geq w(f)$. Hence $w(e) = w(f)$, and $B^* + e - f$ is an optimal base having larger intersection with B than B^* does. Hence in fact $B = B^*$.

8.2.17. *Exercises in axiomatics.*

a) In a hereditary system, the submodularity property implies the weak absorption property. Applying submodularity to $X + e$ and $X + f$ yields $r(X + e + f) + r(X) \leq r(X + e) + r(X + f)$. If $r(X + e) = r(X + f) = r(X)$, then monotonicity of r implies $r(X + e + f) = r(X)$.

b) In a hereditary system, the strong absorption property implies the submodularity property. We use induction on $k = |X \triangle Y|$. If $r((X \cap Y) + e) = r(X \cap Y)$ for all $e \in X \triangle Y$, then $r(X \cup Y) = r(X \cap Y)$, by strong absorption. Monotonicity of r then implies $r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$. This case includes the basis step $k = 0$.

Hence when $k > 0$ we may select $e \in X - Y$ (by symmetry) such that $r((X \cap Y) + e) = r(X \cap Y) + 1$. Let $Y' = Y + e$. By the induction hypothesis, $r(X \cap Y') + r(X \cup Y') \leq r(X) + r(Y')$. The left side equals $r(X \cap Y) + 1 + r(X \cup Y)$ and the right side is bounded by $r(X) + r(Y) + 1$, so subtracting 1 from both sides yields the desired inequality.

c) The base exchange property (B) implies the induced circuit property (J). **Proof 1** (contradiction). For $I \in \mathbf{I}$, if $I + e$ contains distinct circuits C_1, C_2, then each consists of e plus a subset of I. Since $C_1 \neq C_2$, we may choose $a \in C_1 - C_2$. Both $C_1 - a$ and $(C_1 \cup C_2) - e$ are independent; augment them to bases B_1 and B_2, respectively.

Since $C_1 - e \subseteq B_2$, every element of $B_1 - B_2$ except e is outside $C_1 - a$. Using (B), delete such elements from B_1, replacing them with elements of $B_2 - B_1$. This transforms B_1 to a base B such that the only element of $B - B_2$ is e, and still $C_1 - a \subseteq B$ and $a \notin B$.

Since (B) implies that bases have the same size, also $|B_2 - B| = 1$. Since $a \in B_2 - B$, the rest of B_2, including $C_2 - e$, is in B. However, $e \in B$, so $C_2 \in B$, contradicting that B is a base.

Proof 2 (extremality). Since every independent set lies in a base, it suffices to prove for $B \in \mathbf{B}$ that $B + e$ contains exactly one circuit. Let A be a minimal subset of B containing an element of each circuit in $B + e$. Thus $(B - A) + e \in \mathbf{I}$, but $(B - A) + e + a \notin \mathbf{I}$ for all $a \in A$.

Let B' be a base containing $(B - A) + e$; note that $B - B' = A$. If $B' - B$ has an element b other than e, then (B) yields an element $a \in B - B'$ such that $B' - b + a \in \mathbf{B}$, but this contradicts the dependence of $(B - A) + e + a$. Hence $B' - B = \{e\}$, and therefore $|A| = 1$. Since every minimal transversal of the circuits in $B + e$ has one element, there is only one such circuit.

d) The uniqueness of induced circuits (J) implies the weak elimination property (C). Suppose that $C_1, C_2 \in \mathbf{C}$ and $e \in (C_1 capC_2)$. If $(C_1 \cup C_2) - e$ is independent, then adding e creates a unique circuit, which contradicts the distinctness of C_1 and C_2.

e) In a hereditary system, uniqueness of induced circuits (J) implies the augmentation property (I). Choose $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$. We obtain the augmentation by induction on $|I_1 - I_2| = k$. If $I_1 \subseteq I_2$, any element of $I_2 - I_1$ works; this is the basis step $k = 0$.

For $k > 0$, select $e \in I_1 - I_2$. If $I_2 + e \in \mathbf{I}$, then the induction hypothesis allows us to augment I_1 from $I_2 + e$. Hence we may assume that $I_2 + e$ contains a unique circuit C. Choose $f \in C \cap I_2$, and let $I' = I_2 + e - f$; we have $I' \in \mathbf{I}$. Now $\left|I'\right| = |I_2|$ and $\left|I_1 - I'\right| = k - 1$, so the induction hypothesis guarantees an augmentation of I_1 from I'. Any such element is also in I_2.

8.2.18. *A hereditary system is a matroid if and only if it satisfies the following: If $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$ and $|I_1 - I_2| = 1$, then $I_1 + e \in \mathbf{I}$ for some $e \in I_2 - I_1$.* This is a weaker form of the augmentation property, so it suffices to show that this implies the augmentation property. The stated property provides the basis for induction on $k = |I_1 - I_2|$. If $k > 1$, select $x \in I_1 - I_2$, and let $I = I_1 - x$. The induction hypothesis yields $e_1 \in I_2 - I$ such that $I + e_1 \in \mathbf{I}$. Also $|I + e_1| = |I_1| < |I_2|$ and $|I + e_1 - I_2| = k - 1$, so again the induction hypothesis yields $e_2 \in I_2$ such that $I' = I \cup \{e_1, e_2\} \in \mathbf{I}$. Since $\left|I'\right| = |I_1| + 1$ and $I_1 - I' = \{x\}$, the original hypothesis ($k = 1$) yields $e \in \{e_1, e_2\} \subseteq I_2$ such that $I_1 + e \in \mathbf{I}$.

8.2.19. *If $\mathbf{I}$ is the family of independent sets of a matroid on E, and $\mathbf{I}'$ is obtained from $\mathbf{I}$ by deleting the sets that intersect a fixed subset A of E, then $\mathbf{I}'$ is also the family of independent sets of a matroid on E.* If $I \in \mathbf{I}'$, then $I \cap A = \varnothing$, and also $J \cap A = \varnothing$ for $J \subseteq I$. Also $J \in \mathbf{I}$, since $I \in \mathbf{I}$, so J remains in $\mathbf{I}'$. Hence $\mathbf{I}'$ is an ideal. Also $\varnothing \in \mathbf{I}'$, since $\varnothing \in \mathbf{I}$ and $\varnothing \cap A = \varnothing$.

Consider $I_1, I_2 \in \mathbf{I}'$ with $|I_2| > |I_1|$. In fact also $I_1, I_2 \in \mathbf{I}$, since $\mathbf{I}'$ is a subset of $\mathbf{I}$. The augmentation property in $\mathbf{I}$ yields $e \in I_2$ such that $I_1 + e \in \mathbf{I}$. In fact, also $I_1 + e \in \mathbf{I}'$, since I_1 and I_2 are both disjoint from A, so $I_1 + e \cap A = \varnothing$.

8.2.20. Given a matroid on E and $e \notin B \in \mathbf{B}$, let $C(e, B)$ denote the unique circuit in $B \cup e$.

a) For $e \notin B$, the set $B - f + e$ is a base if and only if f belongs to $C(e, B)$. If $f \in C(e, B)$, then $B - f + e$ contains no circuit, because $C(e, B)$ is the only circuit in $B + e$. Hence $B - f + e$ is an independent set of size $r(E)$. By the uniformity property, $B - f + e$ is a base.

If $f \notin C(e, B)$, then $B - f + e$ contains the circuit $C(e, B)$ and hence is not a base.

b) If $e \in C \in \mathbf{C}$, then $C = C(e, B)$ for some base B. The set $C - e$ is independent and hence can be augmented to a base B. This base cannot contain e. Adding e must complete a unique circuit. It completes C, so it completes no other, and hence $C = C(e, B)$.

8.2.21. *If B_1 and B_2 are bases of a matroid such that $|B_1 \triangle B_2| = 2$, then there is a unique circuit C such that $B_1 \triangle B_2 \subseteq C \subseteq B_1 \cup B_2$.* Let e_1 be the element of $B_1 - B_2$, and let e_2 be the element of $B_2 - B_1$. Since $B_1 \cup B_2 = B_2 + e_1$, the union contains a unique circuit, C. Since B_2 is independent, $e_1 \in C$. Furthermore, $e_2 \in C$, since $B_1 \cup B_2 - \{e_2\} = B_1$, which is independent.

8.2.22. *If B_1 and B_2 are bases of a matroid and $X_1 \subseteq B_1$, then there exists $X_2 \subseteq B_2$ such that $(B_1 - X_1) \cup X_2$ and $(B_2 - X_2) \cup X_1$ are both bases of M.* This is easy using Exercise 8.2.24; otherwise that argument must be generalized. We use induction on $|X_1|$; when X_1 is empty the claim is trivial. Otherwise, choose $e \in X_1$ and let $X_1' = X_1 - \{e\}$. By Exercise 8.2.24, there exists $f \in B_2$ such that $B_1 - e + f$ and $B_2 + e - f$ are both bases. Let $B_1' = B_1 - e + f$ and $B_2' = B_2 + e - f$. By the induction hypothesis, there exists X_2' such that $(B_1' - X_1') \cup X_2'$ and $(B_2' - X_2') \cup X_1'$ are both bases. Now let $X_2 = X_2' \cup f$. This set has the desired property, since $(B_1 - X_1) \cup X_2 = (B_1' - X_1') \cup X_2'$ and $(B_2 - X_2) \cup X_1 = (B_2' - X_2') \cup X_1'$.

8.2.23. Consider distinct bases B_1 and B_2 of a matroid M.

a) The B_1, B_2-bigraph G with $e \in B_1$ adjacent to $f \in B_2$ when $B_2 + e - f \in \mathbf{B}$ has a perfect matching. It suffices to verify Hall's condition. Since $|B_1| = |B_2|$, we may verify Hall's Condition for either partite set.

Proof 1: For $S \subseteq B_2$, suppose that $|N(S)| < |S|$. This yields $|B_1 - N(S)| > |B_2 - S|$. Both $B_1 - N(S)$ and $B_2 - S$ are independent, so the augmentation property yields $e \in B_1 - N(S)$ such that $B_2 - S + e \in \mathbf{I}$. Hence S must contain a member of the circuit formed by added e to B_2. This contradicts $e \notin N(S)$, and hence $|N(S)| \geq |S|$.

Proof 2: We seek a transversal of $\{I(e): e \in B_1\}$, where $I(e) \cup e$ is the unique circuit in $B_2 + e$ if $e \in B_1 - B_2$, and $I(e) = \{e\}$ if $e \in B_1 \cap B_2$. For $X \subseteq B_1$, let $Y = \bigcup_{e \in X} I(e)$. Since $e \in \sigma(I(e))$, we have $X \subseteq \sigma(Y)$. Since $X, Y \in \mathbf{I}$, the incorporation property yields $|Y| = r(Y) = r(\sigma(Y)) \geq r(X) = |X|$. Hence Hall's Condition holds.

(Comment: we can similarly establish a bijection $\pi: B_1 \to B_2$ such that $B_1 - e + \pi(e) \in \mathbf{B}$ for all $e \in B_1$.)

b) There is a bijection $\pi: B_1 \to B_2$ such that for each $e \in B_1$, the set $B_2 - \pi(e) + e$ is a base of M. Such a bijection is given by the perfect matching obtained in part (a). Elements of $B_1 \cap B_2$ yield isolated edges in G.

8.2.24. *For any $e \in B_1$, there exists $f \in B_2$ such that $B_1 - e + f \in \mathbf{B}$ and $B_2 - f + e \in \mathbf{B}$.* If $e \in B_1 \cap B_2$, then let $f = e$. Hence we may assume $e \in B_1 - B_2$.

Proof 1 (transitivity of dependence). Let $I(e) + e$ be the unique circuit in $B_2 + e$, so $I(e) = \{f \in B_2: B_2 - f + e \in \mathbf{B}\}$. If $B_1 - e + f \notin \mathbf{B}$ for all $f \in I(e)$, then $I(e) \subseteq \sigma(B_1 - e)$. Since $I(e) + e$ is a circuit, this implies $e \in \sigma(I(e)) \subseteq \sigma(B_1 - e)$, which is impossible since B_1 is independent.

Proof 2 (cocircuits). In $B_2 + e$ there is a unique circuit C containing e. Since $\overline{B}_1$ is a cobase, $\overline{B}_1 + e$ contains a unique cocircuit C^* containing e. Since $|C \cap C^*| = 1$ is forbidden, there exists another element $f \in C \cap C^*$. Hence $B_2 - f + e$ is independent, has size $|B_2|$, and therefore is a base. Similarly $\overline{B}_1 - f + e$ is independent in the dual, has size $|\overline{B}_1|$, and is a cobase. Therefore $B_1 - e + f$ is a base and f is the desired element.

b) There may be no bijection $\pi: B_1 \to B_2$ such that e and $f = \pi(e)$ satisfy part (a) for all $e \in B_1$. Consider the cycle matroid $M(K_4)$. Let B_1 and B_2 be the edge sets of two complementary 4-vertex paths. If e is a pendant edge of B_1, then e can only be matched with the central edge of B_2, since one pendant edge of B_2 completes a triangle with $B_1 - e$, and the other is not in the triangle of $B_2 + e$. This argument applies for both pendant edges, but they cannot both be paired with the one central edge of B_2.

8.2.25. *Every matroid has a fundamental set of circuits* (a collection of $|E| - r(E)$ circuits such that C_i contains $e_{r(E)+i}$ but no higher-indexed element). If the elements $e_1, \ldots, e_r$ form a base B, then addition of any other $e \in E - B$ creates a unique circuit in $B + e$. The set of these generated by the elements of $E - B$ form a fundamental set of circuits.

8.2.26. *If $C_1, \ldots, C_k$ are distinct circuits in a matroid, with none contained in the union of the others, and X is a set with $|X| < k$, then $\bigcup_{i=1}^k C_i - X$ contains a circuit.* We use induction on k. For $k = 1$ the statement is trivial, and for $k = 2$ it is the statement of the weak elimination property (if $x \notin C_1 \cap C_2$, then C_1 or C_2 itself is the desired circuit). For $k > 2$, choose $x \in X$, and let $X' = X - \{x\}$. By the induction hypothesis, $\bigcup_{i=1}^{k-1} C_i - X'$ contains a circuit C'. The case $k = 2$ yields a circuit in $(C' \cup C_k) - x$; this circuit has the desired properties.

8.2.27. $(+)$ For a hereditary system, prove that the weak elimination property implies the strong elimination property, by induction on $|C_1 \cup C_2|$.

8.2.28. *Min-max formula for maximum weighted independent set.* Given weight $w(e) \in \mathbb{N} \cup \{0\}$ for each element e, we prove $\max_{I \in \mathbf{I}} \sum_{e \in I} w(e) = \min \sum_i r(X_i)$, where the minimum is taken over all chains $X_1 \subseteq X_2 \subseteq \cdots$ of sets in E such that each element $e \in E$ appears in at least $w(e)$ sets in the chain (sets may repeat).

Max $\leq$ min. This inequality holds for every I and every acceptable chain $\{X_i\}$. Independence of I implies $r(X_i) \geq |I \cap X_i|$. Now the appearance of each $e \in I$ in at least $w(e)$ sets of $\{X_i\}$ yields $\sum_i |I \cap X_i| \geq \sum_{e \in I} w(e)$.

To establish equality, let I be a maximum weighted independent set, and define a chain by $X_i = \{e \in E: w(e) \geq W + 1 - i\}$, where W is the maximum weight and $1 \leq i \leq W$ (repetition occurs if the weights are not consecutive integers). For this chain, each element e appears in the $w(e)$ largest sets; hence $\sum |I \cap X_i| = \sum_{e \in I} w(e)$.

Thus it suffices to prove that $|I \cap X_i| = r(X_i)$. This holds by induction on i, with trivial basis for $X_0 = \varnothing$. Having selected $I \cap X_i$ of maximum weight and maximum size from the set X_i of elements with weight at least $W + 1 - i$, the greedy algorithm next considers elements of weight $W - i$, adding as many to I as fail to produce a circuit. Hence $|I \cap X_{i+1}| - |I \cap X_i| = r(X_{i+1}) - |I \cap X_i|$, as desired.

8.2.29. *If r and σ are the rank function and span function of a matroid, then $r(X) = \min\{|Y|: Y \subseteq X$ and $\sigma(Y) = \sigma(X)\}$.* Let M be a matroid. Given $Y \subseteq X$ with $\sigma(Y) = \sigma(X)$, two applications of the incorporation property yield $r(X) = r(\sigma(X)) = r(\sigma(Y)) = r(Y) \leq |Y|$. On the other hand, if Y is a maximum independent subset of X, then $\sigma(Y) \subseteq \sigma(X)$, since σ is order-preserving. Now the choice of Y implies $X \subseteq \sigma(Y)$, and transitivity of dependence implies $\sigma(X) \subseteq \sigma(Y)$.

8.2.30. *A matroid of rank r has at least 2^r closed sets.* A base B in such a matroid has size r. For each $X \subseteq B$, the span $\sigma(X)$ is closed. These closed sets are all distinct, because their intersections with B are distinct, since gaining an element of B in $\sigma(X)$ would increase the rank.

8.2.31. *A matroid is simple if and only if no element appears in every hyperplane and every set of two elements intersects some hyperplane exactly once.* If e is a loop, then e is spanned every set. Hence e belongs to every closed set, including all hyperplanes. If e is not a loop, then augment $\{e\}$ to a base B, and let $H = \sigma(B - e)$; now H is a hyperplane avoiding e.

If e and f are parallel, then $\{e, f\}$ is a circuit, so a closed set contains e if and only if it contains f. If $\{e, f\}$ is independent, augment is to a base B, and let $H = \sigma(B - e)$; now H is a hyperplane containing f but not e.

8.2.32. (•) Prove that in a matroid, a set is a hypobase if and only if it is a hyperplane.

8.2.33. *A family of sets is the family of hyperplanes of some matroid if it is an antichain and, for distinct members H_1 and H_2 both avoiding an element e, there is another member H containing $(H_1 \cap H_2) + e$.* A hyperplane is a maximal set containing no base, and hence its complement is a minimal set contained in no cobase, which by definition is a cocircuit. Hence the hyperplanes are the complements of the cocircuits. A family of sets is the set of cocircuits of a matroid M if and only if it is the set of circuits of a matroid, namely M^*. Hence the characterization of families of cocircuits is the same as the characterization of families of circuits. In particular, a family is the set of cocircuits of some matroid if and only if it is an antichain and, for distinct members C_1 and C_2 with a common element e, there is another member contained in $(C_1 \cup C_2) - e$.

Translating this by complementation, let $H_1 = \overline{C_1}$ and $H_2 = \overline{C_2}$, the condition on members H_1 and H_2 of the family, if they both omit e (so that $e \in C_1 \cap C_2$, is the existence of another member H such that $\overline{H} \subseteq (C_1 \cap C_2) - e$. Thus

$$H \supseteq \overline{(\overline{H}_1 \cup \overline{H}_2) - e} = (H_1 \cap H_2) + e.$$

8.2.34. *The closed sets of a matroid are the complements of the unions of its cocircuits.* The closed sets of a matroid are the intersections of its hyperplanes. The hyperplanes are the complements of the cocircuits. Since $\overline{A \cup B} = \overline{A} \cap \overline{B}$, the desired statement holds.

8.2.35. *Closed sets and hyperplanes.*

a) If X and Y are closed sets in a matroid M, with $Y \subseteq X$ and $r(Y) = r(X) - 1$, then there exists a hyperplane H in M such that $Y = X \cap H$. Let Z be a maximal independent subset of Y, and let $e \in X$ be an element that augments Z to a maximal independent subset of X. Augment $Z + e$ to a base B. Let $H = \sigma(B - e)$; we claim H is the desired hyperplane. Since $Z \subseteq B - e$ and $Y = \sigma(Z)$, we have $Y \subseteq H$. Because $B \subseteq H \cup X$, we have $r(H \cup X) = r(M)$. Applying submodularity to H and X yields $r(H \cap X) \le r(X) - 1$. Since $Y \subseteq H \cap X$, we have $r(H \cap X) = r(X) - 1$. Since Y is closed and has rank $r(X) - 1$, we have $H \cap X = Y$.

b) If X is a closed set in a matroid M, then there exist $r(M) - r(X)$ distinct hyperplanes in M whose intersection is X. By induction on $r(M) - k$. If $r(M) - k = 1$, then X is a hyperplane. If $r(M) - k > 1$, than take $e \notin X$ and $Z = \sigma(X + e)$. Now $r(Z) = k + 1$, and by the induction hypothesis there are $r(M) - k - 1$ distinct hyperplanes whose intersection is Z. Since Z is closed, part (a) guarantees an additional hyperplane H such that $X = Z \cap H$.

8.2.36. *Properties of closed sets in a matroid.*

a) The intersection of two closed sets is closed. Let X and Y be closed sets, so $\sigma(X) = X$ and $\sigma(Y) = Y$. Since σ is order-preserving, $\sigma(X \cap Y) \subseteq \sigma(X) = X$ and $\sigma(X \cap Y) \subseteq \sigma(Y) = Y$. Hence $\sigma(X \cap Y) \subseteq X \cap Y$. Equality holds, because σ is expansive. Hence $\sigma(X \cap Y) = X \cap Y$, and $X \cap Y$ is closed.

b) The span of a set is the intersection of all closed sets containing it. Consider $\sigma(X)$, and let X be the intersection of all the closed sets containing X. A closed set has the form $\sigma(Y)$. If $X \subseteq \sigma(Y)$, then $\sigma(X) \subseteq \sigma^2(Y) = \sigma(Y)$, by the order-preserving and idempotence properties of σ. Hence $\sigma(X)$ is contained in all the closed sets containing X, so $\sigma(X) \subseteq Z$. On the other hand, since $\sigma(X)$ itself is a closed set containing X, also $Z \subseteq \sigma(X)$.

c) The union of two closed sets need not be a closed set. Let M be the cycle matroid of a 4-cycle. Any two consecutive edges on the 4-cycle form a closed set in the matroid. The union of two consecutive such sets is not closed, because it spans the remaining edge of the cycle.

8.2.37. *For a matroid M, $M.X$ has no loops if and only if $\overline{X}$ is closed.* An element is a loop in $M.X$ if and only if it completes a circuit with a subset of $\overline{X}$. There is no such element if and only if $\sigma(\overline{X}) = \overline{X}$.

8.2.38. *Bases and cocircuits in matroids.*

a) When e belongs to a base B in a matroid M, there is exactly one cocircuit of M disjoint from $B - e$, and it contains e. The complement of B is a base in M^*. Adding the element e to it creates a unique circuit in M^*. This is the unique cocircuit of M disjoint from $B - e$, and it contains e.

b) If C is a circuit of a matroid M and x, y are distinct elements of C, then there is a cocircuit $C^* \in \mathbf{C}^*$ with $C^* \cap C = \{x, y\}$. Augment the independent set $C - x$ to a base B; this base B contains y but not x. By the first statement, M has a unique cocircuit C^* disjoint from $B - y$, and it contains y. Since a cocircuit cannot intersect a circuit in exactly one element, and x is the only element of C not contained in B, $C^* \cap C = \{x, y, \}$.

c) Part (b) is trivial for cycle matroids. If e and f are edges in a cycle C, then $V(C)$ splits into sets A and B that are the vertex sets of the paths on C connecting e and f. For every minimal edge cut B that separates A and

B, the intersection of the cocircuit B with the circuit C in the cycle matroid is $\{e, f\}$.

8.2.39. *The dual of a simple matroid need not be simple.* The independent sets of a matroid M^* are the complements of the spanning sets of M. Hence an element e is a loop in M^* if and only if its complement is not spanning in M, which means that e belongs to every base in M. Let M be the hereditary system on E in which the only circuit is $E-e$. Since M has only one circuit, M vacuously satisfies weak elimination and is a matroid. If $|E| \geq 4$, then M is simple. The bases of M are the sets of size $|E|-1$ containing e. Since $E-e$ does not span, e is a loop in M^*.

A set of elements in a matroid can be both a circuit and a cocircuit. Suppose C is a both circuit and a cocircuit in M. Hence C is a minimal dependent set in M^*, which means $\overline{C}$ is a hyperplane (maximal nonspanning set) in M. Given $e \in C$, we thus have $C-e$ independent and $\overline{C}+e$ spanning. Since bases of M have the same size, we have $\left|\overline{C}+e\right| \geq |C-e|$, which implies $|C| \leq |E|/2+1$. This suggests considering $U_k(2k)$. Indeed, the dual of $U_k(2k)$ is $U_k(2k)$ itself, in which the circuits and cocircuits are all sets of size $k+1$.

8.2.40. *Proof of Euler's Formula by matroids.* The cycle matroid of a connected n-vertex graph G has rank $n-1$. The dual matroid, also defined on the edge set, has rank $e-n+1$. When the graph is planar, the dual matroid is the cycle matroid of the dual graph G^*. Since G^* has f vertices (one for each face of G), the rank of that matroid is $f-1$. Hence $f-1=e-n+1$, or $n-e+f=2$, as desired.

8.2.41. *Restrictions and contractions of matroids commute.* If e is to be deleted and f is to be contracted away, then we can do these in either order without affecting the resulting matroid. More precisely, for a matroid M on E and $Y \subseteq X \subseteq E$, we use rank functions to prove that $(M|X).Y = (M.\overline{X-Y})|Y$ and $(M.X)|Y = (M|\overline{X-Y}).Y$.

For the first equation, we have two matroids defined on Y. For $Z \subseteq Y$, we have $r_{(M|X).Y}(Z) = r_{M|X}(Z \cup (X-Y)) - r_{M|X}(X-Y)$. Also $r_{M.\overline{X-Y}|Y}(Z) = r_{M.\overline{X-Y}}(Z) = r_M(Z \cup (X-Y)) - r_M(X-Y)$. Since $r_{M|X} = r_M$ on subsets of X, the rank function is the same for the two matroids on Y.

For the second equation, we use the first and duality: For the matroid on the left, $[(M.X)|Y]^* = (M.X)^*.Y = (M^*|X).Y$. For the matroid on the right, similarly $[(M|\overline{X-Y}).Y]^* = (M^*.\overline{X-Y})|Y$. By applying the first equation to M^*, the duals of the two matroids in the second equation are the same, and hence they are the same matroid.

8.2.42. *Rank function for matroid contraction.* We apply the formula for the rank function of the dual and the equality $(M.F)^* = M^*|F$. We compute

$$\begin{aligned} r_{M.F}(X) &= r_{(M^*|F)^*}(X) = |X| - r_{M^*|F}(F) + r_{M^*|F}(F-X) \\ &= |X| - r_{M^*}(F) + r_{M^*}(F-X) \\ &= |X| - [|F| - r_M(E) + r_M(\overline{F})] + |F-X| - r_M(E) + r_M(\overline{F-X}) \\ &= r_M(X \cup \overline{F}) - r_M(\overline{F}) \end{aligned}$$

($\bullet$) Also derive the formula directly by proving that X is independent in $M.F$ if and only if adding X to $\overline{F}$ increases the rank by $|X|$.

8.2.43. *The cycle matroid of a graph G is the column matroid over $\mathbb{Z}_2$ of the vertex-edge incidence matrix of G.* A set of edges is dependent in the cycle matroid if and only if it contains a cycle. A set of columns of the incidence matrix (which correspond to edges) is dependent in the column matroid if and only it contains a subset of columns summing to an even number in each row. If a set of edges contains a cycle C, then the corresponding columns have two 1's in the rows for the vertices of C and no 1's in other rows; hence this set of columns is dependent. If a set of columns is dependent, then the subset with even sum correspondence to a nonempty even subgraph of the graph. Every nonempty even subgraph is a union of cycles; hence the corresponding edges of the graph form a dependent set.

8.2.44. *a) The matrix $\binom{1\,0\,1\,1}{0\,1\,1\,2}$ represents $\mathbf{U}_{2,4}$ over $\mathbb{Z}_3$.* Since the matrix has two rows, there are no three independent columns. Since no column is a multiple of another, all pairs of columns are independent.

b) $\mathbf{U}_{2,4}$ has no representation over $\mathbb{Z}_2$. Suppose that $\mathbf{U}_{2,4} = M(A)$ for some binary matrix A, and let x_1, x_2, x_3, x_4 denote the four column vectors. Since the columns corresponding to circuits must sum to 0, we have $x_1 + x_2 + x_3 = \overline{0}$ and $x_1 + x_2 + x_4 = \overline{0}$, modulo 2. This yields $x_3 + x_4 = \overline{0}$, which contradicts the independence of $\{3, 4\}$.

8.2.45. *The three operations below preserve the cycle matroid of G.*

a) Decompose G into its blocks $B_1, \ldots, B_k$, and reassemble them to form another graph G' with blocks $B_1, \ldots, B_k$.

b) In a block B of G that has a two-vertex cut $\{x, y\}$, interchange the neighbors of x and y in one of the components of $B - \{x, y\}$.

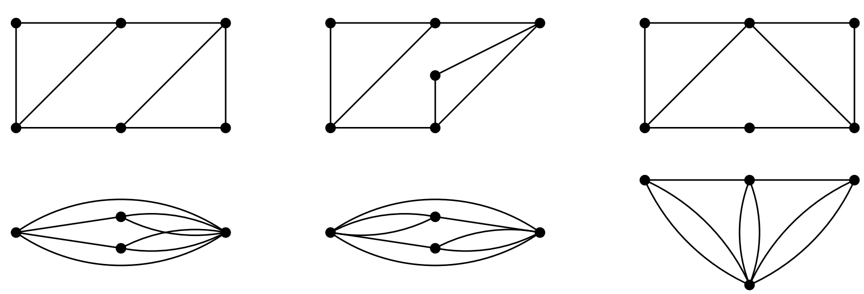
c) Add or delete isolated vertices.

Operation (a) can be described as a succession of "splitting" and "splicing" operations, where a cut-vertex is split into two vertices belonging to separate components or vertices from distinct components are merged. The blocks remain the same. This does not change the cycle matroid, because edge sets of cycles like in a single block, and a matroid is determined by its elements and its circuits. Similarly, adding or deleting isolated vertices does not affect the circuits or the set of elements.

The operation in (b) is a "vertex twist". A vertex twist at $\{x, y\}$ switches the identity of x and y in a component of $G - \{x, y\}$. The sets of edges forming x, y-paths do not change, and every cycle through x and y consists of two internally disjoint x, y-paths. Other cycles do not change. Hence the edge sets forming cycles do not change.

(Comment: Whitney's 2-Isomorphism Theorem [1933b] states that G and H have the same cycle matroid if and only if some sequence of these operations turns G into H. Thus every 3-connected planar graph has only one dual graph, meaning essentially only one planar embedding.)

8.2.46. *An abstract dual that is not a geometric dual.* Among the plane graphs below, H_i is the geometric dual of G_i. Graphs G_1 and G_2 are isomorphic. They are the only distinguishable ways to embed G_1 and G_2 in the plane, so H_1 and H_2 are the only geometric duals of G_1. However, G_3 is obtained from G_1 by an instance of operation (b) of Exercise 8.2.45, so G_1 and G_3 have the same cycle matroid; call it M. Any graph whose cycle matroid is dual to M is an abstract dual of G_1, by Corollary 8.2.37. This includes every geometric dual of G_3. Hence H_3 is an abstract dual of G_1 that is not a geometric dual of G_1.



8.2.47. *For every matroid M, the base exchange graph $\beta(M)$ is Hamiltonian ($\beta(M)$ has a vertex for each base, adjacent when their symmetric difference has size 2).* The proof is by induction on $|E|$, proving the stronger statement that there is a Hamiltonian path connecting the endpoints of any edge. If there is only one base, then the graph is trivial. If there are two bases, then they induce an edge, by the base exchange property.

Suppose that there are more than two bases, which requires $|E| \geq 3$. If M has an element e in no circuit, then every base of M contains e, and $\beta(M)$ is isomorphic to $\beta(M \cdot e)$. Similarly, if e is a loop (itself a circuit), then no base of M contains e, and $\beta(M)$ is isomorphic to $\beta(M - e)$. Hence we may assume that M has no loops and that every element of E belongs to a circuit (no co-loops).

If M has exactly one circuit C, then addition of any element to a base generates C, so every base lacks exactly one element of E, and that element always belongs to C. Thus $\beta(M)$ in this case is a complete graph and has the desired cycle. Hence we may assume that M has more than one circuit.

Let (B_1, B_2) be an arbitrary edge of $\beta(M)$, with $B_1 - B_2 = e$ and $B_2 - B_1 = f$. The subgraph of $\beta(M)$ induced by the bases containing e is isomorphic to $\beta(M \cdot e)$, and the subgraph induced by the bases not containing e is isomorphic to $\beta(M - e)$. The induction hypothesis will yield Hamiltonian paths in these subgraphs starting at B_1 and B_2. We will connect the opposite ends of these paths to obtain a Hamiltonian path from B_1 to B_2 in $\beta(M)$.

Because $B_1 + f$ contains a circuit C but B_1 and $B_1 + f - e$ do not, we conclude that e and f both belong to C and that $C - \{e, f\}$ belongs to both B_1 and B_2. If every element of M that is not in B_1 or B_2 is parallel to e or f, then again $\beta(M)$ is a clique and has the desired cycle. Otherwise, we may select an element h that is not in B_1 or B_2 and is not parallel to e or f.

The set $B_1 + h$ contains a unique cycle C'; because h is not parallel to e we can select an edge $g \neq e, h$ from C'. Note that g cannot be parallel to e or to f, the former because $g, e \in B_1$ and latter because $C - f + g \subseteq B_1$. Let $B_3 = B_1 + h - g$, so (B_1, B_3) is an edge of $\beta(M)$. Now B_1, B_2, B_3 agree outside $\{e, f, g, h\}$, and they intersect $\{e, f, g, h\}$ in $\{e, g\}$, $\{f, g\}$, and $\{e, h\}$, respectively. We claim that the set $B_4 = B_2 + h - g = B_3 + f - e$ that intersects $\{e, f, g, h\}$ in $\{f, h\}$ is also a base of M, so that (B_2, B_4) and (B_3, B_4) are edges of $\beta(M)$.

To show this, keep in mind that $C - f \subseteq B_1$ and $C' - h \subseteq B_1$. Suppose first that $f \notin C'$. In this case $C' - h \subseteq B_2$, so $C' \subseteq B_2 + h$. Since adding h to B_2 introduces a unique circuit, this circuit is C', which contains g, and B_4 is independent.

To eliminate the possibility that $f \in C'$, we use strong elimination. In the situation at hand, we have $f \in C \cap C'$ and $h \in C' - C$, and the strong elimination property guarantees a circuit C'' in $C \cup C' - f$ that contains h. However, $C \cup C' - f \subseteq B_1 + h$. Thus C'' must be the unique circuit C' obtained by adding h to B_1, contradicting the assumption that $f \in C'$.

Hence we can apply the induction hypothesis to $\beta(M \cdot e)$ and $\beta(M - e)$ to obtain paths from B_1 to B_3 and from B_2 to B_4 through all the bases containing e and omitting e, respectively. Adding the edge (B_3, B_4) completes a Hamiltonian path in $\beta(M)$ between B_1 and B_2.

For graphic matroids, the Hamiltonian circuit is a cyclic listing of the maximal forests by changing one edge at a time. For uniform matroids, the result is perhaps more interesting; it guarantees a cyclic listing of the k-sets of an n-set by changing one element at each step. (This can also be done by omitting the non-k-sets from the standard "Gray code" listing of all subsets as produced by Exercise 7.2.17.)

8.2.48. (•) Use weak duality of linear programming to prove the weak duality property for matroid intersection: $|I| \le r_1(X) + r_2(\overline{X})$ for any $I \in \mathbf{I}_1 \cap \mathbf{I}_2$ and $X \subseteq E$. (Hint: Consider the discussion of dual pairs of linear programs in Remark 8.1.7.)

8.2.49. *Common independent and spanning sets in two matroids M_1 and M_2 on E.*

a) The minimum size of a set in E that is spanning in both M_1 and M_2 is $\max_{X\subseteq E}(r_1(E) - r_1(X) + r_2(E) - r_2(\overline{X}))$. A common spanning set contains a base of each matroid. Thus a smallest common spanning set is a smallest union of bases of the two matroids. We minimize $|B_1 \cup B_2|$ by maximizing $|B_1 \cap B_2|$, which is the size of I, a largest common independent set. The minimum size of a common spanning set is thus $|B_1| + |B_2| - |I|$. By the Matroid Intersection Theorem, the size is as claimed.

b) For a U,V-bigraph G without isolated vertices, $\alpha(G) = \beta'(G)$ (König's Other Theorem).

Since G has no isolated vertices, $r_1(E) = |U|$ and $r_2(E) = |V|$. For $X \subseteq E$, let A_1 be the subset of U not touched by X; we have $|A_1| = r_1(E) - r_1(X)$. Similarly, $|A_2| = r_2(E) - r_2(\overline{X})$, where A_2 is the subset of V not touched by $\overline{X}$. If $a_1 \in A_1$ and $a_2 \in A_2$, then every edge of $X \cup \overline{X}$ misses one or the other, so $A_1 \cup A_2$ is a stable set. This holds for all X, so

$$\alpha(G) \ge \max_{X\subseteq E}\{r_1(E) - r_1(X) + r_2(E) - r_2(X)\} = \beta'(G).$$

c) The maximum size of a common independent set plus the minimum size of a common spanning set equals $r_1(E)+r_2(E)$, and thus $\alpha'(G)+\beta'(G) = n(G)$ for a U,V-bigraph G without isolated vertices (Gallai's Theorem). Continuing the argument in part (a), adding $|I|$ to the formula yields $|B_1|+|B_2|$, which equals $r_1(E) + r_2(E)$.

Now let M_1 and M_2 be the partition matroids on $E(G)$ induced by U and V. Since G has no isolated vertices, a set $S \subseteq E$ is spanning in both matroids if and only if it covers all the vertices; hence $\beta'(G) = \min|S|$. As has been remarked repeatedly, a set I is a common independent set if and only if it is a matching. Thus $\alpha'(G) = \max|I|$. We obtain $\alpha'(G) + \beta'(G) = \max|I| + \min|S| = r_1(E) + r_2(E) = |U| + |V| = n(G)$.

Equality holds by letting X be the set of edges with endpoints in $A \cap T$, where A is a maximum stable set. (Alternatively, every edge cover requires at least $\alpha(G)$ edges to cover the vertices in a maximum stable set.)

8.2.50. *In every acyclic orientation of G, the vertices can be covered with at most $\alpha(G)$ pairwise-disjoint paths.* Given an acyclic digraph D, let M_1 and M_2 be the head partition matroid and the tail partition matroid of D. Every common independent set I is the edge set of a family of disjoint paths in D, since D is acyclic. The number of paths is $n(D) - |I|$. By the Matroid Intersection Theorem, the maximum such $|I|$ equals $\min_{X\subseteq E} r_1(X) + r_2(\overline{X})$. For each $X \subseteq E$, the vertices not covered by the head of an edge in X or the tail of an edge in $\overline{X}$ form a stable set of vertices in D. Hence for each $X \subseteq E$ there is a stable set in D of size at least $n - r_1(X) - r_2(\overline{X})$. Hence the maximum size of a stable set in D is at least the minimum number of paths needed to partition the vertices of D. Thus $V(D)$ can be covered using at most $\alpha(D)$ disjoint paths. (Comment: This is the special case of the Gallai–Milgram Theorem for acyclic digraphs.)

8.2.51. *For the transversal matroid M on E induced by subsets $A_1,\dots,A_m$ of E, the rank function r on subsets of E is defined by $r(X) = \min_{Y\subseteq X}\{|X| - (|Y| - |N(Y)|)\}$, where $N(Y)$ indicates the neighborhood in the incidence bigraph with partite sets E and $\{A_1,\dots,A_m\}$.* Let $T = \{A_1,\dots,A_m\}$. Ore's Theorem (Exercise 3.1.32) for an X,T-bigraph H states that $\alpha'(H) = |X| - \max_{Y\subseteq X}(|Y| - |N(Y)|)\}$. In this setting, $r(X) = \alpha'(G[X \cup T])$, so we merely set $H = G[X \cup T]$.

8.2.52. *Hall's Theorem from the rank function.* Let G be a bipartite graph with partite sets E, $[m]$ and with no isolated vertices. For $X \subseteq E$, let $r(X) = \min_{J\subseteq[m]}\{|N(J) \cap X| - |J| + m\}$. The following are equivalent for X.

A) Hall's Condition holds for X ($|N(S)| \ge |S|$ for all $S \subseteq X$).

B) $r(X) \ge |X|$.

C) X is saturated by some matching in G.

C ⇒ A. This is the trivial part of Hall's Theorem; the members of each set $S \subseteq X$ have distinct neighbors in the matching.

A ⇒ B. For $J \subseteq [m]$, we show that $|N(J) \cap X| - |J| + m \ge |X|$. Let $S = X - N(J)$, so $N(S) \subseteq [m] - J$. We are given $|N(S)| \ge |S|$, so $m - |J| \ge |S| = |X| - |N(J) \cap X|$. Moving $|N(J) \cap X|$ yields the desired inequality.

B ⇒ C. By restricting our attention to $G[X \cup [m]]$, we may assume that X is the entire partite set E. Hence we are given $|E| - |N(J)| \le m - |J|$ for all $J \subseteq [m]$. (Comment: Showing that this condition is equivalent to Hall's Condition yields this expression for the rank function of a transversal matroid from Hall's Theorem. The proof of Hall's Theorem from this condition is not very different from the usual proofs of Hall's Theorem.)

Let M be a maximum matching in G. If M does not saturate all of E, let R be the set of vertices reachable from unsaturated vertices of E by paths that alternate between edges not in M and edges in M. Let $S = R \cap E$, and let $T = R \cap [m]$. Alternating paths reach T along edges not in M and continue to S under edges of M. If any vertex of T is unsaturated, then we have an augmenting path and M is not a maximum matching. Hence every vertex of T has a mate in S under M. This yields $|S| > |T|$, since S includes at least one unsaturated vertex.

Now let $J = [m] - T$. Since the edges of M incident to S come from T, the definition of T yields $N(S) \subseteq T$. Hence $N(J) \subseteq E - S$. We obtain $|E| - |N(J)| \geq |S| > |T| = m - |J|$, which contradicts the given inequality.

8.2.53. (!) Let G be an $E, [m]$-bigraph without isolated vertices. For $X \subseteq E$ and $J \subseteq [m]$, let $g(X, J) = |N(J) \cap X| - |J|$, and let $r(X) = \min\{g(X, J) + m \colon J \subseteq [m]\}$. Say that J is *X-optimal* if $r(X) = g(X, J) + m$.

a) Prove that $r(\emptyset) = 0$ and that $r(X) \leq r(X + e) \leq r(X) + 1$.

b) Prove that r satisfies the weak absorption property.

8.2.54. *Restrictions and unions of transversal matroids are transversal matroids.* If M is the transversal matroid on E induced by the bipartite graph G with partite sets E, T, then $M|X$ is the transversal matroid induced on X by the induced subgraph $G' = G[X \cup T]$. By definition, $Y \in \mathbf{I}(M|X)$ if $Y \subseteq X$ and $Y \in \mathbf{I}(M)$. No matching of Y in G uses vertices of $E - X$, so Y has a matching in G if and only if it has a matching in $G[X \cup F]$.

The union of matroids M_1, M_2 on E has X independent in $M_1 \cup M_2$ if $X = X_1 \cup X_2$, where $X_i \in \mathbf{I}(M_i)$. If M_1, M_2 are transversal matroids with set systems $\{A_i\}, \{B_j\}$, then the union of partial transversals of $\{A_i\}$ and $\{B_j\}$ is a partial transversal of the set system $\{A_i\} \cup \{B_j\}$. In the graph context, this is equivalent to identifying corresponding vertices of E in two graphs with partite sets E, F_1 and E, F_2.

Contractions and duals of transversal matroids need not be transversal matroids. We first construct a non-transversal matroid. Define a matroid M of rank 2 on six elements $E = \{a, b, c, d, e, f\}$ by letting the bases of M be all 15 pairs except $\{\{a, b\}, \{c, d\}, \{e, f\}\}$. To verify that M is a matroid, it is easy to check that the remaining twelve pairs satisfy the base axioms, or use the fact that the dual discussed below is a matroid.

Suppose that M is a transversal matroid. Singletons are independent, so a set system $\{A_i\}$ realizing M has each element in some set. A dependent pair appears in only one set. Since each element appears in some dependent pair, each element therefore appears in only one A_i. Hence the elements of any 3-element circuit appear in only two sets. Consider the circuit $\{a, c, e\}$. By symmetry, we may assume a, c appear in the same set. However, a, c appearing in only one set contradicts the independence of $\{a, c\}$.

The dual of this matroid is a transversal matroid of rank 4. The set system realizing it is $A_1 = \{a, b, c, d, e, f\}$, $A_2 = \{a, b\}$, $A_3 = \{c, d\}$, $A_4 = \{e, f\}$. Any set of 4 elements containing one element each from A_2, A_3, A_4 is a transverval. In other words, the only 4-sets of E that are not bases are $\{a, b, c, d\}, \{c, d, e, f\}, \{a, b, e, f\}$. Hence this transversal matroid is M^*.

To define a transversal matroid N whose contraction is M, add an element g. Let $A_1 = \{g, a, b\}$, $A_2 = \{g, c, d\}$, $A_3 = \{g, e, f\}$. Then the tranversal matroid N induced on $E \cup \{g\}$ by A_1, A_2, A_3 has rank 3, and $N \cdot E$

has rank 2. The bases of N are the pairs that with g form a transversal of $\{A_i\}$; these are all pairs except $\{a, b\}, \{c, d\}, \{e, f\}$. Hence $N \cdot E = M$. Note that $N|E$ is a transversal matroid of rank 3.

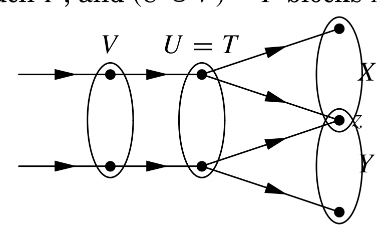
8.2.55. *Gammoids.* Give a digraph D and sets $F, E \subseteq V(D)$, the *gammoid* on E induced by (D, F) is the hereditary system given by $\mathbf{I} = \{X \subseteq E \colon$ there exist $|X|$ pairwise disjoint paths from F to $X\}$; equivalently, $r(X)$ is the maximum number of pairwise disjoint F, X-paths.

a) Every transversal matroid is a gammoid. A transversal matroid M arises from an E, Y-bigraph G by letting the independent sets be the subsets of E that are saturated by matchings. Let D be the orientation of G directing all edges from Y to E, and let $F = Y$. Now M is the gammoid on E induced by (D, F).

b) Every gammoid is a matroid.

Proof 1 (submodularity of the rank function). Say that S *blocks* X if S intersects all F, X-paths. By Menger's Theorem, $r(X)$ is the minimum size of a set blocking X. Let U, V be minimum sets blocking X, Y, respectively. We will obtain a set $T \subseteq U \triangle V$ such that $(U \cap V) \cup T$ blocks $X \cap Y$ and $(U \cup V) - T$ blocks $X \cup Y$. The sizes of such sets sum to $|U| + |V|$, which will yield $r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$.

If $U \cap V$ does not block $X \cap Y$, then for some $z \in X \cap Y$ there is an F, z-path using $U \triangle V$. Let T be the set of vertices that are the last vertex of $U \triangle V$ on some such path. Then $(U \cap V) \cup T$ blocks $X \cap Y$; we need only show that $(U \cup V) - T$ blocks $X \cup Y$. If there is a path P from F to $X \cup Y$ that is not blocked by $(U \cup V) - T$, then all vertices of P in $U \cup V$ belong to T. Let P' be the portion of P up to its first vertex v in T. By the definition of T, for some $z \in X \cap Y$ there is a v, z-path Q that avoids $U \triangle V$ after v. Following P' by Q yields an walk and hence a path from F to $z \in X \cap Y$ that has only one vertex of $U \cup V$, which is a vertex $v \in T$. This path avoids V if $v \in U$ and avoids U if $v \in V$. This is impossible, because U and V each block $X \cap Y$. Hence there is no such P, and $(U \cup V) - T$ blocks $X \cup Y$.

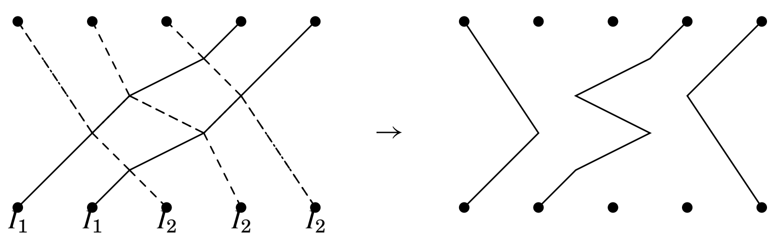


Proof 2 (augmentation property). Let $I_1, I_2 \in \mathbf{I}$ be independent sets with $|I_2| - 1 = |I_1| = k$. We have disjoint F, I_1-paths $P_1, \ldots, P_k$ and disjoint F, I_2-paths $Q_1, \ldots, Q_{k+1}$. Let $U = \{P_i\} \cup \{Q_j\}$. Partition the edges of each path Q_j or P_i into segments that are maximal subpaths for which no internal vertex belongs to another path in U. From these segments we form

$k+1$ disjoint paths with sources in F and sinks I_1+e, for some $e \in I_2$. Since each of $\{P_i\}$ and $\{Q_j\}$ is a set of disjoint paths, each intersection point v is shared by one P_i and one Q_j. Except for sources and sinks, one P-segment and one Q-segment enters v, and one segment of each type leaves v. If v is a source or sink of one path, then the entering or departing segment of that type is missing, and we include v as a trivial segment, declared to meet the other path at its sink or source, respectively. Thus every intersection point has one entering and one departing segment of each type.

Define a bipartite graph H with the segments as vertices. A P-segment and a Q-segment are adjacent in H if they have the same source or have the same sink. A segment from u to v meets at most 1 segment of the other type at each of u, v. Hence every vertex of H has degree at most two, and H consists of alternating paths and alternating (i.e. even) cycles.

Counting the nontrivial segments by endpoints counts each segment twice. Since there is an extra Q-source and Q-sink, the number of Q-segments in H exceeds the number of P-segments by one. Hence there must be some path R that starts and ends at Q-segments. To obtain the augmentation of I_1, we turn each Q-segment along R into a segment of a P-path and delete each P-segment of R, which we view as switching the ownership of the segments along R. This preserves the disjointness of the P-paths and the disjointness of the Q-paths. However, since R starts and ends at Q-segments, we have changed the number of P-paths from k to $k+1$. The sources are still in F, and the sinks are I_1+e for some $e \in I_2$.



8.2.56. *A matroid is a strict gammoid if and only if it is the dual of a transversal matroid,* where a *strict gammoid* is a gammoid on E induced by (D, F) with the additional property that $E = V(D)$.

Given a transversal matroid M of rank n on E, let $G \subseteq K_{E,T}$ be a bipartite graph realizing it. Let $B = \{b_1, \ldots, b_n\} \subseteq E$ be a base of M, and let $L = \{b_1t_1, \ldots, b_nt_n\}$ be a matching of B. Define a directed graph D with vertex set E and edges $e \to b_j$ if and only if $e \leftrightarrow t_j$ in G. Let $F = E - B$. Any path P starting in F starts with a vertex of $E-B$, but thereafter stays in B. Traversal of the edge $e \to b_j$ in D can be interpreted as traversal of the path $e \leftrightarrow t_j \leftrightarrow b_j$ in G. Thus the path P of length k starting in F corresponds to a path Q of length $2k$ that starts outside B and alternates

between edges not in L and edges in L. The symmetric difference $L \Delta P$ is a matching in G in which the source of P is now matched and the sink is not. A set of disjoint paths $\mathbf{P}$ corresponds to paths Q that use disjoint edges in L, so again $L\Delta \cup \mathbf{P}$ is a matching in G in which the sources are matched and the sinks are not.

Let N be the strict gammoid on D with source set F. Since $E = V(D)$, the bases of N are the sink sets of all collections of $|F|$ disjoint paths $\mathbf{P}$ with sources F. As discussed above, for each such collection there is a matching of size n that leaves the sink vertices of $\mathbf{P}$ unmatched. Hence the complement of any base of N is a base of M. Conversely, if L' is a matching in G for a base B' of M, then $L'\Delta L$ is a collection of disjoint alternating paths and cycles, including paths from $B' - B$ to $B - B'$ that alternate between edges of L' and L. These paths collapse to a set of paths in D, half as long, from $F = \overline{B}$ to $\overline{B}'$. Hence also the complement of any base in M is a base in N, and $N = M^*$.

Conversely, given any strict gammoid N on D with vertices E and source set F, we can reverse this construction. First note that $r(N) = |F|$. This means that no edge between vertices of F can be used in paths corresponding to a base, so if we discard or ignore all edges between vertices of F we get the same gammoid (similarly, in the previous construction we ignored edges of G not involving $\{t_1, \ldots, t_n\}$). Letting $B = E - F = \{b_1, \ldots, b_n\}$, we define a bipartite graph $G \subseteq K_{E,T}$, where $T = \{t_1, \ldots, t_n\}$, with edges b_it_i, and also edges et_j if $e \to b_j$ in D. Let M be the transversal matroid induced on E by G. The correspondence between paths in D originating in F and alternating paths is G with respect to the matching $L = \{b_it_i\}$ is the same as above, and once again $N = M^*$.

8.2.57. *If M_1 and M_2 are matroids with spanning sets $\mathbf{S}_1$ and $\mathbf{S}_1$, then the hereditary system $M_1 \wedge M_2$ whose spanning sets are $\{X_1 \cap X_2\colon X_1 \in \mathbf{S}_1, X_2 \in \mathbf{S}_2\}$ is $(M_1^* \cup M_2^*)^*$.* We have X as the intersection of spanning sets X_1, X_2 in M_1 and M_2 if and only if $\overline{X}$ is the union of independent sets $\overline{X}_1, \overline{X}_2$ in M_1^* and M_2^*. Hence $X \in \mathbf{S}_{M_1 \wedge M_2}$ if and only if $\overline{X} \in \mathbf{I}_{M_1^* \cup M_2^*}$, which implies $M_1 \wedge M_2 = (M_1^* \cup M_2^*)^*$.

8.2.58. *Generalized transversal matroids.* Let M be a matroid on E.

a) For $A_1, \ldots, A_m \subseteq E$, the hereditary system M' defined by $\mathbf{I}_{M'} = \{X \subseteq [m]\colon \{A_i\colon i \in X\}$ has a transversal belonging to $\mathbf{I}_M\}$ is a matroid with rank function $r'(X) = \min_{Y \subseteq X}\{|X-Y| + r(A(Y))\}$, where $A(Y) = \bigcup_{i \in Y} A_i$. We show first that r' is the rank function of M'; that is, the family of sets indexed by X has a transversal in $\mathbf{I}_M$ if and only if $|X - Y| + r(A(Y)) \ge |X|$ for all $Y \subseteq X$, or equivalently $r(A(Y)) \ge |Y|$; call this condition $(*)$.

If there is such a transversal T, then its subsets are also independent in M. The subset of T representing the sets indexed by Y has size Y, and hence $r(A(Y)) \geq |Y|$.

For the converse, $(*)$ is given. We use induction on $\sum_{i=1}^{m} |A_i|$ to prove that X has an independent transversal. From $(*)$, $|A_i| \geq 1$ for all i. If equality always holds, then the sets are distinct elements and their union is an independent transversal in M. This completes the basis step.

Hence we may assume that $|A_1| \geq 2$. By the induction hypothesis, it suffices to find $e \in A_1$ such that replacing A_1 with $A_1 - e$ yields a system that also satisfies (*). If not, then for all $x_i \in A_1$ there exists $Y_i \subseteq [m] - \{1\}$ such that $r((A_1 - x_i) \cup A(Y_i)) < |Y_i| + 1$. Taking distinct elements $x_1, x_2 \in A_1$ and applying the submodularity of r and $(*)$ yields

$$\begin{aligned}|Y_1| + |Y_2| &\geq r(A_1 \cup A(Y_1) \cup A(Y_2)) + r((A_1 - \{x_1, x_2\}) \cup A(Y_1 \cap Y_2)) \\ &\geq r(A(\{1\} \cup Y_1 \cup Y_2) + r(A(Y_1 \cap Y_2)) \geq 1 + |Y_1 \cap Y_2| + |Y_1 \cap Y_2|\end{aligned}$$

The contradiction completes the proof that r' is the rank function of the hereditary system M'.

It thus suffices to prove that r' is submodular. For $U \subseteq X$ and $V \subseteq Y$, note (as used in the proof of the Matroid Union Theorem) that $|X - U| + |Y - V| = |(X \cap Y) - (U \cap V)| + |(X \cup Y) - (U \cup V)|$. Also, $A(U) \cup A(V) = A(U \cup V)$, while $A(U) \cap A(V) \supseteq A(U \cap V)$. Using first the submodularity of r, these observations yield

$$\begin{aligned}r'(X) + r'(Y) &= \min_{U \subseteq X}[|X - U| + r(A(U))) + \min_{V \subseteq Y}(|Y - V| + r(A(V))] \\ &\geq \min_{U,V}[|X - U| + |Y - V| + r(A(U) \cup A(V)) + r(A(U) \cap A(V))] \\ &\geq \min_{U,V}[|(X \cup Y) - (U \cup V)| + r(A(U \cup V)) + |(X \cap Y) - (U \cap V)| + r(A(U \cap V)] \\ &\geq \min_{S \subseteq X \cup Y}[|(X \cup Y) - S| + r(A(S))] + \min_{T \subseteq X \cap Y}[|(X \cap Y) - T| + r(A(T)) \geq r'(X \cup Y) + r'(X \cap Y)\end{aligned}$$

b) If f is a function from E to a finite set F, and M' is the hereditary system on F defined by $\mathbf{I}_{M'} = \{f(X) \colon X \in \mathbf{I}_M\}$, then M' is a matroid with rank function $r'(X) = \min_{Y \subseteq X}\{|X - Y| + r(f^{-1}(Y))\}$ when f is surjective. Let $A_i = f^{-1}(i)$ for $i \in F$. Now X is independent in M' if and only if the sets indexed by F have a transversal in $\mathbf{I}_M$, since f is a function. By part (a), M' is a matroid with the specified rank function.

8.2.59. ($\bullet$) Apply matroid sum and Exercise 8.2.58 to prove the Matroid Union Theorem.

8.2.60. *Matroid Intersection from Matroid Union.*

The maximum size of a common independent set in matroids M_1 and M_2 on E is $r_{M_1 \cup M_2^*}(E) - r_{M_2^*}(E)$. Let $a = r_{M_1 \cup M_2^*}(E)$, $b = r_{M_1 \cap M_2}$, and $c = r_{M_2^*}$; we seek $b = a - c$. If $Z \in \mathbf{I}_1 \cap \mathbf{I}_2$ with $|Z| = b$, then $\overline{Z}$ contains a base A of M_2^*, which means that $Z \cup A$ is independent in $M_1 \cup M_2^*$; thus $a \geq b + c$. On the other hand, if $X \in \mathbf{I}_{M_1 \cup M_2^*}$ with $|X| = a$, then we may assume $X = B_1 \cup B_2$, where $B_1 \in \mathbf{B}_1$ and $B_2 \in \mathbf{B}_2^*$. Since $|B_1 \cup B_2| = |B_2| + |B_1 - B_2|$, we have $a \leq c + b$, because $B_1 \in \mathbf{I}_1$ and $B_1 - B_2 \subseteq \overline{B}_2 \in \mathbf{B}_2$.

b) Matroid intersection. Using the rank formula in the Matroid Union Theorem, $a - c = \min_{X \subseteq E}\{|\overline{X}| + r_1(X) + r_2^*(X)\} - r_2^*(E)\}$. Since $r_2^*(X) = |X| - (r_2(E) - r_2(\overline{X}))$ and $r_2^*(E) = |E| - r_2(E)$, the formula for $a - c$ simplifies to $\min_{X \subseteq E}\{r_1(X) + r_2(\overline{X})$, which by part (a) equals $\max\{|I| \colon I \in \mathbf{I}_1 \cap \mathbf{I}_2\}$.

8.2.61. *If G is an n-vertex weighted graph, and $E_1, \ldots, E_{n-1}$ is a partition of $E(G)$ into $n-1$ sets, then there a polynomial-time algorithm to find a spanning tree having exactly one edge in each subset E_i, if one exists.* Such a spanning tree is a set of edges that is independent in both the cycle matroid of G and the partition matroid on $E(G)$ induced by the specified partition. The Matroid Intersection Algorithm finds a common independent set of maximum size, and the size will be $n-1$ if such a spanning tree exists. However, this may not be a common independent set of maximum weight.

8.2.62. *Every $2k$-edge-connected graph G has k pairwise edge-disjoint spanning trees avoiding any specified set of at most k edges.* By Corollary 8.2.59, a necessary and sufficient condition for having k pairwise edge-disjoint spanning trees is that for every vertex partition, the number of edges of G with endpoints in different blocks of the partition is at least $k(p-1)$, where p is the number of blocks.

If G is $2k$-edge-connected and S is a block in the partition, then at least $2k$ edges lie in the edge cut $[S, \overline{S}]$. Summing over all p parts counts each crossing edge twice. Hence at least kp edges crossing between parts. If at most k edges are deleted, then there are still at least $k(r-1)$ crossing edges, which is enough.

The result is sharp, because K_{2k+1} is $2k$-edge-connected but does not have $k+1$ pairwise edge-disjoint spanning trees. Such spanning trees would require $(k+1)2k$ edges, and K_{2k+1} has $(2k+1)k$ edges, which is less.

8.2.63. ($\bullet$) Given matroids $M_1, \ldots, M_k$ on E, the *Matroid Partition Problem* is the problem of deciding whether an input set $X \subseteq E$ partitions into sets $I_1, \ldots, I_k$ with $I_i \in \mathbf{I}_i$.

a) Use the Matroid Union Theorem to show that X is partitionable if and only if $|X - Y| + \sum r_i(Y) \geq |X|$ for all $Y \subseteq X$, and that all maximal partitionable sets are maximum partitionable sets.

b) Let M' be the union of k copies of a matroid M on E, and let X be a maximum partitionable set. Prove that there are disjoint sets $F_1, \ldots, F_k \subseteq X$ such that $\{F_i\} \subseteq \mathbf{I}$ and $\overline{X} \subseteq \sigma(F_1) = \cdots = \sigma(F_k)$.

8.3. RAMSEY THEORY

8.3.1. *Two concentric discs, each with 20 radial sections half red and half blue, can be aligned so that at least 10 sections on the inner disc match color with the corresponding sections on the outer disc.* Over all positions, each section provides 10 agreements, for 200 agreements in total. Since the agreements occur during 20 positions, there must be some position where the number of agreements is at least 200/20, which equals 10.

8.3.2. *Every set of $n+1$ numbers in $[2n]$ contains a pair of relatively prime numbers.* Any two consecutive numbers are relatively prime, since an integer greater than 1 cannot both. Hence it suffices to partition $[2n]$ into the n pairs of the form $(2i-1, 2i)$. Since there are only n such pairs, the pigeonhole principle guarantees that a set of $n+1$ numbers in $[2n]$ must use two from some pair, and these are relatively prime.

The result is best possible, because in the set of n even numbers in $[2n]$, every pair has a common factor. Note that since each pair of even numbers is not relatively prime, a solution to the problem by partitioning $[2n]$ into n classes and applying the pigeonhole principle must put the n even numbers into n different classes.

Every set of $n+1$ numbers in $[2n]$ contains two numbers such that one divides the other. This is best possible in that the n largest numbers in $[2n]$ do not contain such a pair. To apply the Pigeonhole Principle, we partition $[2n]$ into n classes such that for every two numbers in the same class, one divides the other.

Every natural number has a unique representation as an odd number times a power of two. For fixed k, the set $\{(2k-1)2^{j-1}\colon j \in \mathbb{N}\}$ has the desired property; the smaller of any two divides the larger. Since there are only n odd numbers less than $2n$, we have n such classes. The kth class is $\{m \in [2n]\colon m = (2k-1)2^{j-1} \text{ and } j \in \mathbb{N}\}$.

8.3.3. *a) Every set of n integers has a nonempty subset summing to a multiple of n.* Let a_i be the sum of the first i integers in the set. If n divides any a_i, we are finished. So, the a_i fall into $n-1$ congruence classes mod n. Hence there must be two of them in the same class. If these are a_j and a_k, then $a_k - a_j$, which is the sum of the $j+1$th through kth numbers, is divisible by n. So, in fact we have found a consecutive subset summing to a multiple of n. The example showing this is best possible is a set of $n-1$ copies of 1. Since n divides a sum of 1s only if the number of 1s is a multiple of n, the condition fails for this example.

b) At least one of $\{x, \dots, (n-1)x\}$ differs by at most $1/n$ from an integer. Consider the fractional parts of these numbers and the n intervals of the form $[(i-1)/n, i/n)$. If some fractional part falls in the first or last interval, we are done. Otherwise, we have $n-1$ objects in $n-2$ classes, and some pair jx and kx fall in the same interval. Now $(k-j)x$ is within $1/n$ of an integer.

8.3.4. *Private club needing 990 keys.*

990 keys permit every set of 90 members to be housed. Suppose 90 members receive one key apiece, each to a different room, and the remaining 10 members receive keys to all 90 rooms. Each set of 90 members that might arrive consists of k members of the first type and $90-k$ members of the second type. When the k members of the first type go to the rooms for which they have keys, there are $90-k$ rooms remaining, and the $90-k$ members of the second type that are present have keys to those rooms.

No scheme with fewer keys works. If the number of keys is less than 990, then by the pigeonhole principle (every set of numbers has one that is at most the average) there is a room for which there are fewer than 11 keys. Since the number of keys to each room is an integer, there are at most 10 keys to this room. Hence there is a set of 90 of the 100 members that has no one with a key to this room. When this set of 90 members arrives, they have keys to at most 89 rooms among them and cannot all be housed.

8.3.5. *The center of a tree T is a vertex or an edge.* For each vertex v, let v' be a vertex farthest from v in T, and mark the edge incident to v that leaves v on the unique path to v' in T. This makes $n(T)$ marks, so some edge uw has been marked twice.

The graph $T - uw$ consists of two components. If y is a vertex of the component of $T - uw$ containing u, then $d(y, u') > d(u, u')$, since the unique y, u'-path passes through u'. Similarly, if x is a vertex of the other component of $T - uw$, then $d(x, w') > d(w, w')$.

Hence the only candidates for the minimum eccentricity are the adjacent vertices u and w, and the center is a vertex or an edge.

8.3.6. *Every set of $2^m + 1$ integer lattice points in $\mathbb{R}^m$ contains two points whose centroid (mean vector) is also an integer lattice point.* Define 2^m classes by parity; each class is an m-tuple from {odd, even}. With $2^m + 1$ integer points and 2^m classes, there must be two in the same class. When two integer points having the same parity in each coordinate are averaged, the result is an integer point.

8.3.7. *Every red/blue-coloring of $\mathbb{R}^m$ has n integer lattice points with the same color whose centroid also has that color.* From any $2n-1$ lattice points whose coordinates are multiples of n, choose n points $a_1, \dots, a_n$ with the same color, say red; their centroid $\frac{1}{n}\sum a_i$ is also an integer point. Let C denote the centroid. If C is blue, then let $b_j = (n+1)a_j - \sum a_i$ for $1 \le j \le n$.

Now a_j is the centroid of the set obtain by replacing a_j by b_j. If any b_j is red, then we have the desired set in red. Otherwise, $\{b_1, \ldots, b_n\}$ is a blue set with blue centroid, since $\frac{1}{n}\sum b_j = C$.

8.3.8. *If S is a multiset of $n+1$ positive integers with sum k, and $k \le 2n+1$, then S has a subset with sum i for each $i \in [k]$.* The result is sharp, since $n+1$ copies of 2 have sum $2n+2$ but no subset with sum 1.

Proof 1 (counting argument). Let $r = \max S$, and suppose that S has m copies of the number 1. If $m >= r-1$, then we can add 1s successively, increasing the subset sum by one each time until reaching $r-1$. The next set is the number r alone. Then 1s can be introduced again until the next largest number in S can be substituted for them, and so on. To obtain $m >= r-1$, observe that the bound on the sum yields $1\cdot r+(n-m)\cdot 2+m\cdot 1 \le k \le 2n+1$, which simplifies to $m >= r-1$.

Proof 2 (induction on n). For $n = 0$, the only example is $\{1\}$, which works. For $n \ge 1$, if $\max S = 1$ then all sums can be achieved, so we may assume that $\max S = a > 1$. Since a exceeds 1, the sum of the remaining n elements is at most $2n-1$, so we can apply the induction hypothesis to obtain subsets of $S - \{a\}$ summing to all integers from 1 to $k-a$. For $k-a+1 \le i \le k$, adding the element a to a subset summing to $i-a$ will construct a subset of S summing to i, if $i-a >= 0$. This requires $a \le (1+k)/2$. The needed inequality holds, since $2a > k+1$ and $a+n \le k$ would imply $a \ge n+2$ and then $k \ge 2n+2$.

8.3.9. *For even n, Theorem 8.3.4 is sharp, in that there is an ordering of $E(K_n)$ so that the maximum length of an increasing trail is $n-1$.* When n is even, K_n has a 1-factorization. Define the linear ordering on $E(K_n)$ so that for each 1-factor in a specified factorization, the edges of that 1-factor occur consecutive. This ensures that each 1-factor contributes at most one edge to an increasing trail, and hence the maximum length of an increasing trial in this ordering is $n-1$.

8.3.10. *Every set of nine points in the plane with no three collinear contains the vertex set of a convex 5-gon, and this is sharp.* The four points $\{(\pm 1, \pm 2)\}$ and the four points $\{(\pm 2, \pm 3)\}$ together form no convex 5-gon. For nine points, our case analysis forcing a convex pentagon is due to L.H. Mak and D.B. West.

If at least 5 points lie on the convex hull, then we are finished, so we consider the two cases of four points and three points on the hull. The observation that simplifies the analysis is the following LEMMA: The segments ("spokes") from an interior point to the vertices of the hull partition the region into triangles. If some segment between two other interior points crosses two spokes, then these two points and the three endpoints of those spokes form a pentagon.

If the hull is a quadrilateral Q, we may assume that some interior point P is a convex combination of interior points XYZ, else the interior points form a pentagon. Since the triangle XYZ separates P from Q, one of its three edges must cross at least two of the four spokes from P to vertices of Q, yielding a pentagon by the lemma.

If the hull is a triangle T, we may again assume that the hull of the interior points has at most four points, so that at least two of the interior points R, S are convex combinations of the others. Let ABC be the vertices of T, with the line RS cutting AB and AC, oriented with BC horizontal, A above it, and R to the left of S. The segments from R to ABC partition the interior into three triangles. If there is no pentagon, then the lemma implies that any three interior points enclosing R have a point in each of these triangles; in particular, there is a point inside ARB. Similarly, there is a point in ASC. If the point X in ARB is below the line RS, then $BXRSC$ is a pentagon. If the point Y in ASC is below RS, then $BRSYC$ is a pentagon. If both are above, then $AXRSY$ is a pentagon.

8.3.11. *Every nondegenerate set of $R(m, m; 3)$ points in $\mathbb{R}^2$ has m points forming a convex m-gon.*

Proof 1 (stronger result). We can tilt the point set slightly, if necessary, to assume that the horizontal coordinates of the points are distinct. Color each triple of points by whether the line determined by the leftmost and rightmost points is above or below the middle point. With $R(m, m; 3)$ points, there are m points such that all triples have the same color. If all triples have the left-right line above the middle point, then the piecewise-linear function determined by the horizontally consecutive pairs of points is convex, and these points form a convex m-gon in which the line between the leftmost and rightmost is above all the other points. If all triples have the other color, then the piecewise-linear function is concave, and again the points determine a convex m-gon.

Proof 2 (using the full point set). For each triple $T \subseteq S$, color T by the parity of the number of points in $S - T$ that are in the interior of the triangle formed by T. Let Q be a set of four points in S whose triples have the same color. If Q is not convex, then the triangular region R formed by the triple T on the convex hull of Q contains the other three triples. If the three inside triangles are odd, then T is even. If the three inside triangles are even, then T is odd. Hence every homogeneous 4-set is convex. Hence taking at least $R(m, m; 3)$ points yields m points whose 4-sets are all convex, and these m points form a convex m-gon.

8.3.12. *Monotone tournaments: If N is sufficiently large, then every simple digraph with vertex set $[N]$ has an independent set of order m or a monotone tournament of order m or a complete loopless digraph of order m.* Let

$N = R(m,m,m,m)$, the Ramsey number for 4-coloring 2-sets to obtain a homogeneous set of size m. Given a simple digraph D on $[N]$, form a coloring of $E(K_N)$ by letting ij with $i < j$ have color 1 if i and j are nonadjacent in D, color 2 if $i \to j$ in D, color 3 if $j \to i$ in D, and color 4 if D has edges in both directions on this pair. By Ramsey's Theorem, there is a homogeneous set of size m, and this is the vertex set of the desired induced subgraph.

8.3.13. *Given $k > 0$, there exists an integer s_k such that every k-coloring of the integers $1, \ldots, s_k$ yields monochromatic (but not necessarily distinct) x, y, z solving $x + y = z$.* Let $r_k = R_k(3; 2)$. We show that $s_k < r_k$ by showing that every k-coloring f of $[r_k - 1]$ has a monochromatic solution to $x + y = z$. From f, we define a k-coloring f' of $E(K_{r_k})$. Let $V(K_{r_k}) = [r_k]$. Let the color of edge ab in f' be $f(|a - b|)$.

By Ramsey's Theorem, f' yields a monochromatic triangle with vertices a, b, c. We may assume that $a < b < c$. Let $x = b - a$, $y = c - b$, $z = c - a$. Since the triangle is monochromatic, $f(x) = f(y) = f(z)$. By construction, they satisfy $x + y = z$.

b) $s_k \ge 3s_{k-1} - 1$, and hence $s_k \ge (3^k + 1)/2$. Let f be a k-coloring of $[n]$ with no monochromatic $x + y = z$. Define a $(k+1)$-coloring f' of $[3n+1]$ by

$$f'(i) = \begin{cases} f(i) & \text{if } i \le n, \\ f(i - 2n - 1) & \text{if } i \ge 2n + 2, \\ k + 1 & \text{if } n + 1 \le i \le 2n + 1. \end{cases}$$

Under f' there is no monochromatic $x + y = z$. The resulting recurrence $s_k \ge 3s_{k-1} - 1$ yields $s_k \ge (3^k + 1)/2$.

8.3.14. *Application of the lexicographic product (composition) to Ramsey numbers.* The graph $G[H]$ has a copy of H for each vertex of G, with all edges present between copies that correspond to edges of G, and no edges present between copies that correspond to non-adjacent vertices in G.

a) $\alpha(G[H]) \le \alpha(G)\alpha(H)$. Let S be a largest independent set in $G[H]$. No two vertices in S can have first coordinates u and u' with $u \leftrightarrow u'$ in G, since all such vertices in the product are adjacent. So, the vertices u that appear as first coordinates of vertices in S form an independent set in G; hence there are at most $\alpha(G)$ of them.

The vertices of S using a fixed vertex u of G as first coordinate must have as their second coordinates a set of vertices that are independent in H, since these vertices inherit adjacencies from H. Therefore, each $u \in V(G)$ that appears among the first coordinates of vertices in an independent set in $G[H]$ is used at most $\alpha(H)$ times. Since at most $\alpha(G)$ such vertices appear, each at most $\alpha(H)$ times, $\alpha(G[H]) \le \alpha(G)\alpha(H)$. (Actually, equality holds, because $S \times T$ is independent in $G[H]$ whenever S is independent in G and T is independent in H.)

b) The complement of $G[H]$ is $\overline{G}[\overline{H}]$. Nonadjacency in $G[H]$ requires $u \not\leftrightarrow u'$ when $u \ne u'$, and it requires $v \not\leftrightarrow v'$ when $u = u'$. Thus is simply the definition of adjacency in $\overline{G}[\overline{H}]$.

c) $R(pq+1, pq+1) - 1 \ge [R(p+1, p+1) - 1] \cdot [R(q+1, q+1) - 1]$. Let G be a graph on $R(p+1, p+1) - 1$ vertices with no clique or independent set of size $p+1$, and let H be a graph on $R(q+1, q+1) - 1$ vertices with no clique or independent set of size $q + 1$. By part (a), $\alpha(G[H]) \le \alpha(G)\alpha(H) \le pq$ and $\alpha(\overline{G}[\overline{H}]) \le \alpha(\overline{G})\alpha(\overline{H}) = \omega(G)\omega(H) \le pq$. By part (b), $\omega(G[H]) = \alpha(\overline{G[H]}) = \alpha(\overline{G}[\overline{H}]) \le pq$. Thus $G[H]$ has no clique or independent set of size $pq + 1$, which yields the desired bound.

d) $R(2^n + 1, 2^n + 1) \ge 5^n + 1$ for $n \ge 0$, We use induction on n. If $n = 0$, then $R(2, 2) = 2$, as desired. For the induction step, let $k = 2^{n-1}$ and $l = 2$ and apply (c). This yields

$$\begin{aligned} R(2^n + 1, 2^n + 1) = R(kl + 1, kl + 1) &\ge [R(2^{n-1} + 1, 2^{n-1} + 1)][R(3,3) - 1] + 1 \\ &\ge 5^{n-1} \cdot 5 + 1 = 5^n + 1. \end{aligned}$$

To compare this with the nonconstructive lower bound $R(p, p) \le cp2^{p/2}$, let $p = 2^n + 1$. Since $5^n + 1 = 5^{\lg(p-1)} + 1 = 1 + (p-1)^{\lg 5}$. Since $\lg 5 < 2.5$. This construction gives only a low-degree polynomial lower bound, while the nonconstructive lower bound is exponential.

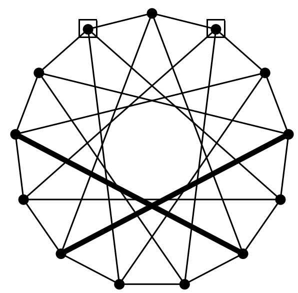
8.3.15. *$R(p, 2) = R(2, p) = p$, and hence $R(p, q) \le \binom{p+q-2}{p-1}$.* In a 2-coloring of $E(K_p)$, all edges have one color or there is an edge with the other color. In $E(K_{p-1})$, making all the edges the first color yields neither a p-set whose edges have the first color nor a single edge of the other color.

For the general upper bound, we use induction on $p + q$. When $\min\{p, q\} = 2$, we have $R(p, 2) = p = \binom{p+2-2}{p-1} = \binom{p}{1}$. Otherwise, we have $R(p, q) \le R(p, q-1) + R(p-1, q) \le \binom{p+q-3}{p-1} + \binom{p+q-3}{p-2} = \binom{p+q-2}{p-1}$.

8.3.16. *$R(3, 5) = 14$.* For the upper bound, $R(3, 5) \le R(3, 4) + R(2, 5) = 9 + 5 = 14$. To show that 13 vertices do not force a red triangle or blue K_5, it suffices to show that the graph G below is triangle-free and has no independent set of size 5.

The graph is vertex-transitive, and the neighborhood of a vertex is independent, so the graph is triangle-free.

Let S be a set of five vertices in G. By the pigeonhole principle, there are two vertices in S separated by distance at most two along the outside cycle. If S is independent, it thus has a pair with distance exactly 2, as marked below. Deleting two such vertices and their neighbors leaves the graph $2K_2$ in bold below. Thus we cannot add three more vertices to obtain an independent 5-set.



8.3.17. *Ramsey numbers for $r = 2$ and multiple colors.*

a) $R(p) \le \sum_{i=1}^{k} R(q_i) - k + 2$, *where* $p = (p_1, \dots, p_k)$ *and* q_i *is obtained from* p *by subtracting 1 from* p_i *but leaving the other coordinates unchanged.* We show that $R(p) \le 2 + \sum(R(q_i) - 1)$. Consider a fixed vertex x in a coloring of the edges of a complete graph on $2 + \sum(R(q_i) - 1)$ vertices. Partition the other vertices into color classes by the color of the edges joining them to x. With k classes and thresholds $R(q_1), \dots, R(q_k)$, by the pigeonhole principle at least one of the thresholds must be met. If this occurs for color j, then the definition of q_j implies that the original coloring, restricted to the edges on the vertices of color j, has a $(p_j - 1)$-clique in color j or a p_i-clique in some other color i. In the former case, x can be added to obtain a p_j-clique in color j, so either way the coloring has a monochromatic complete subgraph of the desired size in some color.

b) $R(p_1 + 1, \dots, p_k + 1) \le \frac{(p_1 + \cdots + p_k)!}{p_1! \cdots p_k!}$. We use induction on $\sum p_i$. If all $p_i = 0$, then $R(1, \dots, 1) = 1 = (\sum 0)!/\prod 0!$. For $\sum p_i > 0$, let $r_i = R(p_1 + 1, \dots, p_{i-1} + 1, p_i, p_{i+1} + 1, \dots, p_k + 1)$. The induction hypothesis gives $r_i \le s_i$, where $s_i = p_i \left(\sum p_j - 1\right)!/\prod p_j!$. Since $k \ge 2$, we now have

$$R(p_1{+}1, \dots, p_k{+}1) \le 2 - k + \sum r_i \le \sum s_i = \frac{\sum p_i \left(\sum p_j - 1\right)!}{\prod p_j!} = \frac{(p_1 + \cdots + p_k)!}{p_1! \cdots p_k!}.$$

8.3.18. *a)* $r_k \le k(r_{k-1} - 1) + 2$, *where* $r_k = R_k(3; 2)$ (the minimum n such that k-coloring $E(K_n)$ forces a monochromatic triangle). Consider a k-coloring with no monochromatic triangle, and let x be some vertex. There are at most $r_{k-1} - 1$ neighbors of x along edges of the ith color, for each i. Otherwise, avoiding a monochromatic triangle (in any other color) within those vertices would force having at least one edge of color i, and its endpoints would form a triangle in color i with x. Thus $1 + k(r_{k-1} - 1)$ is an upper bound on the number of vertices for which it is possible to avoid a monochromatic triangle.

b) $r_k \le \lfloor k!e \rfloor + 1$. Proof by induction. Note that $r_2 = 6$, which satisfies the formula (any irrational number at least 2.5 could be used in place of e). By induction, $r_k \le k(r_{k-1} - 1) + 2 \le k \lfloor (k-1)!e \rfloor + 2$. Since e is irrational, $k!e$ cannot be an integer, so the 2 can be reduced to 1 when k is brought inside the $\lfloor\ \rfloor$.

8.3.19. (•) Prove that $R_k(p; r + 1) \le r + k^M$, where $M = \binom{R_k(p;r)}{r}$.

8.3.20. *Off-diagonal Ramsey numbers.*

a) Let $h(n, p) = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}}$. *For fixed n, If* $h(n, p) < 1$ *for some* $p \in (0, 1)$, *then* $R(k, l) > n$. *Furthermore,* $R(k, l) > n - h(n, p)$ *for all* $n \in \mathbb{N}$ *and* $p \in (0, 1)$. Produce a coloring of $E(K_n)$ at random, by letting each edge be red with probability p, independently. For any k-set, the probability that it induces a red complete graph in the resulting coloring is $p^{\binom{k}{2}}$. Since there are $\binom{n}{k}$ choices of k-sets, the linearity of expectation (see Section 8.5) implies that the expected number of red k-cliques is $\binom{n}{k} p^{\binom{k}{2}}$. Similarly, the expected number of blue l-cliques is $\binom{n}{l}(1-p)^{\binom{l}{2}}$. Letting X be the random variable that counts the monochromatic cliques of threshold size, we have $\mathrm{E}(X) = h(n, p)$.

If $h(n, p) < 1$, then $\mathrm{E}(X) < 1$. This means that in some outcome of the experiment there are no such cliques. That is, there exists a 2-coloring of $E(K_n)$ proving that $R(k, l) > n$. It suffices to have any value of p in $(0, 1)$ with this property.

Similarly, always $R(k, l) > n - h(n, p)$. Since $\mathrm{E}(X) = h(n, p)$, in some outcome of the experiment $X \le h(n, p)$. Deleting one vertex from each bad clique in the resulting coloring yields with at least $n - h(n, p)$ vertices, showing that $R(k, l) > n - h(n, p)$.

b) $R(3, k) > k^{3/2 - o(1)}$. To obtain a lower bound from $R(3, k) > n - \binom{n}{3}p^3 - \binom{n}{k}(1-p)^{\binom{k}{2}}$, we choose p and n in terms of k so that the subtracted terms are less than $n/2$ (constant factors won't matter). Using upper bounds on these terms, we have $R(3, k) > n(1 - \frac{1}{6}n^2p^3 - \frac{1}{k!}(ne^{-pk/2})^{k-1}$. The first term suggests setting $p = cn^{-2/3}$ (it suffices to make c a constant as small as 1). Since k may be large, we also want $e^{pk/2} > n$ (again we can make $e^{pk/2} = c'n$, with c' a constant as large as 1).

Letting $c = c' = 1$ and taking the natural logarithm, we want to choose n so that $k = 2n^{2/3} \ln n$. We want $n^{2/3}$ to cancel $2 \ln n$ and leave k, so we set $n = (\frac{k}{3 \ln k})^{3/2}$ to obtain $2n^{2/3} \ln n = k(1 - c'' \frac{\ln \ln k}{\ln k}$.

This choice of n (and p) yields $R(3, k) > n(1 - \frac{1}{6} - \frac{1}{k!})$. Since $(\ln k)^{-1} = -\ln \ln k / \ln k$, we can write this as

$$R(3, k) > \tfrac{2n}{3} = \tfrac{2}{9\sqrt{3}} k^{(3/2)(1 - \ln \ln k / \ln k)} = k^{3/2 - o(1)}.$$

We could optimize the argument by letting c depend on n and c' depend on k, but this won't improve the exponent.

The first part of (a) yields no useful lower bound on $R(3,k)$. To obtain p such that $\binom{n}{3}p^3 < 1$, we need $p < c/n$. We also need $e^{p(k-1)/2} > n$, which leads to $k > 1 + (2/c)n\ln n$. Unfortunately, this works only when n is smaller than k, and we already know trivially that $R(3,k) > k$.

c) Use part (a) to obtain a lower bound for $R_k(q)$. We have k colors with thresholds all equal to q. We give each edge the ith color with probability $1/k$, for each i, independently, and let X be the number of monochromatic q-cliques. Thus $\mathrm{E}(X) = k\binom{n}{k}k^{-\binom{q}{2}}$. By deleting one vertex of each such clique in an outcome with at most the expected number, we obtain $R_k(q) > n - \mathrm{E}(X)$.

Thus $n - \mathrm{E}(X)$ is a lower bound on $R_k(q)$, for each n. Since $\binom{n}{k} < \left(\frac{ne}{k}\right)^k$, we also have the simpler $n - k\left(\frac{ne}{k}\right)^k k^{-\binom{q}{2}}$ as a lower bound. We seek n to maximize this bound. Differentiating suggests choosing n to satisfy $1 = k\frac{e}{k}\left(\frac{ne}{k}\right)^{k-1}k^{-\binom{q}{2}}$, or $n \approx (k/e)k^{(\binom{q}{2}-1)/(k-1)}$.

At this value of n, $\mathrm{E}(X)$ is near 1, and our lower bound is just a bit less than $(k/e)k^{(\binom{q}{2}-1)/(k-1)}$.

8.3.21. *$R(K_{1,m}, K_{1,n})$ is $m+n$ unless m and n are both even, in which case it is $m+n-1$.* The value of $R(K_{1,m}, K_{1,n})$ is the least N such that, for every N-vertex graph G, either $\Delta(G) \ge m$ or $\Delta(\overline{G}) \ge n$. By the pigeonhole principle, this property occurs when $N = m+n$, because at every vertex there must be m neighbors or n nonneighbors. This is the least such N if and only if there exists an $(m-1)$-regular graph with $m+n-1$ vertices. If m or $m+n$ is odd, then such a graph exists. If m and n are both even, then we are seeking a regular graph of odd order and degree, which does not exist.

8.3.22. *If T is a tree with m vertices, and $m-1$ divides $n-1$, then $R(T, K_{1,n}) = m+n-1$.* With $m+n-1$ vertices, if every vertex in the blue graph has degree at most $n-1$, then every vertex in the red graph has degree at least $m-1$. This implies that the red graph contains every tree with m vertices, by Proposition 2.1.8(easily proved by deleting a leaf and using induction).

For the lower bound, since $m-1$ divides $n-1$, a set of $m+n-2$ vertices splits into sets of size $m-1$. The components of the red graph are K_{m-1}, so there is no red tree with m vertices. Each vertex has $m-2$ red neighbors, so it has $n-1$ blue neighbors, and there is no blue star with n edges.

8.3.23. *$(m-1)(n-1)+1$ is the minimum value of p such that every 2-coloring of $E(K_p)$ in which the red graph is transitively orientable contains a red m-clique or a blue n-clique.* More than $(m-1)(n-1)$ vertices are needed, since $(n-1)K_{m-1}$ is transitively orientable.

Proof 1 (perfect graphs). The red graph is a comparability graph and hence is perfect. If it has no m-clique, then it has a proper $(m-1)$-coloring. If it has more than $(m-1)(n-1)$ vertices, then by the pigeonhole principle some color class has (at least) n vertices. This class yields an n-clique in the blue graph.

Proof 2 (induction on m). Immediate for $m=1$. For $m>1$, consider such a coloring with $(m-1)(n-1)+1$ vertices. Let F be a transitive orientation of the red graph, with sources S. The sources induce a blue clique, so $|S| \le n-1$ if the claim does not hold. However, $F-S$ is then a transitive orientation of a graph G' with more than $(m-2)(n-1)$ vertics. By the induction hypothesis, G has a blue n-clique or a red $m-1$-clique Q. A transitive orientation of such a clique Q has a unique source u. Since u is not a source of F, there is an edge from some $v \in S$ to u, and then transitivity guarantees that v can be added to Q to obtain a red m-clique.

8.3.24. *If T is a tree with m vertices, then $R(T, K_{n_1}, \ldots, K_{n_k}) = (m-1)(R(n_1,\ldots,n_k)-1)+1$.* Let the colors be $0,\ldots,k$ corresponding to T and $K_{n_1},\ldots,K_{n_k}$, respectively.

For the lower bound, begin with a k-coloring of K_q points that has no copy of K_{n_i} in color i for any i, for $1 \le i \le k$. Replace each vertex by a complete graph of order $m-1$ whose edges all get color 0. Each edge in the original graph expands into a copy of $K_{m-1,m-1}$. Give all edges in this subgraph the same color that its original edge had. This coloring has no m-tree in color 0, because the components of the graph in color 0 have only $m-1$ vertices. It meets no clique quota in any other color, because such a monochromatic clique can be collapsed to a monochromatic clique of that size in the original coloring. This holds because the vertices must come from copies of distinct points in the original point set, since copies of the same point are joined by edges of color 0.

For the upper bound, let f be a $(k+1)$-coloring of $E(K_{(m-1)q+1})$, where $q = R(n_1,\ldots,n_k)-1$. Define f' on the same edges by letting an edge be red if it gets color 0 in f and blue if it gets a nonzero color in f. By Chvatál's Theorem for $R(T, Kn)$ (Theorem 8.3.14), this coloring has a red T, in which case we are done, or a blue $(q+1)$-clique. In the latter case, we return to f, restricted to this set of $q+1$ vertices. The definition of the Ramsey number says that f has a monochromatic copy of K_{n_i} in color i on these vertices, for some $i \in [k]$.

8.3.25. $R(C_4, C_4) = 6$.

Claim 1: A 2-colored complete graph with at least 6 vertices containing a monochromatic 5-cycle or 6-cycle also contains a monochromatic 4-cycle. Given a red 5-cycle, avoiding a red 4-cycle makes every chord blue, so the coloring on these 5 points consists of 5-cycles in each color. For a vertex outside these five, its edges to these have at least three in one color, which we may assume by symmetry is red. These three points include two that are nonadjacent on the red 5-cycle. The 2-edge path between them on that cycle, together with the edges to the extra vertex, yield a red 4-cycle.

Given a red 6-cycle, the chords joining opposite vertices on the cycle must be blue, or we already have a red 4-cycle. The remaining chords cannot all be blue, since this yields a blue 4-cycle. consisting entirely of chords. However, if one of them is red, then we have a red 5-cycle and can apply the argument above.

Claim 2: $R(P_4, C_4) = 5$. With four vertices, the color classes may be a triangle and a claw. With five vertices, if there is no monochromatic triangle, then every vertex has two incident vertices of each color. Being 2-regular, each color class is a disjoint union of cycles, which with five vertices can only be C_5. However, C_5 contains P_4.

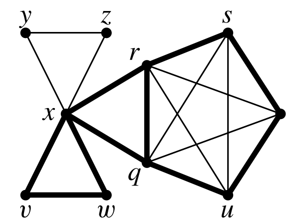
Hence we may assume a monochromatic triangle $[x, y, z]$. If it is red, then any additional red edge incident to the triangle yields a red P_4; if there is no such red edge, then the blue edges from $[x, y, z]$ to the remaining vertices u and v contain a blue 4-cycle. If $[x, y, z]$ is blue, then two blue edges from one of $\{u, v\}$ to $\{x, y, z\}$ would complete a blue 4-cycle. By symmetry, we may thus choose x with no blue edges to $\{u, v\}$, so the path $\langle u, x, v\rangle$ is red. Now vy or vz is red to complete a red P_4.

Claim 3: $R(C_4, C_4) = 6$. Since the 5-cycle and its complement contain no monochromatic C_4, $R(C_4, C_4) \ge 6$. For the upper bound, consider a 2-coloring of $E(K_6)$. Since $R(P_4, C_4) = 5$ by Claim 2, we may assume that the coloring has a red P_4, with vertices u, v, w, x in order. If both of the remaining vertices, y and z, have both edges to $\{u, x\}$ blue, then $[x, y, u, z]$ is a blue C_4. So, we may assume that one of the edges is red and extend the path to a red $P5$, such as u, v, w, x, y. The chords uy, vy, ux must all be blue, else we have a red 4-cycle or 5-cycle, which suffices, by Claim 1. Now consider z. The edges zv and zx cannot be both red or both blue, else $[z, v, w, x]$ is a red 4-cycle or $[z, w, y, u, x]$ is a blue 5-cycle. Hence we may assume by symmetry that zv is blue and zx is red. Now we cannot color zu.

8.3.26. $R(2K_3, 2K_3) = 10$. The lower bound is provided by the construction in Theorem 8.3.15: the red graph is $K_{1,3}+K_5$. For the upper bound, consider a 2-coloring of $E(K_{10})$. Any six vertices contain a monochromatic triangle, and the seven vertices outside that triangle yield another monochromatic triangle. We are finished if they have the same color; if not, then the edges joining them are used to collapse the configuration to a “bow tie” as in the proof of Theorem 8.3.15. Thus we have vertices $\{v, w, x, y, z\}$ such that $[x, v, w]$ is a bold triangle and $[x, y, z]$ is a solid triangle.

To avoid a monochromatic triangle on the remaining five vertices, we must have a bold 5-cycle $[q, r, s, t, u]$ and a solid 5-cycle $[q, s, u, r, t]$, as

shown below.



Vertex x must have three neighbors of the same color in $\{q, r, s, t, u\}$, of which two must be adjacent on the cycle in that set having that color. By symmetry, we may thus assume that $[x, q, r]$ is a bold triangle. If any edge e with endpoints in $\{v, w\}$ and $\{q, r\}$ is solid, then the edges joining the endpoint of e in $\{v, w\}$ to the solid neighbors in $\{s, t, u\}$ of the other endpoint of e (for example, vt and vu when $e = vr$) must be bold to avoid disjoint solid triangles, but this makes disjoint bold triangles ($[x, q, r]$ and $[v, t, u]$ in the example $e = vr$. Hence the edges joining $\{5, 6\}$ to $\{1, 2\}$ are all bold.

Now consider the edges vs and vu. If both are solid, then $[x, y, z]$ and $[v, s, u]$ are disjoint solid triangles. Hence one is bold; we still have symmetry and may assume that it is vu. Now we have disjoint solid triangles $[v, q, u]$ and $[x, w, r]$. Hence a monochromatic $2K_2$ is forced.

8.3.27. $R(mK_2, mK_2) = 3m - 1$. For the lower bound, let the red graph be $K_{2m-1} + \overline{K}_{m-1}$. Every red edge has both endpoints in the $(2m-1)$-clique, so there cannot be m independent red edges. The complementary blue graph is the join of the complete graph on $m-1$ vertices with an independent set on $2m-1$ vertices ($\overline{K}_{2m-1} \vee K_{m-1}$. Every edge has at least one endpoint in the $(m-1)$-clique, so again there cannot be m independent edges.

For the upper bound, we use induction. Note that $R(K_2, K_2) = 2$. For $m > 1$, consider an arbitrary 2-coloring of the edges of K_{3m-1}. There must be incident edges of differing colors, else the entire clique gets one color and has enough points to contain mK_2. Remove the three points hit by these two incident edges of different color, and apply the induction hypothesis. To the resulting monochromatic $(m-1)K_2$, add the edge of the appropriate color from the deleted three vertices.

8.3.28. *If G_i is a graph of order p_i, for $1 \le i \le k$, then $R(m_1G_1, \dots, m_kG_k) \le \sum(m_i - 1)p_i + R(G_1, \dots, G_k)$.* Each m_iG_i is the disjoint union of m_i copies of G_i. Given a 2-coloring with the specified number of vertices, we iteratively extract disjoint monochromatic copies of these graphs in the specified colors. As long as $R(G_1, \dots, G_k)$ vertices remain that have not been touched by the extracted graphs, we can find another monochromatic G_i in color i for some i. If ever we obtain m_i copies of G_i, then we are done. Otherwise, we have obtained at most $m_i - 1$ copies of each G_i, so we have eliminated at most $\sum(m_i - 1)p_i$ vertices from consideration. In this case at least $R(G_1, \dots, G_k)$ vertices remain, and we can continue. Hence the process terminates only by finding $m_i G_i$ in color i, for some i.

8.3.29. *Graphs with n vertices having no clique or independent set with size as large as $2^{c\sqrt{\log n \log\log n}}$ yield a lower bound for $R(p, p)$ in terms of p that grows faster than every polynomial in p but slower than every exponential in p.* The existence of such a graph implies that $R(p, p) > n$, where $p = 2^{c\sqrt{\log n \log\log n}}$. To find the behavior of the lower bound, we need to solve this equation for n in terms of p, but we do not need the complete solution to answer the question.

Taking logs and squaring both sides yields $c'(\log p)^2 = \log n \log\log n$, where $c' = 1/(c \log 2)^2$. To study the form of the function, we express n in terms of p and a parameter t.

First suppose that $n \le p^t$. In this case $c'(\log p)^2 \le (t \log p)(\log t + \log\log p)$. If t is bounded, then this inequality is false. Hence n cannot be bounded by any polynomial function of p.

Now suppose that $n \ge t^p$. In this case $c'(\log p)^2 \ge (p \log t)(\log p + \log\log t)$. Again, this is impossible when t is a constant. Hence n cannot grow faster than any exponential function of p.

8.3.30. *If G is an n-vertex graph such that $\alpha'(\overline{G}) = k$, then $R(P_3, G) = \max\{n, 2n - 2k - 1\}$.* We seek the minimum r such that red/blue-colorings of $E(K_r)$ yield a red P_3 or a blue G.

If $r < n$, then we color all of $E(K_r)$ blue yields $R(P_3, G) > r$, so n is a lower bound. If $r = 2(n-k-1)$, then we color $E(K_r)$ with a perfect matching in red and the rest in blue. The red matching avoids P_3, and every set of n vertices contains at least $k + 1$ pairs from the red matching (if it has s vertices whose mates are omitted and t matched pairs, then $s + 2t = n$ and $s + t \le n - k - 1$, so $n + t \le n - k - 1$). There is no blue G on such a set of vertices, since $\overline{G}$ has no matching of size $k + 1$.

For the upper bound, consider a 2-coloring of $E(K_r)$ with $r = \max\{n, 2n - 2k - 1\}$. If there is no red P_3, then the red graph is restricted to a matching. Thus all edges are blue except for 1) at most $n - k - 1$ pairwise disjoint edges and an isolated vertex if $k < n/2$, or 2) at most $n - k$ pairwise disjoint edges if $k = n/2$. In either case, we choose $n - k$ vertices that span no red edges, and then we augment these with any k other vertices. The coloring induced by these n vertices has at most k red edges, and the red edges are pairwise disjoint. Since $\alpha'(\overline{G}) = k$, the graph G can be mapped into an n-vertex complete graph to avoid any matching of size k, and hence the blue graph on these n vertices contains G.

8.3.31. *If r and s are natural numbers with $r + s \not\equiv 0 \pmod 4$, then every 2-coloring of $E(K_{r,s})$ has a monochromatic connected graph with at least $\lceil r/2 \rceil + \lceil s/2 \rceil$ vertices.* We prove a slightly stronger result: If the edges of $K_{r,s}$ are 2-colored, then there is a monochromatic connected subgraph with at least half the vertices from each side, with one side exceeding half unless each color forms $2K_{r/2,s/2}$.

To prove this, first delete a vertex or two (if necessary) to leave an odd number of vertices on each side. Now let X and Y be the partite sets, and consider an arbitrary edge-coloring. Give each vertex of X the color occuring on a majority of its incident edges. Let blue be the color thus assigned to a majority of the vertices in X. Any two blue vertices in X have a common neighbor in Y along blue edges. Hence the blue vertices in X and their incident blue edges form a connected subgraph; it has more than half of X (by pigeonhole choice of blue) and more than half of Y (by the blue neighbors of each vertex of X).

With the deleted vertex of each original partite set of even size replaced, we have obtained an connected monochromatic subgraph with at least $(r + s)/2$ vertices. Equality requires that all $r/2$ vertices of this subgraph in X have the same $s/2$ neighbors in Y.in Y. Equality forbids additional incident blue edges, so the red edges incident to these vertices form $2K_{r/2,s/2}$. To avoid having a spanning connected red subgraph, all edges not incident to our original blue subgraph must be red, forming another red copy of $K_{r/2,s/2}$.

Every 3-coloring of $E(K_{r+s})$ contains a monochromatic connected subgraph with more than $(r+s)/2$ vertices, except maybe when $r+s \equiv 0 \pmod 4$. Given a 3-coloring of $E(K_n)$, let G be a maximal monochromatic connected subgraph in color 0; let r be its order, with $n = r + s$. On edges joining $V(G)$ and $V(K_n) - V(G)$, only colors 1 and 2 are used. The preceding argument guarantees a monochromatic connected subgraph with more than half the vertices unless r and s are even and the subgraphs between $V(G)$ and $V(K_n) - V(G)$ are $2K_{r/2,s/2}$ in colors 1 and 2.

In the exceptional case, these subgraphs in color 1 (red) and color 2 (blue) partition $V(G)$ into A_0 and A_1 and $V(K_n) - V(G)$ into A_2 and A_3 so that all edges joining A_0 to A_2 or joining A_1 to A_3 are red and all edges joining A_0 to A_3 or joining A_1 to A_2 are blue. If any edge joining A_0 to A_1 or A_2 to A_3 does not have color 0, then we have a monochromatic n-vertex connected subgraph. Otherwise, we have monochromatic connected subgraphs in color 0 with r and s vertices. Hence we have the desired configuration unless $r = s = n/2$, which now implies $n \equiv 0 \pmod 4$.

Furthermore, we have shown that when $n \equiv 0 \pmod 4$ the claim fails only for the following coloring: given the three pairings of four sets A_0, A_1, A_2, A_3 of size $n/4$, assign color i to all edges between groups paired in the ith pairing. Any coloring within each graph can be used; all the monochromatic connected subgraphs have exactly $n/2$ vertices.

8.3.32. *Forcing 4-cycles.*

a) If $\sum_{v\in V(G)}\binom{d(v)}{2} > \binom{n(G)}{2}$, then G contains a 4-cycle. The sum $\sum_{v\in V(G)}\binom{d(v)}{2}$ counts the triples u,v,w such that v is a common neighbor of u and w. If G has no 4-cycle, then every pair of vertices has at most one common neighbor.

b) If $e(G) > \frac{n(G)}{4}(1+\sqrt{4n(G)-3})$, then G contains a 4-cycle. Since $\binom{x}{2}$ is a convex function of x, the minimum of $\sum_{v\in V(G)}\binom{d(v)}{2}$ for fixed $\sum d(v)$ occurs numerically when the values for $d(v)$ are equal (even though this may not be realized by a graph). Since $\sum d(v) = 2e(G)$, we conclude that $\sum_{v\in V(G)}\binom{d(v)}{2} \ge n(G)\binom{2e(G)/n(G)}{2}$. If $e(G)(\frac{2e(G)-n(G)}{n(G)} > \binom{n(G)}{2}$, then the condition of part (a) holds. This inequality reduces to the stated condition.

c) $R_k(C_4) \le k^2+k+2$. If $n > k^2+k+2$, then $\binom{n}{2} > k\frac{n}{4}(1+\sqrt{4n-3})$. Hence some color class is as large as $\frac{n}{4}(1+\sqrt{4n-3})$, and the result of part (b) applies.

8.3.33. *$R(C_m, K_{1,n}) = \max\{m, 2n+1\}$, except possibly if m is even and does not exceed $2n$.* For the lower bound, first consider $m \ge 2n+1$. Form a red clique on $m-1$ vertices; it has no red C_m and no blue edge, hence no vertex with blue degree n. If $m < 2n+1$ and m is odd, form two disjoint blue cliques on n vertices, and let all the edges between them be red. There is no vertex with blue degree n, and all the red cycles have even length. The case with $m < 2n+1$ and m even is unsettled, although the argument for the upper bound is still valid.

For the upper bound, first consider $m \ge 2n+1$ and any 2-coloring of $E(K_m)$. If no vertex has blue degree at least n, then the red degree of every vertex is at least $m-n$, which exceeds $m/2$ since $n < m/2$. We now invoke Bondy's Theorem, stating that if $x \not\leftrightarrow y$ implies $d(x)+d(y) \ge n(G)$, then G is a complete bipartite graph with equal-sized partite sets or G has a cycle of each length from 3 to $n(G)$. The red graph satisfies this hypothesis, so in either case it is Hamiltonian, which yields a red C_m.

If $m < 2n+1$, consider a 2-coloring of $E(K_{2n+1})$. Again having no vertex of blue degree at least n implies that the minimum red degree is at least half the number of vertices. Since $2n+1$ is odd, Bondy's Theorem now yields red cycles of all lengths, including length m.

8.3.34. *Every 2-coloring of $E(K_n)$ contains a monochromatic Hamiltonian cycle or a Hamiltonian cycle consisting of two monochromatic paths.* This is immediate for $n=3$; we proceed by induction. If $n > 3$, consider the coloring on $E(K_n - v)$. If this has a monochromatic cycle, then we can replace an arbitrary edge of the cycle by the edges from its endpoints to v.

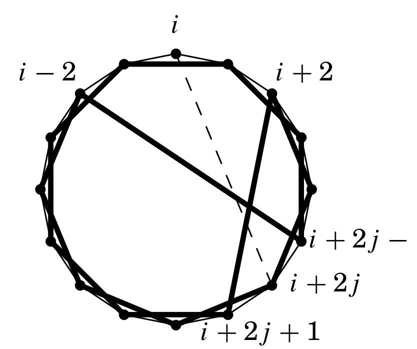
If it has two monochromatic paths whose union is a cycle, then let x,y,z be three consecutive vertices on the cycle with xy red and yz blue. We may assume that yv is red. Now the cycle obtained by replacing yz with $\langle y,v,z\rangle$ has the desired property.

8.3.35. *Ramsey numbers for cycles.*

a) A 2-coloring of $E(K_n)$ that contains a monochromatic C_{2k+1} for some $k \ge 3$ also contains a monochromatic C_{2k}. Let C be a red $(2k+1)$-cycle, with vertices $v_0,\ldots,v_{2k}$ in order. If there is no monochromatic $2k$-cycle, then each x_ix_{i+2} is blue, which yields a blue $2k+1$-cycle C' and implies that each x_ix_{i+4} is red, where indices are mod $2k$. For each i, consider the cycle obtained from C by replacing the path $\langle x_i, x_{i+1},\ldots,x_{i+5}\rangle$ with $\langle x_i, x_{i+3}, x_{i+2}, x_{i+1}, x_{i+5}\rangle$. Skipping x_{i+4}, it has length $2k$, and all edges except x_ix_{i+3} are red; hence x_ix_{i+3} is blue. Now we can replace the path $\langle x_i, x_{i+2}, x_{i+4}, x_{i+6}\rangle$ on C' with $\langle x_i, x_{i+3}, x_{i+6}$ from C to obtain a blue $2k$-cycle. Note that this requires $k \ge 3$.

b) A 2-coloring of $E(K_n)$ that contains a monochromatic C_{2k} for some $k \ge 3$ also contains a monochromatic C_{2k-1} or $2K_k$. Let C be a red $2k$-cycle, with vertices $v_0,\ldots,v_{2k-1}$ in order. We prove that if there is no monochromatic $(2k-1)$-cycle, then all edges of the form x_ix_{i+2j} are blue. This suffices, since it implies that the odd-indexed vertices and even-indexed vertices along C both induce blue copies of K_k.

If there is no monochromatic $(2k-1)$-cycle, then each x_ix_{i+2} is blue, so we may assume that $2 \le j \le k-2$. Replacing $x_{i+2j}x_{i+2j+1}$ and $\langle x_ix_{i+1}x_{i+2}\rangle$ on C with x_ix_{i+2j} and $x_{i+2}x_{i+2j+1}$ yields a cycle of length $2k-1$ in which every edge except the two new edges is red. If x_ix_{i+2j} is red, then $x_{i+2}x_{i+2j+1}$ must therefore be blue to avoid a red $(2k-1)$-cycle. (In the figure below, bold means blue and solid means red.) Similarly, replacing $x_{i+2j}x_{i_2j-1}$ and $\langle x_ix_{i-1}x_{i-2}\rangle$ on C with x_ix_{i+2j} and $x_{i-2}x_{i+2j-1}$ forces $x_{i-2}x_{i+2j-1}$ to be blue. Now replacing $\{x_{i-2}x_i, x_ix_{i+2}, x_{i+2j-1}x_{i+2j+1}\}$ with $\{x_{i-2}x_{i+2j-1}, x_{i+2}x_{i+2j+1}\}$ in the set of edges of the form $\{x_rx_{r+2}\}$ yields a blue $(2k-1)$-cycle avoiding x_i. Hence if there is no monochromatic $(2k-1)$-cycle, then x_ix_{i+2j} is blue.



c) If $m \geq 5$, then $R(C_m, C_m) \leq 2m-1$. (Note that $R(C_3, C_3) = R(C_4, C_4) = 6$; Exercise 8.3.25.) Consider a 2-coloring of $E(K_{2m-1})$. One color has at least half the edges; we may assume it is red. Erdős–Gallai [1959] (Theorem 8.4.35) proved that $e(G) > \frac{1}{2}(m-1)(n(G)-1)$ forces a cycle of length at least m in G. Since $\frac{1}{2}\binom{2m-1}{2} > \frac{1}{2}(m-1)[2m-2]$, we conclude that the coloring has a red cycle of length at least m. By parts (a) and (b), there is also a red m-cycle or two disjoint blue complete graphs of equal order exceeding $m/2$; we may assume the latter.

Let Q_1 and Q_2 be disjoint sets inducing blue complete graphs, each of order exceeding $m/2$, chosen to maximize $|Q_1 \cup Q_2|$. If two nonincident blue edges join Q_1 and Q_2, then we can take $P_{\lceil m/2 \rceil}$ from Q_1 and $P_{\lfloor m/2 \rfloor}$ from Q_2 to form a blue m-cycle with these edges. Hence all blue edges joining Q_1 and Q_2 are incident to a single vertex x, which we may assume is in Q_1. If m is even, then we can now take $m/2$ vertices from each Q_i, avoiding x, and form a red m-cycle using the edges between them.

We may therefore assume that m is odd. Let $T = \overline{Q_1 \cup Q_2}$ and $S = Q_1 - \{x\}$. If there is no blue m-cycle within Q_1 or Q_2, then $|Q_1 \cup Q_2| < 2m-1$, and $T \neq \varnothing$. For $v \in T$, if all edges from v to Q_1 or to Q_2 are blue, then we contradict the maximality of $|Q_1 \cup Q_2|$. Hence there are red edges from v to both Q_1 and Q_2.

Since all the edges joining Q_1 and Q_2 are red except those incident to x, we can complete a red m-cycle through v and alternating between Q_1 and Q_2 unless vx is the only red edge from v to Q_1 and all edges from x to Q_2 are blue (except possibly one incident to the only neighbor of v in Q_2 along a red edge). This is true for all $v \in T$, so every edge from x to T is red and all of $[S, T]$ is blue. Also let R be the subset of Q_2 whose edges to x are blue; R is all of Q_2 except possibly one vertex.

Since $\lceil m/2 \rceil \geq 3$, there are at least two blue edges from x to R. Hence there is a blue cycle spanning $Q_2 \cup \{x\}$, and we may assume that $|Q_2 \cup \{x\}| \leq m-1$. Therefore $|S \cup T| \geq (2m-1)-(m-1) = m$.

We now have a blue m-cycle in the graph induced by $S \cup T$ if $|S| \geq \lceil m/2 \rceil$, so we may assume that $|S| = (m-1)/2$ and $|T| \geq (m+1)/2 \geq 3$. If an edge within T is blue, then we complete a blue cycle by using it and otherwise alternating between S and T. Hence we may assume that all edges induced by T are red.

If there is a blue edge in $[T, R]$, then we can form a blue cycle by following it with any portion of R, then x, then any portion of S. The length is any value from 4 to at least $|Q_1 \cup Q_2|$, which exceeds m. Hence we may assume that all of $[T, R]$ is red. Now we can form a red m-cycle by using a path alternating between S and R using $(m-3)/2$ vertices of S, $(m-1)/2$ vertices of R, and two vertices of T.

8.3.36. *The Ramsey multiplicity of K_3 is 2*, where the *Ramsey multiplicity* of G is the minimum number of monochromatic copies of G in a 2-coloring of $E(K_{R(G,G)})$. To color $E(K_6)$ with only two monochromatic triangles, let the red graph be $K_{3,3}$, which is triangle-free. The complementary graph is $2K_3$, with two triangles.

Now we show that every coloring has at least two monochromatic triangles. Since $R(3,3) = 6$, there is at least one monochromatic triangle T, say in red. If we delete one vertex of T, then there remains a monochromatic triangle on the remaining five vertices unless the color classes on that subgraph are complementary 5-cycles. Let C be the red 5-cycle, and let z be the deleted vertex. To form T, we have edges from z to consecutive vertices on C, which we call x and y. Let u, x, y, v be the consecutive vertices on C including the edge xy. A red edge from z to u or v completes another red triangle, but if uz and vz are both blue they complete a blue triangle with uv.

8.3.37. *Each point in a triangular region has a unique expression as a convex combination of the vertices of the triangle.* We observe first that each point on a segment has a unique expression as a convex combination of the endpoints. Now, given a point x inside the triangle with corners u, v, w, let y be the point at which the ray from u through x reaches the opposite side. Now $y = \lambda v + (1-\lambda)w)$, for a unique λ, and $x = \mu y + (1-\mu)u$, for a unique μ. Hence $x = (1-\mu)u + (\lambda\mu)v + \mu(1-\lambda)w$. The coefficients are uniquely determined in terms of λ and μ, and these constants also are uniquely determined by x and the corners.

8.3.38. *Sperner's Lemma in higher dimensions. In a proper labeling of a simplicial subdivision of a k-dimensional simplex, there is a cell receiving all $k+1$ labels*, where "proper labeling" is a labeling such that label i does not appear at any vertex on the ith outer face.

We prove the stronger result, by induction on k, that there are an odd number of completely labeled cells. When $k = 1$, we have a 0, 1-labeling of a path segment with 0 and 1 on the ends, and there must be an odd number of switches between 0 and 1 along the path.

For $k > 1$, define a graph G with a vertex for each cell plus one vertex v for the outside region. Two vertices of G are adjacent if the corresponding regions share a $(k-1)$-dimensional face with corners having labels $0, \ldots, k-1$. If the vertex for a cell is nonisolated, then the cell has all these k labels among its $k+1$ corners. If it repeats one of the labels, then it has two incident edges in G. Otherwise, it is a completely labeled cell and has degree 1.

Hence the only cells with odd degree are the completely labeled cells. To prove that there are an odd number of them, it suffices to prove that the vertex v also has odd degree. A cell having a $(k-1)$ dimensional face on the ith outside face cannot have label i on it. Therefore, having an edge to v happens only through the $(k+1)$th face, where label $k+1$ is forbidden.

This face is a simplicial subdivision of a $(k-1)$-dimensional simplex. The labeling is proper on this face, as it inherits the needed properties from the full labeling (think of the edges of a triangle in the 2-dimensional case). By the induction hypothesis, this lower-dimensional labeling has an odd number of completely labeled cells. Hence the full-dimensional labeling has an odd number of cells with edges to v.

8.3.39. *The badwidths of P_n, K_n, and C_n are 1, $n-1$, and 2, respectively.* A nontrivial graph has bandwidth 1 if and only if its vertices can be ordered so that no nonconsecutive vertices are adjacent, which means that its components are paths. In ordering, the vertices of K_n, the first and last vertices are adjacent. Since C_n is not a path, its bandwidth is at least 2. To achieve this, number the vertices around the cycle $\ldots, 5, 3, 1, 2, 4, \ldots$ in order, reaching to $n-1$ in one direction and to n in the other direction.

8.3.40. *The bandwidth of $K_{n_1,\ldots,n_k}$ is $n-1-\lfloor n'/2\rfloor$, where $n=\sum_{i=1}^k n_i$ and $n'=\max_i n_i$.* Consider an optimal numbering. If the vertices given labels 1 and n come from different partite sets, then the bandwidth is $n-1$. If they come from the same partite set, with c vertices of this partite set at the beginning and c' at the end of the labeling, then the bandwidth is at least $\max n-c, n-c'$. To minimize this lower bound, we split the largest partite set between the front and back. The lower bound becomes $n-1-\lfloor n'/2\rfloor$. Also, splitting the largest partite set in this way achieves equality for any ordering of the remaining vertices in the remaining middle positions.

8.3.41. *The bandwidth of a tree with k leaves is at most $\lceil k/2\rceil$.* Let $m-\lceil k/2\rceil$. We use the fact that every tree with k leaves is the union of m pairwise intersecting paths (Exercise 2.1.40). We repeat the proof: Let T be a tree with k leaves. By pairing leaves arbitrarily, we form a set of m paths that together cover the leaves. Among all such sets of paths, choose one with maximum total length; we claim it has the desired properties. If some pair of paths is disjoint, say an x,y-path P and a u,v-path Q, consider the path R in T from $V(P)$ to $V(Q)$. Replace P and Q with the x,u-path and the y,v-path in T. The new paths still cover the leaves, and the total length has increased by twice the length of R. If some edge e of T is omitted by the longest covering set of paths, then consider the two components of $T-e$. Each contains a leaf of T, so each contains at least one path in the set. Again making the switch increases the total length.

To prove the upper bound, we provide an injective integer embedding in which the difference along every edge is at most m; the set of labels need not be consecutive. Let $P_0,\ldots,P_{m-1}$ be a set of pairwise-intersecting paths

with union T, and let $T_j=\bigcup_{i=0}^j P_i$. Because the paths are pairwise intersecting, each T_j is connected. Because T_{j-1} is connected and T_j contains no cycle, a traversal of P_j cannot leave T_{j-1} and then return to it.

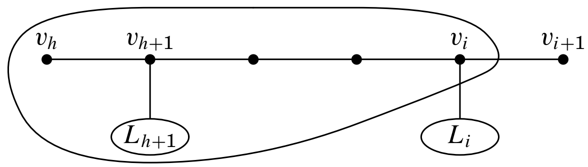
First assign successive multiples of m to the vertices along P_0. For $j>0$, suppose that the vertices of T_{j-1} have received labels congruent to $1,\ldots,j-1$ modulo m so that edges have dilation at most m. We use labels congruent to j on vertices of $V(P_j)-V(T_{j-1})$. Let u,v be the vertices of $V(P_j)\cap V(T_{j-1})$ that are closest to the two ends of P_j (these may be equal). By symmetry, we may assume that $f(u)\le f(v)$. Let a be the largest integer less than $f(u)$ congruent to $j \pmod m$, and let b be the smallest integer greater than $f(v)$ congruent to $j \pmod m$. From the neighbor of u [or v] out to the corresponding leaf of P_j, assign the label $a-(i-1)m$ [or $b+(i-1)m$] to the ith vertex of $V(P_j)-V(T_{j-1})$ encountered. (If k is odd, then for one value of j, one of these subpaths is empty). The new labels are in the new congruence class, and the newly-included edges have difference at most m.

8.3.42. *If G is a caterpillar with $\left\lceil\frac{n(H)-1}{\operatorname{diam} H}\right\rceil\le m$ for all $H\subseteq G$, then $B(G)\le m$.* (Note that the least such m is a lower bound, so equality with hold.)

Let P be the spine of G, having vertices $\langle v_0,\ldots,v_p\rangle$ in order, where v_0 and v_p are leaves. Assign the number im to v_i for $0\le i\le p$. It suffices to show that this allows us to assign numbers to the remaining vertices so that all leaf neighbors of v_i receive numbers between $(i-1)m$ and $(i+1)m$. We can then compress the numbering to eliminate gaps without increasing any edge difference.

Let $L_i=N(v_i)-V(P)$, and let $l_i=|L_i|$. For $1\le i\le k-1$, iteratively label L_i as follows. Use $\min\{l_i,c\}$ labels between $(i-1)m$ and im, where c is the number of labels between $(i-1)m$ and im not already assigned to L_{i-1}. If $l_i>c$, give the remaining vertices l_i-c labels starting with $im+1$.

We show that this works by proving that $l_i-c\le m-1$ at step i. For $j<i$, the algorithm has assigned a label above jm to some vertex of L_j only if it has assigned all labels between $(j-1)m$ and jm. At step i, let h be the least index such that all labels between hm and $(i-1)m$ have been assigned. We have $h\le i-1$; equality is possible. Since the interval between $(h-1)m$ and hm is not "full", no label above hm is assigned to L_h. Let H be the subgraph of G induced by $\{v_{h+1},\ldots,v_i\}$ and their neighbors; note that $\operatorname{diam} H=i+1-h$. Between hm and im, $V(H)$ has received $n(H)-1-l_i$ labels for the vertices circled in the figure below. Hence $c=(i-h)m+1-[n(H)-1-l_i]$. By the local density computation, $n(H)-1\le(i+1-h)m$. Thus $l_i-c=n(H)-1-(i-h)m-1\le m-1$, as desired.



8.3.43. *Bandwidth of grids.*

a) Local density bound for $P_m \square P_n$. We consider only the bound that comes from the subgraph $P_n \square P_n$ and omit some details of that.

It suffices to consider induced subgraphs; adding edges cannot increase diameter. For a subgraph H with diameter $2k$, let u and v be vertices of H at distance $2k$ (the odd case is similar). Let w be a vertex halfway along a shortest u,v-path. Any vertex at distance more than k from w will have distance more than $2k$ from u or v, by the nature of the grid (details omitted). Hence we get the best lower bound by including all the vertices in $P_n \square P_n$ that are within distance k of w. For $k \le (n-1)/2$, the number of vertices in this subgraph is $\sum_{i=1}^{k+1}(2i-1) + \sum_{i=1}^{k}(2i-1)$. This equals $2k^2+2k+1$. Subtracting 1 and dividing by $2k$ yields $B(P_n \square P_n) \ge k+1$.

When k is larger than $(n-1)/2$, the full set we have described does not fit inside the n-by-n grid. We must subtract $4\sum_{i=1}^{k-(n-1)/2}(2i-1)$ vertices after putting w in the center of the grid. The largest subgraph now has $2k(2n-1-k)-(n-1)^2+1$ vertices. After subtracting 1 and dividing by the diameter $2k$, we have a lower bound of $2n-1-[k+(n-1)^2/2k]$. This is maximized by setting k to be about $(n-1)/\sqrt{2}$, where the resulting lower bound is $(n-1)[2-\sqrt{2}]$. This is about $.59n$, which is still short of the desired lower bound of n.

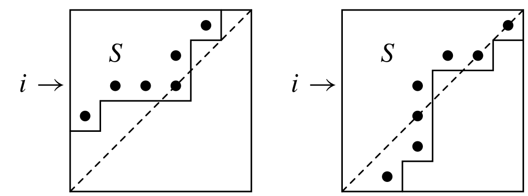
b) Sliding the elements of a vertex subset of $P_n \square P_n$ to the extreme left within their rows does not increase the size of the boundary. Choose $S \subseteq V(P_n \square P_n)$ with a_i vertices in the ith row, for each i. Let T consist of the first a_i in the ith row, for each i. We show that $|\partial T| \le |\partial S|$.

If $a_j = n$, then each set has the same number of boundary elements in row j. Furthermore, no vertex outside row j becomes a boundary element due to an edge to row j. Therefore, we may assume that $a_i < n$ for all i, but by convention we define $a_0 = a_{n+1} = n$. For $1 \le i \le n$, the number of boundary elements in row i of T are the rightmost elements, exactly 1 of them if $a_i = \min\{a_{i-1}, a_i, a_{i+1}\}$, and otherwise the maximum of $a_i - a_{i-1}$ and $a_i - a_{i+1}$. Since $a_i < n$, also S in row i has at least one boundary element due to row i, at least $a_i - a_{i-1}$ boundary elements due to row $i-1$, and at least $a_i - a_{i+1}$ boundary elements due to row $i+1$, since these are lower bounds on the sizes of the set differences. Hence S has at least as many boundary elements in each row as T has.

c) $|\partial S|$ is minimized over k-sets in $V(P_n \square P_n)$ by some S such that $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$, and hence Harper's lower bound for $B(P_n \square P_n)$ is n. By part (b), sliding vertices to the left within their rows does not increase the boundary, and it produces a set whose column populations are in nonincreasing order. By symmetry, sliding vertices to the top within their columns also does not increase the boundary. This leaves the column populations unchanged and produces a set whose row populations also are in nondecreasing order.

To show that the boundary bound is at least n, it suffices to prove that in $P_n \square P_n$ there is some k such that every k-set of vertices has boundary at least n. We choose k such that $\binom{n}{2} < k < \binom{n+1}{2}$. View $V(P_n \square P_n)$ as positions in a matrix. We may restrict our attention to a set S in the upper left as discussed above. Let a_i be the number of vertices of S in the ith row.

If $a_1 = n$ and $a_n = 0$, then S has a boundary element in each column. If $a_1 < n$ and $a_n > 0$, then S has a boundary element in each row. We illustrate the other cases for $n = 6$; the diagonal corresponds to $a_i = n+1-i$.



Case 1: $a_1 < n$ *and* $a_n = 0$. If $a_i \le n-i$ for all i, then $|S| \le \sum(n-i) = \binom{n}{2} < k$. Hence $a_i > n-i$ for some i, and ∂S has distinct elements in rows $1,\ldots,i$ and columns $1,\ldots,a_i-1$. We obtain $|\partial S| \ge n$.

Case 2: $a_1 = n$ *and* $a_n > 0$. If $a_i \ge n+1-i$ for all i, then $|S| \ge \sum(n+1-i) = \binom{n+1}{2} > k$. Hence $a_i \le n-i$ for some i. Now ∂S has distinct elements in rows $i,\ldots,n$ and columns $a_i+1,\ldots,n$, and $|\partial S| \ge n+1$.

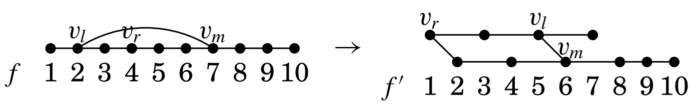
d) $B(P_m \square P_n) = \min\{m,n\}$. Numbering the vertices in successive ranks along the short direction yields maximum difference $\min\{m,n\}$. Since $P_n \square P_n$ is a subgraph of $P_m \square P_n$, it suffices to prove that $B(P_n \square P_n) \ge n$, which is the result of part (c).

Comment: A much shorter proof. For $B(P_n \square P_n)$, consider an optimal numbering f, minimizing the maximum dilation of edges. It suffices to show that some initial segment of f has boundary of size at least n.

Let S be the maximal initial segment of f that does not contain a full row or column. Adding the next element of f completes a row or column, say row r. We claim that S has a boundary element in each column. It has one boundary element in row r. In every column not containing that element it has an element in row r but does not contain all of that column, so it has a boundary element in the column.

8.3.44. *Change in bandwidth under edge addition.* Let G be a simple graph with order n and bandwidth b.

a) If $e \in \overline{G}$, then $B(G+e) \le 2b$. Let f be an optimal numbering of G, let $v_i = f^{-1}(i)$, and let $v_l v_m$ be the added edge e. We define a new numbering f' to prove that $B(G+e) \le 2b$. Let $r = \lfloor (l+m)/2 \rfloor$, and set $f'(v_r) = 1$ and $f'(v_{r+1}) = 2$. Number outward from v_r, setting $f'(v_i) = f'(v_{i+1}) + 2$ if $i < r$ and $f'(v_i) = f'(v_{i-1}) + 2$ if $i > r$ until the vertices on one side of v_r are exhausted. The remaining vertices ($v_{2r}, \dots, v_n$ if $2r \le n$, or $v_{2r-n}, \dots, v_1$ if $2r \ge n+1$) receive the remaining high labels in order. Edges between vertices on the same side of v_r may be stretched by a factor of 2; no other edges stretch as much. Since we began midway between v_l and v_m, we also have $\left| f'(v_l) - f'(v_m) \right| = 1$.



b) If $n \ge 6b$, then $B(G+e)$ can be as large as $2b$. Let G be a maximal n-vertex graph with bandwidth b; that is, G is the graph P_n^b obtained by adding edges to P_n joining any two vertices whose distance in P_n is at most b. Now let e be the edge joining the two vertices that are b positions from the ends of the ordering. Let S be the set of vertices consisting of the first $2b+1$ vertices and last $2b+1$ vertices in the ordering. Note that the subgraph of G induced by S has diameter 3.

Let f' be an optimal numbering of $G+e$. If the vertices labeled 1 and n by f' are both in S, then the path of length at most 3 joining these vertices has some edge whose endpoints differ under f' by at least $(n-1)/3$, which is at least $2b$.

Otherwise, the vertex x assigned number 1 or number n by f' belongs to $G-S$. Now all neighbors of x have numbers lying to one side of $f'(x)$. Since each vertex of $G-S$ has $2b$ neighbors in G, this again forces an edge difference of at least $2b$ under f'.

8.4. MORE EXTREMAL PROBLEMS

8.4.1. *The intersection number of an n-vertex graph G is at most $n^2/4$, using sets of size at most 3.* By Proposition 8.4.2, we need only show $E(G)$ can be covered with $\lfloor n^2/4 \rfloor$ complete subgraphs. The graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ shows that this is best possible. We show by induction that $\lfloor n^2/4 \rfloor$ complete subgraphs suffice to cover $E(G)$; this holds by inspection for $n = 1, 2$ (or $n = 3$). We

use only edges and triangles; this is equivalent to providing an intersection representation by finite sets such that each element appears in at most three sets.

For larger n, select $xy \in E(G)$. The difference between $\lfloor n^2/4 \rfloor$ and $\lfloor (n-2)^2/4 \rfloor$ is $n-1$. By the induction hypothesis, it suffices to show the edges incident to $\{x, y\}$ can be covered by $n-1$ edges and triangles. If $v \notin \{x, y\}$ is incident to exactly one of $\{x, y\}$, we use that edge; if both, we use the triangle $\{v, x, y\}$. If xy is in no triangle, then we add xy itself as a complete subgraph. (Note: the proof can also be phrased in terms of building a discrete intersection representation, without using the equivalence to clique covering.)

8.4.2. *Equivalent conditions for intersection number $\theta'(G)$ when G has no isolated vertices:*
A) $\theta'(G) = \alpha(G)$,
B) $\theta'(G \vee G) = (\theta'(G))^2$,
C) $\theta'(G) = \theta(G)$,
D) Every clique in any minimum clique cover of $E(G)$ contains a simplicial vertex of G.

Let Θ, Θ' denote minimum clique covers of $V(G)$ and $E(G)$, respectively. When discussing $H = G \vee G$, let G_1, G_2 denote the two copies of G in H, and let $\Theta(G_i), \Theta'(G_i)$ denote the copy of Θ, Θ' in G_i. Since G has no isolated vertices, Θ' covers both $E(G)$ and $V(G)$.

A $\Rightarrow$ B. The inequality $\theta'(H) \le (\theta'(G))^2$ always holds, because the join of complete subgraphs in G_1 and G_2 is a complete subgraph in H. Since Θ' covers both $E(G)$ and $V(G)$, $\{Q \vee Q' \colon Q \in \Theta'(G_1), Q' \in \Theta'(G_2)\}$ covers $E(H)$. For equality when $\theta'(G) = \alpha(G)$, consider the two copies in H of a maximum stable set in G. This set induces a complete bipartite subgraph with $\alpha(G)$ vertices in each partite set, so $(\alpha(G))^2$ complete subgraphs of H are needed to cover these edges, and $(\alpha(G))^2 = (\theta'(G))^2$.

B$\Rightarrow$C. Since G has no isolated vertices, $\theta'(G) \ge \theta(G)$. We form a cover of $E(H)$ using fewer than $(\theta'(G))^2$ complete subgraphs if $\theta'(G) > \theta(G)$. To cover $E(G_1) \cup E(G_2)$, for each $Q \in \Theta'$ we take its occurrences in $\Theta'(G_1)$ and $\Theta'(G_2)$ and form their join; this contributes $\theta'(G)$ complete subgraphs. To cover the edges joining $V(G_1)$ and $V(G_2)$, we take $\{Q \vee Q' \colon Q \in \Theta(G_1), Q' \in \Theta(G_2)\}$; this contributes $(\theta(G))^2$ complete subgraphs. Now $\theta'(H) \le \theta'(G) + (\theta(G))^2 < (\theta'(G))^2$, which contradicts the hypothesis.

C$\Rightarrow$D. Let $r = \theta'(G) = \theta(G)$, and let Θ be any set of r complete subgraphs covering $E(G)$ and hence $V(G)$. Every element of Θ has a vertex appearing in no other complete subgraph, else we could omit it and obtain a smaller covering of $V(G)$. A vertex appearing in only one member of a clique cover of $E(G)$ must be simplicial.

D⇒A. Since $\theta'(G) \ge \alpha(G)$ for any graph without isolated vertices, it suffices to obtain a stable set consisting of one vertex from each member of in a minimum clique cover Θ of $E(G)$. We may assume that every member of Θ is a maximal clique. Since a simplicial vertex belongs to only one maximal clique, this implies that each clique of Θ contains a vertex belonging only to that clique. The $\theta'(G)$ vertices from distinct cliques thus selected must be independent, because no edge among them is covered by Θ.

8.4.3. *If $b(G)$ is the minimum number of bipartite graphs needed to partition the edges of G, and $a(G)$ is the minimum number of classes needed to partition $E(G)$ such that every cycle of G contains a non-zero even number of edges in some class, then $b(G) = a(G) = \lceil \lg \chi(G) \rceil$.* We prove $\lg \chi(G) \le b(G) \le a(G) \le \lceil \lg \chi(G) \rceil$. Since $\lg \chi(G)$ and $\lceil \lg \chi(G) \rceil$ differ by less than 1, the integers in this string of inequalities must be the same.

Let $E_1 \cup \cdots \cup E_{b(G)}$ be a minimum partition of $E(G)$ into bipartite subgraphs; we may assume these are spanning subgraphs. We can define a proper $2^{b(G)}$-coloring f by giving each $v \in V(G)$ a binary $b(G)$-sequence $f(v)$ in which $f_i(v)$ indicates which partite set in E_i contains v. Since each edge belongs to some E_i, the endpoints of each edge receive different labels. This proves $\chi(G) \le 2^{b(G)}$, i.e. $\lceil \lg \chi(G) \rceil \le b(G)$.

Let $E_1 \cup \cdots \cup E_{a(G)}$ be a minimum partition having the cycle intersection property defined above. If E_i contains an odd cycle, then this cycle in G does not contain a non-zero even number of edges of any color. Hence each E_i is bipartite, and $b(G) \le a(G)$.

Let f be an optimal vertex coloring of G. Encode the colors in f of G by distinct binary sequences of length $k = \lceil \lg \chi(G) \rceil$. Partition $E(G)$ into $E_1 \cup \cdots \cup E_k$ by using the coordinates of this encoding: put $uv \in E_i$ if i is the first coordinate for which $f_i(u) \ne f_i(v)$. Given any cycle C in G, let j be the lowest-indexed color used on $E(C)$. While traversing C, coordinate j changes a non-zero even number of times, but since every other color on $E(C)$ is higher, when traversing edges of C coordinate j can change only along edges that actually belong to E_j. Hence this partition has the cycle intersection property, and $a(G) \le k$.

8.4.4. (•) Determine all the n-vertex graphs that have product dimension $n-1$. (Lovász–Nešetřil–Pultr [1980])

8.4.5. pdim $G \le 2$ *if and only if G is the complement of the line graph of a bipartite graph.* Given a 2-dimensional encoding of G, define a bipartite graph H with vertices $X \cup Y$ and edges $x_i y_j$ such that (i, j) is one of the vectors in the encoding. Then the vertex for (i, j) is adjacent to the vertex for (k, l) in G if and only if $i \ne k$ and $j \ne l$, which happens if and only if $x_i y_j$ and $x_k y_l$ are not incident, which happens if and only if the vertices for $x_i y_j$ and $x_k y_l$ are adjacent in the complement of the line graph of H. Conversely, if G is the complement of the line graph of a bipartite graph H whose bipartition is $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$, then the vertices of G correspond to the edges of H, and we obtain a two-dimensional encoding by assigning (i, j) to the vertex of G corresponding to $x_i y_j \in E(H)$. This is an encoding because the vertices of G corresponding to $x_i y_j$ and $x_k y_l$ are adjacent if and only if those edges of H are non-incident, which happens if and only if $i \ne j$ and $k \ne l$.

8.4.6. *For $r \ge 2$,* $\text{pdim}\,(K_r + mK_1) = \begin{cases} r & \text{if } 1 \le m \le (r-1)! \\ r+1 & \text{if } m > (r-1)! \end{cases}$.

Allowing $r+1$ coordinates, we can represent this graph using $(i, \ldots, i)$ for the ith vertex of the clique and $(1, \ldots, r, j)$ for the jth vertex of the stable set, with the extra coordinate included to ensure that the vertices of the stable set get distinct encodings. Since $K_r + K_1$ is an induced subgraph, the answer thus is always r or $r+1$.

In every coordinate, the vertices of the clique must have distinct values. By permuting the labels used within a coordinate, we may assume that the code for the ith vertex of the clique is $(i, \ldots, i)$. If pdim $G = r$, then each vertex of the stable set must be encoded by a permutation of $[r]$ in order to establish all non-adjacencies to clique vertices. These permutations must be distinct, and each pair of them must agree in some coordinate to avoid edges in the stable set. Hence no pair of the permutations can be cyclic permutations of each other.

This partitions the $r!$ permutations into $(r-1)!$ classes of size r, from each of which we can take at most 1. Therefore, $\text{pdim}\,(K_r + mK_1) = r$ requires $m \le (r-1)!$. When $m \le (r-1)!$, r coordinates do suffice; give each vertex of the stable set value 1 in coordinate r to prevent edges, and use the $(r-1)!$ distinct permutations of $[r-1]$ in coordinates $1, \ldots, r-1$.

8.4.7. *The product dimension of the three-dimensional cube Q_3 is 2.* Since Q_3 is not a complete graph, we need at least two coordinates, and we can encode it with two coordinates by using the binary triples. Each triple x is a vertex that is adjacent to every vertex of opposite parity except the complement of x. We use coordinate 1 to destroy edges to vertices of the same parity and coordinate 2 to destroy edges between complements. In coordinate 1, assign 0 to each sequence of even weight and 1 to each sequence of odd weight. In coordinate 2, assign 0 to 000 and 111, and assign i to each sequence in which the ith coordinate has a value that appears only once in the sequence. The resulting vectors are $\{(i, j) \colon 0 \le i \le 1,\ 0 \le j \le 3\}$.

8.4.8. *The product dimension of the Petersen graph is 3 or 4.* The Petersen graph is $\overline{L(K_5)}$. It is not the complement of the line graph of a bipartite graph, so by Exercise 8.4.5 its product dimension is at least 3.

The encoding in the table below shows that the product dimension is at most 4. The vertices are named by the 2-element subsets of [5], adjacent when they are disjoint. Hence the codes of two vertices should agree in some coordinate if and only if their names have a common element. In the ith coordinate, value 0 is assigned to the four doubletons that contain element i. The remaining three doubletons that contain element 5 have value 1 in the ith coordinate. The three doubletons not containing 5 or i have value 2.

If two doubletons share an element, then their codes agree in coordinate i (with value 0) if their shared element i is not 5 (and in another coordinate with value 2). If the shared element is 5, then the union omits two elements from [4], and the codes agree in those coordinates (with value 1). If two doubletons are disjoint, then they agree in no coordinate, because in the ith coordinate, value 0 goes only to doubletons sharing element i, value 1 goes only to doubletons sharing element 5, and value 2 goes only to doubletons chosen from the set of three elements outside $\{i, 5\}$, which pairwise intersect.

12: 0022	23: 2002	34:2200
13: 0202	24: 2020	34:1101
14: 0220	25: 1011	34:1110
15: 0111		

8.4.9. *Maximum product of* pdim G *and* pdim $\overline{G}$ *when* G *is an* n*-vertex graph.* Let $f(n)$ be the desired value. From $\max\{\text{pdim}\, G\} = n-1 = \max\{\text{pdim}\, \overline{G}\}$, we have $f(n) \le (n-1)^2$. Form G by identifying one leaf of $K_{1,\lceil n/2\rceil}$ with one vertex of $K_{\lfloor n/2\rfloor}$. Since G contains $K_{\lfloor n/2\rfloor} + K_1$ and $\overline{G}$ contains $K_{\lceil n/2\rceil} + K_1$ as induced subgraphs, we have pdim $G \ge \lfloor n/2\rfloor$ and pdim $\overline{G} \ge \lceil n/2\rceil$, yielding $f(n) \ge (n^2-1)/4$.

8.4.10. *If* $n \ge 4$, *then* pdim $P_n = \lceil \lg(n-1)\rceil$. *If* $n \ge 3$, *then* pdim $C_{2n} = 1 + \lceil \lg(n-1)\rceil$ *and* $1 + \lceil \lg n\rceil \le$ pdim $C_{2n+1} \le 2 + \lceil \lg n\rceil$. Given a path induced by $x_1, \ldots, x_m$ in G, set $u_i = x_i$ and $v_i = x_{i+1}$ for $1 \le i \le m-1$. This yields $u_i \leftrightarrow v_i$ for all i and $u_i \not\leftrightarrow v_j$ for $i < j$. By the LNP lower bound, this yields pdim $G \ge \lceil \lg m - 1\rceil$. For paths, we obtain pdim $P_n \ge \lceil \lg(n-1)\rceil$. Since C_m contains P_{m-1} as an induced subgraph, we obtain pdim $C_m \ge \lceil \lg(m-2)\rceil$. Thus pdim $C_{2n} \ge \lceil \lg(2n-2)\rceil = 1 + \lceil \lg(n-1)\rceil$ and pdim $C_{2n+1} \ge \lceil \lg(2n-1)\rceil = \lceil \lg(2n)\rceil = 1 + \lceil \lg n\rceil$.

We complete the proof for paths by embedding P_{2^k+1} in the weak product of k triangles, beginning with $k = 2$. Let $x_k(i)$ be the encoding of the ith vertex on the path, for $1 \le i \le 2^k$. When $k = 2$, we set $x_2(0) = 00$, $x_2(1) = 11$, $x_2(2) = 02$, $x_2(3) = 10$, $x_2(4) = 01$. For $k > 2$, we obtain $x_k(i)$ from the previous codes, by appending a suitable value in the new coordinate. Here i runs from 0 to 2^k.

index i	parity of i	in first $k-1$ coords	in kth coord
$i < 2^{k-1}$	even	$x_{k-1}(i)$	0
$i < 2^{k-1}$	odd	$x_{k-1}(i)$	1
2^{k-1}	even	$x_{k-1}(2^{k-1})$	2
$i > 2^{k-1}$	odd	$x_{k-1}(2^k - i)$	0
$i > 2^{k-1}$	even	$x_{k-1}(2^k - i)$	1

Codes for consecutive vertices come from codes for consecutive vertices at the previous stage, with distinct values in the new coordinate, so the desired edges exist. The only codes that are distinct throughout the first $k-1$ coordinates are those coming from consecutive vertices at the previous stage. If the distance from the old vertex 2^{k-1} is even, we append a 0 in the first half of the path, a 1 in the second half. If the distance is odd, we append a 1 in the first half, a 0 in the second half. Two vertices whose codes disagree in the first $k-1$ coordinates but are in opposite halves of the path arise from vertices at the previous stages whose distances from the last vertex have opposite parity. Thus their codes agree in the kth coordinate, and the undesired edge is destroyed.

To obtain the encodings for cycles, we need some observations about the above encoding for paths. Since each code is obtained by extending the code of a previous vertex whose index has the same parity, the first coordinate of a code is 1 if and only if the index is odd. For the same reason plus attention to when 2's are introduced, a code contains a 2 at some coordinate after the first if and only if the index of its vertex is even and is not the first or last vertex.

To encode C_{2j+2} in k dimensions, where $1 \le j < 2^{k-1}$, we use $x_k(0), \ldots, x_k(j-1)$ and $x_k(2^k - j + 1), \ldots x_k(2^k)$; these codes induce a disjoint union of two paths. Between $x_k(j-1)$ and $x_k(2^k - j + 1)$ we put $x_{k-1}(j)$ with a 2 appended. Between $x_k(0)$ and $x_k(2^k)$ we put $122\cdots 2$. By the observations above, this encodes C_{2j+2} in k dimensions.

To encode C_{2j+3} in $k+1$ dimensions, where $1 \le j < 2^{k-1}$, we use $u_k(i)$ for $0 \le i \le 2j$, alternately appending 0 and 2, and then complete the cycle using $u_k(2j+1)$ with 1 appended, followed by the code of all 2's.

(Comment: LNP improved the lower bound for odd cycles to agree with the upper bound when the length is one more than an even power of 2. On the other hand, Křivka [1978] showed that the lower bound is the correct answer for asymptotically at least 1/3 of all odd cycles.

8.4.11. *If* $k > 1$, *then* C_{2k+1} *has no isometric embedding in a cartesian product of complete graphs.* In such a cartesian product, the vertices correspond to integer vectors, and the distance between them is the number of coordinates where the vectors differ. Suppose that G is isometrically embeddable. If P is a shortest x,y-path in G, then the distances from x and y change by one with each step along the path, and each coordinate in the encoding therefore changes at most once along the path.

In any isometric embedding of C_{2k+1}, therefore, the edges along any path of length k change distinct coordinates. This implies that along any path of length $k+1$, the last edge changes the same coordinate as the first; repeating any other coordinate violates the previous statement, and changing another new coordinate creates a difference in $k+1$ coordinates for the encoding of two vertices at distance k (along the other part of the cycle). Since $k+1$ is relatively prime to $2k+1$, this implies that all edges change the same coordinate. Hence the only clique product in which C_{2k+1} embeds isometrically has one factor, which implies $k=1$.

8.4.12. $\text{qdim}(C_5)=4$. By Winkler's result (Theorem 8.4.18), it suffices to prove that $\text{qdim}(C_5)>3$. Let f be a 3-dimensional encoding, if one exists. Since C_5 is not bipartite, some code $f(v)$ has a star in some position. No code has more than one star, since each vertex has nonneighbors. By symmetry of 0s and 1s in a given coordinate of the encoding, we may assume that $f(v)=*11$. Let u,v,w,x,y be the vertices in cyclic order. Since the nonneighbors of v are adjacent, their codes must be 000 and 100; by symmetry, let $f(x)=000$ and $f(y)=100$. Since w is farther from y than from x, we have $f_1(w)=0$; similarly, $f_1(u)=1$. To obtain the correct distances from x and y, each of $f(u)$ and $f(w)$ has exactly one 1 after the first position. Since $d(u,v)=d(v,w)=1$, the remaining entry in $f(u)$ and $f(w)$ is 0. Now the distance between $f(u)$ and $f(w)$ is 1 or 3, which contradicts $d(u,w)=2$.

8.4.13. *The squashed-cube dimension of $K_{3,3}$ is 5.* By Theorem 8.4.18, it suffices to assume a 4-dimensional encoding f and obtain a contradiction. Let the partite sets be $A=\{a,b,c\}$ and $B=\{x,y,z\}$. If f encodes some vertex without stars, we may assume $f(a)=0000$. Now each code for B has exactly one 1, and distance two among them forces these 1's to be in distinct coordinates with matching 0's. Hence $\{f(x),f(y),f(z)\}=\{100?,010?,001?\}$, where $?\in\{0,*\}$. Since $d(a,b)=2$, $f(b)$ has exactly two 1's. Placing them in the first three coordinate violates $d(b,w)=1$ for some $w\in B$. Hence $f_4(b)=1$; by symmetry $f(b)=1??1$, where $?\in\{0,*\}$. Now $d(b,y)=d(b,z)=1$ forces $f(b)=1**1$. The same argument shows that $f(c)$ also has no zeros, which violates $d(b,c)=2$.

Hence we may assume that every vertex code has a star. None can have more than two stars, since all have eccentricity 2. Suppose $f(a)=00**$. Now distance two among $\{a,b,c\}$ requires $f(b),f(c)=1101,1110$. Since $B\subseteq N(a)$, each code for B has exactly one 1 in the first two coordinates. By

the pigeonhole principle, we may assume $f_1(x)=f_1(y)=1$ and $f_2(x)\neq 1\neq f_2(y)$. Now $d(x,y)=2$ requires $\{f_3(x),f_3(y)\}=\{f_4(x),f_4(y)\}=\{0,1\}$. To ensure distance 1 to b and c, we now must have $\{f(x),f(y)\}=\{1{*}00,1{*}11\}$. If $f_1(z)=1$, this argument would force $f(z)$ to end both in 00 and in 11. If $f_1(z)=*$, then $d(z,x)=d(z,y)=2$ forces the same result. If $f_1(z)=0$, then $d(z,b)=d(z,c)=1$ implies $f(z)=01**$, which violates $d(z,x)=2$.

Hence we may assume that every vertex code has exactly one star, with $f(a)=000*$. Now each of $f(b),f(c)$ has exactly two 1's in the first three coordinates. Hence they have a common 1, which we may put in the first coordinate. Now $d(b,c)=2$ forces their stars into the same coordinate; we conclude by symmetry that $f(b)=110*$ and $f(c)=101*$. Switching 0 and 1 in the first coordinate yields $f(a),f(b),f(c)=100*,010*,001*$ and restores symmetry. Now no code with one, two, or three 1's in the first three coordinates has distance 1 from each vertex of A. However, if all codes for B have no 1's in the first three coordinates, then we cannot establish distance 2 between any pair of vertices of B.

8.4.14. *Menger's Theorem for edge-disjoint paths in digraphs, from Edmonds' Branching Theorem.* Assume that G is k-edge-connected. Thus at least k edges must be deleted to make some vertex unreachable from another. In particular, at least k edges must be deleted to make some vertex unreachable from x. By Edmonds' Branching Theorem, there is a set of k pairwise edge-disjoint branchings rooted at x. The paths reaching y in these trees are pairwise edge-disjoint. Since x and y were chosen arbitrarily, we have the conclusion of Menger's Theorem: in a k-edge-connected digraph, we can always find k pairwise edge-disjoint x,y-paths.

8.4.15. *The telegraph problem (one-way messages to transmit from each person to every other) requires $2n-2$ message for n people, and this suffices.* Before some person receives all the information, there must be a tree of messages to that person, which requires $n-1$ calls. After that, the remaining $n-1$ people must each receive a message to complete their information.

A tree in and a tree out from the same person completes the transmissions in $2n-2$ messages.

8.4.16. (•) Let D be a digraph solving the telegraph problem in which each vertex receives information from each other vertex exactly once. Prove that in D at least $n-1$ vertices hear their own information. For each n, construct such a D in which only $n-1$ vertices hear their own information, but for each $x\neq y$ there is exactly one increasing x,y-path. (Seress [1987])

8.4.17. *The NOHO property.* Let G be a connected graph with $2n-4$ edges having a linear ordering that solves the gossip problem and satisfies NOHO (no increasing cycle).

a) If $n(G) > 8$ and at most two vertices have degree 2, then the graph obtained by deleting the first calls and last calls of vertices in G has 4 components, of which two are isolated vertices and two are caterpillars having the same size. As argued in Claim 3 of Theorem 8.4.23, the set F of first calls is a matching, as is the set L of last calls. Hence the graph M consisting of the remaining "middle" calls has $n-4$ edges and therefore at least four components.

Let $O(x)$ and $I(x)$ be the trees in the argument of Claim 2 of Theorem 8.4.23, growing the trees that are useful "out from" and "in to" a vertex x. The argument of Claim 2 shows that under the NOHO property, $d(x)-2$ calls are useless to x, none of which are incident to x.

A call in M can be useful to x only if the component containing it also contains a neighbor of x, because a path to x cannot continue after a last call on it, and a path from x cannot start before a first call on it. Since $e(G) < 2n$, there is a vertex of degree at most 3, so there are at most three nontrivial components in M. Since there are at least four components, at least one is an isolated vertex x. Hence there are at most two nontrivial components (containing the first and last neighbor of x), leaving at least two isolated vertices in M. Since the hypothesis specifies no additional vertices of degree 2 in G, there are exactly four components in M, of which two are isolated vertices. Furthermore, the two nontrivial components must be trees since $e(M) = n-4$.

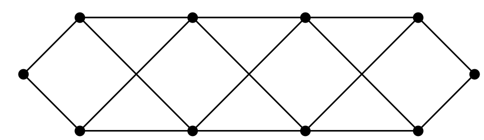
Let x and y be the isolated vertices, with neighbors x_f and y_f in F and neighbors x_l and y_l in L. Since each edge is useful to x, the vertices x_f and y_f are in different components of M. Hence one component is a tree T_1 of paths out of x_f, and the other is a tree T_2 of paths in to x_l. Similarly, one component consists of paths out of y_f and the other is in to y_l.

We claim that y_f cannot lie in T_1. If so, then x_l and y_l both lie in T_2. We claim that this forbids an increasing x,y_l-path. Such a path must start with xx_f, since it cannot continue after xx_l. After x_f, it can only reach T_2 via an edge of L, which it could follow after some increasing path from x_f in T_1. However, this edge of L does not reach y_l, since the last neighbor of y_l is y, and the path cannot continue after a last edge to reach y_l.

Therefore, $x_f, y_l \in V(T_1)$ and $x_l, y_f \in V(T_2)$. The requirement that every edge lies on a path out of x_f and a path into y_l implies that the path from x_f to y_l in T_1 is an increasing path with every edge of T_1 incident to it. In this case T_1 (and similarly T_2) is a caterpillar, and the edges off the spine occur between the incident edges on the spine in the linear ordering of calls. This implies that every two edges in T_1 lie on an increasing path together. Now an edge of F or L joining vertices within one of these components would violate NOHO. After deleting the first and last edges incident to

x and y, the matchings tell us that $n(T_1) = n(T_2)$, so the two nontrivial components are caterpillars of the same size.

b) For even n with $n \ge 4$, there are solutions with $2n-4$ calls that have the NOHO property. Below is a general construction: first perform the matching consisting of diagonal calls with positive slope, then the top path from left to right and the bottom path from right to left, and finally the matching consisting of diagonals with negative slope. There are many other constructions. ∎



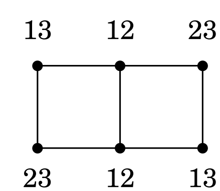
8.4.18. (•) A *NODUP scheme* (NO DUPlicate transmission) is a connected ordered graph that has exactly one increasing path from each vertex to every other.

a) (−) Prove that every NODUP scheme has the NOHO property.

b) Prove that there is no NODUP scheme when $n \in \{6, 10, 14, 18\}$. (Comment: Seress [1986] proved that these are the only even values of n for which NODUP schemes do not exist, constructing them for all other values. For $n = 4k$, West [1982b] constructed NODUP schemes with $9n/4 - 6$ calls, and Seress [1986] proved that these are optimal.)

8.4.19. *Broadcasting can be completed in time $1 + \lceil \lg n \rceil$ in a particular graph with fewer than $2n$ edges.* By having each vertex who knows the information at a given time call a new vertex who does not know it at the next time, broadcasting can be completed from a specified root vertex in $\lceil \lg n \rceil$ steps. Now add edges to make the root adjacent to all other vertices. To broadcast from any other vertex, call the root first, and then finish the job in $\lceil \lg n \rceil$ additional phases. The number of edges in the construction is $2n - 1 - \lceil \lg n \rceil$.

8.4.20. *The graph below is not 2-choosable.* Assign lists as shown. If we put 1 above 2 on the central vertices, then the vertices on the left cannot be properly colored. If we put 2 above 1 on the central vertices, then there is no proper choice for the vertices on the right.



8.4.21. *$K_{k,m}$ is k-choosable if and only if $m < k^k$.* Let the partite sets be $X = \{x_i\}$ of size k and $Y = \{y_j\}$ of size m. For $m \ge k^k$, it suffices to consider $m = k^k$ and a specific collection of k-lists. Let $\{c_{rs}\colon 1 \le r \le k,\ 1 \le s \le k\}$ be a collection of k^2 colors. Let $L(x_i) = \{c_{is}\colon 1 \le s \le k\}$ to x_i. To the vertices of Y, assign the k^k distinct lists obtained by choosing one color with each possible first coordinate. Every choice of colors on X consists of one color with each possible first coordinate. For each such choice, the chosen colors will be precisely the colors in the list for some vertex of Y. No legal color can be chosen for that vertex to complete the coloring.

For $m < k^k$, consider an arbitrary collection of k-lists assigned to the vertices. If some two vertices in X have a common element in their lists, choose that element for them, and choose arbitrarily from the lists for the other vertices of X. This uses at most $k-1$ colors for X, which leaves a color available in the list for each vertex of Y. On the other hand, if the color sets are disjoint, then they can be indexed so that $L(x_i) = \{c_{is}\colon 1 \le s \le k\}$. There are k^k possible choices of one of these colors from each set. Since $m < k^k$, there is at least one such choice that does not occur as a list for vertices of Y. When these colors are chosen for X, for each $y \in Y$ there is a color in $L(y)$ not used on X, and the coloring can be completed.

8.4.22. *Bounds on choosability and edge-choosability.*

$\chi_l(G) \le 1 + \max_{H\subseteq G} \delta(H)$. Order the vertices $v_1, \dots, v_n$ such that v_i is a vertex of minimum degree in the subgraph G_i induced by $v_1, \dots, v_i$ (by selecting the vertices in decreasing order). Consider an arbitrary collection of lists of size $1 + \max_{H\subseteq G} \delta(G)$. Make choices from these lists in the order $v_1, \dots, v_n$. When v_i is considered, there are at most $\delta(G_i)$ neighbors of v_i that have been colored, by the construction of the ordering. Hence there is always a color available in the list for v_i that has not been used on an earlier neighbor.

$\chi_l(G)+\chi_l(\overline{G}) \le n+1$. By part (a), it suffices to show that $\max_{H\subseteq G} \delta(H) + \max_{H\subseteq \overline{G}} \delta(H) \le n-1$. Let H_1 and H_2 be subgraphs of G and $\overline{G}$ achieving the maximums. Let $k_i = \delta(H_i)$. Note that $n(H_i) \ge k_i+1$. If $k_1+k_2 \ge n$, then H_1 and H_2 have a common vertex v. Now v must have at least k_i neighbors in H_i, for each i, but only $n-1$ neighbors are available in total.

$\chi_l'(G) \le 2\Delta(G) - 1$. Place the edges in some order. Each edge is incident to at most $2\Delta(G)-2$ others. If $2\Delta(G)-1$ colors are available at each vertex, then when we reach it there is always a color available not used on the incident edges colored earlier.

8.4.23. *Every chordal graph G is $\chi(G)$-choosable.* We use the reverse of a simplicial elimination ordering. Consider the vertices in the construction order. Cliques are created only as vertices are added, so the clique number is the maximum k such that a vertex belongs to a clique of order k when added. This also equals the chromatic number, by the greedy coloring with respect to this order. The same greedy coloring algorithm establishes k-choosability. When each vertex v is added, it has at most $k-1$ neighbors already present. Here at most $k-1$ colors from the list allowed for v have already been used on its neighbors, and a color remains that can be chosen from the list for v.

8.4.24. *A connected graph G has an L-coloring from any list assignment L such that $|L(v)| \ge d(v)$ for all v if there is strict inequality for at least one vertex y.* Choose a spanning tree of G, and order the vertices descending away from y, meaning that each vertex other than y has a later neighbor. When we reach a vertex v other than y, we have colored fewer than $d(v)$ of its neighbors, and hence a color remains available in $L(v)$ to use on v. When we reach y, we have colored $d(y)$ neighbors, but an extra color still remains available.

8.4.25. *a) Every graph G has a total coloring with at most $\chi_L'(G)+2$ colors.* In a total coloring, colored objects have different colors if they are adjacent vertices, incident edges, or an incident vertex and edge.

Let $k = \chi_L'(G) + 2$. Because $\chi_L'(G) \ge \chi'(G) \ge \Delta(G)$, there exists a proper k-coloring f of G. To each edge uv, assign the list $[k] - \{f(u), f(v)\}$. This assigns each edge a list of $\chi_L'(G)$ colors, from which we can choose a proper edge-coloring to complete a total coloring of G.

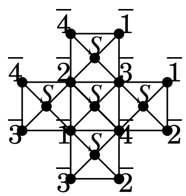
8.4.26. *Non-4-choosable planar graph of order 63.*

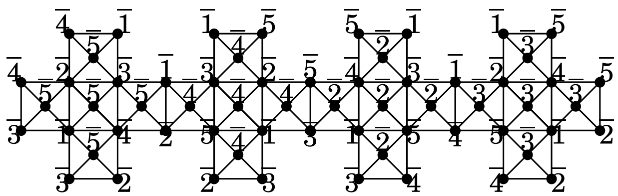
a) With S denoting $[4]$ *and $\bar{i}$ denoting $S - \{i\}$, the given lists for the graph on the left below yield no proper coloring.* The following properties hold for coloring each 4-cycle with distinct lists. (1) the chosen colors cannot be distinct (its center would not be colorable from S). (2) The colors on two consecutive vertices cannot be the colors forbidden from the opposite vertices in the opposite order (those opposite vertices would have to contribute the two remaining colors, violating (1)).

Consider the central 4-cycle C, and view all labels and indices modulo 4. We claim that the vertex in C with list $\bar{i}$ cannot receive color $i+1$. If it does, it forbids $i+1$ from the vertex in C with list $\overline{i-1}$, which by (2) also cannot receive color $i+2$. This leaves only color i for the vertex in C with list $\overline{i-1}$. Repeating the argument leads for each j to color $j+1$ on the vertex in C with list $\bar{j}$, which violates (1).

By making the same argument in the other direction, color $i-1$ on the vertex in C with list $\bar{i}$ would propogate to color i on the vertex in C with list $\overline{i+1}$, again violating (1).

This leaves only the possibility that for each i, color $i+2$ appears on the vertex in C with list $\bar{i}$. This again violates (1).





b) The planar graph G' obtained from G on the right above by adding one vertex with list $\overline{1}$ adjacent to all vertices on the outside face of G has no proper coloring chosen from these lists, where $\overline{i}$ denotes $[5]-\{i\}$. Suppose that G' has such a coloring. When the color chosen for the extra vertex is 5,4,2,3, respectively, the 1st, 2nd, 3rd, or 4th copy of the graph of part (a) in G has lists on its vertices isomorphic to those specified in part (a), via a permutation of the names of the colors. By part (a), there is no way to complete the coloring.

8.4.27. *Equivalence of Dilworth and König–Egerváry Theorems.*

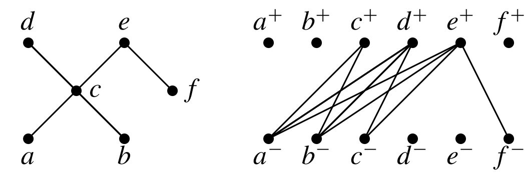
a) Dilworth's Theorem implies the König–Egerváry Theorem. View a bipartite graph G on n vertices as a poset. The vertices of one partite set are maximal elements, the others are minimal, and the edges are cover relations. Chains have one or two elements. Thus every chain-covering of size $n-k$ uses k chains of size 2 and yields a matching of size k in G. Each antichain of size $n-k$ is an independent set in G, and the k remaining vertices are a vertex cover. Hence Dilworth's guarantee of an antichain and a chain-covering of the same size yields a matching and a vertex cover of equal size in G.

b) The König–Egerváry Theorem implies Dilworth's Theorem. Let P be a poset of size n. We apply the König–Egerváry Theorem to a bipartite graph $S(P)$ called the *split* of P. The partite sets of $S(P)$ are $\{x^-: x \in P\}$ and $\{x^+: x \in P\}$. The edge set is $\{x^-y^+: x <_P y\}$.

A matching in $S(P)$ yields a chain-covering in P as follows: if x^-y^+ is in the matching, then y is immediately above x on a chain in the cover. If x^- or x^+ is unmatched, then x is the top or bottom of its chain, respectively. Since each vertex of $S(P)$ appears in at most one edge of the matching, this defines disjoint chains covering P. If the matching has k edges, then the cover has $n-k$ chains, since each added edge links the top of one chain with the bottom of another to form a single chain.

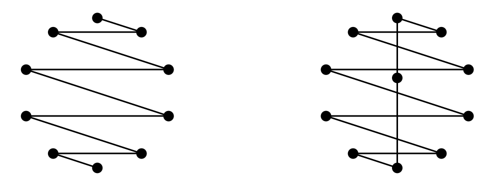
From a minimum vertex cover T of $S(P)$, we obtain an antichain. We show first that T does not use both copies of any element x. By transitivity in P, all of $\{z^-: z \in D(x)\}$ is adjacent in $S(P)$ to all of $\{y^+: y \in U(x)\}$. Covering the edges of this complete bipartite subgraph requires using all of $\{z^-: z \in D(x)\}$ or all of $\{y^+: y \in U(x)\}$. Since these are the neighbor sets of x^+ and x^-, respectively, at least one of $\{x^+, x^-\}$ can be omitted from T.

Now let $A = \{x \in P : x^-, x^+ \notin T\}$; we have shown that $|A| = |P| - |T|$. Also, A is an antichain, since a relation between elements of A would yield an edge of $S(P)$ uncovered by T. Thus a minimum cover of size k yields an of equal size yields an antichain and a chain-covering of equal size.



8.4.28. *K_n decomposes into $\lceil n/2 \rceil$ paths.* When n is even, we can use the decomposition into $n/2$ paths that are rotations of the figure on the left below. Each path uses two edges of each "length" around the circle, and each rotation gives a new pair of each length until all n pairs are obtained.

When n is odd, a bit more care is needed. Putting one vertex in the middle yields a decomposition into $(n-1)/2$ cycles, by rotating the figure on the right below. We can kill one short edge from each cycle to make it into a path, choosing always the short edge on the right side of the picture, and these $(n-1)/2$ leftover edges form a path to complete the decomposition. ■



K_n decomposes into $\lfloor n/2 \rfloor$ cycles when n is odd. We rotate the cycle shown in the figure above.

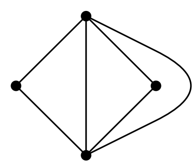
8.4.29. *Decomposition of K_n into spanning connected subgraphs.*

a) If K_n decomposes into k spanning connected subgraphs, then $n \ge 2k$. Each subgraph has at least $n-1$ edges, and they are pairwise edge-disjoint, so $k(n-1) \le \binom{n}{2}$.

b) K_{2k} decomposes into k spanning trees of diameter 3. Such trees are double-stars. Each has $2k-1$ edges, so k of them are needed to cover the $k(2k-1)$ edges of K_{2k}. Partition the vertex set into pairs $\{x_i, y_i\}$ for $i \le i \le n/2$. The ith subgraph consists of x_iy_i along with the edges x_jx_i and y_jy_i for $j < i$ and the edges x_iy_j and y_iy_j for $j > i$.

8.4.30. *Every 2-edge-connected 3-regular simple planar graph decomposes into paths of length 3, as does every simple planar triangulation.* The first statement is a special case of Exercise 3.3.19. For a simple triangulation

G, observe that the dual G^* is 2-edge-connected and 3-regular, so it has a P_4-decomposition. Let e'_1, e'_2, e'_3 be the successive edges in one copy of P_4 in the decomposition. Since G^* is 3-regular, e'_1 and e'_2 are on the same face, as are e'_2 and e'_3. Therefore, the corresponding edges e_1 and e_2 in G are incident, as are e_2 and e_3. Thus e_1, e_2, e_3 form a path or a cycle. It is a cycle if and only if the endpoints of e_1 and e_3 are the same vertex, which requires e'_1 and e'_3 to bound the same face of G^*. However, this face of G^* also shares boundary edges with the faces corresponding to the endpoints of e'_2, since G^* is 3-regular. Therefore, if e_1, e_2, e_3 is a cycle in G, then G must have multiple edges. The triangulation below shows that the prohibition of multiple edges is necessary.



8.4.31. *Theorem 8.4.35 is best possible when $m-1$ divides $n-1$.* Theorem 8.4.35 states that if the number of edges in an n-vertex graph exceeds $m(n-1)/2$, then the circumference exceeds m. We provide an n-vertex graph with exactly $m(n-1)/2$ edges in which the circumference is exactly m. The graph is $(\frac{n-1}{m-1}K_{m-1}) \vee K_1$. Each block is isomorphic to K_m, and every cycle stays within a block, so the circumference is m. The number of edges is $\frac{n-1}{m-1}\binom{m}{2}$, which simplifies to $m(n-1)/2$.

8.4.32. *If G is a graph such that $\overline{G}$ is triangle-free and not a forest, then G has a cycle of length at least $n(G)/2$.* If $\overline{G}$ is triangle-free, then whenever u and v are nonadjacent in G they cannot have a common nonneighbor, and hence $d(u) + d(v) \geq n(G) - 2$. If G is 2-connected, then Theorem 8.4.37 yeilds a cycle of length at least $n(G) - 2$.

If G is not connected, then the prohibition of triangles from $\overline{G}$ implies that G has only two components and that they are complete graphs. Hence one of them has a cycle of length at least $n(G)/2$.

If G is connected and has a cut-vertex v, then $G - v$ again is a disjoint union of two complete graphs. Also v cannot have a nonneighbor in both components. Hence G contains two disjoint complete graphs whose orders sum to $n(G)$, and again there is a cycle of least at least $n(G)/2$.

8.4.33. *Sufficient conditions for spanning cycles in graphs and digraphs.*

Woodall's Theorem implies Ore's Theorem. Ore proved that ($u \nleftrightarrow v \Rightarrow d(u) + d(v) \geq n(G)$) is sufficient for a spanning cycle in a graph G. Woodall

proved that ($u \nleftrightarrow v \Rightarrow d^+(u) + d^-(v) \geq n(G)$) is sufficient for a spanning cycle in a digraph G.

Given a graph satisfying Ore's Condition, let G' be the digraph obtained by replacing each edge of G with two opposing edges having the same endpoints. Now $d^+_{G'}(v) = d^-_{G'}(v) = d_G(v)$. Thus Woodall's Condition holds, and Woodall's Theorem implies that G' has a spanning cycle, which yields a spanning cycle in G.

Meyniel's Theorem implies Woodall's Theorem for strict digraphs. Letting $d(u) = d^+(u) + d^-(u)$, Meyniel proved that ($u \nleftrightarrow v \Rightarrow d(u) + d(v) \leq 2n(G) - 1$) is sufficient for a spanning cycle in a digraph G such that each ordered pair appears at most once as an edge.

Consider a digraph in which each ordered pair of vertices appears at most once as an edge. If Woodall's Condition holds, then when $u \nleftrightarrow v$ we

$$d(u) + d(v) = d^+(u) + d^-(v) + d^-(u) + d^-(v) \geq 2n(G) > 2n(G) - 1.$$

Thus Meyniel's Condition holds if we also show that the digraph is strongly connected. The holds because when $d^+(u) + d^-(v) \geq n(G) - 1$, there is an edge or a path of length 2 from u to v. Thus Meyniel's Theorem applies, and the digraph has a spanning cycle.

8.4.34. *A strict n-vertex digraph has a spanning path if $d(u) + d(v) \geq 2n - 3$ for every pair u, v of distinct nonadjacent vertices.* Given such a digraph, add a vertex w with an edge to and from each of the original vertices. Let G' denote the new digraph, with degree function d'. Now $d'(u) + d'(v) = d(u) + d(v) + 4 \geq 2n + 1 = 2(n+1) - 1$. Also G' is strongly connected, since each vertex can get to and from w. By Meyniel's Theorem, G' has a spanning cycle. Since this passes through w only once, deleting w leaves a spanning path in G.

8.5. RANDOM GRAPHS

8.5.1. *Expectation.*

a) The expected number of fixed points in a random permutation of $[n]$ is 1. Since there are $(n-1)!$ permutations with element i fixed, the probability that i is fixed is $1/n$. Letting X_i be the indicator variable for element i being fixed, we have $\sum X_i$ as the random variable for the number of fixed points. By linearity, $\mathrm{E}(\sum X_i) = \sum \mathrm{E}(X_i) = n(1/n) = 1$.

b) The expected number of vertices of degree k in a random n-vertex graph with edge probability p is $k\binom{n-1}{k}p^k(1-p)^{n-1-k}$. A vertex has degree k when there are k successes among the $n-1$ trial for its incident edges. The probability of this is $\binom{n-1}{k}p^k(1-p)^{n-1-k}$. Letting X_i be the indicator variable for vertex i having degree k, the expected number of vertices of degree k becomes $k\mathrm{P}(X_i = 1)$, by linearity.

8.5.2. *Always $1 - p < e^{-p}$ for $p > 0$.* For $p = 0$, equality holds. Hence it suffices to show that the derivative of e^{-p} exceeds that of $1 - p$ for $p > 0$. We have $(d/dp)e^{-p} = -e^{-p} > -1 = (d/dp)(1-p)$, where the key inequality holds for $p > 0$. (The inequality also holds for $p < 0$, because the terms in the series for e^{-p} are then all positive.)

8.5.3. *The expected number of monochromatic triangles in a random 2-coloring of $E(K_6)$ is $15/4$.* When the edges are given red or blue with probability $1/2$ each, independently, the probability that three vertices produce a monochromatic triangle is $1/4$. There are $\binom{6}{3}$ triples where this may occur. By linearity of expectation, the expected number of occurrences is $15/4$, even though the events are not independent.

8.5.4. *Some 2-coloring of the edges of $K_{m,n}$ has at least $\binom{m}{r}\binom{n}{s}2^{1-rs}$ monochromatic copies of $K_{r,s}$.* We color the edges red or blue with probability $1/2$ each, independently. A particular choice of r vertex in one partite set and s vertices in the other produces a monochromatic copy of $K_{r,s}$ with probability 2^{1-rs}. Since there are $\binom{m}{r}\binom{n}{s}$ ways to choose the vertex sets, by linearity the expected number of copies is $\binom{m}{r}\binom{n}{s}2^{1-rs}$, so some outcome of the experiment is a 2-coloring with that many monochromatic copies of $K_{r,s}$. (Note: The coefficient increases to $\binom{m}{r}\binom{n}{s} + \binom{m}{s}\binom{n}{r}$ if $r \neq s$ and we don't care which partite set contains the bigger part of the subgraph.)

8.5.5. *The statement "$(1-\varepsilon)n \le f(G_n) \le (1+\varepsilon)n$ when $\varepsilon > 0$ for sufficiently large n" is equivalent to "$f(G_n)/n \to 1$ as $n \to \infty$", written as "$f(G_n) \le n + o(n)$".* Let $g(n) = f(G_n)/n$. If $g(n) \to 1$, then for all $\varepsilon > 0$ there exists N such that $n > N$ implies $|g(n) - 1| < \varepsilon$, by the definition of convergence of sequences. The inequality $|g(n) - 1| < \varepsilon$ (for sufficiently large n) is simply the first statement here.

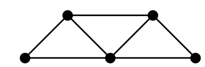
8.5.6. *Probability that the probability that the Hamiltonian closure $C(G)$ of a random graph G with vertex set $[5]$ is complete.* The problem is to determine the fraction of the graphs with vertex set [5] that have complete closure. We describe the graphs, without doing the counting.

If $\delta(G) \le 1$, then $C(G)$ is not complete; a vertex of degree at most 1 never acquires another edge, because every vertex of degree $n(G) - 1$ is already adjacent to it.

If nonadjacent vertices have degree some at least $n(G)$, then all remaining edges are added immediately and the closure is complete.

Therefore, with five vertices, it suffices to consider graphs having nonadjacent vertices of degree 2. Among the remaining graphs, we have $5 \le e(G) \le 7$ for such graphs. All such graphs with five edges are 5-cycles,

which gain no edges. With six edges we can have $K_{2,3}$, which gains one edge and does not become complete, or the union of a 3-cycle and a 4-cycle sharing one edge, whose closure is complete. With seven edges we have only $K_2 \vee \overline{K}_3$ and the graph below; $K_2 \vee \overline{K}_3$ is already closed and gains no edges, but the closure of the graph below is complete.



8.5.7. *If G is a graph with p vertices, q edges, and automorphism group of size s, and $n = (sk^{q-1})^{1/p}$, then some k-coloring of $E(K_n)$ has no monochromatic copy of G.* Produce a k-coloring of the edges at random, with each edge receiving each color with probability $1/k$, independently. A particular copy of G in K_n becomes monochromatic with probability $k \cdot 1/k)^q$. On a given set of p vertices, there are $p!/s$ copies of G. If $\binom{n}{p}\frac{p!}{s}k^{1-q} < 1$, then there is an outcome of the experiment in which no copy of G is monochromatic. Since $\binom{n}{p} < n^p/p!$, the desired inequality holds when $n^p < sk^{q-1}$.

8.5.8. *Bipartite subgraphs.*

a) Every graph has a bipartite subgraph with at least half its edges. Select a random vertex subset A by choosing each vertex with probability $1/2$, independently. Each edge has probability $1/2$ of belonging to the cut $[A, \overline{A}]$, since this is the probability that exactly one of its endpoints lies in A. By linearity of expectation, the expected number of edges in the cut is half the total number of edges. The edges in a cut form a bipartite subgraph, so there is a bipartite subgraph with at least half the edges.

b) If G has m edges and n vertices, then G has a bipartite subgraph with at least $m\frac{\lceil n/2\rceil}{2\lceil n/2\rceil-1}$ edges. Choose A at random from all $\lceil n/2\rceil$-element vertex subsets. The number of these subsets containing exactly one endpoint of a given edge e is $2\binom{n-2}{\lceil n/2\rceil-1}$. Thus e belongs to the cut $[A, \overline{A}]$ with probability $2\binom{n-2}{\lceil n/2\rceil-1}/\binom{n}{\lceil n/2\rceil}$. Since $\binom{n}{\lceil n/2\rceil} = \frac{n}{\lfloor n/2\rfloor}\binom{n-1}{\lceil n/2\rceil} = \frac{n}{\lfloor n/2\rfloor}\;\frac{n-1}{\lceil n/2\rceil}\binom{n-2}{\lceil n/2\rceil-1}$, the probability is $\frac{2\lfloor n/2\rfloor\lceil n/2\rceil}{n(n-1)}$. Since $\frac{n(n-1)}{2\lfloor n/2\rfloor} = 2\lceil n/2\rceil - 1$, linearity of expectation yields $m\frac{\lceil n/2\rceil}{2\lceil n/2\rceil-1}$ as the expected size of the cut, and some cut is at least this large. This fraction of the number of edges is strictly more than .5, so this result improves part (a).

8.5.9. *If in a complete k-ary tree with leaves at distance l from the root, the vertices fail independently with probability p, then the expected number of nodes accessible from the root is $(1-p)\frac{1-(k-kp)^{l+1}}{1-k+kp}$.* There are k^j vertices at depth j (distance j from the root). A vertex at depth j is accessible if and only if it and its ancestors are alive. Thus it is accessible with probability

$(1-p)^{j+1}$. Using linearity of the expectation, the expected number of accessible nodes is $\sum_{j=0}^{l}(1-p)(k-kp)^j$. When $p=1/k$, this is simply $1-p$. Otherwise, the expectation is $(1-p)\frac{1-(k-kp)^{l+1}}{1-k+kp}$.

8.5.10. *The expected number of edges in a matching of size n that are induced by selected k vertices at random is $\frac{n}{2}\frac{k(k-1)}{(n-1)(2n-1)}$.* There are several proofs; using linearity of expectation makes the computations simple. There are $\binom{2n-2}{k-2}$ sets of size k that capture a particular pair of vertices. Hence each edge is captured with probability $\frac{k(k-1)}{(2n-2)(2n-1)}$. By linearity, the expected number of edges in the matching that are captured is $n\frac{k(k-1)}{(2n-2)(2n-1)}$.

8.5.11. *If a graph G has n vertices and m edges, with $m \ge 4n$, then $\nu(G) \ge m^3/[64n^2]$*, where $\nu(G)$ denotes the minimum number of crossings in a drawing of G. Let G have n vertices and m edges, with $m \ge 4n$, and consider a drawing of G in the plane. To obtain a lower bound on $\mathrm{cr}(G)$, we take a random induced subdrawing H, including each vertex independently with probability p. We expect pn vertices and p^2m edges in H. Let Y be the number of edge crossings in the drawing of G that remain in H. We have $E(Y)=p^4\mathrm{cr}(G)$, since all four endpoints of the two edges must be retained to keep the crossing.

Let $X=3n(H)-6+Y-e(H)$. Always $Y \ge e(H)-(3n(H)-6)$, since a planar graph with v vertices has at most $3v-6$ edges, and every edge beyond a maximal plane subgraph of H introduces at least one additional crossing. Thus always $X \ge 0$. We conclude that $E(X) \ge 0$.

By linearity, $E(X)=3np-6+p^4\mathrm{cr}(G)-p^2e(G)$. This yields $3n+p^3\mathrm{cr}(G)-pm>0$. We choose $p=4n/m$, which is feasible since $m \ge 4n$. We now have the inequality $3n+64n^3/m^3\mathrm{cr}(G)>4n$, which yields the desired bound.

8.5.12. *In a random orientation of the vertices of a simple graph G, produced by orienting each edge toward the vertex with higher index in a random permutation, the expected number of sink vertices (outdegree 0) is $\sum_{v\in V(G)}\frac{1}{d(v)+1}$, which ranges from 1 to $n(G)$.*

A vertex is a sink in the resulting orientation if and only if it follows all its neighbors in the permutation. For each vertex v, whether this happens is determined only by whether it is last among the set $N[v]$, which happens with probability $(1+d(v))^{-1}$. By linearity, the expected number of sinks is $\sum_{v\in V(G)}\frac{1}{d(v)+1}$. (Note that the sinks form an independent set, so this is also a lower bound on $\alpha(G)$.)

Given the formula for the expectation, it is minimized by increasing degrees and maximized by reducing degrees, so it is minimized by the complete graph, where the number of sinks is always 1, and it is maximized by the trivial graph, where every vertex is always a sink. Among connected graphs, it is maximized by the path, where the value is $(n+1)/3$.

In order to have only one sink, the last two vertices in the random permutation must be adjacent. When the last vertex has degree d, then with probability $\frac{n-1-d}{n-1}$ the next-to-last vertex is a nonneighbor of it. Thus the probability of having at least two sinks is at least $\frac{1}{n}\sum_{i=1}^{n}\frac{n-1-d_i}{n-1}$, which simplifies to $1-\frac{1}{n}\frac{1}{n-1}\sum_{i=1}^{n}d_i$. Invoking the Degree-Sum Formula, the probability of having only one sink is at most $e(G)/\binom{n}{2}$.

8.5.13. *Bound on choosability of n-vertex bipartite graphs.*

a) Every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colorable. Let H be a k-uniform hypergraph with n edges, where $n<2^{k-1}$. Color vertices by X and Y so that each gets color X with probability $1/2$, independently. The probability that a given edge is monochromatic is $2^{-(k-1)}$. Since $n<2^{k-1}$, the probability that some edge is monochromatic is less than 1. Hence some outcome of the experiment is a proper 2-coloring of H.

Since H has n edges and $n<2^{1+\lfloor \lg n\rfloor}$, the hypergraph H is 2-colorable (in a random coloring of a k-uniform hypergraph with fewer than 2^{k-1} edges, the expected number of monochromatic edges is less than 1). A proper 2-coloring of H partitions its vertices into Type X and Type Y.

b) If each vertex of an n-vertex bipartite graph is given a list of more than $1+\lg n$ usable colors, then a proper coloring can be chosen from the lists. Let G be an X,Y-bigraph with n vertices and such a list assignment. Let H be an auxiliary hypergraph whose vertices are the colors in the lists. Each vertex $v\in V(G)$ generates an edge in H consisting of the colors in $L(v)$. We may reduce the sizes of the lists so that H is k-uniform, where $k=2+\lfloor \lg n\rfloor$. Thus $k-1>\lg n$. By part (a), H is 2-colorable; we call the colors Type X and Type Y.

In choosing an L-coloring for G, we must restrict each color to usage in only one partite set. Colors having Type X in the coloring of X will only be used on partite set X; those of Type Y will only be used on Y. Since H was properly 2-colored, each list has colors of both types. If $v\in X$, then we choose a color of Type X from $L(v)$; if $v\in Y$, then we choose a color of Type Y from $L(v)$. Since each color is chosen on only one partite set in G, we have obtained an L-coloring.

8.5.14. *A graph with n vertices and average degree $d\ge 1$ has an independent set with at least $n/(2d)$ vertices.* Note that G has $nd/2$ edges. Let $S\subseteq V(G)$ be generated at random by including each vertex independently with probability p. If S has X vertices and Y edges, then S contains an independent set of size at least $X-Y$, by deleting a vertex of each induced edge. We will choose p to maximize $E(X-Y)$, since there will be an independent set at least that large.

By linearity of expectation, $E(X-Y)=E(X)-E(Y)$. We have $E(X)=np$. Similarly, the probability that a specified edge of G is induced by S is p^2, since both its endpoints must be included, so $E(Y)=p^2nd/2$. Hence $E(X-Y)=np(1-pd/2)$. We choose $p=1/d$ to maximize this, which is valid since $d\ge 1$, obtaining $E(X-Y)=n/(2d)$.

8.5.15. $ex(n;C_k)\in\Omega(n^{1+1/(k-1)})$. We seek an n-vertex graph with many edges and no k-cycle. We generate a random graph in Model A with some edge probability p. If the expected number $E(Y)$ of k-cycles is much less than the expected number $E(X)$ of edges, then deleting an edge from each k-cycle in some graph where $X-Y$ is large leaves a graph with many edges and no k-cycle.

Given edge probability p, we have $E(X)=\binom{n}{2}p$ and $E(Y)=\binom{n}{k}\frac{1}{2}(k-1)!p^k$. If we can choose p so that $E(Y)<\frac{1}{2}E(X)$, then $\frac{1}{2}E(X)$ will be a lower bound on $ex(n;C_k)$. Using $\binom{n}{k}<n^{k-1}(n-1)/k!$, we have $E(Y)<\frac{1}{2k}(n-1)p(np)^{k-1}$. It suffices to have $\frac{1}{2k}(n-1)p(np)^{k-1})\le\frac{1}{4}n(n-1)p$, which is implied by $n^{k-2}p^{k-1}\le 1$. Hence we choose $p=n^{(k-2)/(k-1)}$. Now there is a C_k-free graph of size at least $\frac{1}{2}\binom{n}{2}p$, which is asymptotic to $\frac{1}{4}n^{1+1/(k-1)}$.

In the particular case $k=4$, this lower bound of $\Omega(n^{4/3})$ compares with an upper bound of $O(n^{3/2})$. A graph with m edges contains C_4 if and only if some pair of vertices has two common neighbors. Recall that the counting argument and the convexity of quadratics yield $\binom{n}{2}\ge\sum_{v\in V(G)}\binom{d(v)}{2}\ge n\binom{2m/n}{2}$, and that the resulting quadratic inequality yields $m\le\frac{n}{4}(1+\sqrt{4n-3})$.

8.5.16. *$R(k,k)>n-\binom{n}{k}2^{1-\binom{k}{2}}$ for all $n\in\mathbb{N}$, and hence $R(k,k)>(1/e)(1-o(1))k2^{k/2}$.* Generate a random 2-coloring of $E(K_n)$; let X be the resulting number of monochromatic copies of K_k. Each k-set contributes to X with probability $2^{1-\binom{k}{2}}$. Since there are $\binom{n}{k}$ of these sets, $\mathrm{E}(X)=\binom{n}{k}2^{1-\binom{k}{2}}$. Some outcome of the experiment has at most $\mathrm{E}(X)$ bad sets, and deleting a vertex from each such set in such an outcome yields a coloring that establishes the lower bound.

8.5.17. *For $n\in\mathbb{N}$, there is a 2-coloring of $E(K_{m,m})$ with no monochromatic copy of $K_{t,t}$ when $m=n-\binom{n}{t}^2 2^{1-t^2}$.* Generate a random 2-coloring of $E(K_{n,n})$; let X be the resulting number of monochromatic copies of $K_{t,t}$. Each choice of t vertices from each partite set counts with probability 2^{1-t^2}. Since there are $\binom{n}{t}^2$ of these sets, $\mathrm{E}(X)=\binom{n}{t}^2 2^{1-t^2}$. In some outcome of the experiment, X has value at most $\mathrm{E}(X)$, and deleting a vertex (in each partite set) from each monochromatic copy of $K_{t,t}$ in such an outcome yields a coloring that establishes the lower bound.

8.5.18. *Off-diagonal Ramsey numbers.* This problem repeats parts (a) and (b) of Exercise 8.3.20.

8.5.19. *For a fixed graph H and constant edge-probability p, almost every G^p contains H as an induced subgraph.* Let k and l be the number of vertices and edges in H. The probability that a given set of k vertices induces H is $\frac{k!}{A}p^l(1-p)^{\binom{k}{2}-l}$, where A is the number of automorphisms of H; let this probability be q. Since k,l,p,A are all constant, q is a constant. Appearances of H at disjoint sets of vertices are independent. Splitting $[n]$ into n/k disjoint sets, the probability that none of them induce H is $(1-q)^{n/k}$. Since q is constant, this probability tends to 0 as $n\to\infty$.

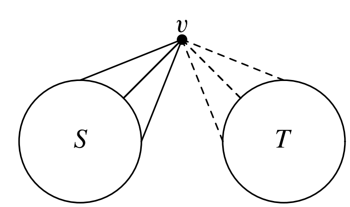
8.5.20. *Common neighbors and nonneighbors.*

a) For constant k,s,t,p, almost every G^p has the following property: for every choice of disjoint vertex sets S and T of sizes s and t, there are at least k vertices that are adjacent to every vertex of S and no vertex of T. Let X be the number of bad choices for S and T in G^p; we need only show that $\mathrm{E}(X)\to 0$. For the ith way to choose $S,T\subseteq V(G)$, define an indicator variable X_i with value 1 when there are fewer than k choices of a vertex v such that $S\subseteq N(v)$ and $T\subseteq\overline{N}(v)$. For $v\notin S\cup T$, failure requires s specified adjacencies and t specified nonadjacencies, so $X_i=1$ requires more than $n-s-t-k$ failures in $n-s-t$ trials when the failure probability is $p^s(1-p)^t$.

We complete the proof only for $k=1$; for larger k a binomail tail bound is needed. When $k=1$, $\mathrm{P}(X_i=1)=(1-p^s(1-p)^t)^{n-s-t}$. Since $X=\sum X_i$, we count the variables X_i (choices of S,T) by the multinomial coefficient to obtain

$$\mathrm{E}(X)=\binom{n}{s,t,n-s-t}(1-p^s(1-p)^t)^{n-s-t}.$$

For fixed s,t,p, the multinomial coefficient is a polynomial in n. It is bounded by n^{s+t}, while $\mathrm{E}(X_i)$ dies exponentially as $n\to\infty$. The logarithm of the product approaches $-\infty$, and thus $\mathrm{E}(X)\to 0$. ■



b) Almost every G^p is k-connected. If the computation for general k is completed, then it suffices to set $s=2$ and $t=0$ to obtain that in almost every graph, every two vertices have k-common neighbors.

c) Almost every tournament has the property that for every choice of disjoint vertex sets S, T of sizes s, t, there are at least k vertices with edges to every vertex of S and from every vertex of T. The argument is essentially the same, using $p = 1/2$ and orienting each edge randomly.

8.5.21. *Random tournaments.*

a) Almost every tournament is strongly connected. This follows by essentially the same computation as part (b): in almost every tournament, for every ordered pair (x, y) of vertices, there is a vertex w such that $x \to w$ and $w \to y$, so every vertex reaches every other. Alternatively, this follows also from the statement of part (b).

b) In almost every tournament, every vertex is a king. The criterion for every vertex being a king is that every vertex be reachable from every other vertex by a path of length at most 2. Let X be the number of ordered pairs of vertices where this fails. For a given pair (x, y) failing to reach y from x by a path of length at most 2 requires that for each other vertex w, the edges xw and wy are not both oriented away from x and toward y. Hence the probability that the ordered pair (x, y) fails is bounded above by $(3/4)^{n-2}$ (the edge xy yields another factor of $1/2$, but this is not important).

Since there are $n(n-1)$ ordered pairs, $E(X) < n^2(3/4)^{n-2}$. The bound tends to 0 as $n \to \infty$, so by Markov's Inequality almost every tournament has no bad pairs and thus has every vertex being a king.

8.5.22. *Edge probability $1/2$ is a sharp threshold for the property that at least half the possible edges of a graph are present.* Let X be the number of edges in G^p. When $p = 1/2$, $E(X) = \frac{1}{2}\binom{n}{2}$. The distribution of X is binomial, and we ask how highly concentrated the distribution is to study the probability of having at least half the edges when we vary p. Although tighter bounds are available, the Chebyshev bound suffices for our purpose. We have $P(|X - E(X)| \ge t) \le V/t^2$, where $V = [E(X^2) - (E(X))^2]$. Direct computation, using the expression of X as a sum of indicator variables, yields $V = Np(1-p)$, where N is the number of trials (here $N = \binom{n}{2}$).

If $p = .5 - \varepsilon$ with ε constant, then $E(X)$ is below $\frac{1}{2}\binom{n}{2}$ by an amount that is quadratic in n. In considering $X \ge \frac{1}{2}\binom{n}{2}$, we are asking for t to be quadratic in n, and the bound on the probability of having at least half the edges tends to 0. Even if we set $p = .5 - c\log n/n$, then still $P(X \ge \frac{1}{2}\binom{n}{2})$ tends to 0. Similarly, if $p = .5 + c\log n/n$, the analogous argument shows that the probability of having at most half the edges goes to 0.

8.5.23. *For $p = 1/n$ and fixed $\varepsilon > 0$, almost every G^p has no component with more than $(1+\varepsilon)n/2$ vertices.* A connected graph with $m+1$ vertices has at least m edges, so it suffices to show that almost every G_p has fewer than $(1+\varepsilon)n/2$ edges. The number of edges is a binomial random

variable; its expectation $\binom{n}{2}p$ equals $(n-1)/2$. The probability that a binomial random variable exceeds its expectation by a constant fraction (here $\varepsilon/2$) is exponentially small in the number of trials. Even so, the weaker Chebyshev bound suffices to show this approaches 0. (We may ignore the .5 in $E(X) = n/2 - .5$ by using a slightly larger choice of ε.) We have $P(|X - E(X)| \ge \varepsilon n/2) \le V/(\varepsilon n/2)^2$, where $V = [E(X^2) - (E(X))^2]$. The expectation of X^2 for $\binom{n}{2}$ independent trials is found by

$$E(X^2) = E(X) + \binom{n}{2}\left(\binom{n}{2} - 1\right)p^2 \sim E(X)^2 = \left(\binom{n}{2}p\right)^2$$

Thus $V = o(n^4p^2) = o(n^2)$. Since the denominator is $\Omega(n^2)$, the ratio bounding the probability approaches 0.

8.5.24. *The smallest connected simple graph that is not balanced is the 5-vertex graph consisting of a kite plus a pendant edge.* If G is unbalanced, then some induced subgraph has larger average degree than G. For the smallest such graph G, we obtain the offending subgraph by deleting one vertex. In Exercise 1.3.44a, we showed that the average degree increases when a vertex x is deleted from an n-vertex graph with average degree a if and only if $d(x) < a/2$ (since $\frac{2e(G-x)}{n-1} = \frac{2[e(G)-d(x)]}{n-1} = \frac{na-2d(x)}{n-1}$).

Since G is connected, every degree is positive. Hence the smallest example will occur by deleting a leaf from a graph with average degree exceeding 2. For average degree exceeding 2, at least four vertices are needed. With four vertices, no graph with at least five edges has a leaf. With five vertices, we need at least six edges in a graph obtained by appending a leaf to a 4-vertex graph with at least five edges. A kite with a pendant edge has this property.

8.5.25. *In terms of the number n of vertices, $n^{-1/\rho(H)}$ is a threshold probability function for the appearance of H as a subgraph of G^p, where $\rho(G) = \max_{G\subseteq H} e(G)/n(G)$.* We extend the second moment argument of Theorem 8.5.23. Let F be a subgraph of H with density $\rho(H)$. This subgraph F is balanced, and the first moment argument in Theorem 8.5.23 shows that if $pn^{\rho(H)} \to 0$, then almost every G^p has no copy of F and hence no copy of H.

To show that $n^{-1/\rho(H)}$ is a threshold probability function for the appearance of H, we must also show that $pn^{\rho(H)} \to \infty$ implies that almost every G^p has a copy of H. An easy modification of the second moment argument in Theorem 8.5.23 (due by Ruciński and Vince) completes the proof.

Let X be the random variable counting the copies of H. We follow the same argument as for balanced graphs, and it suffices to prove that $E(X^2) -> E(X)^2$ when $pn^{1/rho} \to \infty$. The proof of this is the same as for the balanced case Theorem 8.5.23 up to point in the last paragraph where the balance condition is invoked. Replace that portion with the following:

"The desired behavior of $n^{-r}p^{-s}$ is equivalent to $pn^{r/s} \to \infty$. Since s/r is the density of H', we have $s/r \le \rho$. This forces $pn^{r/s} \ge pn^{1/rho} \to \infty$ when $c > 0$."

8.5.26. *Almost every graph (with edge probability p) has the property that for every choice of disjoint vertex sets S, T of size $c\log_{1/(1-p)} n$ with $c > 2$), there is an edge with endpoints in S and T.* (For $p = 1/2$, the formula reduces to $c\lg n$.)

Let X be the number of choices of disjoint sets S and T of this size with no edge between them. By Markov's Inequality, it suffices to show that $E(X) \to 0$ when $c > 2$), because then the probability that Q_k occurs tends to 1. We have $E(X) \sim \frac{1}{k!^2} n^{2k}(1-p)^{k^2}$. Writing this as $c'(n^2(1-p)^k)^k)$, it suffices to have $n^2(1-p)^k < 1$. This requires $k > 2\log_{1/(1-p)} n$. Thus it suffices to choose $c > 2$ in the expression for $|S|$ and $|T|$.

8.5.27. *If $k = \lg n - (2+\varepsilon)\lg\lg n$, then almost every n-vertex tournament has the property that every set of k vertices has a common successor.* The probability that a k-set fails to have a common successor is $(1-2^{-k})^{n-k}$, since this requires that each vertex outside the set is not a common successor. Let X be the number of k-sets with no common successor; we have $\mathrm{E}(X) = \binom{n}{k}(1-2^{-k})^{n-k}$. An upper bound on $\mathrm{E}(X)$ is $\left(\frac{ne}{k}\right)^k e^{-2^{-k}(n-k)}$. If this bound tends to 0 for some choice of k in terms of n, then almost every tournament has the property for this choice of k.

To suggests the appropriate k, we choose k so that $\left(\frac{ne}{k}\right)^k$ grows more slowly than $e^{2^{-k}(n-k)}$. Taking natural logarithms, we want $k(1+\ln n - \ln k) < 2^{-k}(n-k)$. Now taking base-2 logarithms, we want

$$\lg k + \lg\ln n + \lg(1 - \tfrac{(\ln k)-1}{\ln n}) < -k + \lg n + \lg(1 - \tfrac{k}{n}).$$

Roughly speaking, we want $k + \lg k < \lg n - \lg\ln n$. Thus k should be enough less than $\lg n$ that adding $\lg k$ still keeps the value less than $\lg n - \lg\ln n$. Converting from $\ln n$ to $\lg n$ on the right only introduces an additive constant, since the $\ln n$ is inside lg. The ε in the definition of k is more than enough to take care of that.

Setting k as specified above yields $\mathrm{E}(X) \to 0$, and the property almost always holds.

8.5.28. *Transitive subtournaments.*

Every n-vertex tournament has a transitive subtournament with $\lg n$ vertices. We prove by induction on n that every n-vertex tournament has a transitive subtournament with at least $1 + \lfloor \lg n \rfloor$ vertices. The statement holds trivially for $n = 1$.

When $n > 1$, a vertex x with maximum outdegree has outdegree at least $\lfloor n/2 \rfloor$. In the subtournament induced by the successors of x, the induction hypothesis yields a transitive subtournament with at least $1 + \lfloor \lg(\lfloor n/2 \rfloor) \rfloor$ vertices. This equals $\lfloor \lg n \rfloor$ in all cases. Adding x produces a transitive tournament of order $1 + \lfloor \lg n \rfloor$ in the original tournament.

For $c > 1$, almost every tournament has no transitive subtournament with more than $2\lg n + c$ vertices. In the random tournament (each edge is directed toward the lower vertex with probability 1/2), let X be the number of transitive subtournaments of order k. For each set of k vertices, the possible transitive tournaments correspond to the $k!$ linear orderings of the vertices. Hence $E(X) = \binom{n}{k} k! 2^{-\binom{k}{2}}$.

It suffices to show that $E(X) \to 0$ (equivalently, $\lg E(X) \to -\infty$) when k exceeds $1 + 2\lg n$ by any constant. We have $E(X) < n^k 2^{-\binom{k}{2}}$, which we write as $\lg E(X) < k[\lg n - (k-1)/2]$. This bound is a decreasing function of k if $\lg n - (k-1)/2 < 0$. Therefore, we obtain a valid upper bound on $\lg E(X)$ for all larger k if we set $k = 1 + \varepsilon + 2\lg n$.

For further simplification, let $m = 1 + \lg n$, so $k = 2m - 1 + \varepsilon$. The final computation is

$$\lg E(X) < (2m - 1 + \varepsilon)[m - 1 - (m - 1 + \tfrac{\varepsilon}{2})] = (2m - 1 + \varepsilon)(\tfrac{-\varepsilon}{2}) \to -\infty.$$

8.5.29. *Geometric random variable / Coupon Collector.*

a) Under repeated trials of an experiment with success probability p on each trial, independently, the expected number of the trial when the first success occurs is $1/p$. Let X be the random variable for the trial on which the first success occurs.

Proof 1 (computation). The probability of $X = k$ is $(1-p)^{k-1}p$. By the definition of expectation, behavior of geometric series, and differentiation of convergent series,

$$\mathrm{E}(X) = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = p\frac{d}{dp}\sum_{k=0}^{\infty} -(1-p)^k = p\frac{d}{dp}\frac{-1}{1-(1-p)} = p\frac{1}{p^2} = \frac{1}{p}.$$

Proof 2 (conditional expectation). Let $z = \mathrm{E}(X)$. If the first trial is a failure, then the remainder of the experiment is a repetition of the original experiment. Hence $z = p\cdot 1 + (1-p)\cdot(1+z)$, which yields $pz = p + (1-p) = 1$, so $z = 1/p$.

Proof 3 (linearity). Let X_i be the event that the first i trials fail. Now $X = 1 + \sum_{i=1}^{\infty} X_i$, and $\mathrm{E}(X) = 1 + \sum_{i=1}^{\infty} \mathrm{P}(X_i = 1) = \sum_{i=0}^{\infty}(1-p)^i = 1/p$.

b) Given independent trials with n outcomes, each with probability $1/n$, the expected number of trials to obtain all outcomes is $n\sum_{i=1}^{n} 1/i$. Let X be the number of trials taken to obtain all outcomes. Let X_i be the number of trials after $i-1$ of the outcomes have been obtained, up to and including the trial on which for the first time i outcomes have been obtained. We have $X = \sum_{i=1}^n X_i$. When we have obtained $i-1$ of the outcomes, a given trial provides a new outcome with probability $\frac{n-i+1}{n}$. Hence the variable X_i is a geometric random variable with success probability $\frac{n-i+1}{n}$. Using part (a) and linearity of expectation, and reversing the order of summation and letting $j = n+i-i$, we obtain $E(X) = n\sum_{j=1}^n 1/j$.

c) A threshold function $m(n)$ for the number of boxes needed to obtain more than k copies of each prize is given by $m(n) = n\ln n + kn\ln\ln n$. Let X be the number of target points hit at most k times. For each $r \in [n]$, the probability that $f^{-1}(r)$ has size j is $\binom{m}{j}p^j(1-p)^{m-j}$, where $p = 1/n$. The probability that $f^{-1}(r)$ has size at most k is the summation of this up to $j = k$. Let this probability be b; thus $E(X) = nb$.

We claim that the contribution to b from terms with $j < k$ is of lower order than the term when $j = k$. Let $\alpha = (1-p)^{m-k+1}$. We bound the sum by a multiple of a geometric sum. If $mp \to \infty$, this yields

$$b = \sum_{j=0}^{k-1}\binom{m}{j}p^j(1-p)^{m-j} \le \alpha\sum_{j=0}^{k-1}(mp)^i = \alpha\frac{(mp)^k-1}{mp-1} \sim \alpha(mp)^{k-1}$$

On the other hand, $\binom{m}{k}p^k(1-p)^{m-k}$ is bounded below by a constant times $\alpha(mp)^k$. Hence $mp \to \infty$ and k constant yields $b \sim \binom{m}{k}p^k(1-p)^{m-k}$.

We want to choose $m(n)$ so that nb approaches 0 or ∞, depending on the choice of a parameter in $m(n)$. Since k is constant, $(1-p)^k \to 1$ and the binomial coefficient in the top term is asymptotic to $m^k/k!$. Thus $b \sim \frac{1}{k!}m^k p^k(1-p)^m$. Also $np^2 \to 0$, so $1-p$ is asymptotic to e^{-p}.

With $m(n) = n\ln n + cn\ln\ln n$, we have $mp = \ln n + c\ln\ln n \sim \ln n$ and $e^{mp} = n(\ln n)^c$. We now compute

$$E(X) = nb \sim n\frac{(mp)^k}{k!e^{mp}} \sim n\frac{(\ln n)^k}{k!n(\ln n)^c} = \frac{1}{k!}(\ln n)^{k-c}.$$

If $c > k+\varepsilon$, then $E(X) \to 0$, and almost always every target point is hit more than k times. If $c < k-\varepsilon$, then $E(X) \to \infty$. The Second Moment Method then will imply that almost always some target point is hit at most k times if we prove that $E(X^2) \sim E(X)^2$.

Let $X = \sum_{r=1}^n X_r$, where X_r is the event that $\left|f^{-1}(r)\right| \le k$. The probability that $\left|f^{-1}(r)\right| = i$ and $\left|f^{-1}(s)\right| = j$ is $\binom{m}{i,j,m-i-j}p^i p^j(1-2p)^{m-i-j}$, from the multinomial distribution. Thus sum of this over i,j both at most k equals $E(X_rX_s)$. Again because m grows while k is fixed, the sum is asymptotic to the single term with $i = j = k$. Here the multinomial coefficient is asymptotic to $m^{2k}/(k!k!)$. With the other approximations as above, we have

$$E(X^2) = E(X) + \sum_{r<s}E(X_rX_s) \sim E(X) + n^2b^2 \sim E(X)^2.$$

8.5.30. *The length of the longest constant run in a list of n random heads and tails is $(1+o(1))\lg n$.* Let X be the number of runs of length k in a

random list of n flips. A set of k consecutive flips agrees with probability $2\cdot 2^{-k}$. There are $n-k+1$ such sets. Hence $\mathrm{E}(X) = (n-k+1)2^{-k+1}$. Let ε be a fixed small positive constant.

If $k \ge (1+\varepsilon)\lg n$, then $\mathrm{E}(X) \to 0$, which implies that almost every list has no run as long as $(1+\varepsilon)\lg n$.

If $k \le (1-\varepsilon)\lg n$, then $\mathrm{E}(X) \to \infty$. If also $\mathrm{E}(X^2) \to \mathrm{E}(X)^2$, then by the Second Moment Method $\mathrm{P}(X=0) \to 0$, which implies that almost every list has at least one run as long as $(1-\varepsilon)\lg n$. Since X is the sum of $n-k+1$ indicator variables, we have $\mathrm{E}(X^2) = \mathrm{E}(X) + \sum_{i\ne j}X_iX_j$. When the locations corresponding to X_i and X_j are disjoint, the events $X_i = 1$ and $X_j = 1$ are independent. When they overlap, the probability that both are 1 is bounded by 2^{2k-2}.

The essence of the computation is that almost all of the expectation comes from independent events. When the segments overlap, their starting points differ by less than k. There are at most $2(n-j)$ ordered pairs where the difference in the starting locations is j. As j varies, fewer than $2nk$ ordered pairs of k-segments are overlapping. Hence at least $(n-k+1)(n-k)-2nk$ ordered pairs of variables satisfy $\mathrm{P}(X_iX_j) = 2^{2(-k+1)}$.

Since we only need the leading behavior of $\mathrm{E}(X^2)$, we compute $\mathrm{E}(X^2) = n2^{-k+1} + n^2 2^{2(-k+1)} - 4nk2^{2(-k+1)} + O(nk) \sim \mathrm{E}(X)^2$.

Comment: This side of the threshold can also be derived by the first moment method. Using $\lfloor n/k\rfloor$ disjoint segments, where $k = (1-\varepsilon)\lg n$, the constancy of these segments are independent events. Each occurs with probability $2\cdot 2^{-k}$, so the probability p that none occurs is $(1-2\cdot 2^{-k})^{\lfloor n/k\rfloor}$. We have

$$p \le (1-2/n^{1-\varepsilon})^{\lfloor n/k\rfloor} < e^{-2/n^{1-\varepsilon}(2n/k)} = e^{-(4/k)n^\varepsilon} \to 0,$$

so almost every sequence has a run at least this long.

8.5.31. *With $p = (1-\varepsilon)\log n/n$, almost every graph has at least $(1-o(1))n^\varepsilon$ isolated vertices.* Let X be the random variable counting the isolated vertices; we have $\mathrm{E}(X) = n(1-p)^{n-1}$. Let m be a desired threshold, with $m < \mathrm{E}(X)$. By Chebyshev's Inequality,

$$\begin{aligned} P(X < m) &= P(X - \mathrm{E}(X) < m - \mathrm{E}(X)) < P(|X-\mathrm{E}(X)| \ge \mathrm{E}(X) - m) \\ &\le \frac{\mathrm{E}(X^2) - \mathrm{E}(X)^2}{(\mathrm{E}(X)-m)^2}. \end{aligned}$$

We have $X \ge m$ almost always if m is chosen so this bound approaches 0.

If $p = (1-\varepsilon)\log n/n$ for constant ε, then $\mathrm{E}(X) \sim n^\varepsilon$. We have also computed $\mathrm{E}(X^2) \sim n^{2\varepsilon}$; this was what was required of the second moment method to obtain the threshold for disappearance of isolated vertices. Hence $\mathrm{E}(X^2) - \mathrm{E}(X)^2 \in o(\mathrm{E}(X)^2)$, and we may choose $m = (1-\delta)\mathrm{E}(X) =$

$(1-\delta)n^{eps}$ for any $\delta > 0$. It is possible to make m closer to $\mathrm{E}(X)$, but this requires more accurate estimates of $\mathrm{E}(X^2)$ and $\mathrm{E}(X)$, since the leading behavior cancels when $\mathrm{E}(X)^2$ is subtracted from $\mathrm{E}(X^2)$.

8.5.32. *The threshold size k for bad k-sets in G^p, where p is fixed and a k-set S is bad if its vertices have no common neighbor, is $\log_{1/p}\frac{n}{c\ln n}$ with the parameter $c=1$.* This scenario is obtained from that of Exercise 8.5.20 by setting $t=0$, redefining s as k, and turning that k into 0, except that $|S|$ is no longer fixed; we seek a threshold. Below the threshold ($c<1$), k-sets are small enough and leave enough vertices outside so that almost always every k-set has a common neighbor.

Let X be the number of bad k-sets. A vertex v outside a k-set S fails to be a common neighbor with probability $1-p^k$. The probability of having no common neighbor is $(1-p^k)^{n-k}$, so $\mathrm{E}(X)=n(1-p^k)^{n-k} < ne^{-p^k(n-k)}$. If $p^k(n-k)\sim c\ln n$ with $c=1+\varepsilon$, then $\mathrm{E}(X)\to 0$. Hence we set $p^k=\frac{c\ln n}{n}$, which translates to $k=\log_{1/p}\frac{n}{c\ln n}$.

Since $p^k\to 0$, $1-p^k\sim e^{-p^k}$, and hence $\mathrm{E}(X)\to\infty$ when $k=\log_{1/p}\frac{n}{c\ln n}$ with $c=1-\varepsilon$. Now the second moment method can be used to show that $\mathrm{P}(X>0)\to 1$. We need to show $\mathrm{E}(X^2)\sim\mathrm{E}(X)^2$.

Consider the indicator variables for individual k-sets. If X_1 and X_2 correspond to disjoint k-sets, then $\mathrm{E}(X_iX_j)=\mathrm{E}(X_i)^2$. The number of ordered pairs of this sort is the multinomial coefficient $\binom{n}{k,k,n-2k}$, which is asymptotic to $n^{2k}/k!^2$. The terms that come from overlapping k-sets are fewer; the number of them is bounded by a multiple of n^{2k-1}. Since that k in the exponent grows with $\ln n$, there remains work to do, but the idea in the second moment method here is to show that asymptotically all the contribution to $\mathrm{E}(X^2)$ comes from terms that sum to roughly $\mathrm{E}(X)^2$.

8.5.33. *If p is fixed and $k=k(n)\in o(n/\log n)$, then almost every G^p is k-connected.* (sketch) It suffices to show that almost every G^p has the property that any two vertices have k common neighbors. We consider the expected number of vertices failing this. Two vertices fail this with probability $b(n-2,p^2,k-1)$, where $b(m,q,l)$ is the probability of having at most l successes in m independent trials with success probability q.

When $l=o(m)$, $b(m,q,l)$ is bounded by a multiple of the top term in the sum, $\binom{m}{l}q^l(1-q)^{m-l}$ (proof omitted). Applying this enables us to show that $\binom{n}{2}b(n-2,p^2,k-1)\to 0$ when $k\in o(n/\log n)$.

8.5.34. This duplicates Exercise 8.5.31.

8.5.35. A *t-interval* is a subset of $\mathbb{R}$ that is the union of at most t intervals. The *interval number* of a graph G is the minimum t such that G is an intersection graph of t-intervals (each vertex is assigned a set that is the union of at most t intervals). Prove that almost all graphs (edgeprobability 1/2) have interval number at least $(1-o(1))n/(4\lg n)$. (Hint: Compare the number of representations with the number of simple graphs. Comment: Scheinerman [1990] showed that almost all graphs have interval number $(1+o(1))n/(2\lg n)$.) (Erdős–West [1985])

8.5.36. *Threshold for complete matching in random bipartite graph.* Let G be a random labeled subgraph of $K_{n,n}$, with partite sets A,B and independent edge probability $p=(1+\varepsilon)\ln n/n$. Call S a *violated set* if $|N(S)|<|S|$.

a) If $\varepsilon<0$, then almost surely G has no complete matching. Although the probability that no vertex in A is isolated is $[1-(1-p)^n]^n$, it is not easy to show that this approaches 0 when $\varepsilon<0$.

If X is the number of isolated vertices in A, then $\mathrm{E}(X)=n(1-p)^n$. With $p=o(1/\sqrt{n})$, this yields $\mathrm{E}(X)\sim ne^{-np}=n^{-\varepsilon}$. Hence $\mathrm{E}(X)\to\infty$ if $\varepsilon<0$. Because the 0,1-random variables X_i that contribute to X are independent, we have $(\mathrm{E}(X))^2=\mathrm{E}(X)+n(n-1)\mathrm{E}(X_iX_j)\sim n^2(1-p)^{2n}=\mathrm{E}(X)^2$, and the second moment method yields the claim.

b) If S is a minimal violated set, then $|N(S)|=|S|-1$ and $G[S\cup N(S)]$ is connected. If $|N(S)|<|S|-1$, then $S-x$ is a violated set, for any $x\in S$. If $G[S\cup N(S)]$ is not connected, let $\{S_i\}$ be the partition of S induced by its components. Then $\cup N(S_i)=N(S)$, and $\{N(S_i)\}$ are disjoint, so by the pigeonhole principle some S_i is a violated set.

c) If G has no complete matching, then A or B contains a violated set with at most $\lceil n/2\rceil$ elements. If S is a violated subset of A, then $B-N(S)$ is a violated subset of B. If S is a minimal violated subset of A with more than $n/2$ elements, then $|B-N(S)|\le n-(|S|-1)\le\lceil n/2\rceil$.

d) If $\varepsilon>0$, then almost surely G has a complete matching. Let X be the number of spanning trees in subgraphs of the form $G[S\cup N(S)]$, where S is a minimal violated subset of A or B having size at most $\lceil n/2\rceil$. By parts (b) and (c), it suffices to show that $\mathrm{E}(X)\to 0$. The number of spanning subtrees of $K_{r,s}$ is $r^{s-1}s^{r-1}$ if $r,s\ge 1$ (see Exercise 2.2.14). Separating the term due to violated sets of size 1, we have

$$\mathrm{E}(X)=2n(1-p)^n+2\sum_{k=2}^{\lceil n/2\rceil}\binom{n}{k}\binom{n}{k-1}k^{k-2}(k-1)^{k-1}p^{2k-2}(1-p)^{k(n-k+1)}.$$

As seen in part (a), the expected number of isolated vertices approaches 0. Since $\binom{m}{l}<(me/l)^l$, the summation is bounded by $\sum_{k=2}^{\lceil n/2\rceil}(ne)^{2k-1}k^{-2}p^{2k-2}(1-p)^{k(n+1)/2}$. We can ignore the k^{-2} to obtain an upper bound of $(p^2ne)^{-1}\sum_{k\ge 2}[(pne)^2(1-p)^{(n+1)/2}]^k$. The constant ratio in the geometric series is asymptotic to $[e(1+\varepsilon)\ln n]^2n^{-(1+\varepsilon)/2}$, which approaches 0 for $\varepsilon>-1$. The geometric series is bounded by $x^2/(1-x)$, which is asymptotic to x^2 when $x\to 0$. For the bound on the summation,

$$(p^2ne)^{-1}(pne)^4(1-p)^{n+1}\sim p^2n^3e^3e^{-np}=e^3(1+\varepsilon)^2(\ln n)^2n^{-\varepsilon}\to 0.$$

8.5.37. *If* $0 < p < 1$, *and* $k_1, \ldots, k_r$ *are nonnegative integers summing to* m, *then* $\prod_{i=1}^r [1-(1-p)^{k_i}] \le [1-(1-p)^{m/r}]^r$. Since the logarithm function is monotone, it suffices to show that $\sum_{i=1}^r \ln[1-(1-p)^{k_i}] \le r\ln[1-(1-p)^{m/r}]$. This reduces to $\frac{1}{r}\sum_{i=1}^r f(k_i) \le f\left(\frac{1}{r}\sum_{i=1}^r k_i\right)$, where $f(x) = \ln[1-(1-p)^x]$. That is, it suffices to show that f is a concave function for $x \ge 0$.

Rewriting $(1-p)^x$ as $e^{x\ln(1-p)}$ makes it easy to differentiate f twice. The value of the second derivative is $-\left(\frac{\ln(1-p)}{(1-p)^{-x}-1}\right)^2$, which is negative. Hence f is concave.

8.5.38. ($\bullet$) *Tail inequality for binomial distribution.* Let $X = \sum X_i'$, where each X_i' is an indicator variable with success probability $P(X_i' = 1) = .5$, so $\mathrm{E}(X) = n/2$. Applying Markov's Inequality to the random variable $Z = (X - \mathrm{E}(X))^2$ yields $P(|Z| \ge t) \le Var(X)/t^2$. Setting $t = \alpha\sqrt{n}$ yields a bound on the tail probability: $P(|X - np| \ge \alpha\sqrt{n}) \le 1/(2\alpha^2)$. Use Azuma's Inequality to prove the stronger bound that $P(|X-np| > \alpha\sqrt{n}) < 2e^{-2\alpha^2}$. (Hint: Let $Y_i' = X_i' - .5$. Let F_i be the knowledge of $Y_1', \ldots, Y_i'$, and let $Y_i = \mathrm{E}(Y|F_i)$.)

8.5.39. ($\bullet$) *Bin-packing.* Let the numbers $S = \{a_1, \ldots, a_n\}$ be drawn uniformly and independently from the interval $[0,1]$. The numbers must be placed in bins, each having capacity 1. Let X be the number of bins needed. Use Lemma 8.5.36 to prove that $P\big(|X - \mathrm{E}(X)| \ge \lambda\sqrt{n}\big) \le 2e^{-\lambda^2/2}$.

8.5.40. *Azuma's Inequality and the Traveling Salesman Problem.*

a) Azuma's Inequality for general martingales: If $\mathrm{E}(X_i) = X_{i-1}$ *and* $|X_i - X_{i-1}| \le c_i$ *for all* i, *then* $\mathrm{P}(X_n - X_0 \ge \lambda\sqrt{\sum c_i^2}) \le e^{-\lambda^2/2}$. Let $\gamma = \sqrt{\sum c_i^2}$. By translation, we may assume $X_0 = 0$. Markov's Inequality implies $\mathrm{P}(e^{tX_n} \ge e^{t\lambda\gamma}) \le \mathrm{E}(e^{tX_n})/e^{t\lambda\gamma}$ for all $t > 0$. It suffices to prove that $\mathrm{E}(e^{tX_n}) \le e^{t^2\gamma^2/2}$ and then set $t = \lambda/\gamma$. We prove the bound on the expectation by induction on n.

We have

$$\mathrm{E}(e^{tX_n}) = \mathrm{E}(e^{tX_{n-1}}e^{t(X_n-X_{n-1})}) = \mathrm{E}(\mathrm{E}(e^{tX_{n-1}}e^{t(X_n-X_{n-1})}|X_{n-1})) = \mathrm{E}(e^{tX_{n-1}}\mathrm{E}(e^{tY}|X_{n-1})),$$

where $Y = X_n - X_{n-1}$. By hypothesis, $\mathrm{E}(Y) = 0$ and $|Y| \le c_n$. Let Z be the random variable Y/c_n, so $|Z| \le 1$, and let $u = c_n t$. We have $\mathrm{E}(e^{tY}) = \mathrm{E}(e^{uZ}) \le \frac{1}{2}(e^u + e^{-u}) \le e^{u^2/2}$, so the inner expectation is bounded by $e^{t^2c_n^2/2}$. This is a constant, so

$$\mathrm{E}(e^{tX_{n-1}}\mathrm{E}(e^{tY}|X_{n-1})) = e^{t^2c_n^2/2}\mathrm{E}(e^{tX_{n-1}}) = e^{t^2\gamma^2/2},$$

using the induction hypothesis.

b) If Y *is the distance from* $z \in S$ *to the nearest of* n *points chosen uniformly and independently in the unit square* S, *then* $\mathrm{E}(Y) < c/\sqrt{n}$, *for some constant* c. The probability that a random point x lies in region R equals the area of R. Fixing y, z, $\mathrm{P}(d(x,z) > y) \le \pi y^2/4$, with equality when z is in the corner. Hence the probability that the nearest of n points is farther than y from z is bounded by $(1-\pi y^2/4)^n$. Since $\mathrm{E}(Y) = \int_0^\infty \mathrm{P}(Y \ge y)dy$, we have $\mathrm{E}(Y) \le \int_0^\infty (1-\pi y^2/4)^n dy < \int_0^\infty e^{-n\pi y^2/4}dy = 1/\sqrt{n}$.

c) The smallest length of a tour through a random set of n *points in the unit square is highly concentrated around its expectation.* Let X be the actual length of the optimal tour for the random points $Q = \{p_1, \ldots, p_n\}$. Let $F_i = \{p_1, \ldots, p_i\}$, and let $X_i = \mathrm{E}(X|F_i)$, so $\{X_i\}$ is a martingale with $X_0 = \mathrm{E}(X)$ and $X_n = X$. Let W be the length of the optimal tour when p_i is omitted from the set. Let $Y = \mathrm{E}(W|F_{i-1})$ and $Y' = \mathrm{E}(W|F_i)$; note that $Y' = Y$, because p_i does not appear in the tour measured by W. Fixing F_i, the expectation of $X - W$ is bounded by $2/\sqrt{(n-i)}$, by part (b), since we can include p_i in the tour by making a detour from the closest point in $Q - F_i$. Hence $0 \le \mathrm{E}(X - W|F_i) \le 2/\sqrt{n} - i$. Since this bound of $2/\sqrt{n} - i$ is valid for any choice of p_i, we also have the same bounds for $\mathrm{E}(X - W|F_{i-1})$. However, $\mathrm{E}(X - W|F_i) = X_i - Y'$, and $\mathrm{E}(X - W|F_{i-1}) = X_{i-1} - Y$. With $Y = Y'$, we have $X_i - X_{i-1} = \mathrm{E}(X - W|F_i) - \mathrm{E}(X - W|F_{i-1})$, and our bounds on these quantities yield $|X_i - X_{i-1}| \le 2/\sqrt{n} - i$. Since the sum of the reciprocals of the first n natural numbers is asymptotic to $\ln n$, part (a) yields $P[X - \mathrm{E}(X) \ge \lambda(2+\varepsilon)\sqrt{\ln n}] \le e^{-\lambda^2/2}$. The same bound holds for the other tail, because part (a) applies also to $-X$.

8.6. EIGENVALUES OF GRAPHS

8.6.1. *Interpretation of cycle space and bond space.* Consider a graph G.

a) The symmetric difference of two even subgraphs G_1 *and* G_2 *is an even subgraph.* At a given vertex v, let S_1 and S_2 be the sets of incident edges in G_1 and G_2, respectively. In $G_1 \triangle G_2$, the edges incident to v are $S_1 \triangle S_2$. We have $|S_1 \triangle S_2| = |S_1 - (S_1 \cap S_2)| + |S_2 - (S_1 \cap S_2)|$. Since $|S_1|$ and $|S_2|$ have the same parity, the sizes of the differences also have the same parity.

b) The symmetric difference of two edge cuts is an edge cut. This is Exercise 4.1.27.

c) Every edge cut shares an even number of edges with every even subgraph. Every even subgraph decomposes into cycles (Proposition 1.2.27), and every cycle crosses every edge cut an even number of times.

Comment: By parts (a) and (b), the cycle space **C** and bond space **B** of a graph G are binary vector spaces. They are subpaces of the space of dimension $e(G)$ whose vectors are the incidence vectors of subsets of the edges. By part (c), they are orthogonal subspaces of this space, since the dot product of the incidence vector of an even subgraph and the incidence vector of an edge cut is 0.

8.6.2. *For a connected graph G with n vertices and m edges, the cycle space $\mathbf{C}$ has dimension $m-n+1$, and the bond space $\mathbf{B}$ has dimension $n-1$.*

Since the spaces are orthogonal within a space of dimension m, it suffices to show that $\dim \mathbf{C} \ge m-n+1$ and $\dim \mathbf{B} \ge n-1$.

Choose a spanning tree T. Each edge of $E(G)-E(T)$ forms a unique cycle along with edges in T. These cycles are linearly independent in $\mathbf{C}(G)$, since each has a nonzero coordinate outside $E(T)$ that is zero in all other incidence vectors in this set. Hence $\dim \mathbf{C}(G) \ge m-n+1$.

Choose $n-1$ vertices in G. We show that the edge cuts isolating these vertices are linearly independent in $\mathbf{B}$. A nonzero linear combination over $\mathbb{F}_2$ sums a nonempty subset S of these incidence vectors. The resulting coordinate for an edge is even if and only if the edge has an even number of endpoints in S. Edges in $[S,\overline{S}]$ are covered exactly once, and these coordinates remain nonzero. Since S is a nonempty proper subset of the vertices of a connected graph, $[S,\overline{S}] \ne \varnothing$. Hence these vectors are linearly independent, and $\dim \mathbf{B} \ge n-1$.

8.6.3. *a) If in a simple graph G the vertices in $S \subseteq V(G)$ have identical neighborhoods, then 0 is an eigenvalue with multiplicity at least $|S|-1$.* The rows of the adjacency matrix A corresponding to vertices of S are identical. Hence there are at most $n(G)-|S|+1$ linearly independent rows, and the rank is at most $n(G)-|S|+1$, so at least $|S|-1$ eigenvalues are 0.

a) If in a simple graph G the vertices in $S \subseteq V(G)$ have identical closed neighborhoods, then -1 is an eigenvalue with multiplicity at least $|S|-1$. The rows of $A+I$ corresponding to vertices of S are identical, where I is the identity matrix. Hence $A+I$ has 0 as an eigenvalue with multiplicity at least $|S|-1$. However, the spectrum of $A+I$ is shifted up from the spectrum of A by 1, since adding λI to a matrix adds λ to each eigenvalue.

8.6.4. *Counting 3-cycles and 4-cycles using eigenvalues.* Let A be the adjacency matrix of G. Let σ_k be the number of subgraphs of G that are k-cycles. Let L_k and D_k be the sums of the kth powers of the eigenvalues and the vertex degrees, respectively. We have $L_k = \text{Trace} A^k$, by Remark 8.6.2(1) and Proposition 8.6.7. Proposition 8.6.7 also implies that L_k counts the ways to start at a vertex, follow a walk of length k, and end at the same vertex.

$\sigma_3 = \frac{1}{6}L_3$. Every closed walk of length 3 traverses a 3-cycle, and there are six ways to traverse a 3-cycle in in three steps.

$\sigma_4 = \frac{1}{8}L_4 - \frac{1}{4}D_2 - \frac{3}{4}D_1$. A closed walk of length 4 may traverse a 4-cycle, or a path of length 2 (starting at either end or starting in the middle in either direction), or an edge (starting at either end). Hence L_4 counts the copies of C_4 eight times, the copies of P_3 four times, and the copies of P_2 twice. There are σ_4 copies of C_4, there are $\sum_{v\in V(G)} \binom{d(v)}{2}$ copies of P_3, and there are $\sum_{v\in V(G)} d(v)/2$ copies of P_2. Thus $L_4 = 8\sigma_4 + 2D_2 + 2D_1 + D_1$.

8.6.5. *Deletion formulas for the characteristic polynomial.* We write $\phi(G;\lambda)$ as ϕ_G. For a vertex or edge w in G, let $Z(w)$ denote the set of cycles in G containing w. We use Sachs' formula for the characteristic polynomial: $\phi_G = \sum c_i\lambda^{n-i}$, where $c_i = \sum_{H\in \mathbf{H}_i}(-1)^{k(H)}2^{s(H)}$, where $\mathbf{H}_i$ is the set of i-vertex subgraphs of $V(G)$ whose components are edges or cycles, and $k(H)$ and $s(H)$ denote the number of components and number of cycles of H.

a) $\phi_G = \lambda\phi_{G-v} - \sum_{u\in N(v)}\phi_{G-v-u} - 2\sum_{C\in Z(v)}\phi_{G-V(C)}$. Consider the contributions to ϕ_G in Sachs' formula. The subgraphs avoiding v contribute $\lambda\phi G - v$, since these subgraphs are present in both G and $G-v$, but in G the term where they contribute has an extra factor of λ. The subgraphs having a component that is an edge uv contribute $-\phi_{G-v-u}$, since these subgraphs correspond to subgraphs of $G-u-v$ by adding one component that is an edge. The subgraphs having a cycle through v contribute $-2\sum_{C\in Z(v)}\phi_{G-V(C)}$, since these subgraphs correspond to subgraphs of G by adding the vertex set of the cycle, which adds one component that is a cycle and therefore contributes a factor of 2.

b) $\phi_G = \phi_{G-xy} - \phi_{G-x-y} - 2\sum_{C\in Z(xy)}\phi_{G-V(C)}$. Consider the contributions to ϕ_G in Sachs' formula. The subgraphs avoiding xy contribute $\lambda\phi G - v$, since these subgraphs are present in both G and $G-xy$ and the number of vertices used in computing the exponent is the same. The subgraphs having xy as a component contribute $-\phi_{G-x-y}$, since these subgraphs correspond to subgraphs of $G-x-y$ by adding one component that is an edge. The subgraphs having a cycle through xy contribute $-2\sum_{C\in Z(v)}\phi_{G-V(C)}$, by the same reasoning as for the last term in part (a).

8.6.6. *Characteristic polynomial for paths and cycles.*

a) Recurrence. Using Exercise 8.6.5, deleting an endpoint of P_n yields $\phi_{P_n} = \lambda\phi_{P_{n-1}} - \phi_{P_{n-2}}$, with $\phi_{P_0}=1$ and $\phi_{P_1}=\lambda$. Deleting an edge from C_n yields $\phi_{C_n} = \phi_{P_n} - \phi_{P_{n-2}} - 2$.

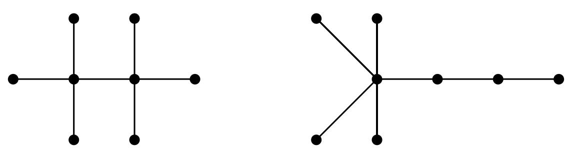
($\bullet$) *b) Without solving the recurrence, prove that $\{2\cos(2\pi j/n)\colon 0 \le j \le n-1\}$ are the eigenvalues of C_n.* This is a matter of designing the appropriate eigenvectors and checking the multiplication.

c) Eigenvalues of C_n^2. If G^2 is obtained from a k-regular graph G by making vertices at distance 2 adjacent, then $A(G^2) = A^2(G) + A(G) - kI$. If x is an eigenvector of $A(G)$ with associated eigenvalue λ, then $A(G^2)x = (\lambda^2+\lambda-k)x$, so x is an eigenvector of $A(G)$ with associated eigenvalue $\lambda^2+\lambda-k$.

8.6.7. *When G is a tree, the coefficient of λ^{n-2k} in the characteristic polynomial is $(-1)^k\mu_k(G)$, where $\mu_k(G)$ is the number of matchings of size k.* By

Corollary 8.6.6, the coefficient c_i of λ^{n-i} in the characteristic polynomial is $\sum(-1)^{k(H)}2^{s(H)}$, where the summation is over all i-vertex subgraphs for which every component is an edge or a cycle, $k(H)$ is the number of these components, and $s(H)$ is the number of cycles. In a tree T, there are no cycles, so $c_i = (-1)^{i/2}\mu_{i/2}(T)$.

Nonisomorphic "co-spectral" 8-vertex trees that both have characteristic polynomial $\lambda^8 - 7\lambda^6 + 9\lambda^4$. We seek two trees on eight vertices that have nine matchings of size 2 and no larger matchings (as trees, they automatically have seven matchings of size 1). The trees appear below. (Comment: As $n \to \infty$, almost no trees are uniquely determined by their spectra.)



8.6.8. *If T is a tree, then $\alpha(T)$ is the number of nonnegative eigenvalues of T.* Let T be an n-vertex tree. In any graph, the vertices outside a maximum independent set form a minimum vertex cover, so $\alpha(T) = n(T) - \beta(T)$, where $\beta(T)$ is the vertex cover number.

Since the characteristic polynomial has the form $\prod(\lambda - \lambda_i)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues, the degree of the last nonzero term is the multiplicity of 0 as an eigenvalue. By Exercise 8.6.7, the coefficient of $n-2k$ is nonzero if and only if T has a matching of size k. Hence the least degree of a nonzero term is $n - 2\alpha'(T)$. Since T is bipartite, this equals $n - 2\beta(T)$, which equals $\alpha(T) - \beta(T)$. Since $\alpha(T) + \beta(T) = n$, there remain $2\beta(T)$ nonzero eigenvalues, which are split equally between positive and negative values, since T is bipartite. We conclude that there are exactly $\alpha(T)$ nonnegative eigenvalues.

8.6.9. *The eigenvalues of a graph with n vertices and m edges are bounded by $\sqrt{2m(n-1)/n}$.* Applying the Cauchy-Schwarz Inequality to the vector of eigenvalues other than the maximum yields

$$(\textstyle\sum_{i=2}^m \lambda_i)^2 \le (n-1)(\sum_{i=2}^m \lambda_i^2).$$

Using $\sum \lambda_i = 0$ on the left and $\sum \lambda_i^2 = 2e$ on the right converts this to $(-\lambda_1)^2 \le (n-1)(2e - \lambda_1^2)$, which is equivalent to $\lambda_1 \le \sqrt{2e(n-1)/n}$.

8.6.10. *The eigenvalues of the cartesian product of graphs G and H are the sums of eigenvalues of G and H.* Let $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ be the eigenvalues of G and H, with adjacency matrices A and B, respectively. The entry in row (i, j) and column (r, s) of C, the adjacency matrix of $A(G \,\Box\, H)$, is $b_{j,s}$ if $i = r$ and $a_{i,r}$ if $j = s$; otherwise it is 0.

Let u and v be eigenvectors for eigenvalues λ and μ of G and H, respectively. Let w be the vector indexed by $[m] \times [n]$ that is defined by $w_{i,j} = u_i v_j$. In Cu, coordinate (i, j) is $\sum_r \sum_s c_{(i,j),(r,s)} w_{r,s}$. The terms in the sum are 0 except when $i = r$ or $j = s$. Note that $c_{(i,j),(i,j)} = 0$. We thus obtain

$$\begin{aligned}(Cw)_{i,j} &= \sum_s c_{(i,j),(i,s)} u_i v_s + \sum_r c_{(i,j),(r,j)} u_r v_j - c_{(i,j),(i,j)} u_i v_j \\ &= u_i \sum_s b_{j,s} v_s + v_j \sum_r a_{i,r} u_r = u_i (Bv)_j + v_j (Au)_i \\ &= u_i \mu v_j + \lambda v_j \lambda u_i = (\mu + \lambda) u_i v_j = (\mu + \lambda) w_{i,j}\end{aligned}$$

This computation shows that w is an eigenvector associated with eigenvalue $\mu + \lambda$. Furthermore, if u and u' are two linearly independent eigenvectors of A associated with λ, then the resulting w and w' are linearly independent eigenvectors of C. Thus the eigenvalues of C are given by the list of all $\lambda_i + \mu_j$ such that $1 \le i \le m$ and $1 \le j \le n$.

The eigenvalues of the k-dimensional hypercube Q_k range from k to $-k$, with $k - 2r$ being an eigenvalue with multiplicity $\binom{k}{r}$, for $0 \le r \le k$. The claim holds by inspection for $k = 1$, where $Q_k = K_2$. For $k > 1$, express Q_k as $Q_{k-1} \,\Box\, K_2$. Since the eigenvalues of K_2 are 1 and -1, each with multiplicity 1, each eigenvalue μ for Q_{k-1} becomes eigenvalues $\mu + 1$ and $\mu - 1$ for Q_k. Thus the multiplicity of $k - 2r$ in the spectrum of Q_k is the sum of the multiplicities of $k - 2r - 1$ and $k - 2r + 1$ in the spectrum of Q_{k-1}. Using the induction hypothesis, the multiplicity is $\binom{k-1}{r} + \binom{k-1}{r-1}$, which by the binomial recurrence equals $\binom{k}{r}$.

8.6.11. (•) Compute the spectrum of the complete p-partite graph $K_{m,\ldots,m}$. (Hint: Use the expression $A(\overline{G}) = J - I - A(G)$ for the adjacency matrix of the complement.)

8.6.12. *If the characteristic polynomial of G is $x^8 - 24x^6 - 64x^5 - 48x^4$, then $G = K_{2,2,2,2}$.* The degree is $n(G)$, and the coefficient of $x^{n(G)-2}$ is $-e(G)$. Since $\binom{8}{2} = 28$, we obtain G by deleting four edges from K_8. Since the coefficient of $x^{n(G)-3}$ is -2 times the number of triangles (Corollary 8.6.6), our graph has 32 triangles. In K_n there are 56 triangles, and deleting an edge kills six triangles. If we kill 24 triangles by deleting four edges, then we must not kill a triangle twice, which means that the four deleted edges are pairwise disjoint. Now $K_{2,2,2,2}$ is the only graph satisfying all these requirements.

8.6.13. (!) Prove that G is bipartite if G is connected and $\lambda_{\max}(G) = -\lambda_{\min}(G)$.

8.6.14. *The squashed-cube dimension (Definition 8.4.12) of a graph G is at least the maximum of the number of positive eigenvalues and the number* *of negative eigenvalues of the matrix $R(G)$ whose whose (i, j)th entry is $d_G(v_i, v_j)$.* Note that $R(K_n) = J - I$, whose eigenvalues are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$. Hence the squashed cube dimension of K_n is at least $n-1$, and equality follows from the construction in Example 8.4.13.

To prove the eigenvalue bound, we encode the distances in a quadratic form. Let $x = (x_1, \ldots, x_n)$, and let $h(x) = \sum_{i,j} d_G(v_i, v_j)x_i x_j = x^T R(G)x$. The combinatorial part of the argument recomputes this sum by accumulating contributions from coordinates in a squashed-cube embedding.

Consider an encoding f, with $f(v_i) = (f_1(v_i), \ldots, f_N(v_i))$. Let V_m^α denote the set of indices i such that $f_m(v_i) = \alpha$. Coordinate m contributes 1 to $d(f(v_i), f(v_j))$ if and only if $i \in V_m^0$ and $j \in V_m^1$, or vice versa. For each such unit contribution, we have a contribution of $x_i x_j + x_j x_i$ to $h(x)$. Hence the grouping by coordinates yields $h(x) = 2\sum_{m=1}^N \left(\sum_{i\in V_m^0} x_i\right)\left(\sum_{j\in V_m^1} x_j\right)$. We have rewritten the quadratic form as a sum of N products of linear combinations. Now Sylvester's Law of Inertia (Lemma 8.6.14) states that expressing a quadratic form as a sum of N products of linear combinations of the variables requires $N \ge r$, where r is the maximum of the number of negative and number of positive eigenvalues of $R(G)$.

8.6.15. (!) The *Laplacian matrix* Q of a graph G is $D - A$, where D is the diagonal matrix of degrees and A is the adjacency matrix. The *Laplacian spectrum* is the list of eigenvalues of Q.

a) Prove that the smallest eigenvalue of Q is 0.

b) Prove that if G is connected, then eigenvalue 0 has multiplicity 1.

c) Prove that if G is k-regular, then $k - \lambda$ is a Laplacian eigenvalue if and only if λ is an ordinary eigenvalue of G, with the same multiplicity.

8.6.16. Given that $\lambda_{\max}(M) + \lambda_{\min}(M) \le \lambda_{\max}(P) + \lambda_{\max}(R)$ for any real symmetric matrix M partitioned as $\left(\begin{smallmatrix} P & Q \\ Q^T & R\end{smallmatrix}\right)$ with P, R square:

a) If A is a real symmetric matrix partitioned into t^2 submatrices $A_{i,j}$ such that the diagonal submatrices A_{ii} are square, then

$$\lambda_{\max}(A) + (t-1)\lambda_{\min}(A) \le \sum_{i=1}^m \lambda_{\max}(A_{i,i}).$$

Let $P = A_{1,1}$, and let R be the matrix obtained by deleting the first row and column of blocks. By the Interlacing Theorem, $\lambda_{\min}(A) \le \lambda_{\min}(R) \le \lambda_{\max}(R) \le \lambda_{\max}(R)$. The desired identity is trivial for $t = 1$. For $t > 1$, we apply induction. Using the given identity and then the induction hypothesis and the Interlacing Theorem,

$$\begin{aligned}\lambda_{\max}(A) + \lambda_{\min}(A) &\le \lambda_{\max}(A_{1,1}) + \lambda_{\max}(R)\\ &\le \lambda_{\max}(A_{1,1}) + \textstyle\sum_{i=2}^t \lambda_{\max}(A_{i,i}) - (t-2)\lambda_{\min}(R)\\ &\le \lambda_{\max}(A_{1,1}) + \textstyle\sum_{i=2}^t \lambda_{\max}(A_{i,i}) - (t-2)\lambda_{\min}(A).\end{aligned}$$

b) $\chi(G) \ge 1 + \lambda_{\max}(G)/(-\lambda_{\min}(G))$ for nontrivial G. Partition the vertices into the $\chi(G)$ color classes of an optimal coloring. With the vertices ordered by color classes, the diagonal submatrices of the adjacency matrix are identically 0, so their eigenvalues are all 0. If G is nontrivial, then the eigenvalues are not all 0, so $\lambda_{\min}(G) < 0$, since the sum is 0. Now part (a) yields $\lambda_{\max}(G) + (\chi(G) - 1)\lambda_{\min}(G) \le 0$, which becomes the desired inequality upon solving for $\chi(G)$.

c) $\lambda_1(G) + 3\lambda_n(G) \le 0$ for planar graphs. Using the Four Color Theorem to set $\chi(G) \le 4$ in part (b) yields the claim.

8.6.17. *The number of spanning trees in $K_{m,m}$ is m^{2m-2}.* Since $K_{m,m}$ is m-regular with $2m$ vertices, Theorem 8.6.28 applies and yields $\tau(K_{m,m}) = (2m)^{-1}\prod_{i=2}^{2m}(m - \lambda_i)$, where $m, \lambda_2, \ldots, \lambda_{2m}$ are the eigenvalues in nonincreasing order. Since the spectrum is Spec $(K_{m,m}) = \left(\begin{smallmatrix} m & 0 & -m \\ 1 & 2m-2 & 1\end{smallmatrix}\right)$ (Example 8.6.3), we obtain $\tau(K_{m,m}) = (2m)^{-1}(m-0)^{2m-2}(2m)^1 = m^{2m-2}$.

8.6.18. *If the columns of a matrix sum to* **0**, *then the cofactors obtained from deletion of a fixed row of A are all equal.* Let A be such a matrix. The cofactor $b_{i,j}$ is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting row i and column j of A. The definition of the determinant by expansion along rows of A yields $A(\mathrm{Adj}\,A) = (\det A)I$ for all A.

If rank $(A) < n-1$, than all cofactors are 0. Otherwise, rank $(A) = n-1$ and $\det A = 0$. Now $A\mathrm{Adj}\,A = 0$ implies that every column of Adj A is in the null-space of A. Since every row-sum of A is 0, we have $(1, \ldots, 1)^T$ in the null-space. Since rank $A = n-1$, every vector in the null-space is a multiple of this. Hence the columns of Adj A are constant-valued, meaning that the cofactors in each row of A are the same.

8.6.19. *Binet-Cauchy Formula:* $\det AB = \sum_{S\in\binom{[n]}{m}} \det A_S \det B_S$, *where A and B are $n\times m$ and $m\times n$ matrices, A_S consists of the columns of A indexed by S, and B_S consists of the rows of B indexed by S.*

Consider the matrix equation $DE = F$ below. Since each of these matrices has $n + m$ rows and columns, the usual product rule applies.

$$\begin{pmatrix} I_m & 0 \\ A & I_n\end{pmatrix}\begin{pmatrix} -I_m & B \\ A & 0\end{pmatrix} = \begin{pmatrix} -I_m & B \\ 0 & AB\end{pmatrix}$$

Since D is lower triangular with diagonal 1, $\det D = 1$. Expanding the determinant of F along the first m columns yields $\det F = (-1)^m \det AB$. Thus it suffices to prove that $\det E = (-1)^m \sum_{S\in\binom{[n]}{m}} \det A_S \det B_S$.

We use the permutation definition of $\det E$. Each nonzero product of $m + n$ elements must use an element from each row of A and an element from each column of B. The remaining $m-n$ elements lie on the diagonal of the upper left block. Thus the columns of the n elements from A have the same indices as the rows of the n elements from B. Furthermore, each term in the expansion of $\det A_S$ is multiplied by each term in the expansion of $\det B_S$ to obtain contributions to $\det E$.

It remains only to consider the signs of the contributions. Two obtain the positions on the main diagonal of A_S and the main diagonal of B_S, we apply n row interchanges from the positions on the main diagonal of E. For other terms in $\det A_S$, the sign corresponds to the parity of the permutation of rows within A_S, and similarly for B_S. Hence we obtain $\det A_S \det B_S$ times $(-1)^n$ for the inital permutation time $(-1)^{m-n}$ for the elements on the diagonal of $-I_m$.

8.6.20. *The incidence matrix of a simple graph G is totally unimodular if and only if G is bipartite.* (The incidence matrix has two $+1$s in each column; a matrix is *totally unimodular* if every square submatrix has determinant in $\{0, 1, -1\}$. Let $S = \{0, 1, -1\}$.)

Proof 1 *Sufficiency*. Given that G is bipartite, we prove by induction on k that the determinant of every k-by-k submatrix of the incidence matrix $M(G)$ is in S. This is certainly true for $k = 1$, since the entries in $M(G)$ are 0 or 1. Suppose $k > 1$, and let A be a k-by-k submatrix. Every column of $M(G)$ has two nonzero entries, so every column of A has at most two. If a column of A is 0, then $\det A = 0$. If a column of A has one nonzero entry, which must be 1, then expanding the determinant down that column expresses $\det A$ and ± 1 times the determinant of a $(k-1)$-by-$(k-1)$ matrix, which by the induction hypothesis is in S. Hence $\det A \in S$.

Finally, suppose that every column of A has two 1s. Each row of A corresponds to a vertex; weight the rows by $+1$ if the corresponding vertices belong to one partite set, -1 if they belong to the other. Since G is bipartite, every edge contains a vertex of each partite set, and hence with this weighting each column of A sums to 0, and $\det A = 0$.

Necessity. If G is not bipartite, then G contains an odd cycle C of length $2k+1$. Consider the rows and columns of $M(G)$ corresponding to the vertices and edges of C. Permuting the rows and columns of this submatrix A may change the sign of $\det A$ but not its magnitude. With $v_1, \ldots, v_{2k+1}$ being the names of the vertices on the cycle in order, permute the rows of A into this order, and permute the columns of A into the order $v_1v_2, v_2v_3, \ldots, v_{2k+1}v_1$. Now A has 1s in positions (i,i) for $1 \le i \le 2k+1$, positions $(i+1, i)$ for $1 \le i \le 2k$, and position $(1, 2k+1)$, with the other positions 0. If we expand the determinant of this matrix along the first row, we have only two nonzero terms. One is a subdeterminant with 1s only on the main diagonal and the first subdiagonal, and the other is a subdeterminant with 1s only on the main diagonal and the first superdiagonal. Because these come from expansion in columns 1 and $2k+1$, which are both odd, they have the same sign. Since each is a triangular matrix with 1s on the diagonal, we have $|\det A| = 2$, and $M(G)$ is not totally unimodular. The expansion is illustrated below for $k = 2$.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Proof 2 (induction on $e(G)$). If $e(G) = 0$, then the determinant of the empty matrix is 0, and G has no odd cycle. If $e(G) > 1$, then consider an arbitrary edge xy of G. The induction hypothesis states that $G - xy$ is bipartite if and only if $M(G - xy)$ is totally unimodular. If $M(G)$ is totally unimodular, then the submatrix $M(G - xy)$ is totally unimodular; hence $G - xy$ is bipartite whenever $xy \in E(G)$. Hence G is bipartite unless every edge of G belongs to every odd cycle of G. This happens only when G itself is an odd cycle; in this case $|\det M(G)| = 2$, as discussed above.

Conversely, if G is bipartite, then every $G - xy$ is bipartite, and the induction hypothesis guarantees every $M(G - xy)$ is totally unimodular. Hence $M(G)$ is totally unimodular unless some submatrix A using all the columns has determinant outside $\{-1, 0, +1\}$. Since $M(G - xy)$ is totally unimodular, expansion on the column indexed by xy forces both 1s in this column to appear in A. Since xy is arbitrary, every 1 in $M(G)$ appears in A. Now the fact that G is bipartite allows us to weight the rows corresponding to one partite set with $+1$ and those corresponding to the other partite set with -1 to obtain a linear dependence among the rows, yielding $\det A = 0$.

8.6.21. *If G is an (n, k, c)-magnifier with vertices $v_1, \ldots, v_n$, and H is the X, Y-bigraph with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $x_i y_j \in E(H)$ if and only if $i = j$ or $v_i v_j \in E(G)$, then H is an $(n, k+1, c)$-expander.* We verify the properties of an $(n, k+1, c)$-expander. The construction yields $d_H(x_i) = d_H(y_i) = d_G(v_i)$, so $\Delta(G) \le k$ yields $\Delta(H) \le k+1$. For $S \subseteq X$ with $|S| \le n/2$, let $S' = \{v_i \in V(G) \colon x_i \in S\}$. We have

$$|N_H(S)| = |S| + \left|N_G(S') - S'\right| \ge (1+c)\,|S| > (1 + c(1 - |S|/n)\,|S|.$$

8.6.22. *Existence of expanders of linear size.* An (n, α, β, d)-**expander** is an A, B-bigraph G with $|A| = |B| = n$, $\Delta(G) \le d$, and $|N(S)| \ge \beta\,|S|$ whenever $|S| \le \alpha n$.

a) If X is the size of the union of d k-subsets of $[n]$ *chosen at random, then* $P(X \le l) \le \binom{n}{l}(l/n)^{kd}$. If $X \le l$, then all the k-sets are confined to some

l-set. By multiplying the probability of this occurrence for a particular l by $\binom{n}{l}$, we obtain a loose upper bound (the events for distinct l-sets are not disjoint). For a particular l-set, we bound the probability it contains any selected k-set in the sequence: $\binom{l}{k}/\binom{n}{k} = \prod_{i=0}^{k-1} \frac{l-i}{n-i} \le (l/n)^k$. We need this event to happen d times.

b) If $\alpha\beta < 1$, then there is a constant d such that, for sufficiently large n, an (n, α, β, d)-expander exists. We generate an A, B-bigraph G by choosing d random complete matchings, discarding extra copies of edges. Hence $\Delta(G) \le d$. We claim that G fails to be an expander with probability less than 1. Let S be a *violated set* if $|N(S)| \le \beta\,|S|$. Let E be the event that a violated set exists; we bound $P(E)$ by a quantity that is less than 1 when n is sufficiently large. The d random matchings provide d random k-sets as neighbors of S when $|S| = k$. By (a), when $|S| < \alpha n$ we have

$$\mathrm{P}(E) < \sum_{k=1}^{\alpha n} \binom{n}{k}\binom{n}{\beta k}\left(\frac{\beta k}{n}\right)^{kd} < \sum_{k=1}^{\alpha n} \left(\frac{ne}{k}\right)^{k}\left(\frac{ne}{\beta k}\right)^{\beta k}\left(\frac{\beta k}{n}\right)^{kd}$$
$$= \sum_{k=1}^{\alpha n}\left[e^{1+\beta}\beta\left(\frac{\beta k}{n}\right)^{d-\beta-1}\right]^k < \sum_{k\ge 1}\left[e^{1+\beta}\beta(\alpha\beta)^{d-\beta-1}\right]^k .$$

If $\alpha\beta < 1$, then we can choose d to make the constant ratio in the geometric series as small as desired. We choose d so $e^{1+\beta}\beta(\alpha\beta)^{d-\beta-1} < \frac{1}{2}$.

c) Conclude the existence of k such that n, k, c-expanders exist for all sufficiently large n. We prove this when $c < 1$. Choose $\alpha = 1/2$ and $\beta = 1 + c$, so $\alpha\beta < 1$. By part (b), there is a constant d such that for sufficiently large n there exists an X, Y-bigraph with $|X| = |Y| = n$ that is an (n, α, β, d)-expander. For $S \subseteq X$ with $|S| \le n/2$, we have $|N(S)| \ge \beta\,|S| = (1 + c)\,|S| > (1 + c(1 - |S|/n))\,|S|$.

8.6.23. *Triangle-free graphs in which every two nonadjacent vertices have exactly two common neighbors.*

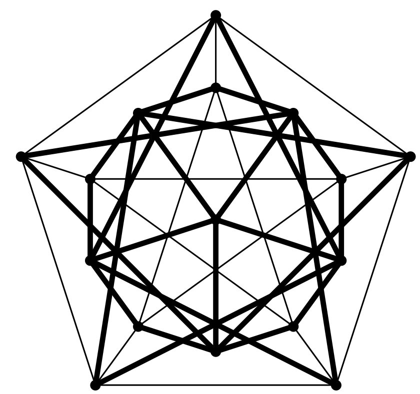
If k is the degree of a vertex in an n-vertex graph G of this sort, then $n = 1 + \binom{k+1}{2}$. For every pair of neighbors of x, there is exactly one nonneighbor of x that they have as a common neighbor. Conversely, every nonneighbor of x has exactly one pair of neighbors of x in its neighborhood, because these are its common neighbors with x. This establishes a bijection between the pairs in $N(x)$ and the nonneighbors of x. Counting x, $N(x)$, and $\overline{N}(x)$, we have $n(G) = 1 + k + \binom{k}{2} = 1 + \binom{k+1}{2}$. Since this argument holds for every $x \in V(G)$, we conclude that G is k-regular. (This is Exercise 1.3.33.)

G is strongly regular. By definition, $\lambda = 0$ (triangle-free) and $\mu = 2$, so G is strongly regular with parameters $(k, 0, 2)$ and $1 + \binom{k+1}{2}$ vertices.

The parameter k must be 1 more than the perfect square of an integer m that is not a multiple of 4. Setting $n - 1 = \binom{k+1}{2}$ and $\lambda = 0$ and $\mu = 2$

in the integrality conditions for strongly regular graphs (Theorem 8.6.33) shows that the two numbers $\frac{1}{2}\left(\binom{k+1}{2} \pm \frac{k(k+1)-2k}{\sqrt{4+4(k-2)}}\right)$ must be integers. The formulas simplify to $\frac{k}{4}(k+1\pm\sqrt{k-1})$. Since these are integers, $k-1$ must be a perfect square. With $m = \sqrt{k-1}$, the numbers $(m^2+1)(m^2+2\pm m)$ must be multiples of 4. This is impossible if m is a multiple of 4.

Examples. The values of k satisfying the necessary conditions are 1, 2, 5, 10, 26, etc. For $k = 1$, we have the degenerate example K_2. For $k = 2$, the graph is C_4. For $k = 5$, the 16-vertex graph is known as the Clebsch graph shown below; deleting any closed neighborhood yields the Petersen graph. For $k = 10$, a realization is known using combinatorial designs; it is called the Gewirtz graph.



8.6.24. *The Petersen graph; spectrum and application to decomposition of K_{10}.* The Petersen graph is regular of degree 3; any pair of adjacent vertices have no common neighbor, while every pair of non-adjacent vertices have one common neighbor. Hence the Petersen graph is strongly regular with $k = 3$, $\lambda = 0$, $\mu = 1$. It eigenvalues are therefore $3, r, s$ where $r+s = \lambda-\mu = -1$ and $rs = -(k-\mu) = -1$, so $r = 1$ and $s = -2$. The multiplicities a and b of r and s satisfy $k + ar + bs = 0$ and $1 + a + b = n$, so $(a, b) = (5, 4)$, and the spectrum is $\binom{3,1,-2}{1,5,\ 4}$.

Without using strong regularity, the following ad hoc argument also yields the spectrum. Consider the $3 + 3^2$ walks of length 1 or 2 that begin at a specified vertex v. Each other vertex is the other end of one of these, and v is the other end of three of them. Hence $P^2 + P = 2I + J$, where J is the all-1 matrix. Factoring $P^2 + P - 2I$ and multiplying $P - 3I$ by both this and J yields $(P - 3I)(P - I)(P + 2I) = 0$. Hence $(\lambda - 3)(\lambda - 1)(\lambda + 2)$ is the minimum polynomial of P, and P has eigenvalues $3, 1, -2$. To determine the multiplicities a, b, c, use the fact that for every $j \geq 0$, $a \cdot 3^j + b \cdot 1^j + c \cdot (-2)^j = \mathrm{trace} P^j$. For $j = 0$, $j = 1$, and $j = 2$, $\mathrm{trace} P^j$ is 10, 0, and 30, respectively, and the resulting three equations in three unknowns yield $(a, b, c) = (1, 5, 4)$.

If K_{10} can be factored into three disjoint copies of the Peterson graph, then we can write $J - I = P_1 + P_2 + P_3$, where P_1, P_2, P_3 are adjacency matrices for the Petersen graph, under various numberings of the vertices by $1, \ldots, 10$. The vector $\overline{1}$ of all ones is an eigenvector for each P_i, and for each P_i there is a five-dimensional space of eigenvectors with eigenvalue 1 that is orthogonal to $\overline{1}$. Since the orthogonal complement of $\overline{1}$ has nine dimensions, P_1 and P_2 have a common eigenvector wv with eigenvalue 1. Being orthogonal to $\overline{1}$, its coordinates sum to 0. Letting both sides of the decomposition operate on it yields $-wv = Jwv - Iwv = \sum P_i wv = wv + wv + P_3 wv$. However, this says that wv is an eigenvector of P_3 with eigenvalue -3, which is impossible. (This result is a special case of a theorem of J. Bosák that no complete graph with fewer than 12 vertices has a decomposition into three spanning subgraphs of diameter 2.)

8.6.25. *If $F = G \square H$, where G and H are simple graphs of order at least 2, and every two nonadjacent vertices in F have exactly two common neighbors, then G and H are complete graphs.* Given distinct vertices $u, v \in V(G)$ and $x, y \in V(H)$, consider the vertices (u, x) and (v, y) in F. By the definition of the cartesian product, these vertices are nonadjacent. Hence they have two common neighbors. The only possible common neighbors are (u, y) and (v, x). The resulting 4-cycle implies that $uv \in E(G)$ and $xy \in E(H)$. Since these vertices were chosen arbitrarily, G and H are complete graphs.

8.6.26. The *subconstituents* of a graph are the induced subgraphs of the form $G[U]$, where $v \in V(G)$ and $U = N(v)$ or $U = \overline{N[v]}$. Vince [1989] defined G to be *superregular* if G has no vertices or if G is regular and every subconstituent of G is superregular. Let $\mathbf{S}$ be the class consisting of $\{aK_b\colon a, b \geq 0\}$ (disjoint unions of isomorphic cliques), $\{K_m \square K_m\colon m \geq 0\}$, C_5, and the complements of these graphs.

a) Every graph in $\mathbf{S}$ *is superregular and every disconnected superregular graph is in* $\mathbf{S}$. Each $G \in \mathbf{S}$ is regular and vertex-transitive; so it suffices to consider any $x \in V(G)$. By induction on $a + b$, we have superregularity for $G = aK_b$, since $G[N(x)] = K_{b-1}$ and $G[\overline{N}(x)] = (a-1)K_b$. For $G = K_m^2$, we also apply induction, since $G[N(x)] = 2K_{m-1}$ and $G[\overline{N}(x)] = K_{m-1}^2$. Finally, for $G = C_5$, $G[N(x)] = 2K_1$ and $G[\overline{N}(x)] = K_2$.

Now suppose that G is superregular and disconnected. If some component of G is not a complete graph, then we may choose vertices x, y, z such that y has distance two from x and z belongs to another component. This implies $d_{G[\overline{N}(x)]}(y) < k = d_{G[\overline{N}(x)]}(z)$, which contradicts the regularity

of $G[\overline{N}(x)]$. If every component of G is a clique, then regularity of G requires equal sizes. (Comment: In fact, every superregular graph is in $\mathbf{S}$, but the complete inductive proof of this requires several pages (Maddox [1996], West [1996])

b) Every superregular graph is strongly regular. If x and y are nonadjacent, then t-regularity of $G[\overline{N}(x)]$ implies that x and y have $k - t$ common neighbors. We have noted that adjacent pairs have s common neighbors. Hence G is strongly regular, with parameters $\lambda = s$ and $\mu = k - t$.

8.6.27. *Automorphisms and eigenvalues.*

a) A permutation σ is an automorphism of G if and only if the permutation matrix corresponding to σ commutes with the adjacency matrix of G; that is, $PA = AP$. Say that P is defined by letting position (j, i) be 1 if $\sigma(i) = j$. That is, $Pe_i = e_j$, where e_k is the kth canonical basis vector. Multiplication by P permutes rows, moving row i to become row j if $P_{j,i} = 1$. The inverse of a permutation matrix moves the rows back again. In order to move row j to become row i, we need $P^{-1}_{i,j} = 1$. That is, we have argued that $P^T P = I$, so $P^{-1} = P^T$.

Multiplication by a permutation matrix Q on the right permutes columns, moving column i to column j if $Q_{i,j} = 1$. To accomplish the renaming by σ on the adjacency matrix, we want to move row i to row j and column i to column j whenever $\sigma(i) = j$. This is accomplished by multiplying by P on the left and by P^T on the right. Thus $PAP^T = A$. Since we have argued that $P^T = P^{-1}$, we have $PA = AP$.

b) If x is an eigenvector of G for an eigenvalue λ of multiplicity 1, and P is the permutation matrix for an automorphism of G, then $Px = \pm x$. Part (a) yields $APx = PAx = P\lambda x = \lambda Px$. Thus Px is also an eigenvector for A with eigenvalue λ. Since λ has multiplicity 1, Px is a multiple of x. Since P merely permutes the entries in x, it cannot change the largest magnitude that appears in x. Hence the new multiple of x must be $\pm x$.

c) When every eigenvalue of G has multiplicity 1, every automorphism of G is an involution (repeating it yields the identity). Part (b) yields $P^2 x = x$ whenever x is an eigenvalue of multiplicity 1. If this is true for every eigenvalue, then when we express any vector w as a linear combination of eigenvalues, we obtain $P^2 w = w$. If $P^2 w = w$ for every vector w, then P^2 is the identity, and hence σ is an involution.

8.6.28. *Every graph has an odd dominating set, meaning a set whose intersection with every closed neighborhood has odd size.* The phrasing in the text, based on Problem 10197 in the *American Mathematical Monthly* (1992), is that lightswitch s_i changes the status of light l_j if and only if s_j changes l_i. Let G be the n-vertex graph with vertices v_i and v_j adjacent if and only switches s_i and s_j affect lights l_j and l_i.

Let S be the set of switches flipped an odd number of times; the flips of other vertices have no effect. Since also s_i changes l_i and l_i begins off, l_i is on at the end if and only if its closed neighborhood in G has an odd number of vertices in S. Thus, the problem is equivalent to finding an odd dominating set S. This form has a combinatorial proof by Gallai (see Lovász [1979], Exercise 5.17).

The proof using linear algebra is much shorter. Let B be the augmented adjacency matrix $A(G) + I$ (add 1 to each diagonal entry). If x is the incidence vector of S in a binary vector space, then Bx is the incidence vector of the set of lights on after flipping the switches at S (since arithmetic is modulo 2). The problem is to show that $1 \in T$, where $T = \{Bx \in \mathbb{Z}_2^n\}$. We prove the more general statement that if B is a symmetric binary matrix, with vector u along the diagonal, then $u \in T$. (see Problem 798, *Nieuw Archief voor Wiskunde* (4) 9 (1991), 117-118) The solution given in the *Monthly* (1993, p. 806) is as follows.

We show that every vector orthogonal to T is also orthogonal to u. This implies that u is in the orthogonal complement of the orthogonal complement of T and hence is in T itself. Thus we want to show that if $\sum_{i=1}^n x_i B_{i,j} = 0$ for all j, then $\sum_{i=1}^n x_i u_i = 0$, where all computation is modulo 2.

Multiplying the vector 0 by x yields $\sum_{j=1}^n \sum_{i=1}^n x_i B_{i,j} x_j = 0$. By the symmetry of B, the off-diagonal entries contribute nothing, and we obtain $\sum_{i=1}^n x_i^2 B_{i,i} = 0$. Since $x_i^2 = x_i$ in binary, this reduces to $0 = \sum_{i=1}^n x_i B_{i,i} = \sum_{i=1}^n x_i u_i$, which completes the proof.